14

Vector Algebra

TOPIC 1

Algebra and Modulus of Vector

01 Let $P_1, P_2, ..., P_{15}$ be 15 points on a circle. The number of distinct triangles formed by points $P_{\hat{i}}, P_{\hat{j}}$ and $P_{\hat{k}}$, such that $\hat{i}+\hat{j}+\hat{k} \neq 15$ is **[2021, 01 Sep. Shift-II]**

(a) 12 (b) 419 (c) 443 (d) 455

Ans. ***(c)***

$\hat{i}+\hat{j}+\hat{k}=15$

where, $\hat{i}=1, \hat{j}+\hat{k}=14$

$\Rightarrow (\hat{j}=2, \hat{k}=12), (\hat{j}=3, \hat{k}=11),$
$(\hat{j}=4, \hat{k}=10),$
$(\hat{j}=5, \hat{k}=9)(\hat{j}=6, \hat{k}=8)$ ... 5 ways

$\hat{i}=2, \hat{j}+\hat{k}=13$

$\Rightarrow (\hat{j}=3, \hat{k}=10), \ldots, (\hat{j}=6, \hat{k}=7)$ ... 4 ways

$\hat{i}=3, \hat{j}+\hat{k}=12$

$\Rightarrow (\hat{j}=4, \hat{k}=8), (\hat{j}=5, \hat{k}=7)$ ... 2 ways

$\hat{i}=4, \hat{j}+\hat{k}=11 \Rightarrow (\hat{j}=5, \hat{k}=6)$ ... 1 way

$\therefore$ Total = 12 ways

Then, number of possible triangles using vertices $P_{\hat{i}}, P_{\hat{j}}, P_{\hat{k}}$ such that $\hat{i}+\hat{j}+\hat{k} \neq 15$ is

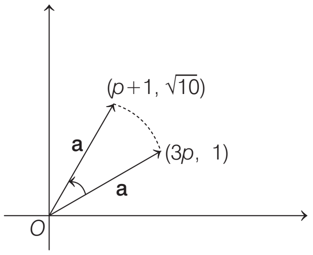
${}^{15}C_3 - 12 = 455 - 12 = 443$

02 A vector **a** has components $3p$ and 1 with respect to rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter clockwise sense. If, with respect to new system, **a** has components $p+1$ and $\sqrt{10}$, then a value of p is equal to **[2021, 18 March Shift-I]**

(a) 1 (b) $-\frac{5}{4}$ (c) $\frac{4}{5}$ (d) -1

Ans. ***(d)***

After counter clockwise (or anti-clockwise) rotation, the length of the vector **a** remains constant.



i.e. $|\mathbf{a}|$ at old position $= |\mathbf{a}|$ at new position

$\Rightarrow (3p)^2 + (1)^2 = (p+1)^2 + (\sqrt{10})^2$

$\Rightarrow 9p^2 + 1 = p^2 + 1 + 2p + 10$

$\Rightarrow 8p^2 - 2p - 10 = 0$

$\Rightarrow 4p^2 - p - 5 = 0$

$\Rightarrow (p+1)(4p-5) = 0$

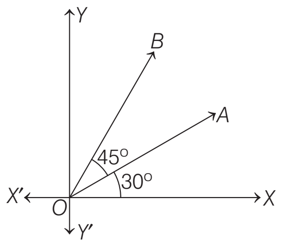
$\Rightarrow p = \frac{5}{4}, -1$

03 Let a vector $\alpha\hat{i}+\beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i}+\hat{j}$ by an angle 45° about the origin in counterclockwise direction in the first quadrant. Then the area of triangle having vertices $(\alpha, \beta), (0, \beta)$ and $(0, 0)$ is equal to **[2021, 16 March Shift-I]**

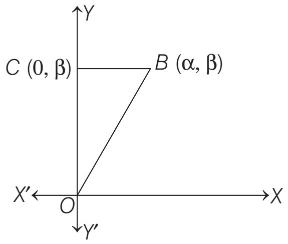
(a) $\frac{1}{2}$ (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $2\sqrt{2}$

Ans. ***(a)***

Let **OA** be $\sqrt{3}\hat{i}+\hat{j}$ and **OB** be $\alpha\hat{i}+\beta\hat{j}$.



As, we can notice in **OA**, $1/\sqrt{3} = \tan 30^\circ$.
So, it makes an angle of 30º with the X-axis.
Now, when **OA** is rotated further by 45ºa nticlockwise, the resultant vector **OB** makes an angle of 75º with the X-axis.
So, $\mathbf{OB} = |\mathbf{OA}|(\cos 75^\circ \hat{i} + \sin 75^\circ \hat{j})$



Let ΔOBC be the required triangle whose area we have to determine.

Area of $\Delta OBC = (1/2) \times (\text{Base}) \times (\text{Height})$

$= 1/2 \times \beta \times \alpha$

$= \frac{1}{2}(2\sin 75^\circ)(2\cos 75^\circ)$

$= 2\sin 75^\circ \cos 75^\circ$

$= \sin 150^\circ$

$= \sin 30^\circ$

$= 1/2$

Hence, the area is 1/2 sq. unit.

04 If vectors $\mathbf{a}_1 = x\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a}_2 = \hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ are collinear, then a possible unit vector parallel to the vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is

[2021, 26 Feb. Shift-II]

(a) $\frac{1}{\sqrt{2}}(-\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(c) $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ (d) $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$

Ans. *(d)*

Given, $\mathbf{a}_1 = x\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a}_2 = \hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ are collinear, then $\frac{x}{1} = \frac{-1}{y} = \frac{1}{z} = \lambda$ (Say)

This gives $x = \lambda$, $y = -\frac{1}{\lambda}$, $z = \frac{1}{\lambda}$

Then, unit vector parallel to vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ will be

$$= \frac{\pm\left((\lambda)\hat{\mathbf{i}} - \left(\frac{1}{\lambda}\right)\hat{\mathbf{j}} + \left(\frac{1}{\lambda}\right)\hat{\mathbf{k}}\right)}{\sqrt{(\lambda)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2}}$$

$$= \pm\frac{(\lambda^2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})\lambda}{\lambda\sqrt{\lambda^4 + 2}} = \pm\frac{(\lambda^2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{\lambda^4 + 2}}$$

Take, $\lambda = 1$ $\quad = \frac{\pm(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{3}}$

05 If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with $\hat{\mathbf{i}}$, $\frac{\pi}{4}$ with $\hat{\mathbf{j}}$ and $\theta \in (0, \pi)$ with $\hat{\mathbf{k}}$, then a value of θ is

[2019, 9 April Shift-II]

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{12}$ (d) $\frac{2\pi}{3}$

Ans. *(d)*

Given unit vector $\mathbf{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{\mathbf{i}}$, $\frac{\pi}{4}$ with $\hat{\mathbf{j}}$ and $\theta \in (0, \pi)$ with $\hat{\mathbf{k}}$.

Now, we know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, where α, β, γ are angles made by the vectors with respectively $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

$\therefore \cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{4}\right) + \cos^2\theta = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$

$\Rightarrow \cos\theta = \cos\left(\frac{\pi}{3}\right)$ or $\cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

So, θ is $\frac{2\pi}{3}$, according to options.

06 Let $\alpha = (\lambda - 2)\mathbf{a} + \mathbf{b}$ and $\beta = (4\lambda - 2)\mathbf{a} + 3\mathbf{b}$ be two given vectors where vectors $\mathbf{a}$ and $\mathbf{b}$ are non- collinear. The value of λ for which vectors α and β are collinear, is **[2019, 10 Jan. Shift-II]**

(a) 4 (b) -3

(c) 3 (d) -4

Ans. *(d)*

Two vectors $\mathbf{c}$ and $\mathbf{d}$ are said to be collinear if we can write $\mathbf{c} = \lambda\mathbf{b}$ for some non-zero scalar λ.

Let the vectors $\alpha = (\lambda - 2)\mathbf{a} + \mathbf{b}$ and $\beta = (4\lambda - 2)\mathbf{a} + 3\mathbf{b}$ are collinear, where $\mathbf{a}$ and $\mathbf{b}$ are non-collinear.

$\therefore$ We can write $\alpha = k\beta$, for some $k \in R - \{0\}$

$\Rightarrow (\lambda - 2)\mathbf{a} + \mathbf{b} = k[(4\lambda - 2)\mathbf{a} + 3\mathbf{b}]$

$\Rightarrow [(\lambda - 2) - k(4\lambda - 2)]\mathbf{a} + (1 - 3k)\mathbf{b} = 0$

Now, as a and b are non-collinear, therefore they are linearly independent and hence

$(\lambda - 2) - k(4\lambda - 2) = 0$ and $1 - 3k = 0$

$\Rightarrow \lambda - 2 = k(4\lambda - 2)$ and $3k = 1$

$\Rightarrow \lambda - 2 = \frac{1}{3}(4\lambda - 2)$ $\left[\because 3k = 1 \Rightarrow k = \frac{1}{3}\right]$

$\Rightarrow 3\lambda - 6 = 4\lambda - 2 \Rightarrow \lambda = -4$

07 Let $\sqrt{3}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} + \sqrt{3}\,\hat{\mathbf{j}}$ and $\beta\hat{\mathbf{i}} + (1 - \beta)\hat{\mathbf{j}}$ respectively be the position vectors of the points A, B and C with respect to the origin O. If the distance of C from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$, then the sum of all possible values of β is **[2019, 11 Jan. Shift-II]**

(a) 1

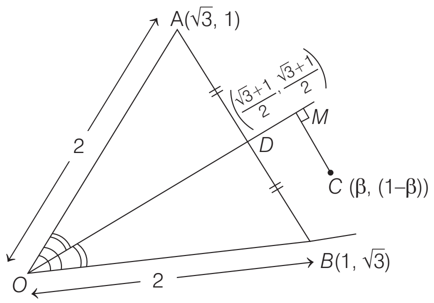
(b) 3

(c) 4

(d) 2

Ans. *(a)*

According to given information, we have the following figure.



Clearly, angle bisector divides the sides AB in $OA : OB$, i.e., $2 : 2 = 1 : 1$

[using angle bisector theorem]

So, D is the mid-point of AB and hence coordinates of D are $\left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}+1}{2}\right)$

Now, equation of bisector OD is

$$(y - 0) = \left(\frac{\frac{\sqrt{3}+1}{2} - 0}{\frac{\sqrt{3}+1}{2} - 0}\right)(x - 0) \Rightarrow y = x$$

$\Rightarrow x - y = 0$

According to the problem,

$$\frac{3}{\sqrt{2}} = CM = \left|\frac{\beta - (1 - \beta)}{\sqrt{2}}\right|$$

[Distance of a point $P(x_1, y_1)$ from the line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$]

$\Rightarrow |2\beta - 1| = 3 \Rightarrow 2\beta = \pm 3 + 1$

$\Rightarrow 2\beta = 4, -2 \Rightarrow \beta = 2, -1$

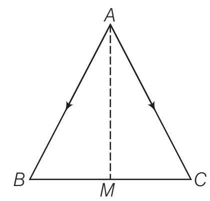
Sum of 2 and -1 is 1.

08 If the vectors $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of a ΔABC, then the length of the median through A is **[JEE Main 2013, 2003]**

(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{45}$

Ans. *(c)*

We know that, the sum of three vectors of a triangle is zero.



$\therefore \mathrm{AB} + \mathrm{BC} + \mathrm{CA} = 0$

$\Rightarrow \mathrm{BC} = \mathrm{AC} - \mathrm{AB}$ $\quad [\because \mathrm{AC} = -\mathrm{CA}]$

$\Rightarrow \mathrm{BM} = \frac{\mathrm{AC} - \mathrm{AB}}{2}$

[$\because M$ is a mid-point of BC]

Also, $\mathrm{AB} + \mathrm{BM} + \mathrm{MA} = 0$

[by properties of a triangle]

$\Rightarrow \mathrm{AB} + \frac{\mathrm{AC} - \mathrm{AB}}{2} = \mathrm{AM}$ $\quad [\because \mathrm{AM} = -\mathrm{MA}]$

$\Rightarrow \mathrm{AM} = \frac{\mathrm{AB} + \mathrm{AC}}{2}$

$$= \frac{3\hat{i} + 4\hat{k} + 5\hat{i} - 2\hat{j} + 4\hat{k}}{2}$$

$= 4\hat{i} - \hat{j} + 4\hat{k}$

$\Rightarrow |\mathrm{AM}| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

09 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a}+3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b}+2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a}+3\mathbf{b}+6\mathbf{c}$ is **[AIEEE 2011]**

(a) $\mathbf{a}+\mathbf{c}$ (b) $\mathbf{a}$ (c) $\mathbf{c}$ (d) $\mathbf{0}$

Ans. *(d)*

As, $a+3b$ is collinear with c.

$\therefore \quad a+3b=\lambda c$ ...(i)

Also, $b+2c$ is collinear with a.

$\Rightarrow \quad b+2c=\mu a$ ...(ii)

From Eq. (i), we get

$a+3b+6c=(\lambda+6)c$ ...(iii)

From Eq. (ii), we get

$a+3b+6c=(1+3\mu)a$ ...(iv)

From Eqs. (iii) and (iv), we get

$\therefore \quad (\lambda+6)c=(1+3\mu)a$

Since, a is not collinear with c.

$\Rightarrow \quad \lambda+6=1+3\mu=0$

From Eq. (iv), we get

$a+3b+6c=0$

10 The non-zero vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are related by $\mathbf{a}=8\mathbf{b}$ and $\mathbf{c}=-7\mathbf{b}$. Then, the angle between $\mathbf{a}$ and $\mathbf{c}$ is **[AIEEE 2008]**

(a) π (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

Ans. *(a)*

Since, $a=8b$ and $c=-7b$

So, a is parallel to b and c is anti-parallel to b.

$\Rightarrow$ a and c are anti-parallel.

So, the angle between a and c is π.

11 If C is the mid-point of AB and P is any point outside AB, then **[AIEEE 2005]**

(a) $\mathbf{PA}+\mathbf{PB}+\mathbf{PC}=\mathbf{0}$

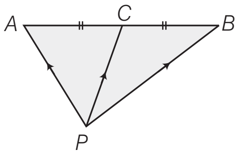
(b) $\mathbf{PA}+\mathbf{PB}+2\mathbf{PC}=\mathbf{0}$

(c) $\mathbf{PA}+\mathbf{PB}=\mathbf{PC}$

(d) $\mathbf{PA}+\mathbf{PB}=2\mathbf{PC}$

Ans. *(d)*

Let P be the origin outside of AB and C is mid-point of AB, then



$$PC=\frac{PA+PB}{2} \Rightarrow 2PC=PA+PB$$

12 If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors such that no two of these are collinear. If the vector $\mathbf{a}+2\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b}+3\mathbf{c}$ is collinear with $\mathbf{a}$ (λ being some non-zero scalar), then $\mathbf{a}+2\mathbf{b}+6\mathbf{c}$ equal to **[AIEEE 2004]**

(a) $\lambda\mathbf{a}$ (b) $\lambda\mathbf{b}$

(c) $\lambda\mathbf{c}$ (d) $\mathbf{0}$

Ans. *(d)*

If $a+2b$ is collinear with c, then

$a+2b=tc$ ...(i)

Also, $b+3c$ is collinear with a, then

$b+3c=\lambda a$

$\Rightarrow \quad b=\lambda a-3c$ ...(ii)

From Eqs. (i) and (ii), we get

$a+2(\lambda a-3c)=tc$

$\Rightarrow \quad (a-6c)=tc-2\lambda a$

On comparing the coefficients of $\mathbf{a}$ and c, we get

$1=-2\lambda \Rightarrow \lambda=-\frac{1}{2}$

and $\quad -6=t \quad \Rightarrow t=-6$

From Eq. (i), we get

$a+2b=-6c$

$\Rightarrow \; a+2b+6c=0$

13 Consider points A, B, C and D with position vectors $7\hat{\mathbf{i}}-4\hat{\mathbf{j}}+7\hat{\mathbf{k}}$, $\hat{\mathbf{i}}-6\hat{\mathbf{j}}+10\hat{\mathbf{k}}$, $-\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and $5\hat{\mathbf{i}}-\hat{\mathbf{j}}+5\hat{\mathbf{k}}$, respectively. Then, $ABCD$ is a **[AIEEE 2003]**

(a) square (b) rhombus

(c) rectangle (d) None of these

Ans. *(d)*

Given that, $OA=7\hat{i}-4\hat{j}+7\hat{k}$,

$OB=\hat{i}-6\hat{j}+10\hat{k}$,

$OC=-\hat{i}-3\hat{j}+4\hat{k}$

and $\quad OD=5\hat{i}-\hat{j}+5\hat{k}$

Now, $AB=\sqrt{(7-1)^2+(-4+6)^2+(7-10)^2}$

$=\sqrt{36+4+9}=\sqrt{49}=7$

$BC=\sqrt{(1+1)^2+(-6+3)^2+(10-4)^2}$

$=\sqrt{4+9+36}=\sqrt{49}=7$

$CD=\sqrt{(-1-5)^2+(-3+1)^2+(4-5)^2}$

$=\sqrt{36+4+1}=\sqrt{41}$

and $\quad DA=\sqrt{(5-7)^2+(-1+4)^2+(5-7)^2}$

$=\sqrt{4+9+4}=\sqrt{17}$

Hence, option (d) is correct.

14 If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$ and vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non- coplanar, then the product abc equal to **[AIEEE 2003]**

(a) 2 (b) −1 (c) 1 (d) 0

Ans. *(b)*

Since,

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}+\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}+abc\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad (1+abc)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}=0$$

$$\left[\because \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}\neq 0\right]$$

$\Rightarrow \quad 1+abc=0$

$\Rightarrow \quad abc=-1$

15 The vector $\hat{\mathbf{i}}+x\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{\mathbf{i}}+(4x-2)\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. The value of x are **[AIEEE 2002]**

(a) $\left\{-\frac{2}{3}, 2\right\}$ (b) $\left(\frac{1}{3}, 2\right)$

(c) $\left\{\frac{2}{3}, 0\right\}$ (d) $\{2, 7\}$

Ans. *(a)*

Since, the vector $\hat{i}+x\hat{j}+3\hat{k}$ is doubled in magnitude, then it becomes

$4\hat{i}+(4x-2)\hat{j}+2\hat{k}$

$\therefore \quad 2|\hat{i}+x\hat{j}+3\hat{k}|=|4\hat{i}+(4x-2)\hat{j}+2\hat{k}|$

$\Rightarrow \quad 2\sqrt{1+x^2+9}=\sqrt{16+(4x-2)^2+4}$

$\Rightarrow \quad 40+4x^2=20+(4x-2)^2$

$\Rightarrow \quad 3x^2-4x-4=0$

$\Rightarrow \quad (x-2)(3x+2)=0$

$\Rightarrow \quad x=2, -\frac{2}{3}$

TOPIC 2

Scalar or Dot Product of Two Vectors and Its Applications

16 Let **a** and **b** be two vectors such that $|2\mathbf{a}+3\mathbf{b}| = |3\mathbf{a}+\mathbf{b}|$ and the angle between **a** and **b** is 60°. If $\frac{1}{8}\mathbf{a}$ is a unit vector, then $|\mathbf{b}|$ is equal to **[2021, 31 Aug. Shift-I]**

(a) 4 (b) 6
(c) 5 (d) 8

Ans. *(c)*

$$|2\mathbf{a}+3\mathbf{b}| = |3\mathbf{a}+\mathbf{b}|$$

$\Rightarrow \quad |2\mathbf{a}+3\mathbf{b}|^2 = |3\mathbf{a}+\mathbf{b}|^2$

$\Rightarrow 4|\mathbf{a}|^2 + 9|\mathbf{b}|^2 + 12\mathbf{a}\cdot\mathbf{b} = 9|\mathbf{a}|^2 + |\mathbf{b}|^2 + 6\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow 5|\mathbf{a}|^2 - 6\mathbf{a}\cdot\mathbf{b} - 8|\mathbf{b}|^2 = 0$

$\frac{\mathbf{a}}{8}$ is a unit vector.

So, $|\mathbf{a}| = 8$

And, $5\cdot 64 - 6\cdot 8|\mathbf{b}|\left(\frac{1}{2}\right) - 8|\mathbf{b}|^2 = 0$

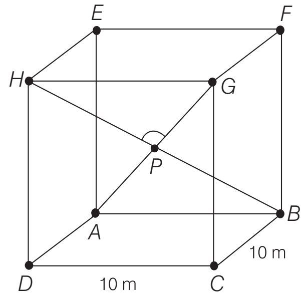
$\Rightarrow \quad |\mathbf{b}|^2 + 3|\mathbf{b}| - 40 = 0$

$\Rightarrow \quad (|\mathbf{b}|+8)(|\mathbf{b}|-5) = 0$

$|\mathbf{b}| = 5$

As, $|\mathbf{b}| = -8$ Not possible.

17 A hall has a square floor of dimension 10m × 10m (see the figure) and vertical walls. If the $\angle GPH$ between the diagonals AG and BH is $\cos^{-1}\frac{1}{5}$, then the height of the hall (in m) is **[2021, 26 Aug. Shift-II]**



(a) 5 (b) $2\sqrt{10}$
(c) $5\sqrt{3}$ (d) $5\sqrt{2}$

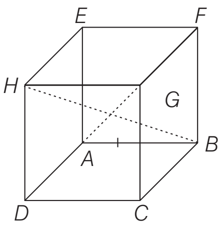
Ans. *(d)*

Let $A = (0, 0, 0)$

$B = (10, 0, 0)$

$G = (10, 10, h)$

$H = (0, 10, h)$



$\mathbf{AG} = 10\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + h\hat{\mathbf{k}}$

$\mathbf{BH} = -10\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + h\hat{\mathbf{k}}$

Since, $\mathbf{AG}\cdot\mathbf{BH} = |\mathbf{AG}||\mathbf{BH}|\cos\theta$

$(-100 + 100 + h^2)$

$= \sqrt{h^2+200}\cdot\sqrt{h^2+200}.\left(\frac{1}{5}\right)$

$\Rightarrow \quad 5h^2 = h^2 + 200 \Rightarrow h^2 = 50$

$\Rightarrow \quad h = 5\sqrt{2}$

18 If the projection of the vector $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ on the sum of the two vectors $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $-\lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is 1, then λ is equal to **[2021, 26 Aug. Shift-II]**

Ans. *(5)*

Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

$\mathbf{c} = -\lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Now, according to the questions,

$$\frac{\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})}{|\mathbf{b}+\mathbf{c}|} = 1$$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} = |\mathbf{b}+\mathbf{c}|$

$\Rightarrow 5 + (-\lambda + 4 + 3) = |(2-\lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}|$

$\Rightarrow \quad (12-\lambda)^2 = (2-\lambda)^2 + 36 + 4$

$\Rightarrow \lambda^2 + 144 - 24\lambda = \lambda^2 - 4\lambda + 4 + 40$

$\Rightarrow \quad \lambda = 5$

19 If $(\mathbf{a}+3\mathbf{b})$ is perpendicular to $(7\mathbf{a}-5\mathbf{b})$ and $(\mathbf{a}-4\mathbf{b})$ is perpendicular to $(7\mathbf{a}-2\mathbf{b})$, then the angle between **a** and **b** (in degrees) is .... **[2021, 25 July Shift-II]**

Ans. *(60)*

If $(\mathbf{a}+3\mathbf{b})$ is perpendicular to $(7\mathbf{a}-5\mathbf{b})$, then $(\mathbf{a}+3\mathbf{b})\cdot(7\mathbf{a}-5\mathbf{b}) = 0$

$\Rightarrow \quad 7|\mathbf{a}|^2 - 15|\mathbf{b}|^2 + 16\mathbf{a}\cdot\mathbf{b} = 0 \quad$ ...(i)

Also, $(\mathbf{a}-4\mathbf{b})$ is perpendicular to $(7\mathbf{a}-2\mathbf{b})$.

$(\mathbf{a}-4\mathbf{b})\cdot(7\mathbf{a}-2\mathbf{b}) = 0$

$\Rightarrow \quad 7|\mathbf{a}|^2 + 8|\mathbf{b}|^2 - 30\mathbf{a}\cdot\mathbf{b} = 0 \quad$ ...(ii)

Eq. (i) × 15 and Eq. (ii) × 8,

$105|\mathbf{a}|^2 - 225|\mathbf{b}|^2 + 240\,\mathbf{a}\cdot\mathbf{b} = 0$

$56|\mathbf{a}|^2 + 64|\mathbf{b}|^2 - 240\mathbf{a}\cdot\mathbf{b} = 0$

$161|\mathbf{a}|^2 - 161|\mathbf{b}|^2 = 0$

$\Rightarrow \quad |\mathbf{a}| = |\mathbf{b}| \quad$ ...(ii)

From (i),

$7|\mathbf{a}|^2 - 15|\mathbf{a}|^2 + 16\mathbf{a}\cdot\mathbf{b} = 0$

$\Rightarrow \quad 16\mathbf{a}\cdot\mathbf{b} = 8|\mathbf{a}|^2$

$\Rightarrow \quad 16|\mathbf{a}||\mathbf{b}|\cos\theta = 8|\mathbf{a}|^2$

$\Rightarrow \quad \cos\theta = \frac{8|\mathbf{a}|}{16|\mathbf{a}||\mathbf{b}|} \quad [\because |\mathbf{a}| = |\mathbf{b}|]$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = 60°$

20 Let a, b, c be three mutually perpendicular vectors of the same magnitude and equally inclined at an angle θ, with the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$. Then, $36\cos^2 2\theta$ is equal to .... **[2021, 20 July Shift-I]**

Ans. *(4)*

$\mathbf{a}\perp\mathbf{b}\perp\mathbf{c}$

$|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$

$\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) = |\mathbf{a}||\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}||\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

$|\mathbf{a}|^2 = |\mathbf{a}||\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

$|\mathbf{a}|(|\mathbf{a}| - |\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta) = 0$

As, $|\mathbf{a}| \neq 0$

So, $|\mathbf{a}| = |\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

Similarly, $|\mathbf{b}| = |\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

$|\mathbf{c}| = |\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

Now, $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{c}$

$= 3|\mathbf{a}|^2$

$\therefore |\mathbf{a}+\mathbf{b}+\mathbf{c}| = \sqrt{3}|\mathbf{a}| = \sqrt{3}|\mathbf{b}| = \sqrt{3}|\mathbf{c}|$

As, $\quad |\mathbf{a}| = |\mathbf{a}+\mathbf{b}+\mathbf{c}|\cos\theta$

$\Rightarrow \quad |\mathbf{a}| = \sqrt{3}|\mathbf{a}|\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{3}}$

$\therefore \quad \cos 2\theta = 2\cos^2\theta - 1$

$= 2\left(\frac{1}{3}\right) - 1 = \frac{-1}{3}$

So, $36\cos^2 2\theta = 36 \times \frac{1}{9} = 4$

21 For $p > 0$, a vector $\mathbf{V}_2 = 2\hat{\mathbf{i}} + (p+1)\hat{\mathbf{j}}$ is obtained by rotating the vector $\mathbf{v}_1 = \sqrt{3p}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$ by an angle θ about origin in counter clockwise direction. If $\tan\theta = \frac{(\alpha\sqrt{3}-2)}{(4\sqrt{3}+3)}$, then the value of α is equal to **[2021, 20 July Shift-II]**

Ans. *(6)*

Since, $\mathbf{V}_2$ is obtained by rotating $\mathbf{V}_1$.

$\therefore \quad |\mathbf{V}_1|^2 = |\mathbf{V}_2|^2$

$$\Rightarrow \quad 3p^2+1=4+(p+1)^2$$
$$\Rightarrow \quad 3p^2+1=4+p^2+1+2p$$
$$\Rightarrow \quad 2p^2-2p-4=0 \Rightarrow p^2-p-2=0$$
$$\Rightarrow \quad (p+1)(p-2)=0 \Rightarrow p=-1 \text{ or } p=2$$

Since, $p>0$ given, then $p=-1$ is discarde(d)

Now, $\cos\theta=\dfrac{\mathbf{V}_1\cdot\mathbf{V}_2}{|\mathbf{V}_1||\mathbf{V}_2|}$

$$=\frac{4\sqrt{3}+3}{\sqrt{13}\cdot\sqrt{13}}=\frac{4\sqrt{3}+3}{13}$$

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{(4\sqrt{3}+3)^2}{(13)^2}}$$

$$=\frac{\sqrt{(13)^2-(4\sqrt{3}+3)^2}}{13}$$

$$\Rightarrow \quad \sin\theta=\frac{\sqrt{112-24\sqrt{3}}}{13}$$

$$\therefore \quad \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sqrt{112-24\sqrt{3}}}{4\sqrt{3}+3}$$

Given, $\tan\theta=\dfrac{\alpha\sqrt{3}-2}{4\sqrt{3}+3}$

$$\therefore \quad \sqrt{112-24\sqrt{3}}=\alpha\sqrt{3}-2$$
$$\Rightarrow \quad 6\sqrt{3}-2=\alpha\sqrt{3}-2$$
$$\therefore \quad \alpha=6$$

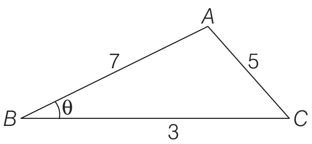
22 In ΔABC, if $|\mathbf{BC}|=3$, $|\mathbf{CA}|=5$ and $|\mathbf{BA}|=7$, then the projection of the vector **BA** on **BC** is equal to

[2021, 20 July Shift-II]

(a) $\dfrac{19}{2}$ (b) $\dfrac{13}{2}$
(c) $\dfrac{11}{2}$ (d) $\dfrac{15}{2}$

Ans. ***(c)***

Given, ΔABC, $|\mathbf{BC}|=3$, $|\mathbf{CA}|=5$, $|\mathbf{BA}|=7$



Now, projection of **BA** on

$\mathbf{BC}=|\mathbf{BA}|\cos\angle ABC$

Now, $\cos\angle ABC=\dfrac{(7)^2+(3)^2-(5)^2}{2\cdot 7\cdot 3}$

$$=\frac{49+9-25}{42}$$
$$=\frac{33}{42}=\frac{11}{14}$$

$\therefore$ Projection $=|\mathbf{BA}|\cdot\dfrac{11}{14}$

$$=7\times\frac{11}{14}=\frac{11}{2}$$

23 In a triangle ABC, if $|\mathbf{BC}|=8$, $|\mathbf{CA}|=7$, $|\mathbf{AB}|=10$, then the projection of the vector **AB** on **AC** is equal to

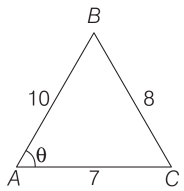
[2021, 18 March Shift-II]

(a) $\dfrac{25}{4}$ (b) $\dfrac{85}{14}$ (c) $\dfrac{127}{20}$ (d) $\dfrac{115}{16}$

Ans. ***(b)***

Projection of AB or $AC=AB\cos\theta$

$$=10\cdot\cos\theta$$



$$=10.\left(\frac{10^2+7^2-8^2}{2\times 7\times 10}\right)=\frac{85}{14}$$
$$\left(\text{using } \cos\theta=\frac{c^2+b^2-a^2}{2bc}\right)$$

24 Let **x** be a vector in the plane containing vectors $\mathbf{a}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}$. If the vector **x** is perpendicular to $(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})$ and its projection on **a** is $\dfrac{17\sqrt{6}}{2}$, then the value of $|\mathbf{x}|^2$ is equal to ....

[2021, 17 March Shift-II]

Ans. ***(486)***

Let $\mathbf{x}=\lambda\mathbf{a}+\mu\mathbf{b}$, where λ and μ are scalars.

$$\Rightarrow \quad \mathbf{x}=\lambda(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})+\mu(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})$$
$$\mathbf{x}=\hat{\mathbf{i}}\,(2\lambda+\mu)+\hat{\mathbf{j}}(2\mu-\lambda)+\hat{\mathbf{k}}(\lambda-\mu)$$

Since, **x** is perpendicular to $(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})$.

Then, $\mathbf{x}\cdot(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})=0$

$$\Rightarrow \quad 3\lambda+8\mu=0 \quad \text{...(i)}$$

Also, given projection of **x** on **a** is $\dfrac{17\sqrt{6}}{2}$.

$$\therefore \quad \frac{\mathbf{x}\cdot\mathbf{a}}{|\mathbf{a}|}=\frac{17\sqrt{6}}{2}$$

$$\Rightarrow 2(2\lambda+\mu)+(\lambda-2\mu)+(\lambda-\mu)=51$$
$$\Rightarrow \quad 6\lambda-\mu=51 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\lambda=8, \mu=-3$$
$$\therefore \quad \mathbf{x}=13\hat{\mathbf{i}}-14\hat{\mathbf{j}}+11\hat{\mathbf{k}}$$
$$\Rightarrow \quad |\mathbf{x}|=\sqrt{(13)^2+(-14)^2+(11)^2}$$
$$\therefore \quad |\mathbf{x}|=(13)^2+(-14)^2+(11)^2=486$$

25 **a**, **b** and **c** be three unit vectors such that $|\mathbf{a}-\mathbf{b}|^2+|\mathbf{a}-\mathbf{c}|^2=8$. Then $|\mathbf{a}+2\mathbf{b}|^2+|\mathbf{a}-2\mathbf{c}|^2$ is equal to ....

[2020, 2 Sep. Shift-I]

Ans. ***(2)***

Given, for unit vectors **a**, **b** and **c**

$$|\mathbf{a}-\mathbf{b}|^2+|\mathbf{a}-\mathbf{c}|^2=8$$
$$\Rightarrow |\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2+|\mathbf{c}|^2-2\mathbf{a}\cdot\mathbf{c}=8$$
$$\Rightarrow \; -2[\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}]=4$$
$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=-2 \quad \text{...(i)}$$

Now, $|\mathbf{a}+2\mathbf{b}|^2+|\mathbf{a}+2\mathbf{c}|^2$

$$=|\mathbf{a}|^2+4|\mathbf{b}|^2+4\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2+4|\mathbf{c}|^2+4\mathbf{a}\cdot\mathbf{c}$$
$$=10+4[\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}]$$
$$=10+4(-2) \quad \text{[from Eq. (i)]}$$
$$=10-8=2$$

26 Let the position vectors of points 'A' and 'B' be $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, respectively. A point 'P' divides the line segment AB internally in the ratio $\lambda:1(\lambda>0)$. If O is the origin and $\mathbf{OB}\cdot\mathbf{OP}-3|\mathbf{OA}\times\mathbf{OP}|^2=6$, then λ is equal to ....

[2020, 2 Sep. Shift-II]

Ans. ***(0.80)***

It is given that $\mathbf{OA}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\quad \mathbf{OB}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

$\therefore$ Point 'P' divides line segment AB internally in the ratio $\lambda:1,(\lambda>0)$, then

$$\mathbf{OP}=\frac{(2\lambda+1)\,\hat{\mathbf{i}}+(\lambda+1)\hat{\mathbf{j}}+(3\lambda+1)\,\hat{\mathbf{k}}}{\lambda+1}$$

$$\therefore \quad \mathbf{OB}\cdot\mathbf{OP}=\frac{1}{\lambda+1}(14\lambda+6)$$

and $\mathbf{OA}\times\mathbf{OP}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{i}} & \hat{\mathbf{k}}\\ 1 & 1 & 1\\ \dfrac{2\lambda+1}{\lambda+1} & 1 & \dfrac{3\lambda+1}{\lambda+1}\end{vmatrix}$

$$=\frac{2\lambda}{\lambda+1}\hat{\mathbf{i}}-\frac{\lambda}{\lambda+1}\hat{\mathbf{j}}-\frac{\lambda}{\lambda+1}\hat{\mathbf{k}}$$

$$\therefore \; |\mathbf{OP}\times\mathbf{OP}|=\frac{\lambda}{\lambda+1}\sqrt{6}$$

$$\Rightarrow |\mathbf{OA}\times\mathbf{OP}|^2=\frac{6\lambda^2}{(\lambda+1)^2}$$

It is given that,

$$\mathbf{OB}\cdot\mathbf{OP}-3|\mathbf{OA}\times\mathbf{OP}|^2=6$$

$$\Rightarrow \quad \frac{14\lambda+6}{\lambda+1}-\frac{18\lambda^2}{(\lambda+1)^2}=6$$

$$\Rightarrow 14\lambda^2+20\lambda+6-18\lambda^2=6\lambda^2+12\lambda+6$$
$$\Rightarrow \quad 10\lambda^2-8\lambda=0 \Rightarrow \lambda=0 \text{ or } 0.8$$
$$\because \; \lambda>0, \quad \therefore \; \lambda=0.8$$

Hence, answer is 0.80

27 Let $a,b,c \in R$ be such that $a^2+b^2+c^2=1$. If $a\cos\theta = b\cos\left(\theta+\frac{2\pi}{3}\right)$ $= c\cos\left(\theta+\frac{4\pi}{3}\right)$, where $\theta=\frac{\pi}{9}$, then the angle between the vectors $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$ and $b\hat{\mathbf{i}}+c\hat{\mathbf{j}}+a\hat{\mathbf{k}}$ is

[2020, 3 Sep. Shift-II]

(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{9}$

Ans. *(a)*

It is given that,

$$a\cos\theta = b\cos\left(\theta+\frac{2\pi}{3}\right) = c\cos\left(\theta+\frac{4\pi}{3}\right) = k \text{ (let)}$$

$$\Rightarrow \quad a=\frac{k}{\cos\theta}, b=\frac{k}{\cos\left(\theta+\frac{2\pi}{3}\right)}$$

and $$c=\frac{k}{\cos\left(\theta+\frac{4\pi}{3}\right)} \quad \text{...(i)}$$

Now angle between vectors $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$ and $b\hat{\mathbf{i}}+c\hat{\mathbf{j}}+a\hat{\mathbf{k}}$, where $a^2+b^2+c^2=1$ for $a,b,c\in R$ is $\alpha=\cos^{-1}|(ab+bc+ca)|$

$\because ab+bc+ca=k^2$

$$\left[\frac{1}{\cos\theta\cos\left(\theta+\frac{2\pi}{3}\right)}+\frac{1}{\cos\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)}+\frac{1}{\cos\left(\frac{4\pi}{3}+\theta\right)\cos\theta}\right]$$

$$=k^2\left[\frac{\cos\left(\frac{4\pi}{3}+\theta\right)+\cos\left(\frac{2\pi}{3}+\theta\right)+\cos\theta}{\cos\theta\cos\left(\frac{2\pi}{3}+\theta\right)\cos\left(\frac{4\pi}{3}+\theta\right)}\right]$$

$$=k^2\frac{2\cos(\pi+\theta)\cos\left(\frac{\pi}{3}\right)+\cos\theta}{\cos\theta\cos\left(\frac{2\pi}{3}+\theta\right)\cos\left(\frac{4\pi}{3}+\theta\right)}$$

$$=k^2\frac{-\cos\theta+\cos\theta}{\cos\theta\cos\left(\frac{2\pi}{3}+\theta\right)\cos\left(\frac{4\pi}{3}+\theta\right)}=0$$

$\therefore \alpha=\cos^{-1}(0)=\pi/2$

Hence, option (a) is correct.

28 Let the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ be such that $|\mathbf{a}|=2, |\mathbf{b}|=4$ and $|\mathbf{c}|=4$. If the projection of $\mathbf{b}$ on $\mathbf{a}$ is equal to the projection of $\mathbf{c}$ on $\mathbf{a}$ and $\mathbf{b}$ is perpendicular to $\mathbf{c}$, then the value of $|\mathbf{a}+\mathbf{b}-\mathbf{c}|$ is ....

[2020, 5 Sep. Shift-II]

Ans. *(6.00)*

It is given that projection of $\mathbf{b}$ on $\mathbf{a}$ is equal to the projection of $\mathbf{c}$ on $\mathbf{a}$, where $|\mathbf{a}|=2$, $|\mathbf{b}|=4$ and $|\mathbf{c}|=4$,

so $\frac{\mathbf{a}\cdot\mathbf{b}}{2}=\frac{\mathbf{a}\cdot\mathbf{c}}{2} \Rightarrow \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$

and $\mathbf{b}$ is perpendicular to $\mathbf{c}$, so $\mathbf{b}\cdot\mathbf{c}=0$

Now, $|\mathbf{a}+\mathbf{b}-\mathbf{c}|^2$

$=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{c}-2\mathbf{a}\cdot\mathbf{c}$

$=4+16+16=36$

$\therefore |\mathbf{a}+\mathbf{b}-\mathbf{c}|=6.$

29 If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then the greatest value of $\sqrt{3}|\mathbf{a}+\mathbf{b}|+|\mathbf{a}-\mathbf{b}|$ is ....

[2020, 6 Sep. Shift-I]

Ans. *(4.00)*

Let angle between unit vectors $\mathbf{a}$ and $\mathbf{b}$ is $\theta\in[0,\pi]$.

Then, $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b}$

$=1+1+2\cos\theta=2(1+\cos\theta)$

$=4\cos^2\frac{\theta}{2}$

$$\Rightarrow \quad |\mathbf{a}+\mathbf{b}|=2\cos\left(\frac{\theta}{2}\right)$$

$$\left[\because \theta\in[0,\pi]\Rightarrow\cos\left(\frac{\theta}{2}\right)\geq 0\right]$$

Similarly, $|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}$

$=1+1-2\cos\theta$

$=2(1-\cos\theta)=4\sin^2\frac{\theta}{2}$

$$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|=2\sin\left(\frac{\theta}{2}\right)$$

So, $\sqrt{3}|\mathbf{a}+\mathbf{b}|+|\mathbf{a}-\mathbf{b}|$

$$=2\left[\sqrt{3}\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right]$$

having greatest value $=2\sqrt{3+1}=4$

[$\because$ greatest value of $a\cos\theta+b\sin\theta$ is $\sqrt{a^2+b^2}$]

30 If $\mathbf{x}$ and $\mathbf{y}$ be two non-zero vectors such that $|\mathbf{x}+\mathbf{y}|=|\mathbf{x}|$ and $2\mathbf{x}+\lambda\mathbf{y}$ is perpendicular to $\mathbf{y}$, then the value of λ is

[2020, 6 Sep. Shift-II]

Ans. *(1.00)*

For two non-zero vectors $\mathbf{x}$ and $\mathbf{y}$, it is given

$$|\mathbf{x}+\mathbf{y}|=|\mathbf{x}|$$

On squaring both sides, we get

$$|\mathbf{x}|^2+2\cdot\mathbf{x}\cdot\mathbf{y}+|\mathbf{y}|^2=|\mathbf{x}|^2$$

$$\Rightarrow \quad |\mathbf{y}|^2+2\mathbf{x}\cdot\mathbf{y}=0 \quad \text{...(i)}$$

Now, as $(2\mathbf{x}+\lambda\mathbf{y})$ is perpendicular to $\mathbf{y}$, so

$$(2\mathbf{x}+\lambda\mathbf{y})\cdot\mathbf{y}=0$$

$$\Rightarrow \quad 2\mathbf{x}\cdot\mathbf{y}+\lambda|\mathbf{y}|^2=0 \quad \text{...(ii)}$$

From Eqs.(i) and (ii), we get

$$\lambda=1$$

31 A vector $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$ $(\alpha,\beta\in\mathbf{R})$ lies in the plane of the vectors, $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$. If $\mathbf{a}$ bisects the angle between $\mathbf{b}$ and $\mathbf{c}$, then

[2020, 7 Jan. Shift-I]

(a) $\mathbf{a}\cdot\hat{\mathbf{i}}+3=0$ (b) $\mathbf{a}\cdot\hat{\mathbf{k}}+2=0$

(c) $\mathbf{a}\cdot\hat{\mathbf{i}}+1=0$ (d) $\mathbf{a}\cdot\hat{\mathbf{k}}+4=0$

Ans. *(b)*

Given vectors $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$.

So, vector bisects the angle between $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{a}=\lambda\left(\frac{\mathbf{b}}{|\mathbf{b}|}\pm\frac{\mathbf{c}}{|\mathbf{c}|}\right)$$

$$\Rightarrow \mathbf{a}=\lambda\left(\frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}}{\sqrt{2}}\pm\frac{\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}}{\sqrt{18}}\right)$$

$$=\frac{\lambda}{3\sqrt{2}}((3\hat{\mathbf{i}}+3\hat{\mathbf{j}})\pm(\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}))$$

$$=\frac{\lambda}{3\sqrt{2}}(4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}})$$

$$=\frac{\lambda}{3\sqrt{2}}(2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-4\hat{\mathbf{k}})$$

On comparing with given $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\beta\hat{k}, (\alpha,\beta\in R)$

and $\frac{\lambda}{3\sqrt{2}}(4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}})$, we have $\lambda=3\sqrt{2}$,

so $\alpha=4$ and $\beta=4$

which not satisfy the given options.

Now, on comparing with $\frac{\lambda}{3\sqrt{2}}(2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-4\hat{\mathbf{k}})$

and $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$,

we have $\lambda=\frac{3\sqrt{2}}{2}$, so $\alpha=1$ and $\beta=-2$

Now, since $\mathbf{a}\cdot\hat{\mathbf{k}}=-2 \Rightarrow \mathbf{a}\cdot\hat{\mathbf{k}}+2=0$.

32 Let $A(3,0,-1)$, $B(2,10,6)$ and $C(1,2,1)$ be the vertices of a triangle and M be the mid-point of AC. If G divides BM in the ratio $2:1$, then $\cos(\angle GOA)$ (O being the origin) is equal to

[2019, 10 April Shift-I]

(a) $\frac{1}{\sqrt{15}}$ (b) $\frac{1}{2\sqrt{15}}$

(c) $\frac{1}{\sqrt{30}}$ (d) $\frac{1}{6\sqrt{10}}$

Ans. *(a)*

Key Idea Use the angle between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ and coordinates of the centroid i.e. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$ of a triangle formed with vertices; $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3).

Given vertices of a ΔABC are $A(3, 0, -1)$, $B(2, 10, 6)$ and $C(1, 2, 1)$ and a point M is mid-point of AC. An another point G divides BM in ratio 2 : 1, so G is the centroid of ΔABC.

$\therefore\ G\left(\frac{3+2+1}{3}, \frac{0+10+2}{3}, \frac{-1+6+1}{3}\right)$

$= (2, 4, 2)$.

Now, $\cos(\angle GOA) = \frac{\mathbf{OG}\cdot\mathbf{OA}}{|\mathbf{OG}||\mathbf{OA}|}$, where O is the origin.

$\because \quad \mathbf{OG} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{OG}| = \sqrt{4+16+4} = \sqrt{24}$

and $\quad |\mathbf{OA}| = 3\hat{\mathbf{i}} - \hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{OA}| = \sqrt{9+1} = \sqrt{10}$

and $\quad \mathbf{OG}\cdot\mathbf{OA} = 6 - 2 = 4$

$\therefore \quad \cos(\angle GOA) = \frac{4}{\sqrt{24}\sqrt{10}} = \frac{1}{\sqrt{15}}$

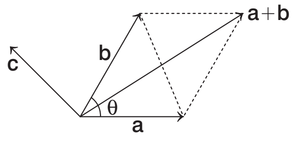
33 Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$, $\mathbf{b} = b_1\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ and $\mathbf{c} = 5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ be three vectors such that the projection vector of $\mathbf{b}$ on $\mathbf{a}$ is $\mathbf{a}$. If $\mathbf{a}+\mathbf{b}$ is perpendicular to $\mathbf{c}$, then $|\mathbf{b}|$ is equal to

[2019, 9 Jan. Shift-II]

(a) 6 (b) 4
(c) $\sqrt{22}$ (d) $\sqrt{32}$

Ans. *(a)*

According to given information, we have the following figure.



Clearly, projection of b on $a = \frac{b\cdot a}{|a|}$

$= \frac{(b_1\hat{i} + b_2\hat{j} + \sqrt{2}\hat{k})(\hat{i} + \hat{j} + \sqrt{2}\hat{k})}{\sqrt{1^2 + 1^2 + (\sqrt{2})^2}}$

$= \frac{b_1 + b_2 + 2}{\sqrt{4}} = \frac{b_1 + b_2 + 2}{2}$

But projection of b on a = |a|

$\therefore \quad \frac{b_1 + b_2 + 2}{2} = \sqrt{1^2 + 1^2 + (\sqrt{2})^2}$

$\Rightarrow \quad \frac{b_1 + b_2 + 2}{2} = 2 \Rightarrow b_1 + b_2 = 2 \quad$ ...(i)

Now,

$a + b = (\hat{i} + \hat{j} + \sqrt{2}\hat{k}) + (b_1\hat{i} + b_2\hat{j} + \sqrt{2}\hat{k})$

$= (b_1 + 1)\hat{i} + (b_2 + 1)\hat{j} + 2\sqrt{2}\hat{k}$

$\because (a+b) \perp c$, therefore $(a+b)\cdot c = 0$

$\Rightarrow \{(b_1+1)\hat{i} + (b_2+1)\hat{j} + 2\sqrt{2}\hat{k}\}$

$(5\hat{i} + \hat{j} + \sqrt{2}\hat{k}) = 0$

$\Rightarrow \quad 5(b_1 + 1) + 1(b_2 + 1) + 2\sqrt{2}(\sqrt{2}) = 0$

$\Rightarrow \quad 5b_1 + b_2 = -10 \quad$ ...(ii)

From Eqs. (i) and (ii), $b_1 = -3$ and $b_2 = 5$

$\Rightarrow \quad b = -3\hat{i} + 5\hat{j} + \sqrt{2}\hat{k}$

$\Rightarrow \quad |b| = \sqrt{(-3)^2 + (5)^2 + (\sqrt{2})^2}$

$= \sqrt{36} = 6$

34 Let $\mathbf{a} = 2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}$ be three vectors such that $\mathbf{b} = 2\mathbf{a}$ and $\mathbf{a}$ is perpendicular to $\mathbf{c}$. Then a possible value of $(\lambda_1, \lambda_2, \lambda_3)$ is

[2019, 10 Jan. Shift-I]

(a) (1, 3, 1) (b) (1, 5, 1)
(c) $\left(-\frac{1}{2}, 4, 0\right)$ (d) $\left(\frac{1}{2}, 4, -2\right)$

Ans. *(c)*

We have, $\mathbf{a} = 2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$;

$\mathbf{b} = 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

and $\quad \mathbf{c} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}$,

such that $\mathbf{b} = 2\mathbf{a}$

Now, $\quad \mathbf{b} = 2\mathbf{a}$

$\Rightarrow \quad 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}} = 2(2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$\Rightarrow \quad 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}} = 4\hat{\mathbf{i}} + 2\lambda_1\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

$\Rightarrow \quad (3 - 2\lambda_1 - \lambda_2)\hat{\mathbf{j}} = \mathbf{0}$

$\Rightarrow \quad 3 - 2\lambda_1 - \lambda_2 = 0$

$\Rightarrow \quad 2\lambda_1 + \lambda_2 = 3 \quad$ ...(i)

Also, as $\mathbf{a}$ is perpendicular to $\mathbf{c}$, therefore $\mathbf{a}\,.\,\mathbf{c} = 0$

$\Rightarrow (2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}) = 0$

$\Rightarrow \quad 6 + 6\lambda_1 + 3(\lambda_3 - 1) = 0$

$\Rightarrow \quad 6\lambda_1 + 3\lambda_3 + 3 = 0$

$\Rightarrow \quad 2\lambda_1 + \lambda_3 = -1 \quad$ ...(ii)

Now, from Eq. (i), $\lambda_2 = 3 - 2\lambda_1$ and from Eq. (ii)

$\lambda_3 = -2\lambda_1 - 1$

$\therefore \quad (\lambda_1, \lambda_2, \lambda_3) \equiv (\lambda_1, 3 - 2\lambda_1, -2\lambda_1 - 1)$

If $\quad \lambda_1 = -\frac{1}{2}$, then $\lambda_2 = 4$, and $\lambda_3 = 0$

Thus, a possible value of

$(\lambda_1, \lambda_2, \lambda_3) = \left(-\frac{1}{2}, 4, 0\right)$

35 Let $\mathbf{u}$ be a vector coplanar with the vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$. If $\mathbf{u}$ is perpendicular to $\mathbf{a}$ and $\mathbf{u}\cdot\mathbf{b} = 24$, then $|\mathbf{u}|^2$ is equal to

[JEE Main 2018]

(a) 336 (b) 315 (c) 256 (d) 84

Ans. *(a)*

Key idea *If any vector* x *is coplanar with the vector* y *and* z, *then* $x = \lambda y + \mu z$

Here, $\mathbf{u}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$.

$\therefore \quad u = \lambda\mathbf{a} + \mu\mathbf{b}$

Dot product with $\mathbf{a}$, we get

$u\cdot\mathbf{a} = \lambda(\mathbf{a}\cdot\mathbf{a}) + \mu(\mathbf{b}\cdot\mathbf{a})$

$\Rightarrow \quad 0 = 14\lambda + 2\mu \quad$ ...(i)

$[\because \mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{u}\cdot\mathbf{a} = 0]$

Dot product with $\mathbf{b}$, we get

$u\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) + \mu(\mathbf{b}\cdot\mathbf{b})$

$24 = 2\lambda + 2\mu \quad$ ...(ii)

$[\because u\cdot\mathbf{b} = 24]$

Solving Eqs. (i) and (ii), we get

$\lambda = -2, \mu = 14$

Dot product with $\mathbf{u}$, we get

$|u|^2 = \lambda(u\cdot\mathbf{a}) + \mu(u\cdot\mathbf{b})$

$|u|^2 = -2(0) + 14(24)$

$\Rightarrow \quad |u|^2 = 336$

36 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors. If the vectors $\mathbf{c} = a + 2\mathbf{b}$ and $\mathbf{d} = 5\mathbf{a} - 4\mathbf{a}$ are perpendicular to each other, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

[AIEEE 2012]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Ans. *(c)*

Given that,

(i) a and b are unit vectors, *i.e.*, $|a| = |b| = 1$

(ii) $c = a + 2b$ and $d = 5a - 4b$

(iii) c and d are perpendicular to each other. *i.e.*, $c\cdot d = 0$

To find Angle between a and b.

Now,

$c\cdot d = 0 \Rightarrow (a + 2b)\cdot(5a - 4b) = 0$

$\Rightarrow \quad 5a\cdot a - 4a\cdot b + 10b\cdot a - 8b\cdot b = 0$

$\Rightarrow \quad 6a\cdot b = 3$

$\Rightarrow \quad a\cdot b = \frac{1}{2}$

So, the angle between a and b is $\frac{\pi}{3}$.

37 Let *ABCD* be a parallelogram such that $\mathbf{AB}=\mathbf{q}$, $\mathbf{AD}=\mathbf{p}$ and $\angle BAD$ be an acute angle. If $\mathbf{r}$ is the vector that coincides with the altitude directed from the vertex *B* to the side *AD*, then $\mathbf{r}$ is given by **[AIEEE 2012]**

(a) $\mathbf{r}=3\mathbf{q}\ \dfrac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{p})}\mathbf{p}$

(b) $\mathbf{r}=-\mathbf{q}+\left(\dfrac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

(c) $\mathbf{r}=\mathbf{q}-\left(\dfrac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

(d) $\mathbf{r}=-3\mathbf{q}+\dfrac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{p})}\mathbf{p}$

Ans. *(b)*

Given

(i) A parallelogram *ABCD* such that AB = q and AD = p.

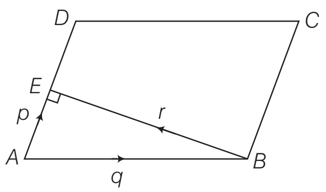
(ii) The altitude from vertex *B* to side *AD* coincides with a vector r.

To find The vector r in terms of p and q.

Let *E* be the foot of perpendicular from *B* to side *A(d)*

AE = Projection of vector q on

$$p=q\cdot p=\frac{q\cdot p}{|p|}$$



$$AE=\text{Vector along } AE \text{ of length } AE$$

$$=|AE|\,AE=\left(\frac{q\cdot p}{|p|}\right)p=\frac{(q\cdot p)p}{|p|^2}$$

Now, applying triangles law in ΔABE, we get

$$AB+BE=AE$$

$$\Rightarrow\quad q+r=\frac{(q\cdot p)p}{|p|^2}$$

$$\Rightarrow\quad r=\frac{(q\cdot p)p}{|p|^2}-q$$

$$\Rightarrow\quad r=-q+\left(\frac{q\cdot p}{p\cdot p}\right)p$$

38 If the vectors $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}=\lambda\hat{\mathbf{i}}+\hat{\mathbf{j}}+\mu\hat{\mathbf{k}}$ are mutually orthogonal, then (λ,μ) is equal to **[AIEEE 2010]**

(a) (– 3, 2) (b) (2, – 3)
(c) (– 2, 3) (d) (3, – 2)

Ans. *(a)*

Since, the given vectors are mutually orthogonal, therefore

$$a\cdot b=2-4+2=0$$

$$a\cdot c=\lambda-1+2\mu=0 \quad \text{...(i)}$$

and $$b\cdot c=2\lambda+4+\mu=0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$\mu=2 \text{ and } \lambda=-3$$

Hence, $(\lambda,\mu)=(-3,2)$

39 The value of *a*, for which the points, *A*, *B*, *C* with position vectors $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}-3\hat{\mathbf{j}}-5\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}$ respectively are the vertices of a right angled triangle with $C=\dfrac{\pi}{2}$ are **[AIEEE 2006]**

(a) –2 and –1 (b) –2 and 1
(c) 2 and –1 (d) 2 and 1

Ans. *(d)*

Since, position vectors of *A*, *B*, *C* are $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $a\hat{i}-3\hat{j}+\hat{k}$, respectively.

Now, $$AC=(a\hat{j}-3\hat{j}+\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$$
$$=(a-2)\hat{i}-2\hat{j}$$

and $$BC=(a\hat{i}-3\hat{j}+\hat{k})-(\hat{i}-3\hat{j}-5\hat{k})$$
$$=(a-1)\hat{i}+6\hat{k}$$

Since, the ΔABC is right angled at *C*, then

$$AC\cdot BC=0$$

$$\Rightarrow \{(a-2)\hat{i}-2\hat{j}\}\cdot\{(a-1)\hat{i}+6\hat{k}\}=0$$

$$\Rightarrow \quad (a-2)(a-1)=0$$

$$\therefore \quad a=1 \text{ and } a=2$$

40 A particle is acted upon by constant forces $4\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ which displace it from a point $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ to the point $5\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}}$. The work done in standard units by the forces is given by **[AIEEE 2004]**

(a) 40 units (b) 30 units
(c) 25 units (d) 15 units

Ans. *(a)*

Total force, $F=(4\hat{i}+\hat{j}-3\hat{k})+(3\hat{i}+\hat{j}-\hat{k})$

$$\therefore \quad F=7\hat{i}+2\hat{j}-4\hat{k}$$

The particle is displaced from $A(\hat{i}+2\hat{j}+3\hat{k})$ to $B(5\hat{i}+4\hat{j}+\hat{k})$.

Now, displacement,

$$AB=(5\hat{i}+4\hat{j}+k)-(\hat{i}+2\hat{j}+3\hat{k})$$
$$=4\hat{i}+2\hat{j}-2\hat{k}$$

$\therefore$ Work done $=F\cdot AB$

$$=(7\hat{i}+2\hat{j}-4\hat{k})\cdot(4\hat{i}+2\hat{j}-2\hat{k})$$
$$=28+4+8=40 \text{ units}$$

41 Let $\mathbf{u},\mathbf{v},\mathbf{w}$ be such that $|\mathbf{u}|=1$, $|\mathbf{v}|=2$, $|\mathbf{w}|=3$. If the projection $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v},\mathbf{w}$ are perpendicular to each other, then $|\mathbf{u}-\mathbf{v}+\mathbf{w}|$ equal to **[AIEEE 2004]**

(a) 2 (b) $\sqrt{7}$ (c) $\sqrt{14}$ (d) 14

Ans. *(c)*

Since, $|u|=1, |v|=2, |w|=3$

The projection of v along u $=\dfrac{v\cdot u}{|u|}$

and the projection of w along u $=\dfrac{w\cdot u}{|u|}$

According to given condition,

$$\frac{v\cdot u}{|u|}=\frac{w\cdot u}{|u|}$$

$$\Rightarrow \quad v\cdot u=w\cdot u \quad \text{...(i)}$$

Since, v, w are perpendicular to each other.

$$\therefore \quad v\cdot w=0 \quad \text{...(ii)}$$

Now, $$|u-v+w|^2=|u|^2+|v|^2+|w|^2-2u\cdot v-2v\cdot w+2u\cdot w$$

$$\Rightarrow |u-v+w|^2=1+4+9-2u\cdot v+2v\cdot u$$ [from Eqs. (i) and (ii)]

$$\Rightarrow |u-v+w|^2=1+4+9$$

$$\Rightarrow |u-v+w|=\sqrt{14}$$

42 $\mathbf{a},\mathbf{b},\mathbf{c}$ are three vectors, such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $|\mathbf{a}|=1$, $|\mathbf{b}|=2$, $|\mathbf{c}|=3$, then $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is equal to **[AIEEE 2003]**

(a) 0 (b) –7 (c) 7 (d) 1

Ans. *(b)*

Given that, $|a|=1, |b|=2, |c|=3$

and $a+b+c=0$

Now, $$(a+b+c)^2=|a|^2+|b|^2+|c|^2+2(a\cdot b+b\cdot c+c\cdot a)$$

$$\Rightarrow \quad 0=1^2+2^2+3^2+2(a\cdot b+b\cdot c+c\cdot a)$$

$$\Rightarrow \quad 2(a\cdot b+b\cdot c+c\cdot a)=-14$$

$$\Rightarrow \quad a\cdot b+b\cdot c+c\cdot a=-7$$

43 Given, two vectors are $\hat{\mathbf{i}}-\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}$, the unit vector coplanar with the two vectors and perpendicular to first is **[AIEEE 2002]**

(a) $\dfrac{1}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ (b) $\dfrac{1}{\sqrt{5}}(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$

(c) $\pm\dfrac{1}{\sqrt{2}}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$ (d) None of these

Ans. *(a)*

Given two vectors lie in *xy*-plane. So, a vector coplanar with them is

$$a = x\hat{i} + y\hat{j}$$

Since, $a \perp (\hat{i} - \hat{j})$

$\Rightarrow (x\hat{i} + y\hat{j}) \cdot (\hat{i} - \hat{j}) = 0$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

$\therefore \quad a = x\hat{i} + x\hat{j}$

and $\quad |a| = \sqrt{x^2 + x^2} = x\sqrt{2}$

$\therefore$ Required unit vector

$$= \frac{a}{|a|} = \frac{x(\hat{i} + \hat{j})}{x\sqrt{2}} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

TOPIC 3

Vector or Cross Product of Two Vectors and Its Applications

44 Let $\mathbf{a} = \hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$ and $\mathbf{c} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ be three vectors such that, $|\mathbf{b} \times \mathbf{c}| = 5\sqrt{3}$ and $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Then, the greatest amongst the values of $|\mathbf{a}|^2$ is **[2021, 27 Aug. Shift-I]**

Ans. *(90)*

Given, $\mathbf{a} = \hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}$

$\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$

and $\quad \mathbf{c} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$\because \quad \mathbf{a} \perp \mathbf{b} \Rightarrow \mathbf{a} \cdot \mathbf{b} = 0$

$\Rightarrow \quad (\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}) = 0$

$\Rightarrow \quad 1 + 15 + \alpha\beta = 0$

or $\quad \alpha\beta = -16 \quad$ ...(i)

Now, $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & \beta \\ -1 & 2 & -3 \end{vmatrix}$

$= \hat{\mathbf{i}}(-9 - 2\beta) - \hat{\mathbf{j}}(-3 + \beta) + \hat{\mathbf{k}}(2 + 3)$

$\mathbf{b} \times \mathbf{c} = \hat{\mathbf{i}}(-9 - 2\beta) + \hat{\mathbf{j}}(3 - \beta) + 5\hat{\mathbf{k}}$

Given, $\quad |\mathbf{b} \times \mathbf{c}| = 5\sqrt{3}$

$\Rightarrow \quad |\mathbf{b} \times \mathbf{c}|^2 = 75$

$\Rightarrow \quad (-9 - 2\beta)^2 + (3 - \beta)^2 + 25 = 75$

$\Rightarrow \quad \beta^2 + 6\beta + 8 = 0 \Rightarrow \beta = -2, -4$

From Eq. (i), we get

For $\beta = -2, \alpha = 8$

For $\beta = -4, \alpha = 4$

For maximum value of $|\mathbf{a}|^2, \alpha = 8$

$\therefore \quad |\alpha|^2 = 1 + 25 + 64 = 90$

45 If $\sum_{r=1}^{50} \tan^{-1}\frac{1}{2r^2} = p$, then the value of $\tan p$ is **[2021, 26 Aug. Shift-II]**

(a) $\frac{101}{102}$ (b) $\frac{50}{51}$

(c) 100 (d) $\frac{51}{50}$

Ans. *(b)*

Given, $\sum_{r=1}^{50} \tan^{-1}\left(\frac{1}{2r^2}\right) = p$

Now, $\Sigma \tan^{-1}\left(\frac{2}{1 + 4r^2 - 1}\right)$

$= \Sigma \tan^{-1}\left[\frac{(2r+1) - (2r-1)}{1 + (2r+1)(2r-1)}\right]$

$= \Sigma [\tan^{-1}(2r+1) - \tan^{-1}(2r-1)]$

$= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \ldots\ldots + \tan^{-1}101 - \tan^{-1}99$

$= \tan^{-1}(101) - \tan^{-1}1$

$= \tan^{-1}\left(\frac{101 - 1}{1 + 101}\right) = \tan^{-1}\left(\frac{50}{51}\right)$

$\therefore \quad \tan^{-1}\frac{50}{51} = p \Rightarrow \tan p = \frac{50}{51}$

46 Let $\mathbf{p} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{q} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ be two vectors. If a vector $\mathbf{r} = (\alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{j}} + \gamma\hat{\mathbf{k}})$ is perpendicular to each of the vectors $(\mathbf{p} + \mathbf{q})$ and $(\mathbf{p} - \mathbf{q})$, and $|\mathbf{r}| = \sqrt{3}$, then $|\alpha| + |\beta| + |\gamma|$ is equal to .... **[2021, 25 July Shift-I]**

Ans. *(3)*

$\mathbf{p} = (2, 3, 1), \mathbf{q} = (1, 2, 1)$

$\mathbf{r}$ is perpendicular to $\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - \mathbf{q}$.

$\mathbf{p} + \mathbf{q} = (3, 5, 2)$

$\mathbf{p} - \mathbf{q} = (1, 1, 0)$

$\mathbf{r}$ is parallel to $(\mathbf{p} + \mathbf{q}) \times (\mathbf{p} - \mathbf{q})$

$\Rightarrow \quad \begin{vmatrix} i & j & k \\ 3 & 5 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (-2, 2, -2)$

$\Rightarrow \quad \mathbf{r} = \lambda(-1, 1, -1)$

$\Rightarrow \quad |\mathbf{r}| = \sqrt{3}$

$\Rightarrow \quad \lambda\sqrt{1 + 1 + 1} = \sqrt{3}$

$\Rightarrow \quad \lambda = 1$

$\therefore \quad \mathbf{r} = (-1, 1, -1)$

$\alpha = -1, \beta = 1$ and $\gamma = -1$

$\therefore \quad |\alpha| + |\beta| + |\gamma| = 3$

47 If $|\mathbf{a}| = 2, |\mathbf{b}| = 5$ and $|\mathbf{a} \times \mathbf{b}| = 8$, then $|\mathbf{a}.\,\mathbf{b}|$ is equal to **[2021, 25 July Shift-II]**

(a) 6 (b) 4

(c) 3 (d) 5

Ans. *(a)*

Given that, $|\mathbf{a}| = 2, |\mathbf{b}| = 5$ and $|\mathbf{a} \times \mathbf{b}| = 8$

$\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\sin\theta = \pm 8$

$\Rightarrow \quad 10\sin\theta = \pm 8$

$\Rightarrow \quad \sin\theta = \pm\frac{4}{5}$

So, $\cos\theta = \pm\frac{3}{5} \quad (\because \sin^2\theta + \cos^2\theta = 1)$

Now, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$= 2 \times 5 \times \left(\pm\frac{3}{5}\right)$

$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = \pm 6$

$\therefore \quad |\mathbf{a} \cdot \mathbf{b}| = 6$

48 Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors perpendicular to each other and $|\mathbf{a}| = |\mathbf{b}|$. If $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|$, then the angle between the vectors $[\mathbf{a} + \mathbf{b} + (\mathbf{a} \times \mathbf{b})]$ and $\mathbf{a}$ is equal to **[2021, 18 March Shift-II]**

(a) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(c) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$

Ans. *(b)*

Given, $\mathbf{a} \perp \mathbf{b} \quad$ ...(i)

$|\mathbf{a}| = |\mathbf{b}| \quad$ ...(ii)

and $\quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|$

$\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\sin 90° = |\mathbf{a}| \quad$ [from Eq. (i)]

$\Rightarrow \quad |\mathbf{b}| = 1 = |\mathbf{a}| \quad$ ...(iii) [from Eq. (ii)]

From Eq. (iii), we can say that $\mathbf{a} \times \mathbf{b}$ are mutually perpendicular unit vectors.

Let $\mathbf{a} = \hat{\mathbf{i}}$ and $\mathbf{b} = \hat{\mathbf{j}}$

$\therefore \quad \mathbf{a} \times \mathbf{b} = \hat{\mathbf{k}}$

Now, $\quad [\mathbf{a} + \mathbf{b} + (\mathbf{a} \times \mathbf{b})] = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$\therefore \quad \cos\theta = \frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \hat{\mathbf{i}}}{\sqrt{3}\sqrt{1}} = \frac{1}{\sqrt{3}}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

49 Let $\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{b} = 7\hat{\mathbf{i}} + \hat{\mathbf{j}} - 6\hat{\mathbf{k}}$. If $\mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{b}, \mathbf{r}.(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = -3$, then. $\mathbf{r}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})$ is equal to **[2021, 17 March Shift-I]**

(a) 12 (b) 8

(c) 13 (d) 10

Ans. *(a)*

$\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\mathbf{b} = 7\hat{\mathbf{i}} + \hat{\mathbf{j}} - 6\hat{\mathbf{k}}$

If $\mathbf{r}\times\mathbf{a}=\mathbf{r}\times\mathbf{b}$

$\Rightarrow \quad \mathbf{r}\times(\mathbf{a}-\mathbf{b})=0$

$\Rightarrow \quad \mathbf{r}=\lambda(\mathbf{a}-\mathbf{b})=\lambda(5\hat{\mathbf{i}}+4\hat{\mathbf{j}}-10\hat{\mathbf{k}})$

Now, $\mathbf{r}\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=-3$

$\Rightarrow \lambda(5\hat{\mathbf{i}}+4\hat{\mathbf{j}}-10\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=-3$

$\Rightarrow \quad \lambda(5+8-10)=-3$

$\Rightarrow \quad \lambda=-1$

$\therefore \mathbf{r}=-5\hat{\mathbf{i}}-4\hat{\mathbf{j}}+10\hat{\mathbf{k}}$

So, $\mathbf{r}\cdot(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$=(-5\hat{\mathbf{i}}-4\hat{\mathbf{j}}+10\hat{\mathbf{k}})(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$=-10+12+10=12$

50 Let $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $\mathbf{b}=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}}$. If $\mathbf{r}\times\mathbf{a}=\mathbf{b}\times\mathbf{r}$, $\mathbf{r}.(\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=3$ and $\mathbf{r}.(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\alpha\hat{\mathbf{k}})=-1, \alpha\in R$, then the value of $\alpha+|\mathbf{r}|^2$ is equal to **[2021, 16 March Shift-II]**

(a) 9 (b) 15

(c) 13 (d) 11

Ans. *(b)*

Given, $\mathbf{a}=(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}})$

and $\mathbf{b}=(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

If $\mathbf{r}\times\mathbf{a}=\mathbf{b}\times\mathbf{r}, \mathbf{r}\cdot(\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=3$

$\mathbf{r}\cdot(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\alpha\hat{\mathbf{k}})=-1$

$\mathbf{r}\times\mathbf{a}=\mathbf{b}\times\mathbf{r}$

$\Rightarrow \quad (\mathbf{r}\times\mathbf{a})=-(\mathbf{r}\times\mathbf{b})$

$\Rightarrow (\mathbf{r}\times\mathbf{a})+(\mathbf{r}\times\mathbf{b})=0$

$\Rightarrow \quad \mathbf{r}\times(\mathbf{a}+\mathbf{b})=0$

$\Rightarrow \quad \mathbf{r}=\lambda(\mathbf{a}+\mathbf{b})$

$\Rightarrow \quad \mathbf{r}=\lambda[(1+2)\hat{\mathbf{i}}+(2-3)\hat{\mathbf{j}}+(-3+5)\hat{\mathbf{k}}]$

$\Rightarrow \quad \mathbf{r}=\lambda(3\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})$

$\mathbf{r}\cdot(\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=3$

$\Rightarrow \lambda(3\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})(\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=3$

$\Rightarrow \quad \lambda(3\alpha-2+2)=3$

$\Rightarrow \quad \alpha\lambda=1$

$\mathbf{r}\cdot(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\alpha\hat{\mathbf{k}})=-1$

$\Rightarrow \lambda(3\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\alpha\hat{\mathbf{k}})=-1$

$\Rightarrow \quad \lambda(6-5-2\alpha)=-1$

$\Rightarrow \quad \lambda(1-2\alpha)=-1$

$\Rightarrow \quad \lambda-2\alpha\lambda=-1$

$\Rightarrow \quad \lambda-2=-1 \Rightarrow \lambda=1$

So, $\alpha=1$

$\mathbf{r}=(3\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}})$

$\Rightarrow \quad |\mathbf{r}|^2=9+1+4=14$

$\therefore \quad \alpha+|\mathbf{r}|^2=1+14=15$

51 Let $\mathbf{c}$ be a vector perpendicular to the vectors $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

If $\mathbf{c}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}})=8$ then the value of $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ is equal to .... **[2021, 16 March Shift-II]**

Ans. *(28)*

Since, $\mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

So, $\mathbf{c}=\lambda(\mathbf{a}\times\mathbf{b})$

$\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$

$\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

Now, $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 1 & -1\\ 1 & 2 & 1\end{vmatrix}$

$=(1+2)\hat{\mathbf{i}}-(1+1)\hat{\mathbf{j}}+(2-1)\hat{\mathbf{k}}$

$=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$\Rightarrow \quad \mathbf{c}=\lambda(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

Now, $\mathbf{c}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}})=8$

$\lambda(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})(\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}})=8$

$\Rightarrow \quad 4\lambda=8$

$\therefore \quad \lambda=2$

So, $\mathbf{c}=2(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

So, $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=2(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})(3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$=2(9+4+1)$

$=28$

52 Let $\mathbf{a}=\hat{\mathbf{i}}+\alpha\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\mathbf{b}=3\hat{\mathbf{i}}-\alpha\hat{\mathbf{j}}+\hat{\mathbf{k}}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\mathbf{a}$ and $\mathbf{b}$ is $8\sqrt{3}$ square units, then $\mathbf{a}\cdot\mathbf{b}$ is equal to .... **[2021, 25 Feb. Shift-II]**

Ans. *(2)*

Area of parallelogram $=|\mathbf{a}\times\mathbf{b}|$

$=|(\hat{\mathbf{i}}+\alpha\hat{\mathbf{j}}+3\hat{\mathbf{k}})\times(3\hat{\mathbf{i}}-\alpha\hat{\mathbf{j}}+\hat{\mathbf{k}})|$

$(64)(3)=16\alpha^2+64+16\alpha^2$

(given, area $=8\sqrt{3}$)

(squaring on both sides)

$\Rightarrow \quad \alpha^2=4$

Now, $\mathbf{a}.\mathbf{b}=3-\alpha^2+3$

$=6-\alpha^2=6-4=2$

53 Let $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}, \mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$ be three given vectors. If $\mathbf{r}$ is a vector such that $\mathbf{r}\times\mathbf{a}=\mathbf{c}\times\mathbf{a}$ and $\mathbf{r}\cdot\mathbf{b}=0$, then $\mathbf{r}\cdot\mathbf{a}$ is equal to .... **[2021, 25 Feb. Shift-I]**

Ans. *(12)*

Given, $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}}, \mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}, \mathbf{c}=\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$

$\mathbf{r}\times\mathbf{a}=\mathbf{c}\times\mathbf{a}$

$\Rightarrow \quad \mathbf{r}\times\mathbf{a}-\mathbf{c}\times\mathbf{a}=0$

$\Rightarrow \quad (\mathbf{r}-\mathbf{c})\times\mathbf{a}=0$

$\therefore \quad \mathbf{r}-\mathbf{c}=\lambda\,\mathbf{a}$

and $\mathbf{r}=\lambda\mathbf{a}+\mathbf{c}$

$\Rightarrow \mathbf{r}\cdot\mathbf{b}=\lambda\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{b}$ (taking dot with $\mathbf{b}$)

$\Rightarrow \quad 0=\lambda\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{b}$ $[\because \mathbf{r}\cdot\mathbf{b}=0]$

$\Rightarrow \lambda(\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}})+(\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}-\hat{\mathbf{j}})=0$

$\Rightarrow \quad \lambda(1-2)+2=0 \Rightarrow \lambda=2$

$\therefore \quad \mathbf{r}=2\mathbf{a}+\mathbf{c}$

$\Rightarrow \quad \mathbf{r}\cdot\mathbf{a}=2\mathbf{a}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}$ [taking dot with $\mathbf{a}$]

$=2|\mathbf{a}|^2+\mathbf{a}\cdot\mathbf{c}$

$=2(1+4+1)+(1-2+1)$

$\Rightarrow \quad \mathbf{r}\cdot\mathbf{a}=12$

54 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. If $\lambda=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ and $\mathbf{d}=\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$, then the ordered pair, $(\lambda, \mathbf{d})$ is equal to **[2020, 7 Jan. Shift-II]**

(a) $\left(\frac{3}{2}, 3\mathbf{b}\times\mathbf{c}\right)$ (b) $\left(-\frac{3}{2}, 3\mathbf{c}\times\mathbf{b}\right)$

(c) $\left(\frac{3}{2}, 3\mathbf{a}\times\mathbf{c}\right)$ (d) $\left(-\frac{3}{2}, 3\mathbf{a}\times\mathbf{b}\right)$

Ans. *(d)*

Given three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ ...(i)

If we do dot product in Eq. (i) with $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, then we get relations

$\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=-1, \mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{c}=-1$

and $\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}=-1$

$\therefore$ On adding above three obtained relations,

we get

$2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=-3 \quad (\because \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a})$

$\Rightarrow \lambda=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-\frac{3}{2}$

If we do cross product in Eq. (i) with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then we get relation

$0+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$

and $\quad \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}=\mathbf{0}$

and $\quad \mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}=0$

$\therefore \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=3(\mathbf{a}\times\mathbf{b})$

$=3(\mathbf{b}\times\mathbf{c})=3(\mathbf{c}\times\mathbf{a})$

$\therefore \quad \mathbf{d}=3\mathbf{a}\times\mathbf{b}=3\mathbf{b}\times\mathbf{c}=3\mathbf{c}\times\mathbf{a}$

From the given options the ordered pair,

$(\lambda, \mathbf{d})=\left(-\frac{3}{2}, 3\mathbf{a}\times\mathbf{b}\right)$

55 Let $\mathbf{a}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ be two vectors. If $\mathbf{c}$ is a vector such that $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{a}$ and $\mathbf{c}\cdot\mathbf{a}=0$, then $\mathbf{c}\cdot\mathbf{b}$ is equal to **[2020, 8 Jan. Shift-II]**

(a) $\frac{1}{2}$ (b) $-\frac{3}{2}$ (c) $-\frac{1}{2}$ (d) -1

Ans. *(c)*

Given vectors, $\mathbf{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\mathbf{b}=\hat{i}-\hat{j}+\hat{k}$ and $\mathbf{c}$, such that $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{a}$

$\Rightarrow \quad \mathbf{b}\times(\mathbf{c}-\mathbf{a})=\mathbf{0}$

$\Rightarrow \quad \mathbf{b}\,||\,\mathbf{c}-\mathbf{a}$

$\Rightarrow \quad \mathbf{c}-\mathbf{a}=\lambda\mathbf{b}$

$\Rightarrow \quad \mathbf{c}=\mathbf{a}+\lambda\mathbf{b}$

$\Rightarrow \quad \mathbf{c}=(1+\lambda)\hat{i}-(2+\lambda)\hat{j}+(1+\lambda)\hat{k}$

$\because \quad \mathbf{c}\cdot\mathbf{a}=0$ (given)

$\Rightarrow (1+\lambda)+2(2+\lambda)+(1+\lambda)=0$

$\Rightarrow \quad 4\lambda+6=0 \Rightarrow \lambda=-\frac{3}{2}$

$\therefore \quad \mathbf{c}=-\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}-\frac{1}{2}\hat{k}$

So $\quad \mathbf{c}\cdot\mathbf{b}=-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}=-\frac{1}{2}$

Hence, option (c) is correct.

56 Let $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+x\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, for some real x. Then $|\mathbf{a}\times\mathbf{b}|=r$ is possible if **[2019, 8 April Shift-II]**

(a) $0<r\le\sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{3}{2}}<r\le3\sqrt{\frac{3}{2}}$

(c) $3\sqrt{\frac{3}{2}}<r<5\sqrt{\frac{3}{2}}$ (d) $r\ge5\sqrt{\frac{3}{2}}$

Ans. *(d)*

Given vectors are $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+x\hat{\mathbf{k}}$

and $\quad \mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$\therefore \quad \mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 2 & x\\ 1 & -1 & 1\end{vmatrix}$

$=\hat{\mathbf{i}}(2+x)-\hat{\mathbf{j}}(3-x)+\hat{\mathbf{k}}(-3-2)$

$=(x+2)\hat{\mathbf{i}}+(x-3)\hat{\mathbf{j}}-5\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}|=\sqrt{(x+2)^2+(x-3)^2+25}$

$=\sqrt{2x^2-2x+4+9+25}$

$=\sqrt{2\left(x^2-x+\frac{1}{4}\right)-\frac{1}{2}+38}$

$=\sqrt{2\left(x-\frac{1}{2}\right)^2+\frac{75}{2}}$

So, $|\mathbf{a}\times\mathbf{b}|\ge\sqrt{\frac{75}{2}}$ [at $x=\frac{1}{2}$, $|\mathbf{a}\times\mathbf{b}|$ is minimum]

$\Rightarrow \quad r\ge5\sqrt{\frac{3}{2}}$

57 Let $\vec{\alpha}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\vec{\beta}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. If $\vec{\beta}=\vec{\beta}_1-\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, then $\vec{\beta}_1\times\vec{\beta}_2$ is equal to

[2019, 9 April Shift-I]

(a) $\frac{1}{2}(3\hat{\mathbf{i}}-9\hat{\mathbf{j}}+5\hat{\mathbf{k}})$ (b) $\frac{1}{2}(-3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

(c) $-3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ (d) $3\hat{\mathbf{i}}-9\hat{\mathbf{j}}-5\hat{\mathbf{k}}$)

Ans. *(b)*

Given vectors $\vec{\alpha}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\vec{\beta}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\vec{\beta}=\vec{\beta}_1-\vec{\beta}_2$ such that $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to α

So, $\quad \vec{\beta}_1=\lambda\alpha=\lambda(3\hat{\mathbf{i}}+\hat{\mathbf{j}})$

Now, $\vec{\beta}_2=\vec{\beta}_1-\vec{\beta}=\lambda(3\hat{\mathbf{i}}+\hat{\mathbf{j}})-(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$=(3\lambda-2)\hat{\mathbf{i}}+(\lambda+1)\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$\because \overrightarrow{\beta}_2$ is perpendicular to α, so $\overrightarrow{\beta}_2\cdot\alpha=0$

[since if non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other, then $\mathbf{a}\cdot\mathbf{b}=0$]

$\therefore \quad (3\lambda-2)(3)+(\lambda+1)(1)=0$

$\Rightarrow \quad 9\lambda-6+\lambda+1=0$

$\Rightarrow \quad 10\lambda=5\Rightarrow\lambda=\frac{1}{2}$

So, $\quad \overrightarrow{\beta}_1=\frac{3}{2}\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}$

and $\quad \overrightarrow{\beta}_2=\left(\frac{3}{2}-2\right)\hat{\mathbf{i}}+\left(\frac{1}{2}+1\right)\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$=-\frac{1}{2}\hat{\mathbf{i}}+\frac{3}{2}\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$\therefore \quad \overrightarrow{\beta}_1\times\overrightarrow{\beta}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{3}{2} & \frac{1}{2} & 0\\ \frac{-1}{2} & \frac{3}{2} & -3\end{vmatrix}$

$=\hat{\mathbf{i}}\left(-\frac{3}{2}-0\right)-\hat{\mathbf{j}}\left(-\frac{9}{2}-0\right)+\hat{\mathbf{k}}\left(\frac{9}{4}+\frac{1}{4}\right)$

$=-\frac{3}{2}\hat{\mathbf{i}}+\frac{9}{2}\hat{\mathbf{j}}+\frac{5}{2}\hat{\mathbf{k}}$

$=\frac{1}{2}(-3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

58 The distance of the point having position vector $-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ from the straight line passing through the point $(2,3,-4)$ and parallel to the vector, $6\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ is

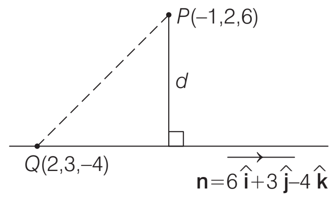
[2019, 10 April Shift-II]

(a) $2\sqrt{13}$ (b) $4\sqrt{3}$

(c) 6 (d) 7

Ans. *(d)*

Let point P whose position vector is $(-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})$ and a straight line passing through $Q(2,3,-4)$ parallel to the vector $\mathbf{n}=6\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$.



$\because$ Required distance d = Projection of line segment PQ perpendicular to vector n.

$=\frac{|\mathbf{PQ}\times\mathbf{n}|}{|\mathbf{n}|}$

Now, $\quad \mathbf{PQ}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}-10\hat{\mathbf{k}}$, so

$\mathbf{PQ}\times\mathbf{n}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & -10\\ 6 & 3 & -4\end{vmatrix}=26\hat{\mathbf{i}}-48\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

So, $\quad d=\frac{\sqrt{(26)^2+(48)^2+(3)^2}}{\sqrt{(6)^2+(3)^2+(4)^2}}$

$=\sqrt{\frac{676+2304+9}{36+9+16}}=\sqrt{\frac{2989}{61}}$

$=\sqrt{49}=7$ units

59 Let $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ be two vectors. If a vector perpendicular to both the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ has the magnitude 12, then one such vector is **[2019, 12 April Shift-I]**

(a) $4(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $4(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})$

(c) $4(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})$ (d) $4(-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$

Ans. *(b)*

Given vectors are

$\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

and $\mathbf{b}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Now, vectors $\mathbf{a}+\mathbf{b}=4\hat{\mathbf{i}}+4\hat{\mathbf{j}}$

and $\quad \mathbf{a}-\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{k}}$

$\therefore$ A vector which is perpendicular to both the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ is

$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}$

$=\hat{\mathbf{i}}(16)-\hat{\mathbf{j}}(16)+\hat{\mathbf{k}}(-8)$

$=8(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})$

Then, the required vector along $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$ having magnitude 12 is

$\pm12\times\frac{8(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})}{8\times\sqrt{4+4+1}}$

$=\pm4(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})$

60 Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c}$ be a vector such that $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is 30°. Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to **[JEE Main 2017]**

(a) $\frac{25}{8}$ (b) 2 (c) 5 (d) $\frac{1}{8}$

Ans. ***(b)***

We have, $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{a}| = \sqrt{4+1+4} = 3$

and $\quad \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \Rightarrow |\mathbf{b}| = \sqrt{1+1} = \sqrt{2}$

Now, $|\mathbf{c} - \mathbf{a}| = 3 \Rightarrow |\mathbf{c} - \mathbf{a}|^2 = 9$

$\Rightarrow \quad (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = 9$

$\Rightarrow \quad |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{c} \cdot \mathbf{a} = 9 \quad$ ...(i)

Again, $\quad |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$

$\Rightarrow |\mathbf{a} \times \mathbf{b}||\mathbf{c}| \sin 30° = 3 \Rightarrow |\mathbf{c}| = \frac{6}{|\mathbf{a} \times \mathbf{b}|}$

But $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\therefore \quad |\mathbf{c}| = \frac{6}{\sqrt{4+4+1}} = 2 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$(2)^2 + (3)^2 - 2\mathbf{c} \cdot \mathbf{a} = 9 \Rightarrow 4 + 9 - 2\mathbf{c} \cdot \mathbf{a} = 9$

$\Rightarrow \quad \mathbf{c} \cdot \mathbf{a} = 2$

61 If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\hat{\mathbf{i}} + \hat{\mathbf{k}})$ and $\mathbf{b} = \frac{1}{7}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$, then value of $(2 - \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is

(a) –3 (b) 5 **[AIEEE 2011]**

(c) 3 (d) –5

Ans. ***(d)***

$a = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$

and $b = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$

$\therefore (2a - b) \cdot \{(a \times b) \times (a + 2b)\}$

$= (2a - b) \cdot \{(a \times b) \times a + (a \times b) \times 2b\}$

$= (2a - b) \cdot \{(a \cdot a) b - (b \cdot a) a + 2 (a \cdot b) b - 2 (b \cdot b) a\}$

$= (2a - b) \cdot \{1 (b) - (0) a + 2 (0) b - 2 (1) a\}$

$[\text{as } a \cdot b = 0 \text{ and } a \cdot a = b \cdot b = 1]$

$= (2a - b)(b - 2a)$

$= -(4|a|^2 - 4a \cdot b + |b|^2) = -\{4 - 0 + 1\}$

$= -5$

62 The vectors $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular and $\mathbf{c}$ and $\mathbf{d}$ are two vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a} \cdot \mathbf{b} = 0$. Then, the vector $\mathbf{d}$ is equal to **[AIEEE 2011]**

(a) $\mathbf{c} + \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} + \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c} - \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} - \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

Ans. ***(c)***

Given, $\quad a \cdot b \neq 0,\ a \cdot d = 0 \quad$ ...(i)

and $\quad b \times c = b \times d$

$\Rightarrow \quad b \times (c - d) = 0$

$\therefore \quad b \parallel (c - d)$

$\Rightarrow \quad c - d = \lambda b$

$\Rightarrow \quad d = c - \lambda b \quad$ ...(ii)

Taking dot product with a, we get

$a \cdot d = a \cdot c - \lambda a \cdot b$

$\Rightarrow \quad 0 = a \cdot c - \lambda (a \cdot b)$

$\therefore \quad \lambda = \frac{a \cdot c}{a \cdot b} \quad$ ...(iii)

$\therefore \quad d = c - \frac{(a \cdot c)}{(a \cdot b)} b$

63 If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and θ is the acute angle between them, then $2\mathbf{u} \times 3\mathbf{v}$ is a unit vector for **[AIEEE 2007]**

(a) exactly two values of θ

(b) more than two values of θ

(c) no value of θ

(d) exactly one value of θ

Ans. ***(d)***

Since, $(2u \times 3v)$ is a unit vector.

$\Rightarrow \quad |2u \times 3v| = 1$

$\Rightarrow \quad 6|u||v|\sin\theta = 1$

$\Rightarrow \quad \sin\theta = \frac{1}{6} \quad [\because |u| = |v| = 1]$

Since, θ is an acute angle, then there is exactly one value of θ for which $(2u \times 3v)$ is a unit vector.

64 For any vector $\mathbf{a}$, the value of $(\mathbf{a} \times \hat{\mathbf{i}})^2 + (\mathbf{a} \times \hat{\mathbf{j}})^2 + (\mathbf{a} \times \hat{\mathbf{k}})^2$ is equal to **[AIEEE 2005]**

(a) $4a^2$ (b) $2a^2$ (c) a^2 (d) $3a^2$

Ans. ***(b)***

Let $\quad a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Then, $\quad a \times \hat{i} = -a_2\hat{k} + a_3\hat{j}$

$a \times \hat{j} = a_1\hat{k} - a_3\hat{i}$

$a \times k = -a_1 j + a_2 i$

$\therefore (a \times i)^2 + (a \times j)^2 + (a \times k)^2$

$= a_2^2 + a_3^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$= 2(a_1^2 + a_2^2 + a_3^2) = 2a^2$

65 Let $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\mathbf{w} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. If $\mathbf{n}$ is a unit vector such that $\mathbf{u} \cdot \mathbf{n} = 0$ and $\mathbf{v} \cdot \mathbf{n} = 0$, then $|\mathbf{w} \cdot \mathbf{n}|$ is equal to **[AIEEE 2003]**

(a) 0 (b) 1 (c) 2 (d) 3

Ans. ***(d)***

Given that, $u = \hat{i} + \hat{j}, \quad v = \hat{i} - \hat{j},$

$w = \hat{i} + 2\hat{j} + 3\hat{k},$

$u \cdot n = 0 \text{ and } v \cdot n = 0$

i.e., $\quad n = \frac{u \times v}{|u \times v|}$

Now, $\quad u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$= 0\hat{i} - 0\hat{j} - 2\hat{k} = -2\hat{k}$

$\therefore \quad |w \cdot n| = \frac{|w \cdot u \times v|}{|u \times v|} = \frac{|-6k|}{|-2k|} = 3$

$[\because w \cdot (u \times w) = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-2\hat{k}) = -6\hat{k}]$

Hence, $\quad |w \cdot n| = 3$

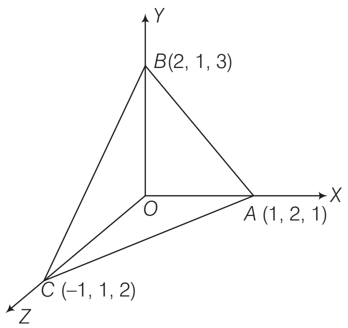
66 A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then, the angle between the faces OAB and ABC will be **[AIEEE 2003]**

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$

(c) 30° (d) 90°

Ans. ***(a)***

Vector perpendicular to face *OAB* is n_1.



$= OA \times OB = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$

$= 5\hat{i} - \hat{j} - 3\hat{k}$

Vector perpendicular to face *ABC* is n_2

$= AB \times AC = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix}$

$= \hat{i} - 5\hat{j} - 3\hat{k}$

Since, angle between faces is equal to the angle between their normals.

$\therefore \quad \cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$

$$= \frac{5\times1+(-1)\times(-5)+(-3)\times(-3)}{\sqrt{5^2+(-1)^2+(-3)^2}\sqrt{1^2+(-5)^2+(-3)^2}}$$

$$= \frac{5+5+9}{\sqrt{35}\sqrt{35}} = \frac{19}{35}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

67 If the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form the sides BC, CA and AB respectively of a ΔABC, then

(a) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{b} = 0$ **[AIEEE 2002]**

(b) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$

(c) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

(d) $\mathbf{a}\times\mathbf{a} + \mathbf{a}\times\mathbf{c} + \mathbf{a}\times\mathbf{a} = 0$

Ans. *(b)*

Since, $\quad a + b + c = 0$

$\Rightarrow \quad a + b = -c$

$\Rightarrow \quad (a+b)\times c = -c\times c$

$\Rightarrow \quad b\times c = c\times a$

Similarly, $\quad a\times b = b\times c$

Hence, $\quad a\times b = b\times c = c\times a$

68 If the vectors $\mathbf{c}, \mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{j}}$ are such that $\mathbf{a}, \mathbf{c}$ and $\mathbf{b}$ form a right handed system, then $\mathbf{c}$ is **[AIEEE 2002]**

(a) $z\hat{\mathbf{i}} - x\hat{\mathbf{k}}$ (b) $\mathbf{0}$

(c) $y\hat{\mathbf{j}}$ (d) $-z\hat{\mathbf{i}} + x\hat{\mathbf{k}}$

Ans. *(a)*

Since, the vectors $a = x\hat{i} + y\hat{j} + z\hat{k}$ and $b = \hat{j}$ are such that a, c and b form a right handed system.

$$\therefore \quad c = b\times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = z\hat{i} - x\hat{k}$$

TOPIC 4
Scalar and Vector Triple Product

69 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors mutually perpendicular to each other and have same magnitude. If a vector $\mathbf{r}$ satisfies.

$\mathbf{a}\times\{(\mathbf{r}-\mathbf{b})\times\mathbf{a}\} + \mathbf{b}\times\{(\mathbf{r}-\mathbf{c})\times b\} + \mathbf{c}\times\{(\mathbf{r}-\mathbf{a})\times\mathbf{c}\} = \mathbf{0}$, then $\mathbf{r}$ is equal to **[2021, 31 Aug. Shift-II]**

(a) $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ (b) $\frac{1}{3}(2\mathbf{a}+\mathbf{b}-\mathbf{c})$

(c) $\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ (d) $\frac{1}{2}(\mathbf{a}+\mathbf{b}+2\mathbf{c})$

Ans. *(c)*

$\mathbf{a}\times[(\mathbf{r}-\mathbf{b})\times\mathbf{a}] + \mathbf{b}\times[(\mathbf{r}-\mathbf{c})\times\mathbf{b}] + \mathbf{c}\times[(\mathbf{r}-\mathbf{a})\times\mathbf{c}] = \mathbf{0}$

$\Rightarrow \mathbf{a}\cdot\mathbf{a}(\mathbf{r}-\mathbf{b}) - (\mathbf{a}\cdot(\mathbf{r}-\mathbf{b}))\mathbf{a} + \mathbf{b}\cdot\mathbf{b}(\mathbf{r}-\mathbf{c}) - (\mathbf{b}\cdot(\mathbf{r}-\mathbf{c}))\mathbf{b} + \mathbf{c}\cdot\mathbf{c}(\mathbf{r}-\mathbf{a}) - (\mathbf{c}\cdot(\mathbf{r}-\mathbf{a}))\mathbf{c} = \mathbf{0}$

$\Rightarrow |\mathbf{a}|^2(\mathbf{r}-\mathbf{b}) - (\mathbf{r}\cdot\mathbf{a})\mathbf{a} + |\mathbf{b}|^2(\mathbf{r}-\mathbf{c}) - (\mathbf{r}\cdot\mathbf{b})\mathbf{b} + |\mathbf{c}|^2(\mathbf{r}-\mathbf{a}) - (\mathbf{r}\cdot\mathbf{c})\mathbf{c} = \mathbf{0}$

[$\because \mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular; $\therefore \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$]

$\Rightarrow |\mathbf{a}|^2[3\mathbf{r} - (\mathbf{a}+\mathbf{b}+\mathbf{c})] - [(\mathbf{r}\cdot\mathbf{a})\mathbf{a} + (\mathbf{r}\cdot\mathbf{b})\mathbf{b} + (\mathbf{r}\cdot\mathbf{c})\mathbf{c}] = \mathbf{0}$

[$\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$]

$\Rightarrow |\mathbf{a}|^2[3\mathbf{r} - (\mathbf{a}+\mathbf{b}+\mathbf{c}) - r] = 0$

$\therefore \quad 3\mathbf{r} - (\mathbf{a}+\mathbf{b}+\mathbf{c}) - \mathbf{r} = 0$

$\Rightarrow \quad \mathbf{r} = \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$

70 Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$. If $\mathbf{c}$ is a vector such that $\mathbf{a}\times\mathbf{c} = \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{c} = 3$, then $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is equal to **[2021, 26 Aug. Shift-I]**

(a) -2 (b) -6

(c) 6 (d) 2

Ans. *(a)*

Given, $\mathbf{a}\times\mathbf{c} = \mathbf{b}$

$\mathbf{a}\times(\mathbf{a}\times\mathbf{c}) = \mathbf{a}\times\mathbf{b}$

$(\mathbf{a}.\mathbf{c})\mathbf{a} - (\mathbf{a}.\mathbf{a})\mathbf{c} = \mathbf{a}\times\mathbf{b}$

We have, $\mathbf{a} = (1,1,1), \mathbf{b} = (0,1,-1), \mathbf{a}\cdot\mathbf{c} = 3$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

So, $3\mathbf{a} - 3\mathbf{c} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$\Rightarrow (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - 3\overline{\mathbf{c}} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

$\Rightarrow 3\mathbf{c} = (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Now, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$$= \left(\frac{1}{3}\right)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 5 & 2 & 2 \end{vmatrix} = \frac{1}{3}(4-5-5) = -2$$

71 Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Then the vector product $(\mathbf{a}+\mathbf{b})\times[(\mathbf{a}\times\{(\mathbf{a}-\mathbf{b})\times\mathbf{b})\}\times\mathbf{b}]$ is equal to **[2021, 27 July Shift-I]**

(a) $5(34\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

(b) $7(34\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

(c) $7(30\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$

(d) $5(30\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$

Ans. *(b)*

Given, $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Vector product

$(\mathbf{a}+\mathbf{b})\times[(\mathbf{a}\times(\mathbf{a}-\mathbf{b})\times\mathbf{b})\times\mathbf{b}]$

$= (\mathbf{a}+\mathbf{b})\times[(\mathbf{a}\times(\mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{b})\times\mathbf{b}]$

$= (\mathbf{a}+\mathbf{b})\times[(\mathbf{a}\times(\mathbf{a}\times\mathbf{b}))\times\mathbf{b}]$

[$\because \mathbf{b}\times\mathbf{b} = 0$]

$= (\mathbf{a}+\mathbf{b})\times[[(\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b}]\times\mathbf{b}]$

$= (\mathbf{a}+\mathbf{b})\times[(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\times\mathbf{b})]$

$= (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}+\mathbf{b})\times(\mathbf{a}\times\mathbf{b})$

$= (\mathbf{a}\cdot\mathbf{b})[[(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}]\mathbf{a} - [(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}]\mathbf{b}]$

$\mathbf{a}\cdot\mathbf{b} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$= -1 + 2 + 6 = 7$ ...(i)

$(\mathbf{a}+\mathbf{b}) = 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ...(ii)

$(\mathbf{a}+\mathbf{b})\cdot\mathbf{a} = (3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 13$ ...(iii)

$(\mathbf{a}+\mathbf{b})\cdot\mathbf{b} = (3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 21$.(iv)

Substituting the values from Eqs. (i), (ii), (iii) and (iv), we get

$(\mathbf{a}\cdot\mathbf{b})[(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}]\,\mathbf{a} - [(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}\mathbf{b}]$

$= 7[21\mathbf{a} - 13\mathbf{b}]$

$= 7[21(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - 13(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})]$

$= 7(34\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

72 Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b}$ and $\mathbf{c} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ be three vectors such that $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ and $\mathbf{a}\cdot\mathbf{b} = 1$. If the length of projection vector of the vector $\mathbf{b}$ on the vector $\mathbf{a}\times\mathbf{c}$ is l, then the value of $3l^2$ is equal to .... **[2021, 27 July Shift-I]**

Ans. *(2)*

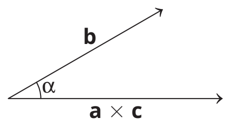
Given, $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$,

$\mathbf{c} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Given, $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ and $\mathbf{a}\cdot\mathbf{b} = 1$

Projection of $\mathbf{b}$ on $\mathbf{a}\times\mathbf{c}$ is

$l = |\mathbf{b}|\cos\alpha$



and $\quad \mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = |\mathbf{b}||\mathbf{a}\times\mathbf{c}|\cos\alpha$

$\therefore \quad l = \mathbf{b}\cdot\frac{(\mathbf{a}\times\mathbf{c})}{|\mathbf{a}\times\mathbf{c}|}$

As, we know that,

$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{b}\,\mathbf{a}\,\mathbf{c}] = [\mathbf{c}\,\mathbf{a}\,\mathbf{b}]$

So, if $\quad \mathbf{a}\times\mathbf{b} = \mathbf{c}$

Then, $\quad \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{c}\cdot\mathbf{c}$

$[\mathbf{c}\,\mathbf{a}\,\mathbf{b}] = |\mathbf{c}|^2$

$$l = \frac{[\mathbf{b}\,\mathbf{a}\,\mathbf{c}]}{|\mathbf{a}\times\mathbf{c}|} = \frac{[\mathbf{c}\,\mathbf{a}\,\mathbf{b}]}{|\mathbf{a}\times\mathbf{c}|} = \frac{|\mathbf{c}|^2}{|\mathbf{a}\times\mathbf{c}|}$$

$$\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$\therefore \quad |\mathbf{a}\times\mathbf{c}| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$\therefore \quad \mathbf{c} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

$|\mathbf{c}| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \quad l = \frac{(\sqrt{2})^2}{\sqrt{6}} = \frac{2}{\sqrt{6}}$

and $\quad 3l^2 = 3\cdot\frac{4}{6} = 2$

73 Let **a**, **b** and **c** be three vectors such that $\mathbf{a} = \mathbf{b}\times(\mathbf{b}\times\mathbf{c})$. If magnitudes of the vectors **a**, **b** and **c** are $\sqrt{2}$, 1 and 2, respectively and the angle between **b** and **c** is $\theta\left(0<\theta<\frac{\pi}{2}\right)$, then the value of $(1+\tan\theta)$ is equal to **[2021, 27 July Shift-II]**

(a) $\sqrt{3}+1$ (b) 2

(c) 1 (d) $\frac{\sqrt{3}+1}{\sqrt{3}}$

Ans. *(b)*

Given that,

$\mathbf{a} = \mathbf{b}\times(\mathbf{b}\times\mathbf{c})$

and $\quad |\mathbf{a}| = \sqrt{2}, |\mathbf{b}| = 1, |\mathbf{c}| = 2$

Now, $\quad \mathbf{a} = \mathbf{b}\times(\mathbf{b}\times\mathbf{c})$

$= (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\,\mathbf{c}$

$= (1\cdot 2\cos\theta)\,\mathbf{b} - (1)\,\mathbf{c}$

$[\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,\ \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}^{-1}|^2\ ]$

$\Rightarrow \quad \mathbf{a} = 2\cos\theta\mathbf{b} - \mathbf{c}$

$|\mathbf{a}|^2 = (2\cos\theta)^2 + 2^2 - 2\cdot 2\cos\theta(\mathbf{b}\cdot\mathbf{c})$

$\Rightarrow \quad 2 = 4\cos^2\theta + 4 - 4\cos\theta(2\cos\theta)$

$\Rightarrow \quad -2 = -4\cos^2\theta$

$\Rightarrow \quad \cos^2\theta = \frac{1}{2} \Rightarrow \sec^2\theta = 2$

$\Rightarrow \quad 1+\tan^2\theta = 2 \Rightarrow \tan^2\theta = 1$

$\Rightarrow \quad \theta = \frac{\pi}{4}\ \left[\text{where}, 0<\theta<\frac{\pi}{2}\right]$

Hence, the value of $1+\tan\theta = 1+1 = 2$.

74 Let $\mathbf{a} = \hat{\mathbf{i}} - \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} - \alpha\hat{\mathbf{k}}$ and $\mathbf{c} = -\alpha\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, where α and β are integers. If $\mathbf{a}\cdot\mathbf{b} = -1$ and $\mathbf{b}\cdot\mathbf{c} = 10$, then $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is equal to .... **[2021, 27 July Shift-II]**

Ans. *(9)*

Let $\quad \mathbf{a} = \hat{\mathbf{i}} - \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$

$\mathbf{b} = 3\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} - \alpha\hat{\mathbf{k}}$

and $\quad \mathbf{c} = -\alpha\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Given that,

$\mathbf{a}\cdot\mathbf{b} = -1$

$\Rightarrow \quad 3 - \alpha\beta - \alpha\beta = -1 \Rightarrow \alpha\beta = 2$

α and β are integers, so possible values of α and β are

If $\quad \alpha = 1 \Rightarrow \beta = 2$

If $\quad \alpha = 2 \Rightarrow \beta = 1$

If $\quad \alpha = -1 \Rightarrow \beta = -2$

If $\quad \alpha = -2 \Rightarrow \beta = -1$

Now, $\mathbf{b}\cdot\mathbf{c} = 10$

$\Rightarrow \quad -3\alpha - 2\beta - \alpha = 10$

$\Rightarrow \quad 2\alpha + \beta + 5 = 0$

$\therefore$ Value of $\alpha = -2$ and $\beta = -1$ satisfy this equation.

So, $\quad \mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}} \Rightarrow \mathbf{b} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\mathbf{c} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$$[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 1 \end{vmatrix}$$

$= 1(-1+4) - 2(3-4) - 1(-6+2)$

$= 3+2+4 = 9$

75 Let the vectors $(2+a+b)\hat{\mathbf{i}} + (a+2b+c)\hat{\mathbf{j}} - (b+c)\hat{\mathbf{k}}$, $(1+b)\hat{\mathbf{i}} + 2b\hat{\mathbf{j}} - b\hat{\mathbf{k}}$ and $(2+b)\hat{\mathrm{i}} + 2b\hat{\mathrm{j}} + (1-b)\hat{\mathrm{k}}$, $a, b, c \in R$ be coplanar. **[2021, 25 July Shift-I]**

Then, which of the following is true?

(a) $2b = a+c$ (b) $3c = a+b$

(c) $a = b+2c$ (d) $2a = b+c$

Ans. *(a)*

Given vectors are coplanar

$$\therefore \begin{vmatrix} a+b+2 & a+2b+c & -b-c \\ b+1 & 2b & -b \\ b+2 & 2b & 1-b \end{vmatrix} = 0$$

Apply, $R_3 \to R_3 - R_2$, $R_1 \to R_1 - R_2$

So,

$$\begin{vmatrix} a+1 & a+c & -c \\ b+1 & 2b & -b \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$= (a+1)2b - (a+c)(2b+1) - c(-2b)$

$= 2ab + 2b - 2ab - a - 2bc - c + 2bc$

$\Rightarrow \quad 2b - a - c = 0$

76 Let three vectors **a**, **b** and **c** be such that $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, $\mathbf{b}\times\mathbf{c} = \mathbf{a}$ and $|\mathbf{a}| = 2$. Then, which one of the following is not true ? **[2021, 22 July Shift-II]**

(a) $\mathbf{a}\times((\mathbf{b}+\mathbf{c})\times(\mathbf{b}-\mathbf{c})) = 0$

(b) Projection of a on $(\mathbf{b}\times\mathbf{c})$ is 2

(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = 8$

(d) $|3\mathbf{a}+\mathbf{b}-2\mathbf{c}|^2 = 51$

Ans. *(d)*

$\mathbf{a}\times\mathbf{b} = \mathbf{c}$

$\mathbf{b}\times\mathbf{c} = \mathbf{a}$

and $\quad |\mathbf{a}| = 2$

(a) $\quad \mathbf{a}\times[(\mathbf{b}+\mathbf{c})\times(\mathbf{b}-\mathbf{c})] = \mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times[\mathbf{b}\times(\mathbf{b}-\mathbf{c}) + \mathbf{c}\times(\mathbf{b}-\mathbf{c})] = \mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times[-\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{b}] = \mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times(-\mathbf{a}-\mathbf{a}) = \mathbf{0}$ (True)

(b) Projection of **a** on $\mathbf{b}\times\mathbf{c}$ is 2.

$\mathbf{a}.\frac{(\mathbf{b}\times\mathbf{c})}{|\mathbf{b}\times\mathbf{c}|} = \frac{\mathbf{a}.\mathbf{a}}{|\mathbf{a}|} = \frac{|\mathbf{a}|^2}{|\mathbf{a}|} = 2$

(c) $\quad [\mathbf{abc}] + [\mathbf{cab}] = 8$

$[\mathbf{abc}] + [\mathbf{abc}] = 8$

$\Rightarrow \quad [\mathbf{abc}] = 4 \Rightarrow \mathbf{a}.(\mathbf{b}\times\mathbf{c}) = 4$

$\Rightarrow \quad \mathbf{a}.\mathbf{a} = 4 \Rightarrow |\mathbf{a}|^2 = 4$ (True)

(d) $|3\mathbf{a}+\mathbf{b}-2\mathbf{c}|^2 = 51$

$(3\mathbf{a}+\mathbf{b}-2\mathbf{c})\cdot(3\mathbf{a}+\mathbf{b}-2\mathbf{c})$

$= 9|\mathbf{a}|^2 + |\mathbf{b}|^2 + 4|\mathbf{c}|^2$

$= (9\times 4) + 1 + (4\times 4)$

$= 36 + 1 + 16 = 53$ (False)

So, (d) is the correct option.

77 Let a vector **a** be coplanar with vectors $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$. If **a** is perpendicular to $\mathbf{d} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $|\mathbf{a}| = \sqrt{10}$. Then a possible value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}] + [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$ is equal to **[2021, 22 July Shift-II]**

(a) -42 (b) -40

(c) -29 (d) -38

Ans. *(a)*

$\left.\begin{array}{l} \mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c} \\ \mathbf{a}\perp\mathbf{d} \\ |\mathbf{a}| = 10 \end{array}\right\}$ Given, $\begin{array}{l} \mathbf{b} = (2,1,1) \\ \mathbf{c} = (1,-1,1) \\ \mathbf{d} = (3,2,6) \end{array}$

$\mathbf{a}\cdot\mathbf{d} = 0$

$\Rightarrow \quad (\lambda\mathbf{b}+\mu\mathbf{c})\cdot\mathbf{d} = 0 \Rightarrow \lambda.\mathbf{b}\cdot\mathbf{d} + \mu\mathbf{c}\mathbf{d} = 0$

$\Rightarrow \lambda(6+2+6) + \mu(3-2+6) = 0$

$\Rightarrow \quad 14\lambda + 7\mu = 0$

$\Rightarrow \quad \mu = -2\lambda$

$\mathbf{a} = (2\lambda, \lambda, \lambda) + (\mu, -\mu, \mu)$

$= (2\lambda+\mu)\hat{\mathbf{i}} + (\lambda-\mu)\hat{\mathbf{j}} + (\lambda+\mu)\hat{\mathbf{k}}$

$= 0\hat{\mathbf{i}} + 3\lambda\hat{\mathbf{j}} - \lambda\hat{\mathbf{k}} = \lambda(3\hat{\mathbf{j}} - \hat{\mathbf{k}})$

Now, $\quad |\mathbf{a}| = \sqrt{10}$

$\Rightarrow \lambda^2(3^2+1^2) = 10$

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$[\mathbf{abc}] + [\mathbf{abd}] + [\mathbf{acd}] = [\mathbf{abc}]\ [\mathbf{a}, \mathbf{b}+\mathbf{c}, \mathbf{d}]$

$\left.\begin{array}{l} \mathbf{a} = \pm 1(3\hat{\mathbf{j}} - \hat{\mathbf{k}}) \\ \mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \\ \mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} \end{array}\right\}, \mathbf{b}+\mathbf{c} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$

$\mathbf{d} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

As, $\mathbf{a}$ is coplanar with $\mathbf{b}$ and $\mathbf{c}$. So, $[\mathbf{abc}]=0$

$$\pm\begin{vmatrix}0 & 3 & -1\\ 3 & 0 & 2\\ 3 & 2 & 6\end{vmatrix}=\pm(-36-6)=\pm 42$$

78 Let $a=2\hat{i}+\hat{j}-2\hat{k}$ and $b=\hat{i}+\hat{j}$. If c is a vector such that $a\cdot c=|c|, |c-a|=2\sqrt{2}$ and the angle between $(a\times b)$ and c is $\frac{\pi}{6}$, then the value of $|(a\times b)\times c|$ is

[2021, 20 July Shift-I]

(a) $\frac{2}{3}$ (b) 4 (c) 3 (d) 3/2

Ans. *(d)*

$\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

$\Rightarrow\ |\mathbf{a}|=\sqrt{4+1+4}=3$

$\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$

$\mathbf{a}\cdot\mathbf{c}=|\mathbf{c}|$

$|\mathbf{c}-\mathbf{a}|=2\sqrt{2}$

$\Rightarrow\ |\mathbf{c}-\mathbf{a}|^2=8$

$\Rightarrow\ (\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})=8$

$\Rightarrow\ |\mathbf{c}|^2-2\mathbf{a}\cdot\mathbf{c}+|\mathbf{a}|^2=8$

$\Rightarrow\ |\mathbf{c}|^2-2|\mathbf{c}|+9=8$

$\Rightarrow\ |\mathbf{c}|^2-2|\mathbf{c}|+1=0$

$\Rightarrow\ (|\mathbf{c}|-1)^2=0$

$\Rightarrow\ |\mathbf{c}|=1$

$|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|=|\mathbf{a}\times\mathbf{b}||\mathbf{c}|\sin\left(\frac{\pi}{6}\right)$

$=\frac{1}{2}(|\mathbf{a}\times\mathbf{b}||\mathbf{c}|)$

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

$|\mathbf{a}\times\mathbf{b}|=\sqrt{4+4+1}=3$

$\therefore\ |(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|=\frac{3}{2}|\mathbf{c}|=\frac{3}{2}$

79 If $\mathbf{a}=\alpha\hat{\mathbf{i}}+\beta\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $\mathbf{b}=-\beta\hat{\mathbf{i}}-\alpha\hat{\mathbf{j}}-\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}}$, such that $\mathbf{a}.\mathbf{b}=1$ and $\mathbf{b}.\mathbf{c}=-3$, then $\frac{1}{3}[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]$ is equal to ....

[2021, 17 March Shift-I]

Ans. *(2)*

$\mathbf{a}=<\alpha,\beta,3>$

$\mathbf{b}=<-\beta,-\alpha,-1>$

$\mathbf{c}=<1,-2,-1>$

$\mathbf{a}\cdot\mathbf{b}=1$

$-\alpha\beta-\alpha\beta-3=1$

$\alpha\beta=-2$

$\mathbf{b}\cdot\mathbf{c}=-3$

$-\beta+2\alpha+1=-3$

$\beta-2\alpha=4$

$\Rightarrow\ \beta-2\left(\frac{-2}{\beta}\right)=4$

$\Rightarrow\ \beta^2+4=4\beta$

$\Rightarrow\beta^2-4\beta+4=0$

$\Rightarrow\ (\beta-2)^2=0$

$\Rightarrow\ \beta=2$

$\alpha\beta=-2\Rightarrow\alpha\cdot 2=-2$

$\Rightarrow\ \alpha=-1$

$\frac{1}{3}[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]$

$\mathbf{a}=<-1,2,3>$

$\mathbf{b}=<-2,1,-1>$

$\mathbf{c}=<1,-2,-1>$

$$(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 2 & 3\\ -2 & 1 & -1\end{vmatrix}$$

$=-5\hat{\mathbf{i}}-7\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=(-5\hat{\mathbf{i}}-7\hat{\mathbf{j}}+3\hat{\mathbf{k}})(\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}})$

$=-5+14-3=6$

$\therefore\ \frac{1}{3}[(\mathbf{a}\times\mathbf{b})-\mathbf{c}]=\frac{1}{3}\times 6=2$

80 Let O be the origin. Let $\mathbf{OP}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{OQ}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3x\hat{\mathbf{k}}$, $x, y\in R, x>0$, be such that $|\mathbf{PQ}|=\sqrt{20}$ and the vector $\mathbf{OP}$ is perpendicular to $\mathbf{OQ}$. If $\mathbf{OR}=3\hat{\mathbf{i}}+z\hat{\mathbf{j}}-7\hat{\mathbf{k}}$, $z\in R$, is coplanar with $\mathbf{OP}$ and $\mathbf{OQ}$, then the value of $x^2+y^2+z^2$ is equal to

[2021, 17 March Shift-II]

(a) 7 (b) 9
(c) 2 (d) 1

Ans. *(b)*

Given, $\mathbf{OP}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}-\hat{\mathbf{k}}$

$\mathbf{OQ}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3x\hat{\mathbf{k}}$

$\mathbf{OR}=3\hat{\mathbf{i}}+z\hat{\mathbf{j}}-7\hat{\mathbf{k}}$

and $|\mathbf{PQ}|=\sqrt{20}$

Now, $|\mathbf{PQ}|=|\mathbf{OQ}-\mathbf{OP}|=|\mathbf{OP}-\mathbf{OQ}|$

$=(x+1)\hat{\mathbf{i}}+(y-2)\hat{\mathbf{j}}-(1+3x)\hat{\mathbf{k}}$

$\Rightarrow\ |\mathbf{PQ}|^2=(\sqrt{20})^2=20$

$\Rightarrow\ (x+1)^2+(y-2)^2+(1+3x)^2=20$

$\Rightarrow\ (x+1)^2+(2x-2)^2+(1+3x)^2=20$

$\left[\begin{array}{l}\because \mathbf{OP}\perp\mathbf{OQ}\\ \therefore \mathbf{OP}\cdot\mathbf{OQ}=0\\ \Rightarrow -x+2y-3x=0\\ \Rightarrow y=2x\end{array}\right]$

$\Rightarrow x^2+1+2x+4x^2+4-8x+1+9x^2+6x=20$

$\Rightarrow 14x^2+6=20\Rightarrow 14x^2=14$

$\Rightarrow x^2=1\Rightarrow x=\pm 1$ but x must be positive as in question conditions i.e. $x>0$.

$\therefore\ x=-1$ (Rejected)

Hence, $x=1$

$\therefore\ y=2x=2\times 1=2$

Now, $\mathbf{OP}$, $\mathbf{OQ}$ and $\mathbf{OR}$ are coplanar.

$\therefore\ [\mathbf{OP}\ \mathbf{OQ}\ \mathbf{OR}]=0$

$$\Rightarrow\ \begin{vmatrix}x & y & -1\\ -1 & 2 & 3x\\ 3 & z & -7\end{vmatrix}=0$$

$$\Rightarrow\ \begin{vmatrix}1 & 2 & -1\\ -1 & 2 & 3\\ 3 & z & -7\end{vmatrix}=0$$

$\Rightarrow\ 1(-14-3z)-2(7-9)-1(-z-6)=0$

$\Rightarrow\ z=-2$

$\therefore\ x^2+y^2+z^2=1+4+4=9$

81 If (1, 5, 35), (7, 5, 5), (1, λ, 7) and (2λ, 1, 2) are coplanar, then the sum of all possible values of λ is

[2021, 26 Feb. Shift-I]

(a) $\frac{39}{5}$ (b) $-\frac{39}{5}$ (c) $\frac{44}{5}$ (d) $-\frac{44}{5}$

Ans. *(c)*

Let $P(1,5,35)$, $Q(7,5,5)$, $R(1,\lambda,7)$, $S(2\lambda,1,2)$

Given P, Q, R, S are coplanar. Then, $\mathbf{PQ}$, $\mathbf{PR}$, $\mathbf{PS}$ lie on the same plane.

$\mathbf{PQ}=(7-1)\hat{\mathbf{i}}+(5-5)\hat{\mathbf{j}}+(5-35)\hat{\mathbf{k}}$

$=6\hat{\mathbf{i}}-30\hat{\mathbf{k}}$

$\mathbf{PR}=(1-1)\hat{\mathbf{i}}+(\lambda-5)\hat{\mathbf{j}}+(7-35)\hat{\mathbf{k}}$

$=(\lambda-5)\hat{\mathbf{j}}-28\hat{\mathbf{k}}$

$\mathbf{PS}=(2\lambda-1)\hat{\mathbf{i}}+(1-5)\hat{\mathbf{j}}+(2-35)\hat{\mathbf{k}}$

$=(2\lambda-1)\hat{\mathbf{i}}-4\hat{\mathbf{j}}-33\hat{\mathbf{k}}$

$\because$ $\mathbf{PQ}$, $\mathbf{PR}$ and $\mathbf{PS}$ lie on same plane, then

$$\begin{vmatrix}6 & 0 & -30\\ 0 & \lambda-5 & -28\\ 2\lambda-1 & -4 & -33\end{vmatrix}=0$$

Expand along first row,

$6[-33(\lambda-5)-112]+30[(2\lambda-1)(\lambda-5)]=0$

$6(-33\lambda+53)+30(2\lambda^2-11\lambda+5)=0$

$60\lambda^2-528\lambda+468=0$

$10\lambda^2-88\lambda+78=0$

$5\lambda^2-44\lambda+39=0$ ... (i)

Possible value of λ are roots of Eq. (i).

Then, sum of all possible values of λ

= Sum of roots of Eq. (i)

$=\frac{-(-44)}{5}=\frac{44}{5}$

[$\because ax^2+bx+c=0$, sum of roots $=-b/a$]

82 If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a}\times(\mathbf{a}\times(\mathbf{a}\times(\mathbf{a}\times\mathbf{b})))$ is equal to

[2021, 26 Feb. Shift-I]

(a) 0 (c) $\frac{1}{2}|\mathbf{a}|^4\mathbf{b}$

(c) $\mathbf{a}\times\mathbf{b}$ (d) $|\mathbf{a}|^4\mathbf{b}$

Ans. *(d)*

$\mathbf{a}\times[\mathbf{a}\times\{\mathbf{a}\times(\mathbf{a}\times\mathbf{a})\}]$

$=\mathbf{a}\times(\mathbf{a}\times[(\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}])$

[Using, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$]

$=\mathbf{a}\times[\mathbf{a}\times((\mathbf{a}\cdot\mathbf{b})\mathbf{a}-|\mathbf{a}|^2\mathbf{b})]$

$=\mathbf{a}\times[(\mathbf{a}\times(\mathbf{a}\cdot\mathbf{b})\mathbf{a})-|\mathbf{a}|^2(\mathbf{a}\times\mathbf{b})]$

$=\mathbf{a}\times[0-|\mathbf{a}|^2(\mathbf{a}\times\mathbf{b})]$ $[\because \mathbf{a}\times\mathbf{a}=0]$

$=-|\mathbf{a}|^2[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]$

$=-|\mathbf{a}|^2[(\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}]$

$=-(\mathbf{a}\cdot\mathbf{b})\mathbf{a}|\mathbf{a}|^2+|\mathbf{a}|^4\mathbf{b}$ $[\because(\mathbf{a}\cdot\mathbf{a})=|\mathbf{a}|^2]$

$=0+|\mathbf{a}|^4\mathbf{b}$ $[\because \mathbf{a}\cdot\mathbf{b}=0]$

$=|\mathbf{a}|^4\mathbf{b}$

83 Let three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be such that $\mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}\cdot\mathbf{c}=7$ and $\mathbf{b}$ is perpendicular to $\mathbf{c}$, where $\mathbf{a}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{k}}$, then the value of $2|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2$ is ....

[2021, 24 Feb. Shift-I]

Ans. *(75)*

Given, $\mathbf{c}$ is co-planar with $\mathbf{a}$ and $\mathbf{b}$

$\mathbf{a}\cdot\mathbf{c}=7$

$\mathbf{b}\perp\mathbf{c}\Rightarrow\mathbf{b}\cdot\mathbf{c}=0$

$\mathbf{a}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$\mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{k}}$

Now, $\mathbf{c}=\lambda[\mathbf{b}\times(\mathbf{a}\times\mathbf{b})]$

[$\because \mathbf{c}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$]

$=\lambda[(\mathbf{b}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{b}]$

$=\lambda[(\sqrt{5})^2(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$-(-2+1)(2\hat{\mathbf{i}}+\hat{\mathbf{k}})]$

$=\lambda[5(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+2\hat{\mathbf{i}}+\hat{\mathbf{k}}]$

$=\lambda(-3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

Now, $\mathbf{c}\cdot\mathbf{a}=7$

$\Rightarrow \lambda(-3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})=7$

$\Rightarrow 3\lambda+5\lambda+6\lambda=7$

$\Rightarrow \lambda=\frac{7}{14}=\frac{1}{2}$

$\therefore 2|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2$

$=2\left|\left(\frac{-3}{2}-1+2\right)\hat{\mathbf{i}}+\left(\frac{5}{2}+1\right)\hat{\mathbf{j}}+(3+1+1)\hat{\mathbf{k}}\right|^2$

[by putting $\mathbf{a},\mathbf{b},\mathbf{c}$]

$=2\left(\frac{1}{4}+\frac{49}{4}+25\right)$

$=25+50=75$

$\therefore$ Required value is 75.

84 Let x_0 be the point of local maxima of $f(x)=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where and $\mathbf{c}=7\hat{\mathbf{i}}-2\hat{\mathbf{j}}+x\hat{\mathbf{k}}$. Then the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ at $x=x_0$ is

[2020, 4 Sep. Shift-I]

(a) 14 (b) -4 (c) -22 (d) -30

Ans. *(c)*

Given vectors $a=x\hat{\mathbf{i}}-2\hat{j}+3\hat{k}$

$b=-2\hat{\mathbf{i}}+x\hat{\mathbf{j}}-\hat{\mathbf{k}}$

and $c=7\hat{\mathbf{i}}-2\hat{\mathbf{j}}+x\hat{\mathbf{k}}$

And, it is given that

$$f(x)=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}x&-2&3\\-2&x&-1\\7&-2&x\end{vmatrix}$$

$=x(x^2-2)+2(-2x+7)+3(4-7x)$

$=x^3-27x+26$

It is also given that $f(x)$ has local maxima at $x=x_0$.

So, $f'(x_0)=0\Rightarrow 3x_0^2-27=0 \Rightarrow x_0^2=9$

$\Rightarrow x_0=\pm3$, but maximum at $x_0=-3$.

Now, $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$

$=-2x-2x-3-14-2x-x+7x+4+3x$

$=3x-13$

$\therefore$ The value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ at $x=x_0=-3$, is -22

Hence, option (c) is correct.

85 If $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, then the value of $|\hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}})|^2+|\hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}})|^2+|\hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})|^2$ is equal to ....

[2020, 4 Sep. Shift-II]

Ans. *(18)*

Given vector $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

$\therefore |\hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}})|^2=|\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}}|^2$

$=|\mathbf{a}-2\hat{\mathbf{i}}|^2=|\hat{\mathbf{j}}+2\hat{\mathbf{k}}|^2=5$

Similarly,

$|\hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}})|^2=|\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}}|^2=|\mathbf{a}-\hat{\mathbf{j}}|^2$

$=|2\hat{\mathbf{i}}+2\hat{\mathbf{k}}|^2=8$

and $|\hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})|^2=|\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}|^2$

$=|\mathbf{a}-2\hat{\mathbf{k}}|^2=|2\hat{\mathbf{i}}+\hat{\mathbf{j}}|^2=5$

$\therefore |\hat{\mathbf{i}}\times(\mathbf{a}\times\hat{\mathbf{i}})|^2+|\hat{\mathbf{j}}\times(\mathbf{a}\times\hat{\mathbf{j}})|^2+|\hat{\mathbf{k}}\times(\mathbf{a}\times\hat{\mathbf{k}})|^2$

$=5+8+5=18$

86 If the volume of a parallelopiped, whose coterminus edges are given by the vectors $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+n\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-n\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+n\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ $(n\geq0)$, is 158 cu units, then

[2020, 5 Sep. Shift-I]

(a) $n=9$ (b) $\mathbf{b}\cdot\mathbf{c}=10$

(c) $\mathbf{a}\cdot\mathbf{c}=17$ (d) $n=7$

Ans. *(b)*

As we know, the volume of parallelopiped, where coterminus edges are given by vectors $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+n\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-n\hat{\mathbf{k}}$

and $\mathbf{c}=\hat{\mathbf{i}}+n\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $(n\geq0)$, is

$$\begin{vmatrix}1&1&n\\2&4&-n\\1&n&3\end{vmatrix}=158 \text{ [given]}$$

$\Rightarrow 1(12+n^2)-1(6+n)+n(2n-4)=158$

$\Rightarrow 3n^2-5n+6=158$

$\Rightarrow 3n^2-5n-152=0$

$\Rightarrow 3n^2-24n+19n-152=0$

$\Rightarrow (3n+19)(n-8)=0$

$\Rightarrow n=8$ as $n\geq0$

$\therefore \mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+8\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-8\hat{\mathbf{k}}$

and $\mathbf{c}=\hat{\mathbf{i}}+8\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

$\therefore \mathbf{a}\cdot\mathbf{c}=1+8+24=33$

and $\mathbf{b}\cdot\mathbf{c}=2+32-24=10$

87 Let the volume of a parallelopiped whose conterminous edges are given by $\mathbf{u}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$, $\mathbf{v}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and $\mathbf{w}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ be 1 cu. unit. If θ be the angle between the edges $\mathbf{u}$ and $\mathbf{w}$, then $\cos\theta$ can be

[2020, 8 Jan. Shift-I]

(a) $\frac{5}{3\sqrt{3}}$ (b) $\frac{7}{6\sqrt{3}}$

(c) $\frac{7}{6\sqrt{6}}$ (d) $\frac{5}{7}$

Ans. *(b)*

Since, the volume of parallelopiped whose coterminous edges are $\mathbf{u}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$, $\mathbf{v}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ and

$$\mathbf{w}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}} \text{ is } [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]=\begin{vmatrix}1&1&\lambda\\1&1&3\\2&1&1\end{vmatrix}=1$$

cubic unit (given)

$\Rightarrow |1(1-3)-1(1-6)+\lambda(1-2)|=1$

$\Rightarrow |-2+5-\lambda|=1$

$\Rightarrow |\lambda-3|=1 \Rightarrow \lambda-3=\pm1$

$\Rightarrow \lambda=2,4$

Since, angle between $\mathbf{u}$ and $\mathbf{w}$ is θ, so

$$\cos\theta=\frac{|2+1+\lambda|}{\sqrt{1+1+\lambda^2}\sqrt{4+1+1}}=\frac{|\lambda+3|}{\sqrt{\lambda^2+2}\sqrt{6}}$$

for $\lambda=2, \cos\theta=\frac{5}{6}$

for $\lambda=4, \cos\theta=\frac{7}{6\sqrt{3}}$

Hence, option (b) is correct.

88 If the vectors $\mathbf{p}=(a+1)\hat{i}+a\hat{j}+a\hat{k}$, $\mathbf{q}=a\hat{i}+(a+1)\hat{j}+a\hat{k}$ and $\mathbf{r}=a\hat{i}+a\hat{j}+(a+1)\hat{k}\,(a\in R)$ are coplanar and $3(\mathbf{p}\cdot\mathbf{q})^2-\lambda|\mathbf{r}\times\mathbf{q}|^2=0$, then the value of λ is ....

[2020, 9 Jan. Shift-I]

Ans. *(1)*

The given vectors are

$$\mathbf{p}=(a+1)\hat{i}+a\hat{j}+a\hat{k},$$
$$\mathbf{q}=a\hat{i}+(a+1)\hat{j}+a\hat{k}$$

and $\mathbf{r}=a\hat{i}+a\hat{j}+(a+1)\hat{k},(a\in R)$ are coplanar, So, $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}]=0$

$$\Rightarrow \begin{vmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{vmatrix}=0$$

$$\Rightarrow (a+1)[(a+1)^2-a^2]-a[a(a+1)-a^2]+a[a^2-a(a+1)]=0$$

$$\Rightarrow (a+1)[2a+1]-2a[a]=0$$

$$\Rightarrow 2a^2+3a+1-2a^2=0$$

$$\Rightarrow a=-\frac{1}{3}$$

So, $\mathbf{p}=\frac{2}{3}\hat{i}-\frac{1}{3}\hat{j}-\frac{1}{3}\hat{k}$, $\mathbf{q}=-\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}$

and $\mathbf{r}=-\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$

So, $(\mathbf{p}\cdot\mathbf{q})^2=\left(-\frac{2}{9}-\frac{2}{9}+\frac{1}{9}\right)^2=\left(\frac{3}{9}\right)^2=\frac{1}{9}$

and $\mathbf{r}\times\mathbf{q}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{vmatrix}$

$$=\hat{i}\left(\frac{1}{9}-\frac{4}{9}\right)-\hat{j}\left(\frac{1}{9}+\frac{2}{9}\right)+\hat{k}\left(-\frac{2}{9}-\frac{1}{9}\right)$$

$$=-\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}-\frac{1}{3}\hat{k}$$

$\therefore |\mathbf{r}\times\mathbf{q}|^2=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3}$

$\because$ It is given that

$$3(\mathbf{p}\cdot\mathbf{q})^2-\lambda|\mathbf{r}\times\mathbf{q}|^2=0$$

$$\Rightarrow 3\left(\frac{1}{9}\right)-\lambda\left(\frac{1}{3}\right)=0$$

$$\Rightarrow \lambda=1$$

Hence, answer is 1.

89 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors such that $|\mathbf{a}|=\sqrt{3}$, $|\mathbf{b}|=5$, $\mathbf{b}\cdot\mathbf{c}=10$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{3}$. If $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}\times\mathbf{c}$, then $|\mathbf{a}\times(\mathbf{b}\times\mathbf{c})|$ is equal to ....

[2020, 9 Jan. Shift-II]

Ans. *(30)*

There are three vectors given $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, such that $|\mathbf{a}|=\sqrt{3}$, $|\mathbf{b}|=5$ and $\mathbf{b}\cdot\mathbf{c}=10$

So, $|\mathbf{b}||\mathbf{c}|\cdot\cos\frac{\pi}{3}=10$

($\because$ it is given angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{3}$)

$$\Rightarrow 5|\mathbf{c}|\left(\frac{1}{2}\right)=10 \Rightarrow |\mathbf{c}|=4$$

Now, as $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}\times\mathbf{c}$, so

$$|\mathbf{a}\times(\mathbf{b}\times\mathbf{c})|=\left||\mathbf{a}||\mathbf{b}\times\mathbf{c}|\sin\frac{\pi}{2}\right|$$

$$=|\mathbf{a}||\mathbf{b}\times\mathbf{c}|=|\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\frac{\pi}{3}$$

$$=(\sqrt{3})(5)(4)\frac{\sqrt{3}}{2}=30$$

Hence answer is 30.

90 If the volume of parallelopiped formed by the vectors $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}}+\hat{\mathbf{k}}$ is minimum, then λ is equal to **[2019, 12 April Shift-I]**

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\sqrt{3}$ (d) $-\sqrt{3}$

Ans. *(b)*

Key Idea Volume of parallelopiped formed by the vectors **a**, **b** and **c** is $V=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Given vectors are $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}}+\hat{\mathbf{k}}$, which forms a parallelopipe(d)

$\therefore$ Volume of the parallelopiped is

$$V=\left\|\begin{matrix} 1 & \lambda & 1 \\ 0 & 1 & \lambda \\ \lambda & 0 & 1 \end{matrix}\right\|=1+\lambda^3-\lambda$$

$$\Rightarrow V=\lambda^3-\lambda+1$$

On differentiating w.r.t. λ, we get

$$\frac{dV}{d\lambda}=3\lambda^2-1$$

For maxima or minima, $\frac{dV}{d\lambda}=0$

$$\Rightarrow \lambda=\pm\frac{1}{\sqrt{3}}$$

and $\frac{d^2V}{d\lambda^2}=6\lambda$

$$=\begin{cases} 2\sqrt{3}>0, & \text{for } \lambda=\frac{1}{\sqrt{3}} \\ 2\sqrt{3}<0, & \text{for } \lambda=-\frac{1}{\sqrt{3}} \end{cases}$$

$\because \frac{d^2V}{d\lambda^2}$ is positive for $\lambda=\frac{1}{\sqrt{3}}$, so volume '$V$' is minimum for $\lambda=\frac{1}{\sqrt{3}}$

91 Let $\alpha\in R$ and the three vectors

$$\mathbf{a}=\alpha\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}},\ \mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\alpha\hat{\mathbf{k}}$$

and $\mathbf{c}=\alpha\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. Then, the set $S=\{\alpha:\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are coplanar$\}$

[2019, 12 April Shift-II]

(a) is singleton.

(b) is empty.

(c) contains exactly two positive numbers.

(d) contains exactly two numbers only one of which is positive.

Ans. *(b)*

Given three vectors are

$$\mathbf{a}=\alpha\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$
$$\mathbf{b}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\alpha\hat{\mathbf{k}}$$

and $\mathbf{c}=\alpha\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

Clearly, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix}$

$$=\alpha(3-2\alpha)-1(6+\alpha^2)+3(-4-\alpha)$$

$$=-3\alpha^2-18=-3(\alpha^2+6)$$

$\because$ There is no value of α for which $-3(\alpha^2+6)$ becomes zero, so

$$=\begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\neq 0$$

$\Rightarrow$ vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not coplanar for any value $\alpha\in R$.

So, the set $S=\{\alpha:\mathbf{a},\ \mathbf{b}$ and $\mathbf{c}$ are coplanar$\}$ is empty set.

92 Let $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$, $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}$ be a vector such that $\mathbf{a}\times\mathbf{c}+\mathbf{b}=\mathbf{0}$ and $\mathbf{a}\cdot\mathbf{c}=4$, then $|\mathbf{c}|^2$ is equal to

[2019, 9 Jan. Shift-I]

(a) 8 (b) $\frac{19}{2}$

(c) 9 (d) $\frac{17}{2}$

Ans. *(b)*

We have, $(a\times c)+b=0$

$\Rightarrow a\times(a\times c)+a\times b=0$

(taking cross product with a on both sides)

$$\Rightarrow (a\cdot c)a-(a\cdot a)c+\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}=\mathbf{0}$$

$(\because a\times(b\times c)=(a\cdot c)b-(a\cdot b)c)$

$$\Rightarrow 4(\hat{i}-\hat{j})-2c+(-\hat{i}-\hat{j}+2\hat{k})=0$$

$[\because a\cdot a=(\hat{i}-\hat{j})(\hat{i}-\hat{j})=1+1=2$ and $a\cdot c=4]$

$\Rightarrow\ 2c = 4\hat{i} - 4\hat{j} - \hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow\ c = \dfrac{3\hat{i} - 5\hat{j} + 2\hat{k}}{2}$

$\Rightarrow\ |c|^2 = \dfrac{9 + 25 + 4}{4} = \dfrac{19}{2}$

93 Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + (\lambda^2 - 1)\hat{\mathbf{k}}$ be coplanar vectors. Then, the non-zero vector $\mathbf{a} \times \mathbf{c}$ is **[2019, 11 Jan. Shift-I]**

(a) $-10\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (b) $-10\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$
(c) $-14\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ (d) $-14\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

Ans. *(a)*

We know that, if a, b, c are coplanar vectors, then $[a\ \ b\ \ c] = 0$

$\therefore\ \begin{vmatrix} 1 & 2 & 4 \\ 1 & \lambda & 4 \\ 2 & 4 & \lambda^2 - 1 \end{vmatrix} = 0$

$\Rightarrow\ 1\{\lambda(\lambda^2 - 1) - 16\} - 2((\lambda^2 - 1) - 8) + 4(4 - 2\lambda) = 0$

$\Rightarrow \lambda^3 - \lambda - 16 - 2\lambda^2 + 18 + 16 - 8\lambda = 0$

$\Rightarrow\ \lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0$

$\Rightarrow\ \lambda^2(\lambda - 2) - 9(\lambda - 2) = 0$

$\Rightarrow\ (\lambda - 2)(\lambda^2 - 9) = 0$

$\Rightarrow\ (\lambda - 2)(\lambda + 3)(\lambda - 3) = 0$

$\therefore\ \lambda = 2,\ 3 \text{ or } -3$

If $\lambda = 2$, then $a \times c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & 4 & 3 \end{vmatrix}$

$= \hat{i}(6 - 16) - \hat{j}(3 - 8) + \hat{k}(4 - 4)$

$= -10\hat{i} + 5\hat{j}$

If $\lambda = \pm 3$, then $a \times c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{vmatrix} = 0$

(because last two rows are proportional).

94 The sum of the distinct real values of μ, for which the vectors, $\mu\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \hat{\mathbf{i}} + \mu\hat{\mathbf{j}} + \hat{\mathbf{k}}, \hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\hat{\mathbf{k}}$ are coplanar, is **[2019, 12 Jan. Shift-I]**

(a) 2 (b) 0 (c) 1 (d) -1

Ans. *(d)*

Given vectors, $\mu\hat{i} + \hat{j} + \hat{k}, \hat{i} + \mu\hat{j} + \hat{k}$, $\hat{i} + \hat{j} + \mu\hat{k}$ will be coplanar, if

$\begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = 0$

$\Rightarrow\ \mu(\mu^2 - 1) - 1(\mu - 1) + 1(1 - \mu) = 0$

$\Rightarrow\ (\mu - 1)[\mu(\mu + 1) - 1 - 1] = 0$

$\Rightarrow\ (\mu - 1)[\mu^2 + \mu - 2] = 0$

$\Rightarrow\ (\mu - 1)[(\mu + 2)(\mu - 1)] = 0$

$\Rightarrow\ \mu = 1 \text{ or } -2$

So, sum of the distinct real values of $\mu = 1 - 2 = -1$.

95 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three unit vectors, out of which vectors $\mathbf{b}$ and $\mathbf{c}$ are non-parallel. If α and β are the angles which vector $\mathbf{a}$ makes with vectors $\mathbf{b}$ and $\mathbf{c}$ respectively and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \dfrac{1}{2}\mathbf{b}$, then $|\alpha - \beta|$ is equal to **[2019, 12 Jan. Shift-II]**

(a) 30º (b) 45º
(c) 90º (d) 60º

Ans. *(a)*

Given, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \dfrac{1}{2}\mathbf{b}$

$\Rightarrow\ (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \dfrac{1}{2}\mathbf{b}$

$[\because \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}]$

On comparing both sides, we get

$\mathbf{a} \cdot \mathbf{c} = \dfrac{1}{2}$ ...(i)

and $\mathbf{a} \cdot \mathbf{b} = 0$ ...(ii)

$\because \mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors, and angle between $\mathbf{a}$ and $\mathbf{b}$ is α and angle between $\mathbf{a}$ and $\mathbf{c}$ is β, so

$|\mathbf{a}||\mathbf{c}|\cos\beta = \dfrac{1}{2}$ [from Eq. (i)]

$\Rightarrow\ \cos\beta = \dfrac{1}{2}$ $[\because |\mathbf{a}| = 1 = |\mathbf{c}|]$

$\Rightarrow\ \beta = \dfrac{\pi}{3}$ ...(iii)

$\left[\because \cos\dfrac{\pi}{3} = \dfrac{1}{2}\right]$

and $|\mathbf{a}||\mathbf{b}|\cos\alpha = 0$ [from Eq. (ii)]

$\Rightarrow\ \alpha = \dfrac{\pi}{2}$ ...(iv)

From Eqs. (iii) and (iv), we get

$|\alpha - \beta| = \dfrac{\pi}{2} - \dfrac{\pi}{3} = \dfrac{\pi}{6} = 30°$

96 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \dfrac{\sqrt{3}}{2}(\mathbf{b} + \mathbf{c})$. If $\mathbf{b}$ is not parallel to $\mathbf{c}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is **[JEE Main 2016]**

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{2}$
(c) $\dfrac{2\pi}{3}$ (d) $\dfrac{5\pi}{6}$

Ans. *(d)*

Given, $|\hat{a}| = |\hat{b}| = |\hat{c}| = 1$

and $\hat{a} \times (\hat{b} \times \hat{c}) = \dfrac{\sqrt{3}}{2}(\hat{b} + \hat{c})$

Now, consider $\hat{a} \times (\hat{b} \times \hat{c}) = \dfrac{\sqrt{3}}{2}(\hat{b} + \hat{c})$

$\Rightarrow\ (\hat{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c} = \dfrac{\sqrt{3}}{2}\hat{b} + \dfrac{\sqrt{3}}{2}\hat{c}$

On comparing, we get

$\hat{a} \cdot \hat{b} = -\dfrac{\sqrt{3}}{2} \Rightarrow |\hat{a}||\hat{b}|\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\Rightarrow\ \cos\theta = -\dfrac{\sqrt{3}}{2}$ $[\because |\hat{a}| = |\hat{b}| = 1]$

$\Rightarrow\ \cos\theta = \cos\left(\pi - \dfrac{\pi}{6}\right) \Rightarrow \theta = \dfrac{5\pi}{6}$

97 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors such that no two of them are collinear and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \dfrac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, then a value of $\sin\theta$ is **[JEE Main 2015]**

(a) $\dfrac{2\sqrt{2}}{3}$ (b) $\dfrac{-\sqrt{2}}{3}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{-2\sqrt{3}}{3}$

Ans. *(a)*

Given, $(a \times b) \times c = \dfrac{1}{3}|b||c|a$

$\Rightarrow\ -c \times (a \times b) = \dfrac{1}{3}|b||c|a$

$\Rightarrow\ -(c \cdot b) \cdot a + (c \cdot a)b = \dfrac{1}{3}|b||c|a$

$\left[\dfrac{1}{3}|b||c| + (c \cdot b)\right]a = (c \cdot a)b$

Since, a and b are not collinear.

$c \cdot b + \dfrac{1}{3}|b||c| = 0$ and $c \cdot a = 0$

$\Rightarrow\ |c||b|\cos\theta + \dfrac{1}{3}|b||c| = 0$

$\Rightarrow\ |b||c|\left(\cos\theta + \dfrac{1}{3}\right) = 0$

$\Rightarrow\ \cos\theta + \dfrac{1}{3} = 0$ $(\because |b| \neq 0, |c| \neq 0)$

$\Rightarrow\ \cos\theta = -\dfrac{1}{3}$

$\Rightarrow\ \sin\theta = \dfrac{\sqrt{8}}{3} = \dfrac{2\sqrt{2}}{3}$

98 If $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = \lambda[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$, then λ is equal to **[JEE Main 2014]**

(a) 0 (b) 1
(c) 2 (d) 3

Ans. *(b)*

Use the formulae

$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$,

$[a\,b\,c] = [b\,c\,a] = [c\,a\,b]$

and $[a\,a\,b] = [a\,b\,b] = [a\,c\,c] = 0$

Further simplify it and get the result.

Now, $[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]$

$= \mathbf{a}\times\mathbf{b}\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}))$

$= \mathbf{a}\times\mathbf{b}\cdot((\mathbf{k}\times\mathbf{c}\times\mathbf{a}))$ [here, $\mathbf{k}=\mathbf{b}\times\mathbf{c}$]

$= \mathbf{a}\times\mathbf{b}\cdot[(\mathbf{k}\cdot\mathbf{a})\mathbf{c}-(\mathbf{k}\cdot\mathbf{c})\mathbf{a}]$

$= (\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\times\mathbf{c}\cdot\mathbf{c})\mathbf{a})$

$= (\mathbf{a}\times\mathbf{b})\cdot([\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\mathbf{c})-0$ $[\because [\mathbf{b}\times\mathbf{c}\cdot\mathbf{c}]=0]$

$= \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\,[\mathbf{b}\,\mathbf{c}\,\mathbf{a}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}][\mathbf{b}\,\mathbf{c}\,\mathbf{a}]$

$= [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$ $\{\because [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\}$

Hence, $[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}] = \lambda[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

$\Rightarrow [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2 = \lambda[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

$\Rightarrow \lambda = 1$

99 If the vectors $p\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+q\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{j}}+r\hat{\mathbf{k}}$ (where, $p\neq q\neq r\neq 1$) are coplanar, then the value of $pqr-(p+q+r)$ is **[AIEEE 2011]**

(a) -2 (b) 2 (c) 0 (d) -1

Ans. *(a)*

Given, $\mathbf{a}=p\hat{i}+\hat{j}+\hat{k}$, $\mathbf{b}=\hat{i}+q\hat{j}+\hat{k}$ and $\mathbf{c}=\hat{i}+\hat{j}+r\hat{k}$ are coplanar and $p\neq q\neq r\neq 1$.

Since, a, b and c are coplanar.

$$\Rightarrow [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=0 \Rightarrow \begin{vmatrix} p & 1 & 1\\ 1 & q & 1\\ 1 & 1 & r\end{vmatrix}=0$$

$\Rightarrow p(qr-1)-1(r-1)+1(1-q)=0$

$\Rightarrow pqr-p-r+1+1-q=0$

$\therefore pqr-(p+q+r)=-2$

100 Let $\mathbf{a}=\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$. Then, the vector b satisfying $\mathbf{a}\times\mathbf{b}+\mathbf{c}=0$ and $\mathbf{a}\cdot\mathbf{b}=3$, is **[AIEEE 2010]**

(a) $-\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

(c) $\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Ans. *(a)*

We have, $\mathbf{a}\times\mathbf{b}+\mathbf{c}=0$

$\Rightarrow \mathbf{a}\times(\mathbf{a}\times\mathbf{b})+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow (\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow 3\mathbf{a}-2\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow 2\mathbf{b}=3\mathbf{a}+\mathbf{a}\times\mathbf{c}$

$\Rightarrow 2\mathbf{b}=3\hat{j}-3\hat{k}-2\hat{i}-\hat{j}-\hat{k}$

$= -2\hat{i}+2\hat{j}-4\hat{k}$

$\therefore \mathbf{b}=-\hat{i}+\hat{j}-2\hat{k}$

101 If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\mathbf{u}\;p\mathbf{v}\;p\mathbf{w}]-[p\mathbf{v}\;\mathbf{w}\;q\mathbf{u}]-[2\mathbf{w}\;q\mathbf{v}\;q\mathbf{u}]=0$ holds for **[AIEEE 2009]**

(a) exactly two values of (p, q).

(b) more than two but not all values of (p, q).

(c) all values of (p, q).

(d) exactly one value of (p, q).

Ans. *(d)*

Since, $[3\mathbf{u}\;p\mathbf{v}\;p\mathbf{w}]-[p\mathbf{v}\;\mathbf{w}\;q\mathbf{u}]-[2\mathbf{w}\;q\mathbf{v}\;q\mathbf{u}]=0$

$\therefore 3p^2[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})]-pq[\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})]-2q^2[\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})]=0$

$\Rightarrow (3p^2-pq+2q^2)[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})]=0$

But $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]\neq 0$

$\Rightarrow 3p^2-pq+2q^2=0$

$\therefore p=q=0$

102 The vector $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$ lies in the plane of the vectors $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and bisects the angle between $\mathbf{b}$ and $\mathbf{c}$. Then, which one of the following gives possible values of α and β? **[AIEEE 2008]**

(a) $\alpha=1, \beta=1$

(b) $\alpha=2, \beta=2$

(c) $\alpha=1, \beta=2$

(d) $\alpha=2, \beta=1$

Ans. *(a)*

Given that, $\mathbf{b}=\hat{i}+\hat{j}$ and $\mathbf{c}=\hat{j}+\hat{k}$.

The equation of bisector of b and c is

$$\mathbf{r}=\lambda(\mathbf{b}+\mathbf{c})=\lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}+\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)$$

$$=\frac{\lambda}{\sqrt{2}}(\hat{i}+2\hat{j}+\hat{k}) \quad \text{...(i)}$$

Since, vector a lies in plane of b and c.

$\therefore \mathbf{a}=\mathbf{b}+\mu\mathbf{c}$

$\Rightarrow \frac{\lambda}{\sqrt{2}}(\hat{i}+2\hat{j}+\hat{k})=(\hat{i}+\hat{j})+\mu(\hat{j}+\hat{k})$

On equating the coefficient of i both sides, we get

$\frac{\lambda}{\sqrt{2}}=1 \Rightarrow \lambda=\sqrt{2}$

On putting $\lambda=\sqrt{2}$ in Eq. (i), we get

$\mathbf{r}=\hat{i}+2\hat{j}+\hat{k}$

Since, the given vector a represents the same bisector equation r.

$\therefore \alpha=1$ and $\beta=1$

Alternate Solution

Since, a, b and c are coplanar.

$$\Rightarrow \begin{vmatrix}\alpha & 2 & \beta\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=0$$

$\Rightarrow \alpha(1-0)-2(1-0)+\beta(1-0)=0$

$\Rightarrow \alpha+\beta=2$, which is possible for $\alpha=1, \beta=1$.

103 Let $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{c}=x\hat{\mathbf{i}}+(x-2)\hat{\mathbf{j}}-\hat{\mathbf{k}}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then x equal to **[AIEEE 2007]**

(a) 0 (b) 1 (c) -4 (d) -2

Ans. *(d)*

Since, given vectors a, b and c are coplanar.

$$\therefore \begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ x & x-2 & -1\end{vmatrix}=0$$

$\Rightarrow 1\{1-2(x-2)\}-1(-1-2x)+1(x-2+x)=0$

$\Rightarrow 1-2x+4+1+2x+2x-2=0$

$\Rightarrow 2x=-4 \Rightarrow x=-2$

104 If $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are any three vectors such that $\mathbf{a}\cdot\mathbf{b}\neq 0$, $\mathbf{b}\cdot\mathbf{c}\neq 0$, then $\mathbf{a}$ and $\mathbf{c}$ are **[AIEEE 2006]**

(a) inclined at an angle of $\frac{\pi}{6}$ between them.

(b) perpendicular.

(c) parallel.

(d) inclined at an angle of $\frac{\pi}{3}$ between them.

Ans. *(c)*

Since, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\therefore (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\Rightarrow (\mathbf{b}\cdot\mathbf{c})\mathbf{a}=(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\Rightarrow \mathbf{a}=\frac{(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{b}\cdot\mathbf{c})}\cdot\mathbf{c}$

Hence, a is parallel to c.

105 Let $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{k}}$, $\mathbf{b}=x\hat{\mathbf{i}}+\hat{\mathbf{j}}+(1-x)\hat{\mathbf{k}}$ and $\mathbf{c}=y\hat{\mathbf{i}}+x\hat{\mathbf{j}}+(1+x-y)\hat{\mathbf{k}}$. Then, $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ depends on **[AIEEE 2005]**

(a) Neither x nor y (b) Both x and y

(c) Only x (d) Only y

Ans. *(a)*

Given, vectors are

$\mathbf{a}=\hat{i}-\hat{k}$, $\mathbf{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$

and $\mathbf{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$

$$\therefore [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=\begin{vmatrix}1 & 0 & -1\\ x & 1 & 1-x\\ y & x & 1+x-y\end{vmatrix}$$

Applying $C_3\to C_3+C_1$, we get

$$=\begin{vmatrix}1 & 0 & 0\\ x & 1 & 1\\ y & x & 1+x\end{vmatrix}=1(1+x)-x=1$$

Thus, $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ depends upon neither x nor y.

106 Let a, b and c be distinct non-negative numbers. If the vectors $a\hat{\mathbf{i}}+a\hat{\mathbf{j}}+c\hat{\mathbf{k}}, \hat{\mathbf{i}}+\hat{\mathbf{k}}$ and $c\hat{\mathbf{i}}+c\hat{\mathbf{j}}+b\hat{\mathbf{k}}$ lie in a plane, then c is **[AIEEE 2005]**

(a) the harmonic mean of a and b.
(b) equal to zero.
(c) the arithmetic mean of a and b.
(d) the geometric mean of a and b.

Ans. ***(d)***

Since, the given vectors lie in a plane.

$$\therefore \quad \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_2$, we get

$$\begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix} = 0$$

$$\Rightarrow \quad -1(ab - c^2) = 0$$

$$\Rightarrow \quad c^2 = ab$$

Hence, c is GM of a and b.

107 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors and λ is a real number, then $[\lambda(\mathbf{a}+\mathbf{b}) \;\; \lambda^2\mathbf{b} \;\; \lambda\mathbf{c}] = [\mathbf{a} \;\; \mathbf{b}+\mathbf{c} \;\; \mathbf{b}]$ for **[AIEEE 2005]**

(a) exactly two values of λ.
(b) exactly three values of λ.
(c) no value of λ.
(d) exactly one value of λ.

Ans. ***(c)***

Given that,
$[\lambda(a+b) \;\; \lambda^2 b \;\; \lambda c] = [a \;\; b+c \;\; b]$

$$\therefore \begin{vmatrix} \lambda(a_1+b_1) & \lambda(a_2+b_2) & \lambda(a_3+b_3) \\ \lambda^2 b_1 & \lambda^2 b_2 & \lambda^2 b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\Rightarrow \lambda^4 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad \lambda^4 = -1$$

So, no real value of λ exists.

108 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a}+2\mathbf{b}+3\mathbf{c}, \lambda\mathbf{b}+4\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ are non-coplanar for

(a) all values of λ. **[AIEEE 2004]**
(b) all except one value of λ.
(c) all except two values of λ.
(d) no value of λ.

Ans. ***(c)***

The three vectors $(a+2b+3c), (\lambda b+4c)$ and $(2\lambda-1)c$ are non-coplanar, if

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & 2\lambda-1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad (2\lambda-1)(\lambda) \neq 0 \Rightarrow \lambda \neq 0, \frac{1}{2}$$

So, these three vectors are non-coplanar for all except two values of λ.

109 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is an acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ is equal to **[AIEEE 2004]**

(a) $\frac{1}{3}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$

Ans. ***(d)***

Given that, $\frac{1}{3}|b||c|a = (a\times b)\times c$

We know that, $(a\times b)\times c = (a\cdot c)b - (b\cdot c)a$

$$\therefore \quad \frac{1}{3}|b||c|a = (a\cdot c)b - (b\cdot c)a$$

On comparing the coefficients of a and b, we get

$$\frac{1}{3}|b||c| = -b\cdot c \text{ and } a\cdot c = 0$$

$$\Rightarrow \quad \frac{1}{3}|b||c| = -|b||c|\cos\theta$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{3} \Rightarrow 1-\sin^2\theta = \frac{1}{9}$$

$$\Rightarrow \quad \sin^2\theta = \frac{8}{9}$$

$$\therefore \quad \sin\theta = \frac{2\sqrt{2}}{3} \quad \left[\because 0 \le \theta \le \frac{\pi}{2}\right]$$

110 If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are three non-coplanar vectors, then $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot[(\mathbf{u}-\mathbf{v})\times(\mathbf{v}-\mathbf{w})]$ equal to **[AIEEE 2003]**

(a) 0 (b) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$
(c) $\mathbf{u}\cdot\mathbf{w}\times\mathbf{v}$ (d) $3\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$

Ans. ***(b)***

$$(u+v-w)\cdot[(u-v)\times(v-w)]$$
$$= (u+v-w)\cdot[u\times v - u\times w - v\times v + v\times w]$$
$$= u\cdot(u\times v) - u\cdot(u\times w) + u\cdot(v\times w) + v\cdot(u\times v) - v\cdot(u\times w) + v\cdot(v\times w) - w\cdot(u\times v) + w\cdot(u\times w) - w\cdot(v\times w)$$
$$= u\cdot v\times w - v\cdot u\times w - w\cdot u\times v \quad \{[a\ a\ b] = 0\}$$
$$= u\cdot v\times w + w\cdot u\times v - w\cdot u\times v = u\cdot v\times w$$