CHAPTER 21

FUNCTIONS

21.1 DEFINITION OF FUNCTION

Let X and Y be two non-empty sets. Then a function 'f' from set X to set Y is denoted as $f : X \to Y$ or $y = f(x)$; $x \in X$ and $y \in Y$. A function f(x) from X (domain) to Y (co-domain) is defined as a relation f from set X to set Y such that each and every element of X is related with exactly one element of set Y.

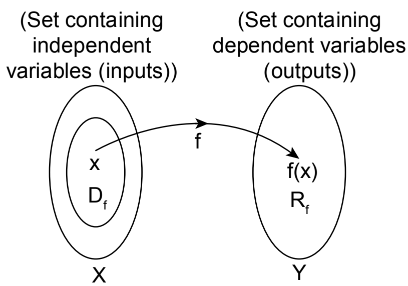
Image and Pre-image: Let f be a function from set X to set Y, i.e., $f : X \to Y$ and let an element x of set X be associated to the element y of set Y through the rule 'f', then $(x, y) \in f$, i.e., $f(x) = y$, then y is called 'image of x under f' and x is called 'pre-image of y under f'.

Natural Domain: The natural domain of a function is the largest set of real number inputs that give real number outputs of the function.

Co-domain: Set Y is called co-domain of function f.

Range of Function: If $f : D_f (\subseteq X) \to Y$ is a function with domain D_f, then the set of images y (output $\in Y$) generated corresponding to input $x \in D_f$ is called range of function, and it is denoted by R_f.

i.e., $R_f = \{f(x): x \in D_f\} \subseteq Y$.



Remarks:

(i) Every function is a relation but every relation read not be a function

(ii) A relation $R : A \to B$ is a function if its domain = A and it is not one-many i.e., either one-one or many-many.

(iii) *To find domain of function, we need to know when does a function become undefined and when it is defined*

i.e., we need to find those values of x where f(x) is finite and real and those values of x where f(x) is either infinite or imaginary.

(iv) *When its value tends to infinity (∞).*

e.g., $y = \frac{1}{x^2 - 1}$ *at x = ±1; f(x) is not defined at x = ±1 and defined ∀ x ∈ ℝ except for ±1; therefore domain of f(x) = ℝ ~ {1, –1}*

(v) *When it takes imaginary value. e.g.,* $y = \sqrt{x-1}$ *at x ∈ (–∞, 1); f(x) is not defined on (–∞, 1) and defined on [1, ∞); therefore domain of f(x) = [1, ∞).*

(vi) *When it takes indeterminate form, i.e., becomes of the form* $\frac{0}{0}, \frac{\infty}{\infty}, 1^{\infty}, \infty^0, 0^0, \infty - \infty$, *etc.*

21.2 REPRESENTATION OF A FUNCTION

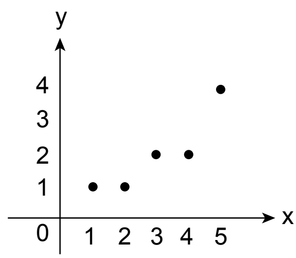
A fanction can be represented analytcally as orduced pass, parametrically wita arrow diagram praphibly.

Remarks:

All function cannot be represented by all the above methods.

(i) *The Drichlet-Function which is defined as f(x) =* $\begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$ *cannot be graphed since there exist infinite number of rationals as well as irrationals between any two real numbers.*

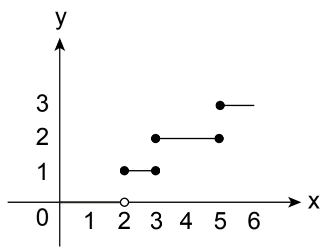
(ii) *Consider the Euler's totient or Euler's phi function φ(n) = Number of positive integers less than or equal to n and co-prime to n; where n is a natural number.*



The domain of φ is the set of positive integers. Its range is the set of positive integers {1, 2, 3, …}.

We cannot represent this function analytically. A portion of the graph of φ(n) as shown here for understanding of the function.

(iii) *Consider another function called prime number function defined by f(x) = number of prime numbers less than or equals to x; where x is non-negative real number.*



Then domain of f(x) is (0, ∞) and range is the set of non-negative integers, i.e., {0, 1, 2, 3, ….}.

The graph of function is shown here.

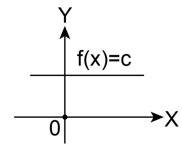
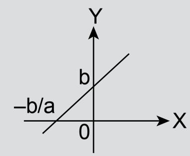
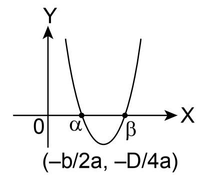
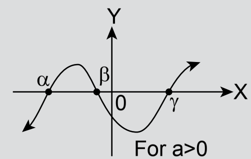
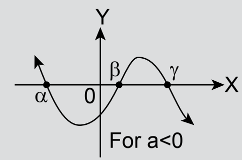
As x increases, the function f(x) remains constant until x reaches a prime, at which the graph of function jumps by 1.Therefore, the graph of f consists of horizontal line segments. This is an example of a class of function called step functions.

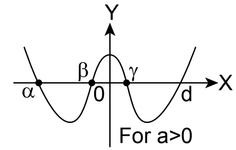
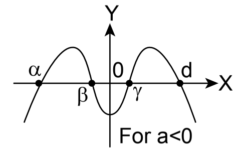
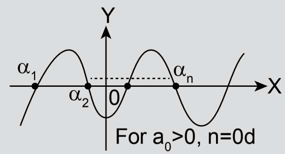
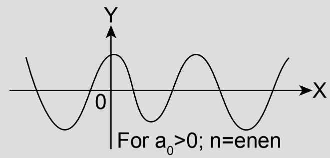
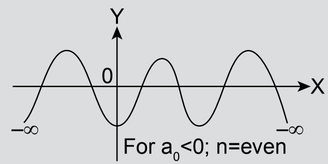
(iv) *Another function, which is so complicated that it is impossible to draw its graph,*

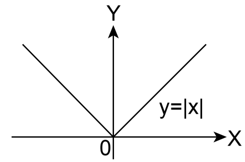
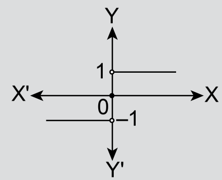
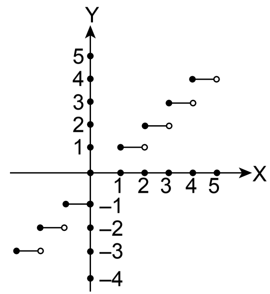
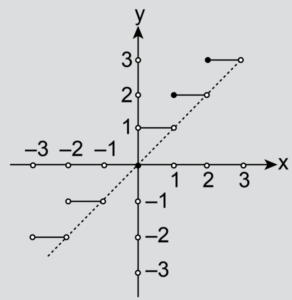
$$h(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -\frac{x}{2} & \text{if } x \text{ is irrational} \end{cases}.$$

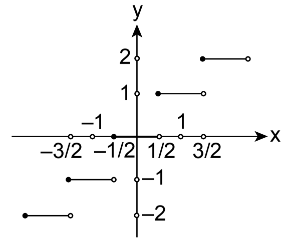
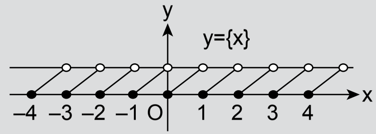
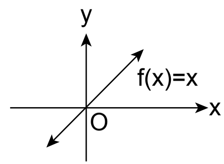
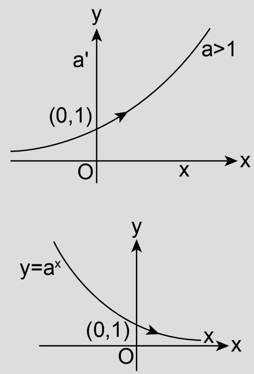
As we know that between any two real numbers there lie infinitely many relations and irrational number, so it is impossible to draw its graph.

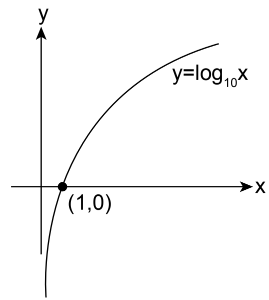
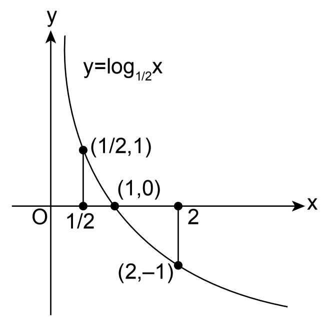
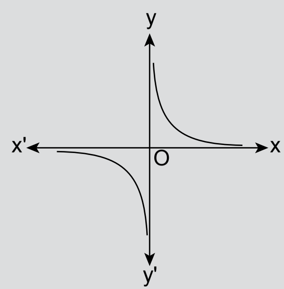
21.3 SOME STANDARD FUNCTION

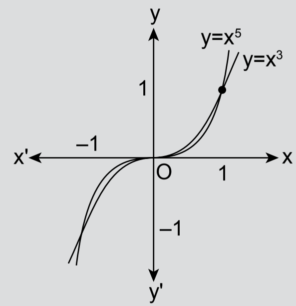
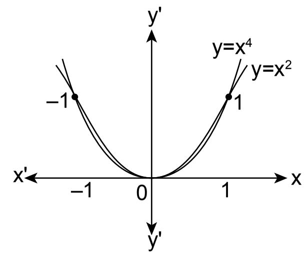
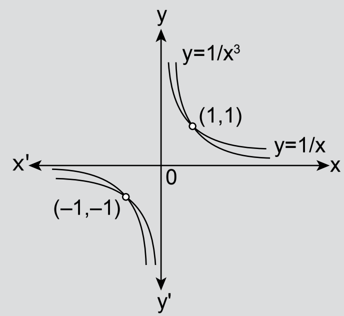
S. NO.	Standard Function	Basic Definition	Domain	Range	Form of Curve Function
1.	Constant function	$y = c;\ c \in \mathbb{R}$ is a fined read number	R	{c}	
2.	line as functions	$y = ax^2 + b;\ a, b \in \mathbb{R};\ a \neq 0$	R	R	
3.	Quadeatic function	$y = ax^2 + bx + c;\ a, b, c, \in \mathbb{R};\ a \neq 0$	R	$\left[-\frac{D}{4a}, \infty\right)$; where $D = b^2 - 4ac$	
4.	Cubic function	$y = ax^3 + bx^2 + cx + d;\ a, b, c, d \in \mathbb{R};$ $a \neq 0$	R	R	

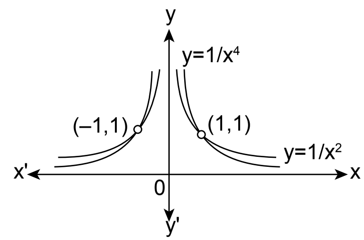
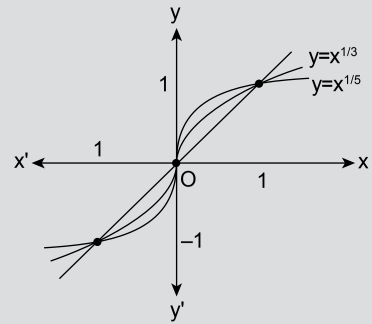
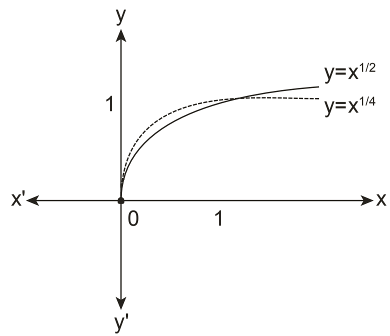
5.	Biquadratic function	$y = ax^4 + bx^3 + cx^2 + dx + e$; a, b, c, d, e $\in \mathbb{R}$; $a \neq 0$	R	$[f(k), \infty]$; for $a > 0$; where K is the point of local minima having least image and $(-\infty, f(k)]$; for $a < 0$; where k is the point of local maxima having greatest image.	 
6.	Polynomial function of nth degree	$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots. + a_{n-1}\, x + a_n$; $a_i \in \mathbb{R}$; $a_0 \neq 0$, $n \in \mathbb{N}$	R	$= \mathbb{R}$ if n is odd; $= [f(k), \infty]$ for $a_0 > 0$; k is point of local minima having least image if n is even; $= (-\infty, f(k)]$ for $a_0 > 0$; k is point of local maxima having greatest image if n is even.	  

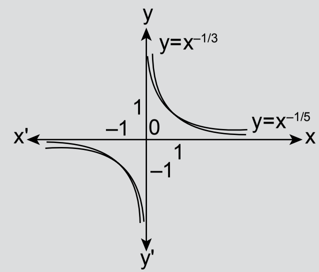
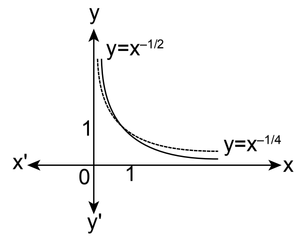
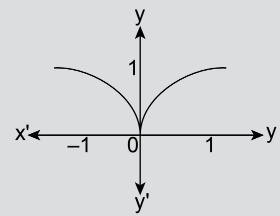
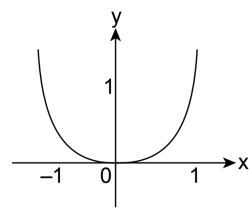
7.	Modulus function	$f(x)=\lvert x\rvert=\begin{cases} x; x\ge 0 \\ -x; x<0 \end{cases}$	R	$[0, \infty)$	
8.	Signum function	$f(x)=\text{sgn}(x)\begin{cases} -1 \text{ for } x<0 \\ 0 \text{ for } x=0 \\ 1 \text{ for } x>0 \end{cases}$	R	$\{-1, 0, 1\}$	
9.	Greatest integer function	$f(x)=[x]=\begin{cases} x \text{ if } x\in\mathbb{Z} \\ k \text{ if } k<x<k+1; \\ k\in\mathbb{Z} \end{cases}$ i.e., [x] = greatest among the integers less than or equal to x	R	$\mathbb{Z}$ = set of all integers	
10.	Least integer function or ceiling of x	$f(x)=\lceil x\rceil=\begin{cases} x \text{ if } x\in\mathbb{Z} \\ k+1 \text{ if } k<x<k+1 \end{cases}$ i.e., $\lvert x\rvert$ = least among the integers greater than or equal to x.	R	$\mathbb{Z}$ = set of all integers	

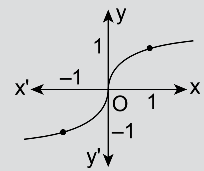
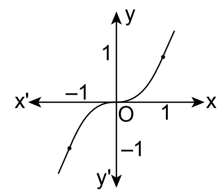
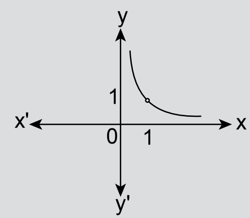
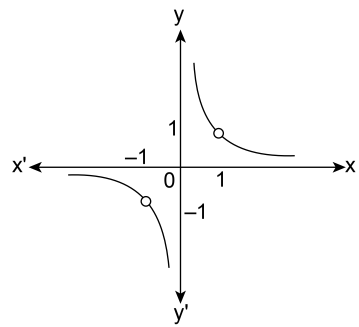
11.	Nearest integer function	$f(x) = (x) = \begin{cases} x \text{ if } k - \frac{1}{2} \le x < k + \frac{1}{2} \\ k+1 \text{ if } k + \frac{1}{2} \le x < k + \frac{3}{2} \end{cases}$ i.e., (x) = integer nearest to x and if x is of the form $k + \frac{1}{2} : k \in \mathbb{Z}$, then (x) = k + 1	R	$\mathbb{Z}$ = Set of integers.	
12.	Fractional part function	$f(x) = \{x\} = \begin{cases} 0 \text{ if } x \in \mathbb{Z} \\ f \text{ if } x = k + f \\ \text{and } f \in (0,1), k \in \mathbb{Z} \end{cases}$	R	[0, 1)	
13.	Identity function	f(x) = x	R	R	
14.	Exponential function	$f(x) = a^x$; a > 0; a ≠ 1; a is fixed and x varies over set of read numbers	R	(0, ∞)	

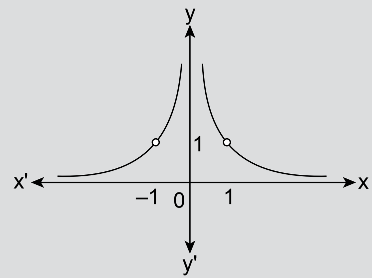
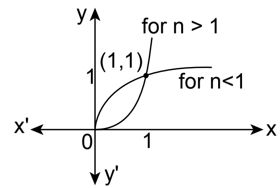
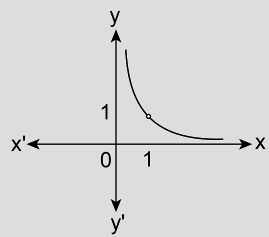
15.	Logarithmic function	$f(x) = \log_a x$; $a > 0$; $a \neq 1$ and a is fixed real number x vories over set of real numbers	$(0, \infty)$	R	 
16.	Reciprocal function or rectangular hyperbola	$y = \dfrac{1}{x}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$	

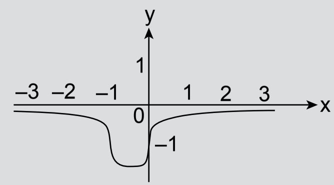
17.	Pour function	$y = x^k; k \in \mathbb{R}$			
		(i) $k = (2n + 1); n \in \mathbb{N}$	R	R	
		(ii) $k = 2n; n \in \mathbb{N}$	R	$[0, \infty)$	
		(iii) $k = -(2x + 1); n \in \mathbb{N}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$	

(iv) $k = -(2n); n \in \mathbb{N}$	$\mathbb{R} - \{0\}$	$(0, \infty)$	
(v) $k = \dfrac{1}{2n+1}; n \in \mathbb{N}$	R	R	
(vi) $k = \dfrac{1}{2n}; n \in \mathbb{N}$	$[0, \infty)$	$[0, \infty)$	

(vii) $k = -\frac{1}{(2n+1)}; n \in \mathbb{N}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$	
(viii) $k = -\frac{1}{2n}; n \in \mathbb{N}$	$(0, \infty)$	$(0, \infty)$	
(ix) $k = \frac{2n}{2n+(2m-1)}; n, m \in \mathbb{N}$	R	$[0, \infty)$	
(x) $k = \frac{2n}{2n-(2m-1)}; n, m \in \mathbb{N}$	R	$[0, \infty)$	

(xi) $k = \frac{2n-1}{2m-1}; n, m \in \mathbb{N};$ $n < m \frac{2n-1}{2m-1} \in (0,1)$	R	R	
(xii) $k = \frac{2n-1}{2m-1}; n, m \in \mathbb{N}$ and $n > m; k > 1$	R	R	
(xiii) $k = -\frac{(2n-1)}{2m}; n, m \in \mathbb{N}$	$(0, \infty)$	$(0, \infty)$	
(xiv) $k = -\frac{(2n-1)}{(2n-1)}; n, m \in \mathbb{N}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$	

		(xv) $k = -\frac{2n}{(2m-1)}; n, m \in \mathbb{N}$	R – {0}	(0, ∞)	
		(xvi) k = a regalive irrational number	(0, ∞)	(0, ∞)	
		(xviii) k = a negalive irrational number	(0, ∞)	(0, ∞)	
18.	Trigonometry functions	y = f(sin x, cos x, tan x, cot x, sec x, cosec x) e.g., f(x) = sin x + cos x $f(x) = 1 - \cos x + \sec^2 x$	Common domain of trigonometric functions involved	Can be found using properties functions like continuity monotonicity bounded here etc.	Depends upon the trigonometric ration involved

19.	Algebraic functions	Functions consisting of finite number of terns involving powers and lots of independent variable and the four fundamental operations (+, –, ×, ÷)	Depends on function, e.g., $f(x)=\frac{\sqrt{x-1}}{x^{3/2}}$ has its domain $[1, \infty)$	Depends on function and can be found using calculus	Depends upon the function
20.	Transcndien-tal function	The functions which are not algebraic e.g., $f(x)=\sqrt{\ell nx-\sin^{-1}x}$ etc.	Depends on function	can be found using calculus	Depends upon the function
21.	Rational function	$y=f(x)=\frac{P(x)}{Q(x)}$; P(x) and Q(x) are polynomial function	$\mathbb{R}-\{x : Q(x)=0\}$	Express x in terms of y; and by the knowledge of quadiatic equation, those values of y for which x is real and belong to domain	e.g., graph of $f(x)=\frac{(x-1)}{(x-2)(x-3)}$ is shown below.
		(i) If $f(x)=\frac{ax+b}{cx+b}; 0\neq 0$	$\mathbb{R}\sim\left\{-\frac{d}{c}\right\}$	$\mathbb{R}\sim\left\{\frac{a}{c}\right\}$	
22.	Irrational Function	The algebrenc functions having rational (non-integer) poures of x are couled irrational functions e.g., $f(x)=\sqrt{x+1}; f(x)=\sqrt[3]{x}$, $f(x)=\frac{\sqrt{x^3+1}-\sqrt{x-1}}{\sqrt{x^2+x+1}}$ etc.,	Depends upon the function	Can be found by using calculus	Depends upon the function, e.g., $f(x)=\sqrt[3]{x}-\sqrt[3]{x+1}$ has following wave form 

21.4 EQUAL OR IDENTICAL FUNCTIONS

Two functions f and g are said to be equal if:

1. The domain of f = the domain of g.
2. The range of f = the range of g.
3. $f(x) = g(x)$ for every x belonging to their common domain, e.g., $f(x) = 1/x$ and $g(x) = x/x^2$ are identical functions.
   $f(x) = \log(x^2)$ and $g(x) = 2\log(x)$ are not-identical functions as domain of $f(x) = (-\infty, \infty) \sim \{0\}$ whereas that of $g(x) = (0, \infty)$.

Remark:

Graphs of trigonometric function and inverse trigonometric functions with their domain and range are givenin the same book under corresponding topics.

21.5 PROPERTIES OF GREATEST INTEGER FUNCTION (BRACKET FUNCTION)

(i) Domain of $[x] : \mathbb{R}$; Range of $[x] : \mathbb{Z}$

(ii) $[[x]] = [x]$

(iii) $[x + m] = [x] + m$ provided $m \in \mathbb{Z}$

(iv) $[x + [y + [z]]] = [x] + [y] + [z]$

(v) $[x] > n;\ n \in \mathbb{Z}$

$\Rightarrow$ $[x] \in \{n + 1, n + 2, n + 3, \ldots\}$

$\Rightarrow$ $x \in [n + 1, \infty)$

(vi) $[x] \geq n \quad \Rightarrow \quad x \in [n, \infty)$

(vii) $[x] < n \quad \Rightarrow \quad x \in (-\infty, n)$

(viii) $[x] \leq n \Rightarrow n \in (-\infty, n + 1)$

(ix) $$[-x] = \begin{cases} -[x] = -x & \text{if } x \in \mathbb{Z} \\ -1 - [x] & \text{if } x \notin \mathbb{Z} \end{cases}$$

(x) $x - 1 < [x] \leq x$; equality holds iff $x \in \mathbb{Z}$

(xi) $[x] \leq x < [x] + 1$

(xii) $\left[\frac{[x]}{c}\right] = \left[\frac{x}{c}\right]$ for $c \in \mathbb{N}$ and $x \in \mathbb{R}$

(xiii) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(xiv) $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right] \ \forall\ x \in \mathbb{R}$

(xv) The number of positive integers less than or equal to n and divisible by m is given by $\left[\frac{n}{m}\right]$; m and n are positive integers.

(xvi) If p is a prime number and e is the largest exponent of p such that, p^e divides n!, then $e = \sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$

21.5.1 Properties of Least Integer Function

1. The domain of the function is: $(-\infty, +\infty)$
2. The range is the set of all integers.
3. $[x]$ converts $x = (I + f)$ into I while $\lceil x \rceil$ converts it into I + 1.
   E.g., If $x = 2.4$, then $2 < x < 3 \Rightarrow \lceil x \rceil = 3 = I + 1$
4. When x is an integer $[x] = x = \lceil x \rceil$
5. $\lceil x + n \rceil = \lceil x \rceil + n$, where n is an integer.

21.5.2 Properties of Fractional Part Function

(i) Domain of fractional part function = $D_f = \mathbb{R}$; Range of fractional part function = $R_f = [0, 1)$
(ii) $\{x\}$ is periodic function with period 1.
(iii) $[\{x\}] = 0$
(iv) $\{[x]\} = 0$
(v) $\{\{x\}\} = \{x\}$; this result is true when fractional part function is applied on x on left hand side more than or equal to twice.
(vi) $\{-x\} = \begin{cases} 0; & x \in \mathbb{Z} \\ 1 - \{x\}; & x \notin \mathbb{Z} \end{cases}$
(vii) $[x + y] = \begin{cases} [x] + [y]; & 0 \le \{x\} + \{y\} < 1 \\ [x] + [y] + 1; & 1 \le \{x\} + \{y\} < 2 \end{cases}$

21.5.3 Properties of Nearest Integer Function

(i) $(x) = \begin{cases} [x] & \text{if } 0 \le \{x\} < \frac{1}{2} \\ [x]+1 & \text{if } \frac{1}{2} \le \{x\} < 1 \end{cases}$

(ii) $(x + n) = (x) + n$ if $n \in \mathbb{Z}$

(iii) $(-x) = \begin{cases} -(x); & \forall x \in \mathbb{R} \sim \left\{ x = \left(\frac{2n+1}{2}\right); n \in \mathbb{Z} \right\} \\ -(x) + 1; & \text{for } x = \left(\frac{2n+1}{2}\right); n \in \mathbb{Z} \end{cases}$

(iv) $(x) = \begin{cases} [x] = n & \text{if } n \le x < n + \frac{1}{2} \\ [x] + 1 = n + 1 & \text{if } n + \frac{1}{2} \le x < n + 1 \end{cases}$

Properties of Modulus of a real number:

1. $|x_1 . x_2 . x_3, \ldots, x_n| = |x_1| . |x_2| . |x_3| \ldots |x_n|\ \forall\ x_i \in \mathbb{R}$
2. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ \forall\ x, y \in \mathbb{R}$ and $y \neq 0$.
3. $|x^n| = |x|^n\ \forall\ n \in \mathbb{Z}$
4. $|-x| = |x|\ \forall\ x \in \mathbb{R}$
5. $|x| = \delta \Rightarrow x = \delta$ or $x = -\delta$
6. $|x| < \delta \Rightarrow x \in (-\delta, \delta)$; and $|x| > \delta$
7. $|x - a| < \delta \Rightarrow x \in (a - \delta, a + \delta)$

8. $| x - a | = \delta \quad \Rightarrow \quad x = a + \delta$ or $a - \delta$

9. $| x - a | > \delta \quad \Rightarrow \quad x > a + \delta$ or $x < a - \delta$

10. $\sqrt{x^2} = | x | \; \forall \; x \in \mathbb{R}$

11. $|x| = \max.\{-x, x\} \; \forall \; x \in \mathbb{R}$

12. $|x| = |y| \Leftrightarrow x^2 = y^2$

13. $|x + y|$ is not always equal to $| x | + | y |$.

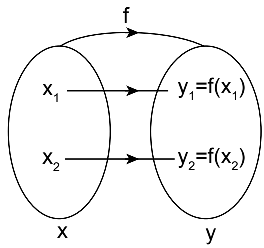
14. (Triangle inequality) $| x + y | \le | x | + | y |$ for all real x and y, inequality holds if $x.y < 0$, i.e., x and y are of opposite signs, equality holds if $x.y \ge 0$, i.e., x and y are of same sign or at least one of x and y is zero.

15. $|x - y| \le |x| + |y|$ for real x and y, inequality holds if $x.y > 0$, i.e., x and y are of same sign, equality holds if $x.y \le 0$, i.e., x and y are of opposite sign or at least one of x and y is zero.

16. $||x| - |y|| \le |x + y|$ for real x and y. Equality holds if x and y are of opposite signs and for same sign inequality holds.

17. $||x| - |y|| \le |x - y|$ for real x and y. Equality holds if x and y are of same sign and for opposite signs inequality holds.

21.5.3.1 Methods of testing a relation to be a function



Method 1: When the relation to be tested is represented analytically: A relation $f : X \to Y$ defined as $y = f(x)$ will be function iff $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$, since otherwise, an element of X would have two different image

Method 2: When the relation to be tested is represented as a set of ordered pairs:

A relation $f : X \to Y$ represented as a set of ordered pairs will be function from X to Y iff

- ❑ Set of abscissa of all ordered pairs is equal to X.
- ❑ No two ordered pairs should have same abscissa.

Remark:

Because f is a relation from $X \to Y$, therefore abscissa of ordered pairs must belong to X where as ordinates of ordered pairs must belong to Y.

Method 3: When the relation to be tested is represented graphically: relation $f : X \to Y$; $y = f(x)$ is function iff all the straight line $x = \alpha$; $\forall \; \alpha \in X$ intersect the graph of function exactly once as shown below.

A relation $f : X \to Y$ will not be a function in following two conditions.

1. If for some $\alpha \in X$, line $x = \alpha$ does not cut the curve $y = f(x)$, e.g., in the graph of function shown below the line $x = \alpha$ does not cut the graph of function and $\alpha \in X$ $(D_f) = [a, b]$, i.e., no output for input $x = \alpha$.

$\Rightarrow$ f(x) is not a function from X to Y.

2. If for atleast one $\alpha \in X$, line $x = \alpha$ intersects $y = f(x)$ more than once, i.e., there exists an input having more than one output say at (α, y_1) , (α, y_2) and (α, y_3).

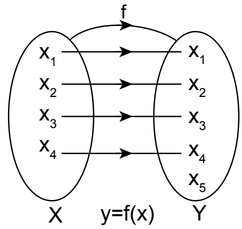
$\Rightarrow$ For input $x = \alpha$, f(x) has three outputs y_1 , y_2 as well as y_3.

Hence, f(x) is not function.

Method 4: When the relation to be tested is represented diagrammatically: A relation $f : X \to Y$ is a function if no input has two or more outputs in Y and no $x \in X$ is un-related.

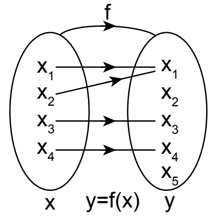
21.6 CLASSIFICATION OF FUNCTIONS

21.6.1 One-one (Injective) Function



$f : X \to Y$ is called injective, when different elements in set X are related with different elements of set Y, i.e., no two elements of domain have same image in co-domain. In other words we can also say that, no element of co-domain is related with two or more elements of domain.

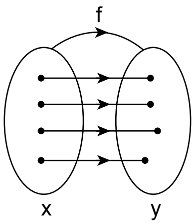
21.7 MANY-ONE FUNCTIONS

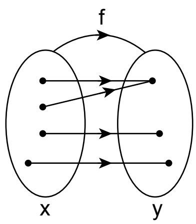


$f : X \to Y$ is many-one, when there exist at least two elements in the domain set X which are related with same element of co-domain Y.

21.7.1 Onto (Surjective) Function

A function $f : X \to Y$ is called surjective only when each element in the co-domain is f-image of at least one element in the domain, i.e., $f : X \to Y$ is onto iff $y \in Y$ there exists $x \in X$ such that $f(x) = y$ i.e., iff R_f = co-domain (Y).





∴ Surjective f: $X \to Y$ reduces the co-domain set to range of function.

21.8 METHOD OF TESTING FOR INJECTIVITY

(a) **Analytical Method:** A function $f : X \to Y$ is injective (one-one) iff whenever two images are equal then it means that they are outputs of same pre-image, i.e., $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$. Or, by using contra-positive of the above condition, i.e., $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2) \ \forall \ x_1, x_2 \in X$.

Notes:

1. *If f (x) is not one-one, then it is many-one function. If we go according to definition consider $f(x_1) = f(x_2)$. $\Rightarrow x_1$ is not necessarily equal to x_2.*

   i.e., If two f-images are equal then their pre-images may or may not be equal.

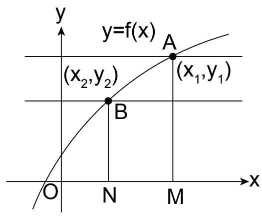
2. *To test injectivity of f(x), consider $f(x_1) = f(x_2)$ and solve the equation and get x_2 in terms of x_1. If $x_2 = x_1$ is only solution, then function f is injective, but if other real solutions also exist, then f is many-one function.*

(b) **Graphical Method:** For one-one, every line parallel to x-axis, $y = k \in R_f$ cuts the graph of function exactly once, then the function is one-one or injective.

For many-one If there exists a line parallel to x-axis which cuts the graph of function at least twice, then the function is many one.

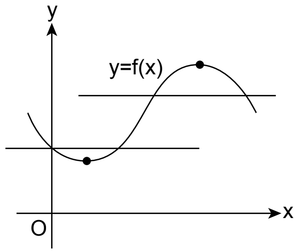
(c) **Method of Monotonicity:**

for one-one: If a function f(x) is continuous and monotonic ($f'(x) \geq 0$, $f'(x) = 0$ occures at isolated points) on an interval I, then it is always one-one on interval I because any straight line parallel to x-axis $y = k \in I$ intersects the graph of such functions exactly once.

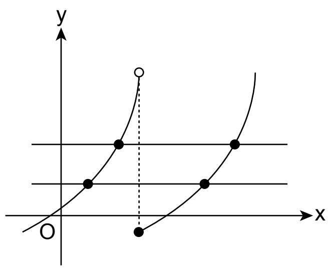


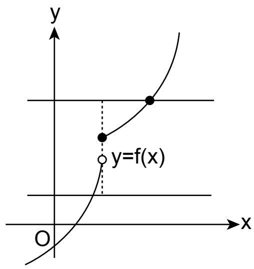
For many-one:

(i) If a function is continuous and non-monotonic on interval I, then it must be many-one on interval I



(ii) If a function is discontinuous and monotonic on interval I, then it can be one-one or many-one on I as is clear from the figures given below:



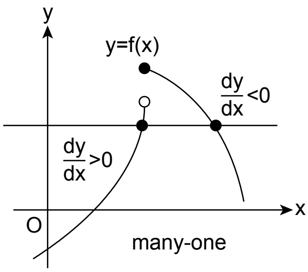


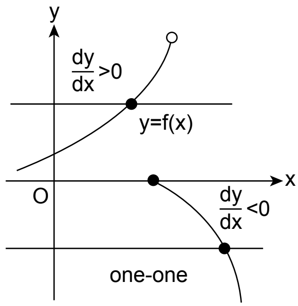
(iii) Even functions and periodic functions are always many-one in their natural domains, whereas they are one-one in their principal domain. They can be made one-one by restricting the domain.

e.g., cosx is many one on $\mathbb{R}$, but is one-one on $[0, \pi]$ or $\left[0, \frac{\pi}{2}\right]$. Similarly, fraction part function

$\{x\}$ is periodic function with period 1. It is many one on $\mathbb{R}$, but one-one on $[n, n + 1)$ for each integer n.

(iv) If a function is discontinuous and non-monotonic on an interval I, then it can be one-one or many one on I. It can be understood well by the graph shown as follows:





(v) All polynomials of even degree defined in ℝ have at least one local maxima or minima and hence are many-one in the domain ℝ. Polynomials of odd degree can be one-one or many-one in ℝ.

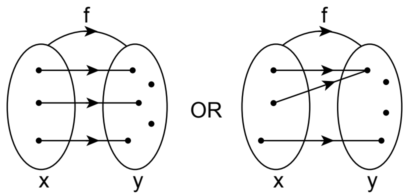
(d) **Hit and trial method to test many-one functions:**
It is possible to find an element in the range of function which is f image of two or more than two elements in the domain of function.

21.9 INTO (NON-SURJECTIVE) FUNCTION

While defining function we have mentioned that there may exist some element in the co-domain which are not related to any element in the co-domain.

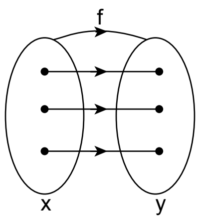
$f : X \to Y$ is into iff there exists at least one $y \in Y$ which is not related with any $x \in X$.

Thus, the range of the into function is proper subset of the co-domain, i.e., range $\subset$ co-domain (properly)



21.10 ONE-ONE ONTO FUNCTION (BIJECTIVE FUNCTION)

If a function is both one-one as well as onto, then f(x) is set to be bijective function or simply bijection.



21.11 TESTING OF A FUNCTION FOR SURJECTIVE

Method 1: The equality of range of function to co-domain forms the condition to test surjectivity of function. For instance to test surjectivity of $f: [0, \infty) \to [2, \infty)$ such that $f(x) = x^2 + 2$.

Using the analytic formula we obtain the rule of function for argument x in terms of y as shown below:

$\because \quad y = x^2 + 2;\ x^2 = y - 2$, i.e., $|x| = \sqrt{y-2}$

$\Rightarrow \quad x = \sqrt{y-2} \qquad \because \quad x \geq 0$

Now, we check whether the expression of x in forms of y is valid for all elementary co-domain. If it is so, then f is surjective, otherwise it is non-surjective

Thus x to be real and positive RHS, i.e., $\sqrt{y-2}$ must be real and positive, thus y ∈ [2, ∞). Hence, the given function f is onto.

Method 2: Hit and Trial Method: Sometimes we choose an element of co-domain which may not be an image of any element in domain and we test it for same. If it comes out to be true, then f is into function.

Remark:

In order to convert a function from many-one to an injective function its domain must be transformed to principal domain. In order to convert a function from into to onto, the co-domain of function must be replaced by its range.

21.12 NUMBER OF RELATIONS AND FUNCTIONS

Number of Relations: No. of relations = Number of subsets of A × B = $2^{n(A\times B)} = 2^{nm}$

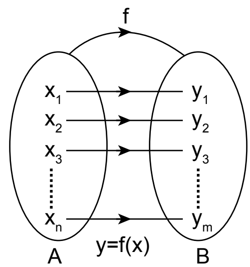
Number of Functions: Since each element of set A can be mapped in m ways

⇒ Number of ways of mapping all n elements of A

$$=\underbrace{m\times m\times m\times....\times m}_{\text{n times}}\text{ ways}=m^n\text{ ways}$$

Conclusion: $2^{nm} \geq m^n\ \forall\ m, n \in \mathbb{N}$

21.12.1.1 Number of one-one function (injective)



⇒ Number of injective functions

$$= m(m-1)(m-2)\ \dots\ (m-n+1) = \begin{cases} {}^mP_n & m \geq n \\ 0 & m < n \end{cases}$$

Conclusion: ${}^mP_n \leq m^n$ total number of functions.

21.12.1.2 Number of non-surjective functions (into functions)

Number of into function (N) = Number of ways of distributing n different objects into m distinct boxes, so that at least one box is empty; N = $\sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n$

21.12.1.3 Number of surjective functions

Number of surjective functions = Total number of functions – Number of into functions.

$$= m^n - \sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n = \sum_{r=0}^{m} {}^mC_r(-1)^r(m-r)^n$$

Conclusion: In case when n(A) = n(B) the onto function will be bijection

Number of onto function = Number of one-one function

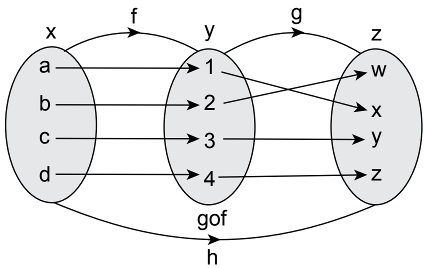
$$\Rightarrow \sum_{r=0}^{n} {}^nC_r(-1)^r(n-r)^n = n!$$

Remarks:

1. *If n(X) < n(Y), then after mapping different elements of X to different elements of Y, we are left with at least one element of Y which is not related with any element of X, and hence there will be no onto function from X to Y, i.e., all the functions from X to Y will be into.*
2. *If f from X to Y is a bijective function, then n(X) = n(Y).*

21.12.1 Composite of Uniformly Defined Functions

Given two functions f : X → Y and g : Y → Z, then there exists a function h = gof : X → Z such that h(x) = (gof) (x) = g(f(x)) ∀ x ∈ X. It is also called as 'function of function' or 'composite function of g and f' or 'g composed with f 'and diagrammatically shown as:



21.13 COMPOSITION OF NON-UNIFORMLY DEFINED FUNCTIONS

If $f(x)=\begin{cases}2x-1; & 0\le x<2\\ x^2+1; & 2\le x\le 4\end{cases}$ and $g(x)=\begin{cases}x+1; & -1\le x<1\\ 2x; & 1\le x\le 3\end{cases}$; then $fog(x)=\begin{cases}2x+1; & -1\le x<1\\ 4x^2+1; & 1\le x\le 2\end{cases}$

21.14 PROPERTIES OF COMPOSITION OF FUNCTION

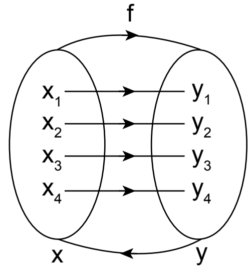
(a) fog(x) is not necessarily equal to gof(x), i.e., generally not commutative.
(b) The composition of functions is associative in nature, i.e., fo(goh) = (fog) oh.
(c) The composite of two bijections is a bijection.
(d) If gof is one-one, then f is one-one and g need not be one-one.
(e) If gof is onto, then g is onto but f need not be onto.
(f) If f(x) and g(x) are both continuous functions, then g(f(x)) is also continuous.
(g) Monotonicity of composite function: Composition of two functions having same monotonicity is a monotonically increasing function.
(h) Composition of two functions having opposite monotonicity is a decreasing function.

21.14.1 Definition of Inverse of a Function

A function f: X → Y is said to be invertible iff there exists another function
g : Y → X such that f(x) = y ⇔ g(y) = x, ∀ x ∈ X and y ∈ Y.
Then, g : Y → X is called inverse of f : X → Y and is denoted by f^{-1}.
⇒ $g = f^{-1} = \{(f(x), x): (x, f(x)) \in f\}$

21.15 CONDITION FOR INVISIBILITY OF A FUNCTION

For a function to be invertible it should be one-one and onto, i.e., bijective function.



f^{-1} is a function

21.15.1 Method to Find Inverse of a Given Function

Step 1: Check the injectivity (one-one): Take $f(x_1) = f(x_2)$ and show that $x_1 = x_2$ or show that f is continuous and monotonic on its domain.

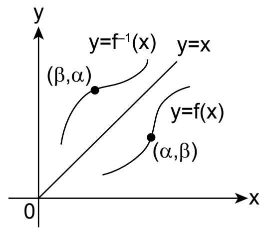
Step 2: Surjectivity (onto):
Find the Range of the function (R_f) and compare it with co-domain.
If R_f = Co-domain, then f is onto.

Step 3: Using equation y = f(x) express x in terms of y. …. (1)

Step 4: Replace x by y and y by x in the obtained relation (1) to get $y = f^{-1}(x)$.

Remark:

Since to each $(x, y) \in f$ there exists $(y, x) \in f^{-1}$ and (y, x) and (x, y) are mirror images of each other in the line y = x, therefore the graph of $f^{-1}(x)$ is obtained by reflecting the graph of f(x) in the line y = x as shown below.



21.16 PROPERTIES OF INVERSE OF A FUNCTION

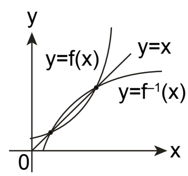
(i) The inverse of a bijection is unique.

(ii) The inverse of a bijection is also a bijection.

(iii) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

(iv) Inverse of inverse of a given function is the given function itself, i.e., $(f^{-1})^{-1} = f$

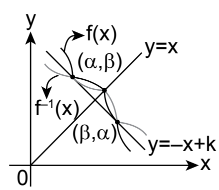
(v) f(x) and $f^{-1}(x)$ if intersect, then the point of intersection should be on the line y = x or y = –x + k for some real value of k.



(vi) f(x) and $f^{-1}(x)$ have same monotonic nature, i.e., either both increasing or both decreasing.

(vii) If f(x) is increasing function, then $f^{-1}(x)$ is also an increasing function, but f(x) and $f^{-1}(x)$ have opposite curvatures.

(viii) If f(x) is a decreasing function, then $f^{-1}(x)$ is also a decreasing function, but f(x) and $f^{-1}(x)$ have same curvatures.



(ix) If the graph of a function f(x) is symmetric about the line y = x, then f(x) and $f^{-1}(x)$ are equal functions, i.e., f(x) will be self invertible function or (involution).

(x) If $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is an inverse function of f, then $f^{-1}of = I_A$ and $fof^{-1} = I_B$. Here, I_A is an identity function on set A, and I_B is an identity function on set B.

21.17 EVEN FUNCTION

A function $f : X \to Y$ is said to be an even function iff $f(-x) = f(x)\ \forall\ x, -x \in X$ (Domain).
i.e., $f(x) - f(-x) = 0$
e.g., x^{2n}, $\sin^2 x$, $\cos x$, $\sec x$, $2^x + 2^{-x}$

21.17.1 Properties of even functions

(i) Graph of even function is symmetric about y-axis.
(ii) For any function f(x) if $g(x) = f(x) + f(-x)$, then g(x) is always an even function.
(iii) The domain of even function must be symmetric about zero.
(iv) Even functions are non invertible as they can not be strictly monotonic when taken in their natural domain, however, even functions can be made invertible by restricting their domains.
(v) If f(x) is even function, then $f'(x)$ is odd function.
(vi) $f(x) = c$; where 'c' is a constant defined on symmetrical domain is an even function.

21.17.2 Odd Function

A function $f : X \to Y$ is said to be an odd function iff $f(-x) = -f(x)\ \forall\ x, -x \in X$.
i.e., $f(x) + f(-x) = 0\ \forall\ x, -x \in X$.
e.g., x^3, $\sin x$, $\tan x$, $2^x - 2^{-x}$ are odd functions.

21.17.3 Properties of Odd Functions

(i) Graph of odd function is symmetric about origin. Also known as symmetric in opposite quadrants.
(ii) For any function f(x) if $g(x) = f(-x) - f(x)$, then g(x) is always an odd function.
(iii) The domain of odd function must be symmetric about zero.
(iv) f(x) is odd, then $f'(x)$ is an even function.
(v) If $x = 0$ lies in the domain of an odd function, then $f(0) = 0$.

21.18 ALGEBRA OF EVEN-ODD FUNCTIONS

1. $f(x) = 0$ (identically zero function) is the only function which is both, an odd and an even function, provided it is defined in a symmetric domain.
2. A linear combination of two or more even functions is an even function, i.e., in particular for two even functions f(x) and g(x), the function $(\alpha f + \beta g)$ is an even function; where $\alpha, \beta \in \mathbb{R}$.
3. A linear combination of two or more odd functions is an odd function, i.e., in particular for two odd functions f(x) and g(x), the function $(\alpha f + \beta g)$ is an odd function; where $\alpha, \beta \in \mathbb{R}$.
4. The product of two or more even functions is an even function.
5. The product of an odd and an even function is an odd function.
6. The quotient of two even functions (or two odd functions) is an even function.
7. The nature (odd or even) of product of odd functions depends upon the number of functions taken in the product, i.e., product of odd number of odd functions is an odd function and that of even number of odd functions is an even function.
8. Composition of several functions, f(g(h...(p(x))...))) is odd iff all are odd functions.

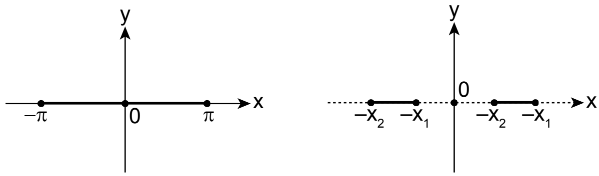
9. Composition of several functions is even iff at least one function is even, provided the function composed of either even or odd functions after that even function.

10. Any function f(x) can always be written as sum of an even function and an odd function.

Remarks:

(i) The functions having no symmetry like odd/even functions are called as 'neither even nor odd functions'.

(ii) Before testing the even/odd symmetry of the function it is essential to observe whether the domain of function is symmetric about y-axis, i.e., if the domain is of the type $[-x_0, x_0]$ *or* $[-x_2, -x_1] \cup [x_1, x_2]$*, etc.*



21.19 EVEN EXTENSION OF FUNCTION

Extending the domain of function f(x) and defining such that the function obtained is even.

i.e., $h(x) = \begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

21.20 ODD EXTENSION OF FUNCTION

Extending the domain of function and redefining it such that the new function obtained becomes odd.

i.e., $h(x) = \begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ -f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

21.20.1 Definition of Periodic Function

A function f(x) is said to be a periodic function if there exists a real positive and finite constant T independent of x such that f(x + T) = f(x), $\forall\, x \in D_f$ provided, $(x + T) \in D_f$ (domain).

The least positive value of such T (if exists), is called the period/principal period or fundamental period of f(x).

e.g., f(x) = tan x, f(x) = sin x are periodic functions with period π and 2π, respectively.

21.21 FACTS AND PROPERTIES REGARDING PERIODICITY

(a) **Trigonometric functions:** The function sin x, cos x, sec x, cosec x are periodic with period 2π. Whereas tan x, cot x are periodic functions with period π.

(b) **There may be periodic functions which have no fundamental period, e.g.**,

(i) **Dirichlet function:** $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases}$

(ii) **Constant function:** Consider a function f(x) = c

(c) No rational function (except constant function) can be a periodic function.

(d) **Algebraic function (Except Constant Function) cannot be a periodic function.**

(i) If f (x) is periodic with period T, then a f (x + k) + b is also periodic with same period T; where a, b are real constants and a > 0.

(ii) If f(x) is periodic with period T, then f(kx + b) is periodic with period $\frac{T}{|k|}$ provided 'k' is non-zero real number and $b \in \mathbb{R}$.

21.22 PERIOD OF COMPOSITE FUNCTIONS

Theorem: If f(x) is periodic function with fundamental period T and g(x) is monotonic function, over the range of f(x), then g(f(x)) is also periodic with fundamental period T.

If f(x) is periodic with period T, then:

(i) $\frac{1}{f(x)}$ is also periodic with same period T.

(ii) $\sqrt{f(x)}$ is also periodic with same period T.

Notes:

1. *Composition of a non-monotonic function g(x) over a periodic function f(x) having period T is always a periodic function with period T. (But fundamental period may be less than T).*
   e.g., if $g(x) = x^2$ and $f(x) = \cos x$, then $gof(x) = \cos^2 x$ is periodic with period 2π, But its fundamental period is π.
2. *Composition of a non-periodic function g(x) with a periodic function f(x) may be a periodic function. e.g., if g(x) = [x] and $f(x) = \cos\pi x$, then fog(x) = cos[x] is periodic with period 2.*
3. *Composition of two non-periodic functions may be a periodic function*

   e.g., consider g(x) = 3[x] – 2 and $f(x) = \begin{cases} \dfrac{x^3-8}{x-2}; & x \notin \mathbb{Z} \\ 3\left(\sin^2 x + \cos^2 x\right); & x \in \mathbb{Z} \end{cases}$*, we have* $fog(x) = 3\ \forall\ x \in \mathbb{R}$.

   which is a periodic function indeed.

21.23 PERIODICITY OF MODULUS/POWER OF A FUNCTION

(i) **Period if $f(x)^{2n+1}$:** If the fundamental period of f(x) is T, then the fundamental period of $f(x)^{2n+1}$; $n \in \mathbb{Z}$ will also be T. i.e., the fundamental period of function remains same on raising it to an odd integer power.

(ii) **Period of $f(x)^{2n}$:** If the fundamental period of f(x) is T, then the fundamental period of $f(x)^{2n}$; $n \in \mathbb{Z}$ may not be T.
i.e., the fundamental period of function may change on raising it to an even integer power.
For example, we know that the period of the functions sin x, cos x, sec x, cosec x is 2π and that of tanx, cotx is π, whereas the period of the functions $(\sin x)^{2n}$, $(\cos x)^{2n}$, $(\sec x)^{2n}$, $(\mathrm{cosec}\,x)^{2n}$, $(\tan x)^n$, $(\cot x)^n$, |sinx|, |cosx|, |tanx|, |cotx|, |secx|, |cosecx| is π.

(iii) If f (x) be periodic with period T_1 and g(x) with period T_2 such that LCM of T_1 and T_2 exist and is equal to T, then a.f(x) + b.g(x) is a periodic function with period T (a and b are non-zeros).

Remarks:

(i) *L.C.M. of two or more fractional numbers* = *L.C.M of* $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{L.C.M.\ of\ (a,c,e)}{H.C.F.\ of\ (b,d,f)}$

e.g., the L.C.M of $\frac{7}{30}$ *and* $\frac{3}{20}$ *is* $\frac{\text{L.C.M of 7 and 3}}{\text{H.C.F of 30 and 20}} = \frac{21}{10}$.

(ii) *L.C.M of rational and irrational number does not exist.*

e.g., The function $\{x\} + \cos x$ *is non-periodic, because the period of* $\{x\}$ *is 1 and the period of cosx is* 2π

And the L.C.M $(1, 2\pi)$ *does not exist.*

Also the function = $\sin x + \tan \pi x + \sin x/3$ *is not periodic; because L.C.M. of* $(2\pi, 1, 6\pi)$ *does not exists.*

(iii) *The L.C.M. of two irrational quantities may or may not exist.*

1. *The sum/difference of a periodic and an non-periodic function can be periodic.*
2. *The sum/difference of two non-periodic functions can be periodic function.*
3. *The product/quotient of a periodic and an non-periodic function can be periodic.*

   e.g., consider $f(x) = \cot x$ *and* $g(x) = \begin{cases} 1, & x = \mathbb{R} - \{0\} \\ 3, & x = 0 \end{cases}$*, then the function* $f(x) \cdot g(x)$ *and* $\frac{f(x)}{g(x)}$ *are periodic.*

   Clearly f(x) is periodic with period π *but g(x) is non-periodic function.*

   The domain of $f(x) \cdot g(x)$ *and* $\frac{f(x)}{g(x)}$ *is* $\mathbb{R} \sim \{n\pi;\ n \in \mathbb{Z}\}$*; hence* $f(x).\ g(x) = \frac{f(x)}{g(x)} = \cot x$ *which is periodic function in its natural domain with fundamental period* π.
4. *The product/quotient of two non-periodic functions can be periodic function*

   e.g., consider $f(x) = \begin{cases} 1; & x < 0 \\ -1; & x \geq 0 \end{cases}$ *and* $g(x) = \begin{cases} -1; & x < 0 \\ 1; & x \geq 0 \end{cases}$ *then the function f(x).g(x) and* $\frac{f(x)}{g(x)} = -1$ *which being a constant function is a periodic function with no fundamental period.*

21.24 EXCEPTION TO LCM RULE

Case I: If f (x) be periodic with period T_1 and g(x) with period T_2, such that LCM of T_1 and T_2 exist and is equal to T and f (x) and g (x) can be interchanged by adding a least positive constant K (< T).

i.e., f(x + K) = g(x) and g(x + K) = f(x), then K is period of f(x) + g(x), otherwise period will be T.

Case II: If f (x) be periodic with period T_1 and g(x) with period T_2 such that LCM of T_1 and T_2 exist and is equal to T, then the period of F(x) = f (x) ± g (x) or f(x).g(x) or $\frac{f(x)}{g(x)}$ is necessarily T, but the fundamental period can be given by a positive constant K (< T) if F(x) gets simplified to an equivalent function F(x + K) = F(x).

21.25 PERIODICITY OF FUNCTIONS EXPRESSED BY FUNCTIONAL EQUATIONS

(i) If a function f(x) is defined, such that f(x + T) = –f(x); where T is a positive constant, then f is periodic with period 2T. (Converse is not true).

(ii) If a function f(x) is defined, such that $f(x + T) = \frac{1}{f(x)}$ or $f(x + T) = \frac{-1}{f(x)}$; where T is a positive constant, then f is periodic with period 2T. (Converse is not true)

(iii) If $f(x + \lambda) = g(f(x))$, such that $\underbrace{g\left(g\left(g\left(...\left(g(x)\right)\right)...\right)\right)}_{\text{composed n times}} = x$, then prove that f(x) is periodic with period $n\lambda$ (where λ is fixed positive real constant)

21.26 TIPS FOR FINDING DOMAIN AND RANGE OF A FUNCTION

If f(x) and g(x) be two functions such that f(x) has domain D_f and g(x) has domain D_g, then the following results always hold good.

Rule 1: Domain (k . f(x)) = D_f for all k ∈ set of non-zero real numbers.

Rule 2: Domain $\left(\frac{1}{f(x)}\right) = D_f \sim \{x : f(x) = 0\}$

Rule 3: Domain (f(x) ± g(x)) = $D_f \cap D_g$.

Rule 4: Domain (f(x) . g(x)) = $D_f \cap D_g$

Rule 5: Domain f(g(x)) = $\{x : x \in D_g \text{ and } g(x) \in D_f\} = D_g \sim \{x: g(x) \notin D_f\}$.

Rule 6: Domain of even root of f(x) = $\sqrt[2m]{f(x)} = D_f \sim \{x : f(x) < 0\}$

Rule 7: Domain $\sqrt[2m+1]{f(x)} = D_f$

Rule 8: Domain (log f(x)) = $D_f \sim \{x : f(x) \le 0\}$

Rule 9: Domain of composite exponential function: $y = [f(x)]^{g(x)} = \{x \in \mathbb{R} : x \in D_f \cap D_g \text{ and } f(x) > 0\}$.

Remarks:

$y = \sqrt[x]{f(x)}$ is defined for x ∈ {2, 3, 4, ...} and f(x) > 0 where as $y = (f(x))^{1/x}$ is defined for x ≠ 0 and f(x) > 0.

Rule 10: Methods to find Range of Functions: Given a function $f : X \to Y$ where y = f(x)

Method I:

Step 1: Find domain of f(x), say $\alpha \le x \le \beta$.

Step 2: Express x in terms of y using equation of function, i.e., $x = f^{-1}(y)$.

Step 3: Apply the domain restriction, i.e., $\alpha \le x \le \beta \Rightarrow \alpha \le f^{-1}(y) \le \beta$.

Step 4: Find the set of all possible y satisfying above inequality.

Method II:

For composition of continuous functions.

Step 1: Identify the function as composite function of constituent functions f, g and h, say $\phi(x) = h(f(g(x)))$.

Step 2: Test the monotonicity of f and g and h say g(–(increasing)), f(↓ (decreasing)), h(↓ (decreasing)).

Step 3: Find domain of h(f(g(x))) say $\alpha \le x \le \beta$.

Step 4: $\because \quad \alpha \le x \le \beta \Rightarrow R_\phi = [h(f(g(\alpha))), h(f(g(\beta)))]$

Rule 5: If domain is a set having only finite number of points, then the range will be the set of corresponding values of f (x).

Rule 6: If domain of y = f (x) is R or R–{some finite points} or an infinite interval, then with the help of given relation, express x in terms of y and from there find the values of y for which x is defined and belongs to the domain of the function f (x). The set of corresponding values of y constitute the range of function.

Rule 7: If domain is not an infinite interval, find the least and the greatest values of f (x) using monotonicity. (This method is applicable only for continuous functions and is the most general method.)

Rule 8: For the quadratic function $f(x) = ax^2 + bx + c$, domain is $\mathbb{R}$, and range is given by $R_f = \begin{cases} \left[-\frac{D}{4a}, \infty\right) & \text{for } a > 0 \\ \left(-\infty, \frac{-D}{4a}\right] & \text{for } a < 0 \end{cases}$.

Rule 9: For the quadratic function $f(x) = \sqrt{ax^2 + bx + c}$ domain is given by $D_f = \begin{cases} \mathbb{R} \text{ for } a > 0, D < 0 \\ \phi \text{ for } a < 0, D < 0 \end{cases}$

and range is given by
$$R_f = \begin{cases} [0, \infty) & \text{for } D \ge 0, a > 0 \\ \left[\sqrt{\frac{-D}{4a}}, \infty\right) & \text{for } D < 0, a > 0 \\ \left[0, \sqrt{\frac{-D}{4a}}\right] & \text{for } D \ge 0, a < 0 \\ \phi & \text{for } D < 0, a < 0 \end{cases}$$

Rule 10: For odd degree polynomial domain and range both are $\mathbb{R}$.

Rule 11: For even degree polynomial domain is $\mathbb{R}$ and range is given by $[k, \infty)$ if the leading coefficient is positive, where k is the minimum value of polynomial occurring at one of the points of local minima, whereas range is $(-\infty, k]$ if the leading coefficient is negative where k is maximum value of polynomial occurring at one of the points of local maxima.

Rule 12: For $\frac{\text{Quadratic}}{\text{Quadratic}}$ or $\frac{\text{Linear}}{\text{Quadratic}}$ or $\frac{\text{Quadratic}}{\text{Linear}}$ expression put $y = \frac{Q}{Q}$; cross-multiply, convert into a quadratic and use the knowledge of quadratic equations.

Rule 13: For discontinuous functions, only method is to draw the graph and find the range known as graphical method of finding out range.

Rule 14: Range of function f(x) = asinx + bcosx is $\left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}\right]$.

Rule 15: (i) If f(x) and g(x) are increasing functions in their respective domain, then gof(x) is also an increasing function in its domain. Further, if both f(x) and g(x) are continuous in their respective domain, then gof is also continuous in its domain. Now, if common domain of f(x) and gof(x) is [α, β] or (α, β), then range of f(x) is [f(α), f(β)], or (f(α), f(β)) which in turn is domain of g(x). Then range of fog(x) will be [g(f(α), g(f(β))] or (g(f(α), g(f(β))).

(ii) If f(x) and g(x) both are decreasing functions in their respective domain, then gof is also a decreasing function. Further, if both f(x) and g(x) are continuous in their respective domain, then gof is continuous and increasing function in its domain. If common domain of f(x) and gof(x) is [α, β] or (α, β), then range of f(x) is [f(β), f(α)] or (f(β), f(α)) which in turn in domain of g(x) which is decreasing and continuous function. Thus, range of gof will be [g(f(α)), g(f(β))] or (g(f(α)), g(f(α))).

(iii) If f(x) and g(x) are functions of opposite monotonicity in their respective domain, then gof is a decreasing function on its domain. Further, if f(x) and g(x) are continuous functions, then gof is continuous and decreasing function. If [α, β] or (α, β) is common domain of gof(x) and decreasing function f(x) (say), then range of f(x) is [f(β), f(α)] or (f(β), f(α)) which in turn in domain of g(x). g(x) being continuous and increasing (say), range of gof(x) will be [g(f(β)), g(f(α))] or (g(f(β)), g(f(α))). Same will be the range of gof(x) if f(x) is increasing and g(x) is decreasing.

(iv) If f(x) is an increasing and continuous function in its domain, and g(x) is non-monotonic having range [α, β] or (α, β), then the range of fog(x)) will be [f(α), f(β)] or (f(α), f(β)). Similarly, if f(x) is decreasing and continuous function in its domain and g(x) is non-monotonic having range [α, β] or (α, β), then the range of fog(x) will be [f(β), f(α)] or (f(β), f(α)).

(v) If f(x) is non-monotonic function and continuous in its domain and g(x) is any function (monotonic or non-monotonic) for which the composition function fog is defined, then range of fog can be obtained by analyzing the behaviour of function f(x) on the range set of function g(x). i.e., by determining the intervals of monotonicity, l.u.b., g.u.b. of f(x) in range set of g(x).

(vi) If f(x) is monotonic and continuous in its domain and g(x) is non-monotonic for which fog(x) is defined and range of g(x) is [α, β] or (α, β), then the range of fog(x) will be [f(α), f(β)] or (f(α), f(β)), if f(x) is increasing and it will be [f(β), f(α)] or (f(β), f(α)) if f(x) is decreasing.

(vii) If f(x) and g(x) both are non-monotonic and continuous, for which fog(x) is defined, then the range of f(x) can be obtained by analyzing the behaviour of f(x) on the range set of g(x), i.e., by determining the intervals of monotonicity, l.u.b. and g.l.b. of f(x) in the range set of g(x).