CHAPTER FOUR

Determinants

EVALUATION OF DETERMINANTS

A determinant of order two is written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad (a_{ij} \in \mathbf{C} \ \forall \ i, j)$$

and is equal to $a_{11}\, a_{22} - a_{12}\, a_{21}$.

A determinant of order three is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (a_{ij} \in \mathbf{C} \ \forall \ i, j)$$

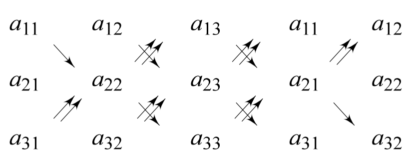
and is equal to

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{12}\,(a_{21}\, a_{33} - a_{23}\, a_{31}) + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31})$$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{32}\, a_{21} - a_{13}\, a_{31}\, a_{22} - a_{32}\, a_{23}\, a_{11} - a_{12}\, a_{21}\, a_{33}$$

A determinant of order 3 can also be evaluated by using the following diagram, due to **Sarrus:**



The product of the three terms on each of the three single arrows are prefixed by a positive sign and the product of the three terms on each of the three double arrows are prefixed by a negative sign.

Remark

This method does not work for higher order determinants.

MINORS AND COFACTORS

Minor

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n. Then the minor M_{ij} of the element a_{ij} of the matrix A is the determinant of the square sub-matrix of order $(n - 1)$ obtained by deleting ith row and jth column of matrix A.

Illustration 1

Minor of element a_{23} in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactor

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n. Then the cofactor of the element a_{ij} of the matrix A is denoted by C_{ij} and is equal to $(-1)^{i+j}\, M_{ij}$ where M_{ij} is the minor of the element a_{ij} of the matrix A.

Note that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\, M_{11} - a_{12}\, M_{12} + a_{13}\, M_{13} = a_{11}\, C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

If
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then
$$\Delta = a_{i1}\, C_{i1} + a_{i2}\, C_{i2} + a_{i3}\, C_{i3} \qquad i = 1, 2, 3$$
$$= a_{1j}\, C_{1j} + a_{2j}\, C_{2j} + a_{3j}\, C_{3j} \qquad j = 1, 2, 3$$

Note

$$a_{i1}\, C_{j1} + a_{i2}\, C_{j2} + a_{i3}\, C_{j3} = 0 \qquad i \neq j$$

and
$$a_{1i}\, C_{1j} + a_{2i}\, C_{2j} + a_{3i}\, C_{3j} = 0 \qquad i \neq j$$

Remark

The above results remain true for determinants of every order.

PROPERTIES OF DETERMINANTS

1. Reflection Property

The determinant remains unaltered if its rows are changed into columns and the columns into rows.
In other words, if A is a square matrix, then $|A| = |A'|$ where A' is transpose of A.

2. All-zero Property

If all the elements of a row (column) are zero, then the determinant is zero.

3. Proportionality [Repetition] Property

If the elements of a row (column) are proportional [identical] to the element of the some other row (column), then the determinant is zero.

4. Switching Property

The interchange of any two rows (columns) of the determinant changes its sign.

5. Scalar Multiple Property

If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

6. Property of Invariance

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix}$$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$, where $j, k \neq i$, or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$, where $j, k \neq i$.

7. Sum Property

$$\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Remark

It is one of the most under used property. But evaluation of some of the determinants beame very easy when we use it.

Illustration 2

To evaluate

$$\Delta = \begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$$

write $\Delta = \Delta_1 + a\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & 1+b & c \\ 0 & b & 1+c \end{vmatrix}$$

$$= (1 + b)(1 + c) - bc = 1 + b + c$$

and

$$\Delta_2 = \begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix}$$

Using $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$, we get

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1$$

$\therefore \quad \Delta = 1 + a + b + c$

8. Factor Property

If a determinant Δ becomes zero when we put $x = \alpha$, then $(x - \alpha)$ is a factor of Δ.

9. Triangle Property

If all the elements of a determinant above or below the main diagonal consists of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

10. Product of Two Determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Here we have multiplied rows by rows. We can also multiply rows by columns, or columns by rows, or columns by columns.

11. Conjugate of a Determinant

If $a_i, b_i, c_i \in \mathbf{C}$ $(i = 1, 2, 3)$, and

$$Z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then } \bar{Z} = \begin{vmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 \end{vmatrix}$$

12. Differentiation of a Determinant

If each $a_i(x)$ is differentiable function

and $\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) \\ a_3(x) & a_4(x) \end{vmatrix}$

then $\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2(x) \\ a_3'(x) & a_4(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2'(x) \\ a_3(x) & a_4'(x) \end{vmatrix}$

If we write $\Delta(x) = [C_1, C_2]$, where C_i denotes ith column, then

$\Delta'(x) = [C_1', C_2] + [C_1, C_2']$ where C_1' denotes the column which contains the derivative of all the functions in the ith column C_i. Similarly, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad \text{then} \quad \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$$

Next, if $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$

then $\Delta'(x)$

$$= \begin{vmatrix} a_{11}'(x) & a_{12}(x) & a_{13}(x) \\ a_{21}'(x) & a_{22}(x) & a_{23}(x) \\ a_{31}'(x) & a_{32}(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}'(x) & a_{13}(x) \\ a_{21}(x) & a_{22}'(x) & a_{23}(x) \\ a_{31}(x) & a_{32}'(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}'(x) \\ a_{21}(x) & a_{22}(x) & a_{23}'(x) \\ a_{31}(x) & a_{32}(x) & a_{33}'(x) \end{vmatrix}$$

$= [C_1', C_2, C_3] + [C_1, C_2', C_3] + [C_1, C_2, C_3']$

Similarly, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$$

Corollary (Differentiation and Integration of Determinant)

If $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

where $a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$ and a_{33} are constants, then

$$\Delta'(x) = \begin{vmatrix} a_{11}'(x) & a_{12}'(x) & a_{13}'(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ and}$$

$$\int \Delta(x)dx = \begin{vmatrix} \int a_{11}(x)dx & \int a_{12}(x)dx & \int a_{13}(x)dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

In general, for any positive integer m

$$\Delta^m(x) = \begin{vmatrix} a_{11}^m(x) & a_{12}^m(x) & a_{13}^m(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

13. Determinant of Cofactor Matrix

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then $\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$

where C_{ij} denotes the co-factor of the element a_{ij} in Δ.

SOME TIPS FOR QUICK EVALUATION OF DETERMINANTS

1. If Δ is a skew symmetric determinant of odd order, then $\Delta = 0$

Illustration 3

Let $\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

Using the reflection property, write

$$\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Taking –1 common from R_1, R_2 and R_3 we get

$$\Delta = (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = -\Delta$$

$\Rightarrow \quad 2\Delta = 0 \quad \Rightarrow \quad \Delta = 0$

2. If a_1, a_2, a_3 are in A.P.; b_1, b_2, b_3 are in A.P. and c_1, c_2, c_3 are also in A.P. Then

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Use $C_1 \to C_1 + C_3 - 2C_2$

3. If a_1, a_2, a_3 are in G.P.; b_1, b_2, b_3 are also in G.P., with the same common ratio, then

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

[c_1, c_2, c_3 can be any three complex numbers.]

Some Frequently used Determinants

1. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$$
$$= (a - b)(b - c)(c - a)$$

2. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$$

3. $$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(bc + ca + ab)$$

4. $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$
$$= (a + b + c)(bc + ca + ab - a^2 - b^2 - c^2)$$
$$= -\frac{1}{2}(a+b+c)\left[(b-c)^2+(c-a)^2+(a-b)^2\right]$$

LINEAR EQUATIONS

The system of linear homogeneous equations

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

has a non-trivial solution (i.e., at least one of the x, y, z is different from zero) if and only if $\Delta = 0$, where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $\Delta \neq 0$, then the only solution of the above system of equations is $x = 0$, $y = 0$ and $z = 0$.

Corollary If at least one of x, y, z is non-zero and x, y and z are connected by the three given equations, then the elimination of x, y and z leads to the relation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

CRAMER'S RULE

If $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

then the solution of the system of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

is given by $x = \dfrac{\Delta_1}{\Delta}, \; y = \dfrac{\Delta_2}{\Delta}, \; z = \dfrac{\Delta_3}{\Delta}$

where

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: Let

$$\Delta(x, y) = \begin{vmatrix} 1 & x & y \\ 1 & x + y & y \\ 1 & x & x + y \end{vmatrix}$$

Then $\Delta(-3, 2)$ equals

(a) 13 (b) – 6
(c) 12 (d) – 5

Ans. (b)

◎ **Solution:** Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get

$$\Delta(x, y) = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = xy$$

$\therefore \quad \Delta(-3, 2) = -6$

◉ **Example 2:** Suppose a, b, c are distinct real numbers and

$$\Delta = \begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = 0$$

Then $a + b + c$ equals

(a) -1 (b) 2
(c) 0 (d) -5

Ans. (b)

◎ **Solution:** Using $C_3 \to C_3 + C_1$ and taking $(a + b + c)$ common from C_3 we get

$$\Delta = (a + b + c)\,\Delta_1 \qquad (1)$$

where

$$\Delta_1 = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta_1 = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b & 0 \\ 1 & b+c & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$$

[Expand along C_3]

$$= (a-b)(b-c)(c-a) \qquad (2)$$

From (1) and (2)

$$\Delta = (a+b+c)(a-b)(b-c)(c-a)$$

As $\Delta = 0$ and a, b, c are distinct,

we get $a + b + c = 0$

◉ **Example 3:** Suppose $A = (a_{ij})_{3\times 3}$, where $a_{ij} \in \mathbf{R}$. If det (adj A) = 25, then |det (A)| equals:

(a) 5 (b) 12.5
(c) $5\sqrt{5}$ (d) $5^{2/3}$

Ans. (a)

◎ **Solution:** Using det (adj A) = $(\det(A))^2$, we get

$(\det(A))^2 = 25 \quad \Rightarrow \quad |\det(A)| = 5$

◉ **Example 4:** Let

$$\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}, \text{ then}$$

Δ equals:

(a) 0 (b) abc
(c) $a^2 + b^2 + c^2$ (d) $bc + ca + ab$

Ans. (a)

◎ **Solution:** Interchanging the rows and columns, we get

$$\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$$

Taking -1 common from each of R_1, R_2, R_3 we get

$$\Delta = (-1)^3\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$$

$\Rightarrow \quad \Delta = -\Delta \Rightarrow 2\Delta = 0$ or $\Delta = 0$

Alternative Solution

Δ is a skew symmetric determinant of odd order, therefore $\Delta = 0$

◉ **Example 5:** Let $A = (a_{ij})_{3\times 3}$, where $a_{ij} \in \mathbf{C}$ the set of complex numbers. If det (A) = $2 - 3i$, then det (A^{-1}) equals:

(a) $\frac{1}{13}(2 - 3i)$ (b) $\frac{1}{13}(2 + 3i)$
(c) $2 - 3i$ (d) $2 + 3i$

Ans. (b)

◎ **Solution:** $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{2-3i}$

$$= \frac{2+3i}{2^2+3^2} = \frac{1}{13}(2+3i)$$

◉ **Example 6:** In a triangle ABC, if

$$\Delta = \begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$$

then $\sin^2 A + \sin^2 B + \sin^2 C$ is:

(a) $\frac{3\sqrt{3}}{2}$ (b) $\frac{5}{4}$
(c) $\frac{9}{4}$ (d) 2

Ans. (a)

◎ **Solution:** Evaluating along C_1, we get

$\Delta = (c^2 - ab) + (b^2 - ac) + (a^2 - bc) = 0$

$\Rightarrow \quad (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\Rightarrow \quad a = b = c \Rightarrow A = B = C = \pi/3$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = \dfrac{3\sqrt{3}}{2}$.

Example 7: If

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B,$$

then A is equal to:

(a) 12 (b) 18

(c) 24 (d) 30

Ans. (c)

Solution: Applying $R_2 \to R_2 - R_1 - R_3$, we get

$$Ax + B = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Expanding along R_2, we get

$$Ax + B = (-1)(-4)\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & x-2 \\ 0 & 2x-1 \end{vmatrix}$$

$$= 4(3)(2x - 1) = 24x - 12$$

Thus, $A = 24$

Example 8: Suppose a, b, c are three integers such $a < b < c$ and p is a prime number.

Let $$\Delta = \begin{vmatrix} a & a^2 & p+a^3 \\ b & b^2 & p+b^3 \\ c & c^2 & p+c^3 \end{vmatrix}$$

If $\Delta = 0$, then which one of the following is **not** true

(a) $a = -1, b = 1$ (b) $b = 1, c = p$

(c) $a = 0, c = p$ (d) $abc + p = 0$

Ans. (c)

Solution: Write $\Delta = p\Delta_1 + \Delta_2$
where

$$\Delta_1 = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

write

$$\Delta_2 = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc(-1)^2\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= abc\,\Delta_1$$

Thus, $\Delta = (p + abc)\,\Delta_1$.

Using $R_2 \to R_2 - R_3$, $R_1 \to R_1 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$$

$$= (a - b)(b - c)(c - a)$$

As $a < b < c$, $\Delta_1 \neq 0$. Therefore,

$$\Delta = 0 \quad \Rightarrow \quad p + abc = 0$$

$$\Rightarrow \quad p = -abc$$

As p is prime and a, b, c are integers such that $a < b < c$, we must have $-a = 1$, $b = 1$, $c = p$.

$$\Rightarrow \quad a = -1, b = 1, c = p.$$

Example 9: Suppose

$$P(x) = \begin{vmatrix} x & -51 & -71 \\ 51 & x & -73 \\ 71 & 73 & x \end{vmatrix}$$

Product of zeros of $P(x)$ is

(a) 0 (b) 195

(c) –195 (d) –264333

Ans. (a)

Solution:

TIP

Suppose $P(x) = x^3 + ax^2 + bx + c$, then product of zeros of $P(x)$ is $-c = -P(0)$.

Now, $$-c = -P(0) = -\begin{vmatrix} 0 & -51 & -71 \\ 51 & 0 & -73 \\ 71 & 73 & 0 \end{vmatrix}$$

$$= 0$$

[$\because$ Skew symmetric determinant of odd order]

Example 10: Let

$$P(x) = \begin{vmatrix} x & -3+4i & 3-4i \\ x & -7i & 5+6i \\ -x & 7-2i & -7-2i \end{vmatrix}$$

The number of values of x for which $P(x) = 0$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, we get

$$P(x) = \begin{vmatrix} x & -3+4i & 3-4i \\ 0 & 3-11i & 2+10i \\ 0 & 4+2i & -4-6i \end{vmatrix}$$

As $P(x)$ is a linear polynomial, $P(x) = 0$ for exactly one value of x.

Example 11: Let

$$\Delta(\theta) = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}, 0 \le \theta \le 2\pi$$

Solution of $\Delta(\theta) = 3$ is

(a) $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ (b) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

(c) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (d) $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}$

Ans. (b)

Solution: Using $C_1 \to C_1 + C_3$, we get

$$\Delta(\theta) = \begin{vmatrix} 2 & \sin\theta & 1 \\ 0 & 1 & \sin\theta \\ 0 & -\sin\theta & 1 \end{vmatrix}$$

$$= 2\,(1 + \sin^2\theta)$$

$\therefore \quad \Delta(\theta) = 3 \quad \Rightarrow \quad 2\sin^2\theta = 1$

$\Rightarrow \quad \sin\theta = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Example 12: Suppose $a \in \mathbf{R}$ and $x \ne 0$. Let

$$\Delta(x) = \begin{vmatrix} 1-x & a & a^2 \\ a & a^2-x & a^3 \\ a^2 & a^3 & a^4-x \end{vmatrix}$$

Number of values of x for which $\Delta(x) = 0$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: Using the sum property, write

$$\Delta(x) = \Delta_1 - x\Delta_2$$

where

$$\Delta_1 = \begin{vmatrix} 1 & a & a^2 \\ a & a^2-x & a^3 \\ a^2 & a^3 & a^4-x \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} 1 & a & a^2 \\ 0 & a^2-x & a^3 \\ 0 & a^3 & a^4-x \end{vmatrix}$$

$$= (a^2 - x)(a^4 - x) - a^6$$

$$= -(a^2 + a^4)x + x^2$$

In Δ_1, use $C_2 \to C_2 - aC_1$, $C_3 \to C_3 - a^2C_1$ to obtain

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ a & -x & 0 \\ a^2 & 0 & -x \end{vmatrix} = x^2$$

Thus, $\Delta(x) = x^2 + (a^2 + a^4)\,x^2 - x^3$

$= (1 + a^2 + a^4)\,x^2 - x^3$

As $x \ne 0$, $\Delta(x) = 0 \quad \Rightarrow \quad x = \frac{1}{1 + a + a^2}$.

Thus, $\Delta(x) = 0$ for exactly one value of x.

Example 13: If

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = k\lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$\lambda \ne 0$, then k is equal to:

(a) $4\lambda abc$ (b) $-4\lambda abc$

(c) $4\lambda^2$ (d) $-4\lambda^2$

Ans. (c)

Solution: Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a\lambda + \lambda^2 & 2b\lambda + \lambda^2 & 2c\lambda + \lambda^2 \\ -4a\lambda & -4b\lambda & -4c\lambda \end{vmatrix}$$

Take -4λ common from R_3, and applying $R_2 \to R_2 - 2\lambda R_3$, we get

$$\Delta = -4\lambda^3 \begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$= \lambda(4\lambda^2) \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore \quad k = 4\lambda^2$

Example 14: Let

$$f(\theta) = \begin{vmatrix} 1 & \cos\theta & -1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$$

Suppose A and B are respectively maximum and minimum value of $f(\theta)$. Then (A, B) is equal to:

(a) (2, 1) (b) (2, 0)

(c) $(\sqrt{2}, 1)$ (d) $\left(2, \frac{1}{\sqrt{2}}\right)$

Ans. (b)

Solution: Using $C_1 \to C_1 + C_3$, we get

$$f(\theta) = \begin{vmatrix} 0 & \cos\theta & -1 \\ -(\sin\theta + \cos\theta) & 1 & -\cos\theta \\ 0 & \sin\theta & 1 \end{vmatrix}$$

Evaluating along C_1, we get

$$f(\theta) = (\sin\theta + \cos\theta)\begin{vmatrix}\cos\theta & -1\\ \sin\theta & 1\end{vmatrix}$$

$$= (\sin\theta + \cos\theta)^2$$

$$= \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)\right]^2$$

$$= 2\sin^2\left(\frac{\pi}{4} + \theta\right)$$

As $0 \le \sin^2\left(\frac{\pi}{4} + \theta\right) \le 1$, we get

$$0 \le f(\theta) \le 2$$

$$A = 2 \quad \text{for} \quad \theta = \frac{\pi}{4}$$

and $B = 0$ for $\theta = -\frac{\pi}{4}$

Thus, $(A, B) = (2, 0)$

Example 15: If a, b, c are non-zero real numbers and if the system of equations

$(a - 1)x = y + z$,

$(a - 1)y = z + x$,

$(a - 1)z = x + y$,

has a non-trivial solution, then $ab + bc + ca$ equals:

(a) $a + b + c$ (b) abc

(c) 1 (d) –1

Ans. (b)

Solution: As the given system of equations has a non-trivial solution,

$$\Delta = \begin{vmatrix}a-1 & -1 & -1\\ -1 & b-1 & -1\\ -1 & -1 & c-1\end{vmatrix} = 0$$

Write

$$\Delta = \begin{vmatrix}a & -1 & -1\\ 0 & b-1 & -1\\ 0 & -1 & c-1\end{vmatrix} - \begin{vmatrix}1 & -1 & -1\\ 1 & b-1 & -1\\ 1 & -1 & c-1\end{vmatrix}$$

$$= a[(b-1)(c-1) - 1] - \begin{vmatrix}1 & 0 & 0\\ 1 & b & 0\\ 1 & 0 & c\end{vmatrix}$$

[use $C_2 \to C_2 + C_1$
$C_3 \to C_2 + C_1$]

$$= a(bc - b - c) - bc$$

As $\Delta = 0$, we get

$$ab + bc + ca = abc$$

LEVEL 1

Straight Objective Type Questions

Example 16: Let

$$P(x) = \begin{vmatrix}7 & 6 & x-10\\ 2 & x-10 & 5\\ x-10 & 3 & 4\end{vmatrix}$$

sum of zeros of $P(x)$ is

(a) 30 (b) 28

(c) 27 (d) 25

Ans. (a)

Solution: Using $R_1 \leftrightarrow R_3$, write

$$P(x) = -\begin{vmatrix}x-10 & 3 & 4\\ 2 & x-10 & 5\\ 7 & 6 & x-10\end{vmatrix}$$

TIP

Observe $P(x)$ is of the form
$P(x) = -(x - 10)^3 + a(x - 10) + b$
where a, b are some real numbers.

∴ sum of zeros of $P(x)$

$$= -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = 30.$$

Example 17: Let

$$P(x) = \begin{vmatrix}x^2-13 & 4 & 2\\ 3 & x^2-13 & 7\\ 6 & 5 & x^2-13\end{vmatrix}$$

If $x = -2$ is a zero of $P(x)$, then sum of the remaining five zeros is

(a) –2 (b) 0

(c) 2 (d) 3

Ans. (c)

Solution:

TIP

Observe
$P(x) = (x^2 - 13)^3 + a(x^2 - 13) + b.$
where a, b are some real numbers.

As coefficient of x^5 in $P(x)$ is 0, sum of six zeros of $P(x)$ is 0.

⇒ sum of the remaining five zeros + (–2) = 0

$\Rightarrow$ sum of the remaining five zeros = 2.

Example 18: Suppose α, β are two real numbers and $f(n) = \alpha^n + \beta^n$. Let

$$\Delta = \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

If $\Delta = k(\alpha-1)^2(\beta-1)^2(\alpha-\beta)^2$, then k is equal to

(a) 1 (b) $4\alpha\beta$
(c) 9 (d) $\alpha^2\beta^2$

Ans. (a)

Solution:

$$\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1^2$$

where $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-\alpha \\ 1 & \alpha^2-1 & \beta^2-\alpha^2 \end{vmatrix}$$

$$= (\alpha-1)(\beta-\alpha)\begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+\alpha \end{vmatrix}$$

$$= (\alpha-1)(\beta-1)(\beta-\alpha)$$

Thus, $\Delta = (\alpha-1)^2(\beta-1)^2(\alpha-\beta)^2$

$\therefore$ $k = 1$

Example 19: Suppose a, b and c are distinct real numbers. Let

$$\Delta = \begin{vmatrix} a & a+c & a-b \\ b-c & b & a+b \\ c+b & c-a & c \end{vmatrix} = 0$$

Then the straight line $a(x-5) + b(y-2) + c = 0$ passes through the fixed point

(a) (5, 2) (b) (6, 2)
(c) (6, 3) (d) (5, 3)

Ans. (c)

Solution: Applying $C_1 \to aC_1 + bC_2 + cC_3$, we get

Let $\Delta = \dfrac{1}{a}\begin{vmatrix} a(a+b+c) & a+c & a-b \\ b(a+b+c) & b & a+b \\ c(a+b+c) & c-a & c \end{vmatrix}$

$$= \frac{1}{a}a(a+b+c)\Delta_1, \text{ where}$$

$$\Delta_1 = \begin{vmatrix} a & a+c & a-b \\ b & b & a+b \\ c & c-a & c \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta_1 = \begin{vmatrix} a & c & -b \\ b & 0 & a \\ c & -a & 0 \end{vmatrix} = a(a^2+b^2+c^2)$$

$\therefore$ $\Delta = (a+b+c)(a^2+b^2+c^2) = 0$

As a, b, c are distinct real numbers, $a^2+b^2+c^2 \neq 0$, therefore $a+b+c = 0$.

$\Rightarrow$ the line $a(x-5) + b(y-2) + c = 0$ passes through (6, 3).

Example 20: Suppose a, b, c and x are real numbers. Let

$$\Delta = \begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix}$$

Then Δ is independent of

(a) a, b, c (b) x
(c) a, b, c, x (d) none of these

Ans. (c)

Solution: Write $\Delta = \Delta_1 + \Delta_2$, where

$$\Delta_1 = \begin{vmatrix} 1 & 1+ax & 1+ax^2 \\ 1 & 1+bx & 1+bx^2 \\ 1 & 1+cx & 1+cx^2 \end{vmatrix}$$

and $\Delta_2 = \begin{vmatrix} a & ax & ax^2 \\ b & bx & bx^2 \\ c & cx & cx^2 \end{vmatrix} = 0$

[$\because$ C_1 and C_2 are propotional]

In Δ_1, use $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ to obtain

$$\Delta_1 = \begin{vmatrix} 1 & ax & ax^2 \\ 1 & bx & bx^2 \\ 1 & cx & cx^2 \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are proportional]

Thus, $\Delta = 0$ and hence independent of a, b, c, x.

Example 21: Suppose $a, b, c > 1$ and

$$f(x) = \begin{vmatrix} a^{-x} & a^x & x \\ b^{-3x} & b^{3x} & 3x^3 \\ c^{-5x} & c^{5x} & 5x^5 \end{vmatrix}, x \in \mathbf{R}$$

then f is

(a) a constant function
(b) a polynomial of degree 5
(c) an odd function
(d) an even function

Ans. (d)

◎ **Solution:**

$$f(-x) = \begin{vmatrix} a^x & a^{-x} & -x \\ b^{3x} & b^{-3x} & -3x^3 \\ c^{5x} & c^{-5x} & -5x^5 \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} a^{-x} & a^x & x \\ b^{-3x} & b^{3x} & 3x^3 \\ c^{-5x} & c^{5x} & 5x^5 \end{vmatrix}$$

$$= f(x)$$

Thus, f is an even function.

◉ **Example 22:** Suppose n, m are natural numbers and

$$f(x) = \begin{vmatrix} 1 & (1+x)^m & (1+mx)^{mn} \\ (1+mx)^n & 1 & (1+nx)^{mn} \\ (1+nx)^m & (1+x)^n & 1 \end{vmatrix}$$

constant term of the polynomial $f(x)$ is:

(a) 1 (b) $m + n$
(c) $m - n$ (d) 0

Ans. (d)

◎ **Solution:** Constant term of polynomial $f(x)$ is $f(0)$, and

$$f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

◉ **Example 23:** Suppose a, b, c are sides of a scalene triangle. Let

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Then

(a) $\Delta \le 0$ (b) $\Delta < 0$
(c) $\Delta > 0$ (d) $\Delta \ge 0$

Ans. (b)

◎ **Solution:** Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (a + b + c)\,\Delta_1$$

where

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= -(b-c)^2 - (a-b)(a-c)$$

$$= -(a^2 + b^2 + c^2 - bc - ca - ab)$$

$$\Rightarrow \quad \Delta_1 = -\frac{1}{2}\,[(b-c)^2 + (c-a)^2 + (a-b)^2] < 0$$

As $a + b + c > 0$, we get

$$\Delta = (a + b + c)\,\Delta_1 < 0$$

◉ **Example 24:** Suppose A, B, C are angles of a triangle, and let

$$\Delta = \begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$$

Then value of Δ is

(a) – 1 (b) – 4
(c) 0 (d) 4

Ans. (b)

◎ **Solution:** Taking e^{iA}, common from R_1, e^{iB} from R_2 and e^{iC} from R_3, we get

$$\Delta = e^{i(A+B+C)}\,\Delta_1$$

where

$$\Delta_1 = \begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

But $A + B + C = \pi$, so that $e^{i(A+B+C)} = e^{i\pi}$

$$= \cos \pi + i \sin \pi = -1. \text{ Also,}$$

$$A + C = \pi - B \quad \Rightarrow \quad e^{-i(A+C)} = e^{-\pi i}\, e^{iB} = -e^{iB}.$$

Thus, $$\Delta_1 = \begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{-iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

$$= e^{i(A+B+C)}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2$, we get

$$\Delta_1 = (-1)\begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = (-1)(-2)(2) = 4$$

Therefore, $\Delta = (-1)\,\Delta_1 = -4$

◉ **Example 25:** Suppose x_1, x_2, x_3 are real numbers such that $x_1 x_2 x_3 \ne 0$. Let

$$\Delta = \begin{vmatrix} x_1 + a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & x_2 + a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & x_3 + a_3 b_3 \end{vmatrix}$$

Then $\dfrac{\Delta}{x_1x_2x_3}-1$ equals:

(a) $\dfrac{a_1b_1}{x_1}+\dfrac{a_2b_2}{x_2}+\dfrac{a_3b_3}{x_3}$

(b) -1

(c) $\dfrac{a_1a_2a_3+b_1b_2b_3}{x_1x_2x_3}$

(d) 0

Ans. (a)

◎ **Solution:** Using the sum property, write

$$\Delta = x_1\,\Delta_1 + b_1\,\Delta_2$$

where

$$\Delta_1 = \begin{vmatrix} 1 & a_1b_2 & a_1b_3 \\ 0 & x_2+a_2b_2 & a_2b_3 \\ 0 & a_3b_2 & x_3+a_3b_3 \end{vmatrix}$$

$$= (x_2 + a_2\,b_2)\,(x_3 + a_3\,b_3) - a_3\,b_2\,a_2\,b_3$$

$$= x_2\,x_3 + x_2\,a_3\,b_3 + x_3\,a_2\,b_2$$

and $$\Delta_2 = \begin{vmatrix} a_1 & a_1b_2 & a_1b_3 \\ a_2 & x_2+a_2b_2 & a_2b_3 \\ a_3 & a_3b_2 & x_3+a_3b_3 \end{vmatrix}$$

Using $C_2 \to C_2 - b_2\,C_1$, $C_3 \to C_3 - b_3\,C_1$, we get

$$\Delta_2 = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & x_2 & 0 \\ a_3 & 0 & x_3 \end{vmatrix} = a_1x_2x_3$$

Thus,

$$\Delta = x_1[x_2\,x_3 + x_2\,a_3\,b_3 + x_3\,a_2\,b_2] + a_1\,b_1\,x_2\,x_3$$

$$\Rightarrow \frac{\Delta}{x_1x_2x_3}-1 = \frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{x_3}$$

◉ **Example 26:** Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$

$$= \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda-3 \\ \lambda-1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$$

where p, q, r, s and t are constants. Then value of t is

(a) 0 (b) -1

(c) 2 (d) 3

Ans. (a)

◎ **Solution:** Putting $\lambda = 0$, we obtain

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = 0$$

as it is a skew symmetric determinant of odd order.

◉ **Example 27:** Let $\Delta = \begin{vmatrix} 1 & -4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix}$. Solution set of $\Delta = 0$ is

(a) $\{-2, 3\}$ (b) $\{-3, 4\}$

(c) $\{4, -6\}$ (d) $\{-2, -1\}$

Ans. (d)

◎ **Solution:** Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get

$$\Delta = \begin{vmatrix} 1 & -4 & 20 \\ 0 & 2 & -15 \\ 0 & 2\,(x+2) & 5\,(x^2-4) \end{vmatrix} = 2(x+2)\begin{vmatrix} 1 & -15 \\ 1 & 5\,(x-2) \end{vmatrix}$$

$$= 10(x + 2)\,(x + 1)$$

Now, $\Delta = 0 \Rightarrow x = -2, -1.$

◉ **Example 28:** Let $\Delta = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$, then Δ is equal to

(a) 0 (b) $a + b + c$

(c) $\dfrac{1}{2}(a^2 + b^2 + c^2)$ (d) none of these

Ans. (a)

◎ **Solution:** Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (b-a)\,(c-a)\begin{vmatrix} 1 & a & a^2-bc \\ 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \end{vmatrix} = 0$$

[$\because$ R_2 and R_3 are identical]

◉ **Example 29:** Suppose a, b, $c > 0$ and a, b, c are the pth, qth, rth terms of a G.P. Let

$$\Delta = \begin{vmatrix} 1 & p & \log a \\ 1 & q & \log b \\ 1 & r & \log c \end{vmatrix}$$

then numerical value of Δ is

(a) -1 (b) 2

(c) 0 (d) none of these

Ans. (c)

◎ **Solution:** Let $a = AR^{p-1}$, $b = AR^{q-1}$ and $c = AR^{r-1}$

$\Rightarrow$ $\log a = \alpha + (p-1)\beta$, $\log b = \alpha + (q-1)\beta$ and $\log c = \alpha + (r-1)\beta$

where $\alpha = \log a$, $\beta = \log R$.

Now,

$$\Delta = \begin{vmatrix} 1 & p & \alpha+(p-1)\,\beta \\ 1 & q & \alpha+(q-1)\,\beta \\ 1 & r & \alpha+(r-1)\,\beta \end{vmatrix}$$

Using $C_3 \to C_3 - (\alpha - \beta)C_1 - \beta C_3$, we get $\Delta = 0$.

Example 30: Let $\omega = \frac{1}{2}\left(-1+\sqrt{3}i\right)$ and

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$ then Δ equals

(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

Ans. (b)

Solution: Using $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$, and applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} = 3(\omega^2 - \omega) = 3\omega(\omega - 1)$$

Example 31: Let a, b, and c be respectively the pth, qth and rth terms of a harmonic progression and

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ bc & ca & ab \end{vmatrix}$$

Then numerical value of Δ is

(a) 0 (b) – 1
(c) 1 (d) none of these

Ans. (a)

Solution: $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are the pth, qth, rth terms of an A.P.

Let $\frac{1}{a} = A + (p-1)D$, $\frac{1}{b} = A + (q-1)D$,

and $\frac{1}{c} = A + (r-1)D$.

Now

$$\Delta = abc \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ A+(p-1)D & A+(q-1)D & A+(r-1)D \end{vmatrix}$$

Using $R_3 \to R_3 - (A - D)R_1 - DR_2$, we get
$\Delta = 0$.

Example 32: Let $\omega \neq 1$ be complex cube root of unity and n be a natural number and

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$

Then Δ equals

(a) 0 (b) 1
(c) ω (d) ω^2

Ans. (a)

Solution: If n is a multiple of 3, we get each element of Δ becomes 1.

$\therefore$ $\Delta = 0$.

If $n = 3k + 1$, then

$$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

If $n = 3k + 2$, then

$$\Delta = \begin{vmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

Example 33: Let

$$\Delta = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix},$$

then

(a) $\Delta = 2\Delta_1$ (b) $\Delta = -2\Delta_1$
(c) $\Delta = 4\Delta_1$ (d) $\Delta = -4\Delta_1$

Ans. (a)

Solution: Using $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Taking 2 common from R_1 and applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we obtain

$$\Delta = 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = 2(-1)(-1)\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = -2\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$$

$$= 2\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} = 2\Delta_1$$

Example 34: If $x = -2$, and $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

then numerical value of Δ is

(a) 8 (b) – 8
(c) 4 (d) – 4

Ans. (b)

◎ **Solution:** Taking x common from C_2 and C_3 we obtain

$$\Delta = x^2\begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix} = x^3\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix}$$

[using $C_1 \to C_1 - yC_2$]

Next using $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we obtain

$$D = x^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 5 & 3 \end{vmatrix} = x^3\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = x^3$$

As $x = -2, \Delta = -8.$

◉ **Example 35:** If $a = \omega \neq 1$, is a cube root of unity, $b = -785$, $c = 2008\, i$, and

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

then Δ equals

(a) $-i$ (b) i
(c) 1 (d) $1 - wi$

Ans. (c)

◎ **Solution:** Write

$$\Delta = a\begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix}$$

and apply $C_2 \to C_2 - bC_1$, $C_3 \to C_3 - cC_1$ to obtain

$$\Delta = a\begin{vmatrix} 1 & a & a+b \\ 2 & 3a & 4a+3b \\ 3 & 6a & 10a+6c \end{vmatrix}$$

Take a common from C_2 and apply $C_3 \to C_3 - bC_2$ to obtain

$$\Delta = a^2\begin{vmatrix} 1 & 1 & a \\ 2 & 3 & 4a \\ 3 & 6 & 10a \end{vmatrix} = a^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Apply $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$ to obtain

$$\Delta = a^3\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 3 & 4 \end{vmatrix} = a^3 = \omega^3 = 1.$$

◉ **Example 36:** Let x, y, z be positive and $x, y, z \neq 1$. Let

$$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix},$$

then numerical value of Δ is

(a) -1 (b) 0
(c) 1 (d) none of these

Ans. (b)

◎ **Solution:** Using change of base formula

$$\log_a b = \frac{\log b}{\log a}, \; a, b > 0, \; a, b \neq 1,$$

we can write

$$\Delta = \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0.$$

◉ **Example 37:** Let $\omega \neq 1$ be a cube root of unity and

$$\Delta = \begin{vmatrix} 1-\omega-\omega^2 & 2 & 2 \\ 2\omega & \omega-\omega^2-1 & 2\omega \\ 2\omega^2 & 2\omega^2 & \omega^2-1-\omega \end{vmatrix}$$

then Δ equals

(a) $-\omega$ (b) $3\omega(1-\omega)$
(c) 0 (d) $1-\omega^2$

Ans. (c)

◎ **Solution:** Using $R_1 \to R_1 + R_2 + R_3$ and $1 + \omega + \omega^2 = 0$, we obtain

$$\Delta = 0.$$

◉ **Example 38:** Suppose $x = -\frac{1}{3}\left(1+\sqrt{7}i\right)$ and

$$y = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}.$$

Let $$\Delta = \begin{vmatrix} 1 & x & x \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix},$$

then Δ equals

(a) $-\sqrt{7}$ (b) 7
(c) i (d) $-1.$

Ans. (c)

◎ **Solution:** Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & x & x \\ 0 & y & y-x \\ 0 & 0 & y \end{vmatrix} = y^2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$$

[Using De Moivre's Theorem]

◉ **Example 39:** Let $x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

then numerical value of Δ is

(a) 0 (b) -1
(c) 8 (d) -4

Ans. (c)

Solution: Use $C_3 \to C_3 - xC_2$ and $C_2 \to C_2 - xC_1$.

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^3 & 0 \\ 1 & x^2 - x & 1-x^3 \end{vmatrix}$$

$$= (1-x^3)^2 = [1 - \cos\pi - i\sin\pi]^2 = 8$$

[De Moivre's Theorem]

Example 40: Let $f(x) = \left[2^{-x^2}\left[2^{x^2}\right]\right]$, $x \in \mathbf{R}$. ([ ] denotes the greatest integer function.) Let $x_1 = 0$, $x_2 = \log_2 3$ and $x_3 = \sqrt{2}$. Suppose $0 < x_4 < 1$.

$$\Delta = \begin{vmatrix} f(x_1) & f(x_2) & f(x_3) \\ f(x_4) & f(x_4) & f(x_2) \\ f(x_2) & f(x_3) & f(x_1) \end{vmatrix}$$, then Δ is equal to:

(a) –1 (b) 0
(c) 1 (d) $2 \log_2{}^3$

Ans. (b)

Solution: $0 \le [2^{x^2}] \le 2^{x^2} \Rightarrow 0 \le \dfrac{[2^{x^2}]}{2^{x^2}} \le 1$. Therefore, range $f(x)$ is $\{0, 1\}$

Also $[2^{x^2}] = 2^{x^2} \Leftrightarrow 2^{x^2}$ is a positive integer.

Thus, $f(x_1) = 1, f(x_2) = 1, f(x_3) = 1$ and $f(x_4) = 0$.

Therefore,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Example 41: If $\omega \ne 1$ is a cube root of unity and

$$\Delta = \begin{vmatrix} x+\omega^2 & \omega & 1 \\ \omega & \omega^2 & 1+x \\ 1 & x+\omega & \omega^2 \end{vmatrix} = 0$$, then value of x is

(a) 0 (b) 1
(c) –1 (d) none of these

Ans. (a)

Solution: Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+\omega^2+\omega+1 & \omega & 1 \\ \omega+\omega^2+1+x & \omega^2 & 1+x \\ 1+x+\omega+\omega^2 & x+\omega & \omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & \omega & 1 \\ x & \omega^2 & 1+x \\ x & x+\omega & \omega^2 \end{vmatrix}$$ using $1 + \omega + \omega^2 = 0$

Δ is clearly equal to 0 for $x = 0$.

Example 42: Let $f : \mathbf{N} \to \mathbf{N}$ be defined by

$$f(x) = (x+1)^2 + x - \left[\sqrt{(x+1)^2 + (x+1)}\right]^2$$

([ ] denotes the greatest integer function). Suppose *a, b, c* are three distinct natural numbers. Let

$$\Delta = \begin{vmatrix} f(a) & a^2 & a \\ f(b) & b^2 & b \\ f(c) & c^2 & c \end{vmatrix}$$

The Δ is equal to:

(a) $-(a+b+c)$ (b) $a+b+c$
(c) –1 (d) 0

Ans. (d)

Solution: If $n \in \mathbf{N}$, then

$$n^2 < n^2 + n < (n+1)^2$$

$$\Rightarrow \quad n < \sqrt{n^2+n} < n+1$$

$$\Rightarrow \quad \left[\sqrt{n^2+n}\right] = n$$

$$\Rightarrow \quad \left[\sqrt{(x+1)^2+(x+1)}\right] = x + 1 \ \forall\ x \in \mathbf{N}.$$

$\therefore \quad f(x) = (x+1)^2 + x - (x+1)^2 = x \ \forall\ x \in \mathbf{N}$.

Thus,

$$\Delta = \begin{vmatrix} a & a^2 & a \\ b & b^2 & b \\ c & c^2 & c \end{vmatrix} = 0$$

[$\because$ C_1 and C_3 are identical]

Example 43: If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then the other two roots are

(a) 3, 7 (b) 2, 7
(c) 3, 6 (d) 2, 6

Ans. (b)

Solution: Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

Taking $x + 9$ common from R_1 and applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ we get

$$\Delta = (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix}$$

$$= (x+9)(x-2)(x-7)$$

Thus, $\Delta = 0 \Rightarrow x = -9, 2, 7$

$\therefore$ the other two roots are 2 and 7.

Example 44: If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$, then

(a) $\Delta_1 = 3\,(\Delta_2)^2$ (b) $\dfrac{d}{dx}(\Delta_1) = 3\Delta_2$

(c) $\dfrac{d}{dx}(\Delta_1) = 3\Delta_2^2$ (d) $\Delta_1 = 3\Delta_2^{3/2}$

Ans. (b)

Solution: We have

$$\frac{d\Delta_1}{dx} = \begin{vmatrix} 1 & b & b \\ 0 & x & b \\ 0 & a & x \end{vmatrix} + \begin{vmatrix} x & 0 & b \\ a & 1 & b \\ a & 0 & x \end{vmatrix} + \begin{vmatrix} x & b & 0 \\ a & x & 0 \\ a & b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix}$$

$= 3\Delta_2$.

Example 45: If $x \,\varepsilon\, \mathbf{R}$ and $n \,\varepsilon\, \mathbf{I}$, then the determinant

$$\Delta = \begin{vmatrix} \sin(n\pi) & \sin x - \cos x & \log\tan x \\ \cos x - \sin x & \cos[(2n+1)\pi/2] & \log\cot x \\ \log\cot x & \log\tan x & \tan(n\pi) \end{vmatrix}$$

equals

(a) 0 (b) $\log\tan x - \log\cot x$

(c) $\tan(\pi/4 - x)$ (d) none of these

Ans. (a)

Solution: We can write Δ as

$$\Delta = \begin{vmatrix} 0 & \sin x - \cos x & \log\tan x \\ -(\sin x - \cos x) & 0 & -\log\tan x \\ -\log\tan x & \log\tan x & 0 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 0 & -(\sin x - \cos x) & -\log\tan x \\ \sin x - \cos x & 0 & \log\tan x \\ \log\tan x & -\log\tan x & 0 \end{vmatrix}$$

[taking – 1 common from R_1, R_2 and R_3]

$= -\Delta$ [using the reflection property]

$\Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$.

Example 46: Let $\Delta_1 = \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$, then $\Delta_1 - \Delta_2$ equals

(a) $(x-1)(y-1)(z-1)$

(b) $(x-y)(y-z)(z-x)$

(c) $abc\,(x-y)(y-z)(z-x)$

(d) 0

Ans. (d)

Solution: Multiplying R_3 of Δ_1 by xyz, we get

$$\Delta_1 = \frac{1}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \Delta_2$$

[taking x, y, z common from C_1, C_2, C_3 respectively]

$\Rightarrow \quad \Delta_1 - \Delta_2 = 0.$

Example 47: If

$$\Delta(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$$

then $\Delta(100)$ equals

(a) 0 (b) – 100

(c) 100! (d) – 100!

Ans. (a)

Solution: Taking x common from C_2, $x + 1$ from C_3 and $x - 1$ from R_3, we get

$$\Delta(x) = x(x+1)(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$, we get

$$\Delta(x) = x(x+1)(x-1)\begin{vmatrix} 0 & 0 & 1 \\ x & -1 & x \\ 2x & -2 & x \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are proportional]

Thus, $\quad \Delta(100) = 0$

Example 48: If $\Delta(x) = \begin{vmatrix} 1 & 1 & 1 \\ (e^x + e^{-x})^2 & (\pi^x + \pi^{-x})^2 & 2 \\ (e^x - e^{-x})^2 & (\pi^x - \pi^{-x})^2 & -2 \end{vmatrix}$

then $\Delta(x)$ equals

(a) x^2 (b) $x^2 - 1$

(c) $e^{x^2} - \pi^{x^2}$ (d) 0

Ans. (d)

Solution: Using $R_2 \to R_2 - R_3$ and $(a^x + a^{-x})^2 - (a^x - a^{-x})^2 = 4a^x a^{-x} = 4$, we get

$$\Delta(x) = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ (e^x - e^{-x})^2 & (\pi^x - \pi^{-x})^2 & -2 \end{vmatrix} = 0$$

[since R_1 and R_2 are proportional]

Example 49: If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$,

then $\int_0^{\pi/2} \Delta(x)\, dx$ equals

(a) 1/4 (b) 1/2
(c) 0 (d) – 1/2

Ans. (d)

Solution: Using $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta(x) = \begin{vmatrix} 0 & \cos x & 1-\cos x \\ 0 & \cos x & 1+\sin x-\cos x \\ -1 & \sin x & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} \cos x & 1-\cos x \\ \cos x & 1+\sin x-\cos x \end{vmatrix}$$

$$= (-1)\cos x\,[1 + \sin x - \cos x - 1 + \cos x]$$

$$= -\cos x \sin x = -\frac{1}{2}(\sin 2x)$$

$$\therefore \int_0^{\pi/2} \Delta(x)\, dx = -\frac{1}{2}\frac{(-1)}{2} \cos 2x \Big]_0^{\pi/2}$$

$$= \frac{1}{4}(\cos \pi - \cos 0) = -\frac{1}{2}$$

Example 50: The determinant

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$$ equals zero, if

(a) a, b, c are in A.P.
(b) a, b, c are in H.P.
(c) $x - \alpha$ is a factor of $ax^2 + 2bx + c$
(d) $x - \alpha$ is a factor of $ax^2 + bx + c$

Ans. (c)

Solution: Applying $C_3 \to C_3 - \alpha C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -(a\alpha^2+2b\alpha+c) \end{vmatrix}$$

$$= -(a\alpha^2 + 2b\alpha + c)\begin{vmatrix} a & b \\ b & c \end{vmatrix} \quad \text{[expanding along } C_3]$$

$$= -(a\alpha^2 + 2b\alpha + c)(ac - b^2)$$

= 0 if $x - \alpha$ is a factor of $ax^2 + 2bx + c$.

Example 51: If $\theta \,\varepsilon\, \mathbf{R}$, then the determinant

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin(\theta+2\pi/3) & \cos(\theta+2\pi/3) & \sin(2\theta+4\pi/3) \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

equals

(a) $-\sin\theta - \cos\theta$ (b) $\sin 2\theta$
(c) $1 + \sin 2\theta - \cos 2\theta$ (d) 0

Ans. (d)

Solution: Applying $R_2 \to R_2 + R_3$, and using $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$ and $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$, we get

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos(2\pi/3) & 2\cos\theta\cos(2\pi/3) & 2\sin 2\theta\cos(4\pi/3) \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

since $\cos(2\pi/3) = \cos(\pi - \pi/3) = -\cos(\pi/3) = -1/2$

and $\cos(4\pi/3) = \cos(\pi + \pi/3) = -\cos(\pi/3) = -1/2$

$\Rightarrow \Delta = 0$ [$\because R_1$ and R_2 are proportional.]

Example 52: If α, β, γ are the roots of $x^3 + px^2 + q = 0$, where $q \neq 0$, and

$$\Delta = \begin{vmatrix} 1/\alpha & 1/\beta & 1/\gamma \\ 1/\beta & 1/\gamma & 1/\alpha \\ 1/\gamma & 1/\alpha & 1/\beta \end{vmatrix}$$

then Δ equals

(a) $-p/q$ (b) $1/q$
(c) p^2/q (d) 0

Ans. (d)

Solution: We have $\beta\gamma + \gamma\alpha + \alpha\beta = 0$. We can write Δ as

$$\Delta = \frac{1}{\alpha^3\beta^3\gamma^3}\begin{vmatrix} \beta\gamma & \gamma\alpha & \alpha\beta \\ \gamma\alpha & \alpha\beta & \beta\gamma \\ \alpha\beta & \beta\gamma & \gamma\alpha \end{vmatrix}$$

$$= \frac{1}{\alpha^3\beta^3\gamma^3}\begin{vmatrix} \beta\gamma+\gamma\alpha+\alpha\beta & \gamma\alpha & \alpha\beta \\ \gamma\alpha+\alpha\beta+\beta\gamma & \alpha\beta & \beta\gamma \\ \alpha\beta+\beta\gamma+\gamma\alpha & \beta\gamma & \gamma\alpha \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

$$= \frac{1}{\alpha^3\beta^3\gamma^3}\begin{vmatrix} 0 & \gamma\alpha & \alpha\beta \\ 0 & \alpha\beta & \beta\gamma \\ 0 & \beta\gamma & \gamma\alpha \end{vmatrix} = 0 \text{ [all zero property]}$$

Example 53: If A, B, C are the angles of a triangle, and

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

then Δ equals

(a) 0 (b) – 1
(c) 2 cos *A* cos *B* cos *C* (d) none of these

Ans. (a)

Solution: Applying $C_1 \to aC_1 + bC_2 + cC_3$, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} -a+b\cos C+c\cos B & \cos C & \cos B \\ a\cos C-b+c\cos A & -1 & \cos A \\ a\cos B+b\cos A-c & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

[using projection formulae and all zero property.]

Example 54: If a, b, c are three complex numbers such that $a^2 + b^2 + c^2 = 0$ and

$$\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = ka^2\,b^2\,c^2,$$

then the value of k is

(a) 1 (b) 2
(c) – 2 (d) 4

Ans. (*d*)

Solution: Using $a^2 + b^2 + c^2 = 0$, we can write Δ as

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

[taking a, b, c common from C_1, C_2, C_3 respectively]

$$= a^2\,b^2\,c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

[taking a, b, c common from R_1, R_2, R_3 respectively]

$$= a^2\,b^2\,c^2\begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} = -2a^2\,b^2\,c^2\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 4a^2\,b^2\,c^2$$

[applying $C_1 \to C_1 + C_3$ and $C_3 \to C_2 + C_3$]

Thus, $k = 4$.

Example 55: If $\theta, \phi \,\varepsilon\, \boldsymbol{R}$, then the determinant

$$\Delta = \begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 1 \\ \cos(\theta+\phi) & -\sin(\theta+\phi) & 0 \end{vmatrix}$$

lies in the interval

(a) $\left[-\sqrt{2}, \sqrt{2}\right]$ (b) [– 1, 1]
(c) $\left[-\sqrt{2}, 1\right]$ (d) $\left[-1, \sqrt{2}\right]$

Ans. (a)

Solution: Applying $R_3 \to R_3 - \cos\varphi\, R_1 + \sin\varphi\, R_2$, we get

$$\Delta = \begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 1 \\ 0 & 0 & \sin\varphi - \cos\phi \end{vmatrix}$$

$$= (\sin\varphi - \cos\varphi)(\cos^2\theta + \sin^2\theta)$$

$$= \sqrt{2}\left\{\frac{1}{\sqrt{2}}\sin\varphi - \frac{1}{\sqrt{2}}\cos\varphi\right\} = \sqrt{2}\,\sin(\phi - \pi/4)$$

As $-1 \le \sin(\phi - \pi/4) \le 1$, $-\sqrt{2} \le \sqrt{2}\,\sin(\varphi - \pi/4) \le \sqrt{2}$ or $-\sqrt{2} \le \Delta \le \sqrt{2}$.

Example 56: If $\Delta_1 = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then $\Delta_1 - \Delta_2$ equal

(a) 0 (b) 3*abc*
(c) 6*abc* (d) $2(a^3 + b^3 + c^3)$

Ans. (a)

Solution: Using $C_1 \to C_1 + C_3$ and $C_2 \to C_2 - C_3$ in Δ_1, we get

$$\Delta_1 = \begin{vmatrix} a+b+c & -b & a \\ a+b+c & -c & b \\ a+b+c & -a & c \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 - C_3$, we get

$$\Delta_1 = \begin{vmatrix} c & -b & a \\ a & -c & b \\ b & -a & c \end{vmatrix} = -\begin{vmatrix} a & -b & c \\ b & -c & a \\ c & -a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$\Rightarrow \Delta_1 = \Delta_2 \Rightarrow \Delta_1 - \Delta_2 = 0.$

Example 57: If x, y, z are different from zero and

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0,$$

then the value of the expression $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z}$ is

(a) 0 (b) –1
(c) 1 (d) 2

Ans. (d)

Solution: Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(c-z)\begin{vmatrix}-x & y\\ -x & 0\end{vmatrix} + z\begin{vmatrix}a & b-y\\ -x & y\end{vmatrix} = 0$$

$\Rightarrow \quad (c-z)(xy) + z(ay + bx - xy) = 0$

$\Rightarrow \quad cxy - xyz + ayz + bxz - xyz = 0$

$\Rightarrow \quad ayz + bzx + cxy = 2xyz$

$\Rightarrow \quad \dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = 2$

Example 58: If $p + q + r = a + b + c = 0$, then the determinant

$$\Delta = \begin{vmatrix}pa & qb & rc\\ qc & ra & pb\\ rb & pc & qa\end{vmatrix} \text{ equals}$$

(a) 0 (b) 1
(c) $pa + qb + rc$ (d) none of these

Ans. (a)

Solution: We have

$\Delta = pqr(a^3 + b^3 + c^3) - abc(p^3 + q^3 + r^3)$

But $a + b + c = 0 \Rightarrow (a + b)^3 = -c^3$

$\Rightarrow \quad a^3 + b^3 + 3ab(a + b) + c^3 = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = -3ab(-c) = 3abc$

Similarly, $p^3 + q^3 + r^3 = 3pqr$

Thus, $\Delta = pqr(3abc) - abc(3pqr) = 0$

Example 59: The system of equations

$$\lambda x + y + z = 0$$
$$-x + \lambda y + z = 0$$
$$-x - y + \lambda z = 0$$

will have a non-trivial solution if real values of λ are

(a) 0, 1 (b) 0, – 1
(c) 0, 2 (d) 0

Ans. (d)

Solution: The system of equation will have a non-trivial solution if and only if

$\Delta = 0$ where

$$\Delta = \begin{vmatrix}\lambda & 1 & 1\\ -1 & \lambda & 1\\ -1 & -1 & \lambda\end{vmatrix} = \lambda^3 + 3\lambda = \lambda(\lambda^2 + 1)$$

Thus, $\Delta = 0$, $\lambda \in \mathbf{R} \Rightarrow \lambda = 0$

Example 60: The values of k for which the system of equations

$$x + ky - 3z = 0 \qquad (1)$$
$$3x + ky - 2z = 0 \qquad (2)$$
$$2x + 3y - 4z = 0 \qquad (3)$$

has a non-trivial solution is (are)

(a) $\dfrac{21}{10}$ (b) $\dfrac{31}{10}$
(c) – 5 (d) 4

Ans. (a)

Solution:

From (1) and (2), $2x + z = 0 \Rightarrow 2x = -z$.

From (3), we get $3y = 5z$

$\therefore \quad x = -\dfrac{1}{2}z,\ y = \dfrac{5}{3}z$

Substituting this in (1) we get

$$-\frac{1}{2}z + \frac{5k}{3}z - 3z = 0. \Rightarrow \left(\frac{5k}{3} - \frac{7}{2}\right)z = 0$$

Now, $\quad z \neq 0 \Rightarrow k = \dfrac{21}{10}$.

Example 61: If $a \in \mathbf{R}$ and the system of equations $x + ay = 0$, $az + y = 0$, $ax + z = 0$ has infinite number of solution then the value of a is

(a) – 1 (b) 1
(c) 0 (d) no real value

Ans. (a)

Solution: We have $x = -ay$, therefore

$-a^2y + z = 0 \Rightarrow z = a^2y$.

Thus, $\quad az + y = 0 \Rightarrow (a^3 + 1)z = 0$.

For the system of equations to have infinite number of solutions $a^3 + 1 = 0$

$\Rightarrow \quad a = -1 \qquad [\because a \text{ is real}]$

Example 62: Given $2x - y + 2z = 2$,

$$x - 2y + z = -4,$$
$$x + y + \lambda z = 4$$

then the value of λ such that the given system of equations has no solution, is

(a) 3 (b) 1
(c) 0 (d) – 3

Ans. (b)

Solution: From the first two equations, we get

$x + y = 6 - z$

Putting this value in the last equation we get

$6 + (-1 + \lambda)z = 4 = (\lambda - 1)z = -2$.

For the system of equations to have no solutions, $\lambda = 1$

Example 63: If the system of linear equations

$$x + 2ay + az = 0$$
$$x + 3by + bz = 0$$
$$x + 4cy + cz = 0$$

has a non-zero solution, then a, b, c

(a) are in G.P.
(b) are in H.P.

(c) satisfy $a + 2b + 3c = 0$
(d) are in A.P.

Ans. (b)

◎ **Solution:** The system will have a non-zero solution if

$$\Delta = \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

Using $C_2 \to C_2 - 2C_3$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix} = 0$$

$$\Rightarrow 1(bc - 2bc) + a(2c - b) = 0 \Rightarrow 2ac = bc + ab$$

$$\Rightarrow \frac{2ac}{a + c} = b \Rightarrow a, b, c \text{ are in H.P.}$$

Example 64: The system of homogenous equations

$$(a - 1)x + (a + 2)y + az = 0$$
$$(a + 1)x + ay + (a + 2)z = 0$$
$$ax + (a + 1)y + (a - 1)z = 0$$

has a non-trivial solution if a equals

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 2 (d) – 1.

Ans. (b)

◎ **Solution:** The system of equations will have a non-trivial solution if $\Delta = 0$ where

$$\Delta = \begin{vmatrix} a-1 & a+2 & a \\ a+1 & a & a+2 \\ a & a+1 & a-1 \end{vmatrix}$$

Using $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we obtain

$$\Delta = \begin{vmatrix} -1 & 2 & a \\ -1 & -2 & a+2 \\ 1 & 2 & a-1 \end{vmatrix}$$

Using $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} -1 & 2 & a \\ -1 & -2 & a+2 \\ 0 & 0 & 2a+1 \end{vmatrix} = (2a + 1)(2 + 2) = 4(2a + 1)$$

Now, $\Delta = 0 \Rightarrow a = -1/2$.

Example 65: The system of equations

$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$

has no solution, if α equals

(a) – 2 (b) 1
(c) – 2 (d) either – 2 or 1

Ans. (a)

◎ **Solution:** Let

$$\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (\alpha + 2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha + 2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha - 1 & 0 \\ 0 & 1 & \alpha - 1 \end{vmatrix}$$

[using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$\therefore$ $\Delta = (\alpha + 2)(\alpha - 1)^2$

For no solution, $\Delta = 0 \Rightarrow \alpha = -2, 1$.

For $\alpha = 1$, the system of equations has infinite number of solutions.

For $\alpha = -2$, on adding the three equations we obtain

$$0 = -9$$

Thus, system of equations has no solution for $\alpha = -2$.

Example 66: Suppose $a, b, c, \alpha \in \mathbf{R}$ and $abc\,\alpha \neq 0$. If the system of equations:

$$(a + \alpha)x + \alpha y + \alpha z = 0 \quad (1)$$
$$\alpha x + (b + \alpha)y + \alpha z = 0 \quad (2)$$
$$\alpha x + \alpha y + (\alpha + c)z = 0 \quad (3)$$

has a non-trivial solution, then $\alpha\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ is equal to

(a) – 1 (b) 0
(c) abc (d) $bc + ca + ab$

Ans. (a)

◎ **Solution:** From (1) and (2)

$$ax - by = 0$$

and from (2) and (3)

$$ax - cz = 0$$

$\therefore$ $ax = by = cz$

$$\Rightarrow \frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c}$$

Putting in (1) we get

$$\frac{a + \alpha}{a} + \frac{\alpha}{b} + \frac{\alpha}{c} = 0$$

$$\Rightarrow \alpha\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = -1$$

Example 67: Suppose $a, b, \in \mathbf{R}$ and $a, b \neq 1$. If the system of equations:

$$ax + y + z = 0 \quad (1)$$
$$x + by + z = 0 \quad (2)$$
$$x + y + 2z = 0 \quad (3)$$

has a non-trivial solution, then

(a) $a + b = 2$ (b) $a + b = ab$

(c) $a+\dfrac{1}{b}=2$ (d) $a + b = 0$

Ans. (a)

Solution: From (1) and (3)

$(a - 1)x - z = 0$

and from (2) and (3)

$(b - 1)y = z$

$$\therefore \quad \frac{x}{\frac{1}{(1-a)}} = \frac{y}{\frac{1}{(1-b)}} = \frac{z}{1}$$

Putting in (1) we get

$$\frac{a}{1-a}+\frac{1}{1-b}+1 = 0$$

$$\Rightarrow \quad \frac{a}{1-a} = -\frac{1}{1-b}$$

$$\Rightarrow \quad 1 - b = -1 + a$$

$$\Rightarrow \quad a + b = 2$$

Example 68: If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

then $f(x)$ is a polynomial of degree

(a) 3 (b) 2

(c) 1 (d) 0

Ans. (b)

Solution: Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

[using $a^2 + b^2 + c^2 = -2$]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = (1-x)^2$$

which is a polynomial of degree 2.

Example 69:

$$\text{Let} \quad f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

then value of $\int_{\pi/4}^{\pi/2} f(x)dx$ is

(a) 0 (b) $\pi/48$

(c) $-\dfrac{\pi}{2}-\dfrac{\pi}{15\sqrt{2}}$ (d) none of these

Ans. (d)

Solution: Applying $R_2 \to R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ -\sin^2 x & 0 & 0 \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= -(-\sin^2 x)\begin{vmatrix} \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= \sin^2 x\,[\cos x \operatorname{cosec}^2 x - \cos^2 x\,(\sec^2 x + \cot x \operatorname{cosec}^2 x)]$$

$$= \cos x - \sin^2 x - \frac{\cos^3 x}{\sin x}$$

$$= \cos x - \frac{1}{2}(1 - \cos 2x) - \left(\frac{1}{\sin x} - \sin x\right)\cos x$$

$$\text{Thus, } \int_{\pi/4}^{\pi/2} f(x)\,dx = \int_{\pi/4}^{\pi/2} \cos x\,dx - \frac{1}{2}\int_{\pi/4}^{\pi/2} dx + \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx - \int_{\pi/4}^{\pi/2}\left(\frac{1}{\sin x} - \sin x\right)\cos x\,dx$$

$$= 1-\frac{1}{\sqrt{2}}-\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)+\frac{1}{4}(0-1)-\left[\left(\log|t|-\frac{t^2}{2}\right)\right]_{1/\sqrt{2}}^{1}$$

where $t = \sin x$

$$= 1-\frac{1}{\sqrt{2}}-\frac{\pi}{8}-\frac{1}{2}\log 2$$

Example 70: If $a + b + c = 0$, then a root of the equation

$$\Delta = \begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \text{ is}$$

(a) 1 (b) -1

(c) $a^2 + b^2 + c^2$ (d) 0

Ans. (d)

Solution: Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix}$$

$$= \begin{vmatrix} -x & c & b \\ -x & b-x & a \\ -x & a & c-x \end{vmatrix} \quad [\because a+b+c=0]$$

Δ clearly equals 0 when $x = 0$.

It is unnecessary to evaluate the determinant further.

Example 71: A root of the equation

$$\Delta = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0 \text{ is}$$

(a) $\frac{1}{2}(a+b+c)$ (b) 0

(c) –1 (d) 1

Ans. (b)

Solution: When we substitute $x = 0$, Δ becomes

$$\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

which is equal to 0 as Δ is skew symmetric determinant of odd order.

Alternative Solution

Evaluating Δ along R_1, we get

$$-(x-a)\begin{vmatrix} x+a & x-c \\ x+b & 0 \end{vmatrix} + (x-b)\begin{vmatrix} x+a & 0 \\ x+b & x+c \end{vmatrix} = 0$$

$\Rightarrow$ $(x-a)(x+b)(x-c) + (x-b)(x+a)(x+c) = 0$

$\Rightarrow$ $x^3 + (-a+b-c)x^2 + (-ab+ac-bc)x + abc + x^3 + (a-b+c)x^2 + (-ab+ac-bc) - abc = 0$

$\Rightarrow$ $2x^3 - 2(ab-ac+bc)x = 0$

$\Rightarrow$ $2x[x^2 - (ab-ac+bc)] = 0$

$\Rightarrow$ $x = 0$ or $x = \pm\sqrt{ab-ac+bc}$

$\therefore$ one of the roots of the equation is 0.

Example 72: If α, β, γ are three real numbers such that $\alpha+\beta+\gamma = 0$, then

$$\Delta = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} \text{ equals}$$

(a) – 1 (b) 0

(c) 1 (d) $\cos\alpha\cos\beta\cos\gamma$

Ans. (b)

Solution: Let A, B and C be three real numbers such that $\alpha = B - C$, $\beta = C - A$ and $\gamma = A - B$, clearly $\alpha + \beta + \gamma = 0$. We have

$$\Delta = \begin{vmatrix} 1 & \cos(A-B) & \cos(C-A) \\ \cos(A-B) & 1 & \cos(B-C) \\ \cos(C-A) & \cos(B-C) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2 A + \sin^2 A & \cos A\cos B + \sin A\sin B & \cos C\cos A + \sin A\sin C \\ \cos A\cos B + \sin A\sin B & \cos^2 B + \sin^2 B & \cos B\cos C + \sin B\sin C \\ \cos C\cos A + \sin C\sin A & \cos B\cos C + \sin B\sin C & \cos^2 C + \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}\begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} = (0)(0) = 0.$$

Example 73: If a, b, c are the sides of a ΔABC opposite angles A, B, C respectively, and

$$\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos(B-C) \\ c\sin A & \cos(B-C) & 1 \end{vmatrix}, \text{ then } \Delta \text{ equals}$$

(a) $\sin A - \sin C \sin B$ (b) abc

(c) 1 (d) 0

Ans. (d)

Solution: By the law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow a = k\sin A$, $b = k\sin B$, $c = k\sin C$. Now

$$\Delta = \begin{vmatrix} a^2 & ab/k & ac/k \\ ab/k & 1 & \cos(B-C) \\ ac/k & \cos(B-C) & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos(B-C) \\ \sin C & \cos(B-C) & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin(A+C) & \sin(A+B) \\ \sin(A+C) & 1 & \cos(B-C) \\ \sin(A+B) & \cos(B-C) & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} \sin A & \cos A & 0 \\ \cos C & \sin C & 0 \\ \cos B & \sin B & 0 \end{vmatrix}\begin{vmatrix} \sin A & \cos A & 0 \\ \cos C & \sin C & 0 \\ \cos B & \sin B & 0 \end{vmatrix}$$

$$= a^2(0) = 0$$

Example 74: If a, b, c are distinct, and $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

then $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$

vanishes if

(a) $x = \frac{1}{3}(a+b+c)$

(b) $x = \frac{2}{3}(a+b+c)$

(c) $x = a+b+c$

(d) none of these

Ans. (a)

◎ **Solution:** Multiplying C_1 by $(x-a)$, C_2 by $(x-b)$ and C_3 by $(x-c)$, we get

$$\Delta = \frac{1}{ABC}\begin{vmatrix} A & B & C \\ A^3 & B^3 & C^3 \\ ABC & ABC & ABC \end{vmatrix}$$

where $A = x-a$, $B = x-b$, $C = x-c$.

$$\Delta = \begin{vmatrix} A & B & C \\ A^3 & B^3 & C^3 \\ 1 & 1 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ A & B & C \\ A^3 & B^3 & C^3 \end{vmatrix}$$

$$= (B-C)(C-A)(A-B)(A+B+C)$$
$$= (c-b)(a-b)(b-a)[3x-(a+b+c)]$$

Note that Δ become 0 when $x = \frac{1}{3}(a+b+c)$

◉ **Example 75:** The equation

$$\Delta = \begin{vmatrix} x-a & x-b & x-c \\ x-b & x-c & x-a \\ x-c & x-a & x-b \end{vmatrix} = 0$$

is satisfied when

(a) $x = \frac{1}{3}(a+b+c)$ (b) $x = \frac{1}{2}(a+b+c)$

(c) $x = a+b+c$ (d) $x = 0$

Ans. (a)

◎ **Solution:** Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3x-(a+b+c) & x-b & x-c \\ 3x-(a+b+c) & x-c & x-a \\ 3x-(a+b+c) & x-a & x-b \end{vmatrix}$$

Note that Δ becomes 0 when $x = \frac{1}{3}(a+b+c)$.

◉ **Example 76:** If α, β, γ are different from 1 and are the roots of $ax^3 + bx^2 + cx + d = 0$ and $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta) = 25/2$, then the determinant

$$\Delta = \begin{vmatrix} \frac{\alpha}{1-\alpha} & \frac{\beta}{1-\beta} & \frac{\gamma}{1-\gamma} \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \text{ equals}$$

(a) $\frac{25\,d}{2a}$ (b) $\frac{25\,d}{a}$

(c) $\frac{-25d}{a+b+c+d}$ (d) none of these

Ans. (d)

◎ **Solution:** Taking α, β, γ common from C_1, C_2, C_3 respectively, we get

$$\Delta = \alpha\beta\gamma\begin{vmatrix} \frac{1}{1-\alpha} & \frac{1}{1-\beta} & \frac{1}{1-\gamma} \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix}$$

$$= \alpha\beta\gamma\begin{vmatrix} \frac{1}{1-\alpha} & \frac{1}{1-\beta}-\frac{1}{1-\alpha} & \frac{1}{1-\gamma}-\frac{1}{1-\alpha} \\ 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \end{vmatrix}$$

[using $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$]

$$= \frac{\alpha\beta\gamma(-1)(\beta-\alpha)(\gamma-\alpha)}{(1-\alpha)(1-\beta)(1-\gamma)}\begin{vmatrix} 1-\gamma & 1-\beta \\ 1 & 1 \end{vmatrix}$$

$$= \frac{\alpha\beta\gamma(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{(1-\alpha)(1-\beta)(1-\gamma)}$$

As α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$,

$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$

and $\alpha\beta\gamma = -d/a$

Thus, $\Delta = \frac{(-d/a)(25/2)}{(a+b+c+d)/a} = -\frac{25d}{2(a+b+c+d)}$

◉ **Example 77:** Let $P = [a_{ij}]$ be a 3×3 matrix and $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(a) 2^{10} (b) 2^{11}

(c) 2^{12} (d) 2^{13}

Ans. (d)

◎ **Solution:**

$$\det(Q) = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= (2^2)(2^3)(2^4)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2 a_{31} & 2^2 a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$= 2^9\,(2)\,(2^2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2^{12}\det(P) = 2^{13}$$

Example 78: If x is a positive integer, and $\Delta(x) = \begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix}$, then $\Delta(x)$ is equal to

(a) $2x!\,(x + 1)!$
(b) $2x!\,(x + 1)!\,(x + 2)!$
(b) $2x!\,(x + 3)!$
(d) $2(x + 1)!\,(x + 2)!\,(x + 3)!$

Ans. (b)

Solution: Taking $x!$ common from R_1, $(x + 1)!$ from R_2 and $(x + 2)!$ from R_3 to obtain

$$\Delta(x) = x!\,(x+1)!\,(x+2)!\,\Delta_1(x) \text{ where}$$

$$\Delta_1(x) = \begin{vmatrix} 1 & x+1 & (x+1)(x+2) \\ 1 & x+2 & (x+2)(x+3) \\ 1 & x+3 & (x+3)(x+4) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta_1(x) = \begin{vmatrix} 1 & x+1 & (x+1)(x+2) \\ 0 & 1 & 2(x+2) \\ 0 & 1 & 2(x+3) \end{vmatrix} = 2$$

Example 79: Let a, b, c be such that $b(a + c) \neq 0$.

$$\text{If } \Delta = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

$$+ \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$$

Then the value of n is

(a) any odd integer
(b) any integer
(c) zero
(d) any even integer

Ans. (a)

Solution: Let $\Delta_2 = \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix}$

$$= \begin{vmatrix} a+1 & a-1 & (-1)^{n+2}a \\ b+1 & b-1 & (-1)^{n+1}b \\ c-1 & c+1 & (-1)^n c \end{vmatrix}$$

[using the reflection property]

$$= (-1)^2\begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^n c & c-1 & c+1 \end{vmatrix}$$

Thus,

$$\Delta = \begin{vmatrix} a+(-1)^{n+2}a & a+1 & a-1 \\ -b+(-1)^{n+1}b & b+1 & b-1 \\ c+(-1)^n c & c-1 & c+1 \end{vmatrix}$$

The first column consists of all 0's if n is any odd integer.
$\therefore\ \Delta = 0$ if n is any odd integer.

Example 80: Let ω be the complex number $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex number z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is

(a) 1 (b) 0
(c) 2 (d) 3

Ans. (a)

Solution: Denote the given determinant by Δ.
Using $C_1 \to C_1 + C_2 + C_3$ and $1 + \omega + \omega^2 = 0$, we get

$$\Delta = z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = z\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix}$$

$$= z[(z + \omega^2 - \omega)(z + \omega - \omega^2) - (1 - \omega)(1 - \omega^2)]$$
$$= z[z^2 - (\omega^2 - \omega)^2 - (1 - \omega - \omega^2 + 1)]$$
$$= z[z^2 - (\omega^4 + \omega^2 - 2\omega^3) - 3] = z(z^2) = z^3$$

Thus, $z^3 = 0 \Rightarrow z = 0$

Thus, there is just one value of z, satisfying the given equation.

Assertion-Reason Type Questions

Example 81: Let $\Delta_n = \begin{vmatrix} a_n & a_{n+3} & a_{n+6} \\ a_{n+1} & a_{n+4} & a_{n+7} \\ a_{n+2} & a_{n+5} & a_{n+8} \end{vmatrix}$

Statement-1: If $a_k > 0\ \forall\ k \ge 1$ and $a_1, a_2, a_3, \ldots$ are in G.P. then

$$\Delta_n = 0\ \forall\ n \ge 1.$$

Statement-2: If $a_1, a_2, a_3 \ldots$ are in A.P. then

$$\Delta_n = 0\ \forall\ n \ge 1.$$

Ans. (b)

Solution: Let $a_k = ar^{k-1}\ \forall\ k \ge 1$, then

$$\Delta_n = a_n a_{n+3}\, a_{n+6} \begin{vmatrix} 1 & 1 & 1 \\ r & r & r \\ r^2 & r^2 & r^2 \end{vmatrix} = 0$$

Next, if $a_k = b + (k-1)d$, then using $C_2 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_n = \begin{vmatrix} a_n & 3d & 3d \\ a_{n+1} & 3d & 3d \\ a_{n+2} & 3d & 3d \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

Example 82: Let $f(x) = \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$.

Statement-1: If $\sin 2x = 1$, then $f(x) = 2/3$

Statement-2: $f(x) = 0$ if $\sin x = \cos x$

Ans. (d)

Solution: Multiplication of two determinants leads us

$$f(x) = \begin{vmatrix} 1 & -y & y \\ -y & 1 & y \\ y & y & 1 \end{vmatrix}$$

where $y = \sin x \cos x$

Using $C_1 \to C_1 - C_2,\ C_2 \to C_2 + C_3$, we get

$$f(x) = (1+y)^2 \begin{vmatrix} 1 & 0 & y \\ -1 & 1 & y \\ 0 & 1 & 1 \end{vmatrix} = (1+y)^2 \begin{vmatrix} 1 & 0 & y \\ 0 & 1 & 2y \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1+y)^2 (1-2y)$$

When $\sin 2x = 1$, $y = 1/2$ and $f(x) = 0$

When $\sin x = \cos x$, $1 - 2y = 1 - 2\sin^2 x = \cos 2x = 0$

$\therefore$ $f(x) = 0$.

Example 83: Suppose $x > 0, y > 0, z > 0$ and

$$\Delta(a, b, c) = \begin{vmatrix} x\log 2 & 3 & 15 + \log(a^x) \\ y\log 3 & 5 & 25 + \log(b^y) \\ z\log 5 & 7 & 35 + \log(c^z) \end{vmatrix}$$

Statement-1: $\Delta(8, 27, 125) = 0$

Statement-2: $\Delta\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) = 0$

Ans. (b)

Solution: Using $\log(b^c) = c \log b$ and applying $C_3 \to C_3 - 5C_2$ we get

$$\Delta(a, b, c) = \begin{vmatrix} x\log 2 & 3 & x\log a \\ y\log 3 & 5 & y\log b \\ z\log 5 & 7 & z\log c \end{vmatrix}$$

$\Delta(8, 27, 125) = \Delta(2^3, 3^3, 5^3) = 0$ as in this case C_1 and C_3 are proportional. Similarly, $\Delta(1/2, 1/3, 1/5) = \Delta(2^{-1}, 3^{-1}, 5^{-1}) = 0$.

Example 84: Let $a \ne 0, p \ne 0$ and

$$\Delta = \begin{vmatrix} a & b & c \\ 0 & p & q \\ p & q & 0 \end{vmatrix}$$

Statement-1: If the equations

$$ax^2 + bx + c = 0 \text{ and } px + q = 0$$

have a common root, then $\Delta = 0$.

Statement-2: If $\Delta = 0$, then the equations

$$ax^2 + bx + c = 0 \text{ and } px + q = 0$$

have a common root.

Ans. (b)

Solution: If λ is a common root of

$$ax^2 + bx + c = 0 \text{ and } px + q = 0, \text{ then}$$

$$a\lambda^2 + b\lambda + c = 0,\ p\lambda + q = 0 \text{ and } p\lambda^2 + q\lambda = 0$$

Eliminating λ, we obtained $\Delta = 0$.

For statement-2, expanding Δ along C_1 we obtain

$-aq^2 + p(bq - cp) = 0$

or $$a\left(-\frac{q}{p}\right)^2 + b\left(-\frac{q}{p}\right) + c = 0$$

Thus, $ax^2 + bx + c = 0$ and $px + q = 0$ have a common root.

Example 85: Statement-1:

$$\Delta = \begin{vmatrix} \sin\pi & \cos(x + \pi/4) & \tan(x - \pi/4) \\ \sin(x - \pi/4) & -\cos(\pi/2) & \log(x/y) \\ \cot(\pi/4 + x) & \log(y/x) & \tan\pi \end{vmatrix} = 0$$

Statement-2: A skew symmetric determinant of odd order equals 0.
Ans. (a)

◎ **Solution:** For statement-2, see theory.
Now, using

$$\cos\left(x+\frac{\pi}{4}\right) = \cos\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-x\right)\right]$$
$$= \sin\left(\frac{\pi}{4}-x\right) = -\sin\left(x-\frac{\pi}{4}\right);$$
$$\cot\left(\frac{\pi}{4}+x\right) = \cot\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-x\right)\right]$$
$$= \tan\left(\frac{\pi}{4}-x\right) = -\tan\left(x-\frac{\pi}{4}\right),$$

and $\log(x/y) = -\log(y/x)$
we find Δ is a skew symmetric determinant of odd order.

Example 86: Let $f: Q \to [-1, 1]$ by
$f(x) = \sin x$
Let x_1, x_2, x_3 be three distinct rational numbers. Let $a = f(x_1)$, $b = f(x_2)$, $c = f(x_3)$.

Statement-1:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

Statement-2: f is a one-to-one function.
Ans. (a)

◎ **Solution:** Let $x_1, x_2 \in \mathbf{Q}$ be such that

$f(x_1) = f(x_2)$

$\Rightarrow \quad \sin x_1 = \sin x_2$

$\Rightarrow \quad x_1 = n\pi + (-1)^n x_2, n \in \mathbf{I}$

$\Rightarrow \quad x_1 + (-1)^{n+1} x_2 = n\pi, n \in \mathbf{I}.$

But LHS is rational and RHS is irrational except when $n = 0$.

$\therefore \quad x_1 = x_2$

Thus, f is one-to-one.

Also, $\Delta = (a-b)(b-c)(c-a) \neq 0$

as x_1, x_2, x_3 are distinct and hence a, b, c are distinct.

Example 87: Let

$$f(x) = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix}$$
$$= ax^4 + bx^3 + cx^2 + dx + e$$

Statement-1: $a = -1$
Statement-2: $e = f(0)$
Ans. (a)

◎ **Solution:** $e = f(0) = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix}$
$= 0$

[skew symmetric determinant of odd order]

$\therefore$ Statement-2 is true.

To obtain a, replace x by $\frac{1}{x}$, so that

$$\begin{vmatrix} \left(\frac{1}{x}\right)^2+\frac{3}{x} & \frac{1}{x}-1 & \frac{1}{x}+3 \\ \frac{1}{x}+1 & -\frac{2}{x} & \frac{1}{x}-4 \\ \frac{1}{x}-3 & \frac{1}{x}+4 & \frac{3}{x} \end{vmatrix}$$
$$a\left(\frac{1}{x}\right)^4 + b\left(\frac{1}{x}\right)^3 + c\left(\frac{1}{x}\right)^2 + d\left(\frac{1}{x}\right) + e$$

Take x^2 common from C_1, x from C_2 and C_3

$$= \frac{1}{x^4}\begin{vmatrix} 1+3x & 1-x & 1+3x \\ x+x^2 & -2 & 1-4x \\ x-3x^2 & 1+4x & 3x \end{vmatrix}$$
$$= \frac{1}{x^4}\left[a+bx+cx^2+dx^3+ex^4\right]$$

Cancel $1/x^4$ and put $x = 0$ to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = a$$

$\Rightarrow \quad a = -1$

Example 88: **Statement-1:** The system of line equations

$x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$

$x + (\cos\alpha)\, y + (\sin\alpha)\, z = 0$

$x - (\sin\alpha)\, y - (\cos\alpha)\, z = 0$

has a non-trivial solution for only one value of α lying in the interval $(0, \pi/2)$.

Statement-2: The equation in α

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

has only one solution lying in the interval $(0, \pi/2)$.
Ans. (b)

◎ **Solution:** Using $C_1 \to C_1 - C_3$ in Δ, we get

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ 0 & \cos\alpha & \sin\alpha \\ 2\cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix}$$
$$= 2\cos\alpha\,(\sin^2\alpha - \cos^2\alpha)$$

$= -2\cos\alpha\cos 2\alpha$

$\Delta = 0$ for $\alpha = \pi/4 \in (0, \pi/2)$.

This is the only value of α lying in $(0, \pi/2)$ for which $\Delta = 0$.

The system of linear equations will have a non-trivial solution if and only if

$$\Delta_1 = \begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Using $R_2 \to R_2 + R_1$, we get

$$\Delta_1 = \begin{vmatrix} 2 & 0 & 0 \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-\cos^2\alpha + \sin^2\alpha) = 0$

$\Rightarrow \quad -2\cos 2\alpha = 0$

This is true for only one value of $\alpha \in (0, \pi/2)$ viz, $\alpha = \pi/4$.

Thus, statement-1 is also true. However, statement-2 is not a correct reason for statement-1.

LEVEL 2

Straight Objective Type Questions

Example 89: Consider the system of equations in x, y, z:

$$(\sin 3\theta)x - y + z = 0$$
$$(\cos 2\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

The values of θ for which the system has a non-trivial solution are given by

(a) $\pi\left(n + \frac{1}{3}(-1)^n\right)$ (b) $\pi\left(n + \frac{1}{4}(-1)^n\right)$

(b) $\pi\left(n + \frac{1}{6}(-1)^n\right)$ (d) $\frac{n\pi}{2}$

where n is an integer

Ans. (c)

Solution: For the system of equations to have a non-trivial solution we must have

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Using $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$ we get

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta + 4\sin 3\theta & 0 & 7 \\ 2 + 7\sin 3\theta & 0 & 14 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(\cos 2\theta + 4\sin 3\theta) - (2 + 7\sin 3\theta) = 0$

$\Rightarrow \quad \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta - 4\sin^2\theta = 0$

$\Rightarrow \quad \sin\theta(2\sin\theta + 3)(2\sin\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\sin\theta = 1/2 \qquad [\because |\sin\theta| \le 1]$

$\Rightarrow \quad \theta = m\pi$ or $\theta = n\left(\pi + \frac{(-1)^n}{6}\right)$ where m and n are integers

Example 90:

Let $$\Delta(x) = \begin{vmatrix} 3 + 2\sin^4 x & 2\cos^4 x & \sin^2 2x \\ 2\sin^4 x & 3 + 2\cos^4 x & \sin^2 2x \\ 2\sin^4 x & 2\cos^4 x & 3 + \sin^2 2x \end{vmatrix}$$

then $\int_{-\pi/2}^{\pi/2} x\Delta(x)dx$ equals

(a) π^2 (b) $\pi(\pi - 1)$

(c) 1 (d) 0

Ans. (d)

Solution: Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta(x) = f(x)\begin{vmatrix} 1 & 2\cos^4 x & \sin^2 2x \\ 1 & 3 + 2\cos^4 x & \sin^2 2x \\ 1 & 2\cos^4 x & 3 + \sin^2 2x \end{vmatrix}$$

where $f(x) = 3 + 2\sin^4 x + 2\cos^4 x + \sin^2 2x$

$= 3 + 2(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x)$

$= 3 + 2(\cos^2 x + \sin^2 x)^2 = 5$

Applying $C_2 \to C_2 - (2\cos^4 x)C_1$ and $C_3 \to C_3 - (\sin^2 2x)C_1$, we get

$$\Delta(x) = 5\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 45$$

Thus, $\int_{-\pi/2}^{\pi/2} x\Delta(x)dx = 45\int_{-\pi/2}^{\pi/2} x\,dx = 0$

Example 91: If $f(x)$, $g(x)$and $h(x)$ are three polynomials of degree 3 then

$$\phi(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

is a polynomial of degree

(a) 3 (b) 4
(c) 5 (d) none of these

Ans. (d)

◎ **Solution:** We have

$$\phi'(x) = \begin{vmatrix} f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} +$$

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ f^{iv}(x) & g^{iv}(x) & h^{iv}(x) \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ 0 & 0 & 0 \end{vmatrix} = 0$$

since f, g, h are polynomials of degree 3, $f^{iv}(x) = g^{iv}(x) = h^{iv}(x) = 0$

$\Rightarrow \phi(x)$ must be a constant.

Example 92: The determinant

$$\Delta = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

equals

(a) $9b^2(a+b)$ (b) $9a^2(a+b)$
(c) $9(a+b)^3$ (d) $9ab(a+b)$

Ans. (a)

◎ **Solution:** Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 3(a+b)\begin{vmatrix} 1 & a+b & a+2b \\ 1 & a & a+b \\ 1 & a+2b & a \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$\Delta = 3(a+b)\begin{vmatrix} 1 & a+b & a+2b \\ 0 & -b & -b \\ 0 & b & -2b \end{vmatrix}$$

$$= 3(a+b)\begin{vmatrix} -b & -b \\ b & -2b \end{vmatrix} = 9b^2(a+b)$$

Example 93: Let

$$\Delta(x) = \begin{vmatrix} \sin x & \cos x & \sin 2x + \cos 2x \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

then $\Delta'(x)$ vanishes at least once in

(a) $(0, \pi/2)$ (b) $(\pi/2, \pi)$
(c) $(0, \pi/4)$ (d) $(-\pi/2, 0)$

Ans. (a)

◎ **Solution:** The function $\Delta(x)$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$. Also $\Delta(0) = 0$ and $\Delta(\pi/2) = 0$. Thus, by the Rolle's theorem there exists at least one $c \in (0, \pi/2)$ such that $\Delta'(c) = 0$.

Example 94: Let $\Delta(x) = \begin{vmatrix} \cos^2 x & \cos x \sin x & -\sin x \\ \cos x \sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$

then $\int_0^{\pi/2} [\Delta(x) + \Delta'(x)]\, dx$ equals

(a) $\pi/3$ (b) $\pi/2$
(c) 2π (d) $3\pi/2$

Ans. (b)

◎ **Solution:** Applying $C_1 \to C_1 - \sin x\, C_3$ and $C_2 \to C_2 + \cos x\, C_3$, we get

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

Applying $R_3 \to R_3 - \sin x\, R_1 + \cos x\, R_2$, we get

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ 0 & 0 & \cos^2 x + \sin^2 x \end{vmatrix} = 1$$

$$\Rightarrow \quad \Delta'(x) = 0$$

Thus, $\int_0^{\pi/2} [\Delta(x) + \Delta'(x)]\, dx = \int_0^{\pi/2} dx = \dfrac{\pi}{2}$.

Example 95: The determinant

$$\Delta = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix} \text{ equals}$$

(a) $15\sqrt{2} - 25\sqrt{3}$ (b) $25\sqrt{3} - 15\sqrt{2}$
(c) $3\sqrt{5}$ (d) $-15\sqrt{2} + 7\sqrt{3}$

Ans. (a)

◎ **Solution:** Taking $\sqrt{5}$ common from C_2 and C_3, we get

$$\Delta = \left(\sqrt{5}\right)^2 \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+\sqrt{26} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Applying $C_1 \to C_1 - \sqrt{13}\ C_3 - \sqrt{3}\ C_2$, we get

$$\Delta = (5)\begin{vmatrix} -\sqrt{3} & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix} = 5\left(-\sqrt{3}\right)\left(5-\sqrt{6}\right)$$

$$= 5\left(\sqrt{18}\right) - 25\sqrt{3} = 15\sqrt{2} - 25\sqrt{3}$$

Example 96: The values of λ for which the system of equations

$$x + y - 3 = 0$$
$$(1 + \lambda)x + (2 + \lambda)y - 8 = 0$$
$$x - (1 + \lambda)y + (2 + \lambda) = 0$$

has a non-trivial solution, are

(a) $-5/3, 1$ (b) $2/3, -3$
(c) $-1/3, -3$ (d) 0

Ans. (a)

Solution: The given system of equations has a non trivial solution if

$$\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & -5+3\lambda \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(5 + \lambda) + (2 + \lambda)(3\lambda - 5) = 0$

$\Rightarrow$ $5 + \lambda + 6\lambda - 10 + 3\lambda^2 - 5\lambda = 0$

$\Rightarrow$ $3\lambda^2 + 2\lambda - 5 = 0 \Rightarrow (3\lambda + 5)(\lambda - 1) = 0$

$\Rightarrow$ $\lambda = -5/3$ or $\lambda = 1$.

Example 97: The values of m for which the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfying the conditions $x > 0$, $y > 0$ are given by the set

(a) $\{m : m < -13/2\}$
(b) $\{m : m > 17/2\}$
(c) $\{m : m < -13/2 \text{ or } m > 17/2\}$
(d) none of these

Ans. (d)

Solution: Here $\Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -15 - 2m$

$$\Delta_x = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m$$

and $\Delta_y = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60 - 2m$

If $\Delta = 0$, then $m = -15/2$. But for this value of m, $\Delta_x \neq 0$ and $\Delta_y \neq 0$. Thus, in this case, the system of equations is not consistent.

If $\Delta \neq 0$, then $x = \dfrac{25m}{2m+15}$ and $y = \dfrac{2m-60}{2m+15}$, by the Cramer's rule.

Now, $x > 0, y > 0 \Leftrightarrow 25m > 0, 2m - 60 > 0, 2m + 15 > 0$
or $25m < 0, 2m - 60 < 0, 2m + 15 < 0$
$\Rightarrow m > 30$ or $m < -15/2$.

Example 98: If $a + b + c \neq 0$, the system of equations

$$(b + c)(y + z) - ax = b - c$$
$$(c + a)(z + x) - by = c - a$$
$$(a + b)(x + y) - cz = a - b$$

has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) finitely many solutions

Ans. (a)

Solution: We can write the above system of equations

$$(a + b + c)(y + z) - a(x + y + z) = b - c$$
$$(a + b + c)(z + x) - b(x + y + z) = c - a$$
$$(a + b + c)(x + y) - c(x + y + z) = a - b$$

Adding the above equations, we obtain

$$2(a + b + c)(x + y + z) - (a + b + c)(x + y + z) = 0$$

$\Rightarrow$ $(a + b + c)(x + y + z) = 0$

$\Rightarrow$ $x + y + z = 0$ $[\because a + b + c \neq 0]$

$\Rightarrow$ $y + z = -x$

$\therefore$ $(b + c)(-x) - ax = b - c \Rightarrow x = \dfrac{c-b}{a+b+c}$

Similarly, $y = \dfrac{a-c}{a+b+c}$, $z = \dfrac{b-a}{a+b+c}$.

Example 99: Let a, b, c be positive real numbers. The following system of equations in x, y and z.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \ \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ \frac{-x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has

(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) finitely many solutions

Ans. (d)

Solution: Adding all the equations, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3$$

Subtracting first equation from it we get $\dfrac{2z^2}{c^2} = 2 \Rightarrow z^2 = c^2$
$\Rightarrow z = \pm c$. Similarly, $x = \pm a$, $y = \pm b$. Thus, the given system of equations has eight solutions.

Example 100: If the system of equations

$$x - ky - z = 0, \ kx - y - z = 0, \ x + y - z = 0$$

has a non-zero solution, then the possible values of k are

(a) $-1, 2$ (b) $1, 2$
(c) $0, 1$ (d) $-1, 1$

Ans. (d)

◎ **Solution:** As the given system has a non-zero solution,

$$0 = \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1+k & -k-1 & -1 \\ 1+k & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

[using $C_1 \to C_1 - C_2$, $C_2 \to C_2 + C_3$]

$\Rightarrow \quad 0 = (-1)\,[(1+k)(-2) - (1+k)(-k-1)]$

$\Rightarrow \quad 0 = (1+k)(-2+k+1) \Rightarrow k = -1, 1$

◉ **Example 101:** If the system of equations

$\lambda x_1 + x_2 + x_3 = 1,\ x_1 + \lambda x_2 + x_3 = 1,\ x_1 + x_2 + \lambda x_3 = 1$

is inconsistent, then λ equals

(a) 5 (b) – 2/3

(c) – 3 (d) – 2

Ans. (d)

◎ **Solution:** Let $\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 & 1 \\ \lambda+2 & \lambda & 1 \\ \lambda+2 & 1 & \lambda \end{vmatrix}$

[$C_1 \to C_1 + C_2 + C_3$]

$$= (\lambda+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ 1 & 0 & \lambda-1 \end{vmatrix}$$

[using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$= (\lambda+2)(\lambda-1)^2$

If $\Delta = 0$, then $\lambda = -2$ or $\lambda = 1$.

But when $\lambda = 1$, the system of equation becomes $x_1 + x_2 + x_3 = 1$ which has infinite number of solutions. When $\lambda = -2$, by adding three equations, we obtain $0 = 3$ and thus, the system of equations is inconsistent.

◉ **Example 102:** If $p \neq a$, $q \neq b$, $r \neq c$ and the system of equations

$$px + ay + az = 0$$
$$bx + qy + bz = 0$$
$$cx + cy + rz = 0$$

has a non-trivial solution, then the value of

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} \text{ is}$$

(a) – 1 (b) 0

(c) 1 (d) 2

Ans. (d)

◎ **Solution:** As the given system of equations has a non-trivial solution

$$\Delta = \begin{vmatrix} p & a & a \\ b & q & b \\ c & c & r \end{vmatrix} = 0$$

Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} p & a-p & a-p \\ b & q-b & 0 \\ c & 0 & r-c \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(a-p)\begin{vmatrix} b & q-b \\ c & 0 \end{vmatrix} + (r-c)\begin{vmatrix} p & a-p \\ b & q-b \end{vmatrix} = 0$$

$\Rightarrow (a-p)(-c)(q-b) + (r-c)\{p(q-b) - b(a-p)\} = 0$

$\Rightarrow (p-a)(q-b)\,c + p\,(r-c)(q-b) + b\,(r-c)(p-a) = 0$

Dividing by $(p-a)(q-b)(r-c)$ we get

$$\frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = \frac{q-b}{q-b} + \frac{r-c}{r-c} = 2$$

◉ **Example 103:** Let λ and α be real. Relation between λ and α for which the system of equations

$$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$$
$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$
$$-x + (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non-trivial solution is

(a) $\lambda = \sin 2\alpha + \cos 2\alpha$

(b) $\lambda = |\sin 2\alpha|$

(c) $\lambda = |\sin 2\alpha - \cos 2\alpha|$

(d) $\lambda = \cos 2\alpha$

Ans. (a)

◎ **Solution:** The system of equations will have a non-trivial solution if and only if

$$0 = \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (\lambda+1)\sin\alpha & (1-\lambda)\cos\alpha \\ 0 & \cos\alpha + \sin\alpha & \sin\alpha - \cos\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

[using $R_1 \to R_1 + \lambda R_3$, $R_2 \to R_2 + R_3$]

$= (\lambda+1)\sin\alpha\,(\sin\alpha - \cos\alpha) + (\lambda-1)\cos\alpha(\cos\alpha + \sin\alpha)$

$= \lambda(\sin^2\alpha + \cos^2\alpha) + [\sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha]$

Thus, $\lambda = \sin 2\alpha + \cos 2\alpha$

◉ **Example 104:** If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$

and $$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

then $F'(a)$ is

(a) – 1 (b) a

(c) 0 (d) none of these

Ans. (c)

◎ **Solution:** Differentiating w.r.t. x we get $F'(x)$

$$= \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

Since $f_r(a) = g_r(a) = h_r(a)$ for $r = 1, 2, 3$, we get each of the three determinants on the right side becomes zero when x is replaced by a. (In each case two rows become identical.) Thus, $F'(a) = 0$.

◉ **Example 105:** The number of real values of a for which the system of equations

$$x + ay - z = 0,\ 2x - y + az = 0,\ ax + y + 2z = 0$$

has a non-trivial solution, is

(a) 3 (b) 1
(c) 0 (d) infinite

Ans. (a)

◎ **Solution:** Since the given system of equations has a non-trivial solution,

$$\Delta = \begin{vmatrix} 1 & a & -1 \\ 2 & -1 & a \\ a & 1 & 2 \end{vmatrix} = 0$$

Using $C_1 \to C_1 + C_3$, $C_2 \to C_2 + aC_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & -1 \\ 2+a & -1+a^2 & a \\ 2+a & 1+2a & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1)\begin{vmatrix} 2+a & -1+a^2 \\ 2+a & 1+2a \end{vmatrix} = 0$$

$$\Rightarrow \quad (2 + a)(1 + 2a + 1 - a^2) = 0$$

$$\Rightarrow \quad a = -2,\ 1 \pm \sqrt{3}.$$

Thus, there are three real values of a for which the system of equations has a non-trivial solution.

◉ **Example 106:** The solution set of

$$x + 2y + z = 1$$
$$2x - 3y + w = 2$$

subject $\quad x \ge 0, y \ge 0, z \ge 0, w \ge 0$ is

(a) $x = \frac{1}{7}(y - w), \quad y \ge w \ge 0, z \ge 0$

(b) $x = \frac{1}{8}(y + w), \quad z = \frac{1}{3}(y - w), y \ge w \ge 0$

(c) $x = \frac{1}{7}(y - w), \quad z = \frac{1}{3}(y + w), y \ge 0$

(d) $x = 1, y = 0, \quad z = 0, w = 0.$

Ans. (d)

◎ **Solution:** We have

$$1 - (x + 2y) = z \ge 0$$

and $\quad 2x - 3y - 2 = -w \le 0.$

$\Rightarrow \quad x + 2y \le 1$ and $2x - 3y \le 2$, $x \ge 0$, $y \ge 0$

The only point where the two shaded region intersect is (1, 0). Thus, $x = 1$, $y = 0$ for these values $z = 0$, $w = 0$.

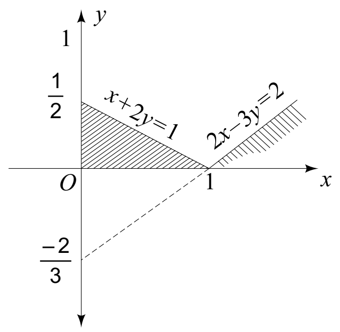


Fig. 4.1

◉ **Example 107:** The number of values of k for which the system of equations

$$(k + 1)x + 8y = 4k$$
$$kx + (k + 3)y = 3k - 1$$

has infinitely many solutions is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (b)

◎ **Solution:** Let $\Delta = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix}$

$$= k^2 - 4k + 3 = (k - 1)(k - 3)$$

If $\Delta \ne 0$, the system of equations has a unique solution.

For the system of equations to have an infinite number of solutions

$$\Delta = 0 \Rightarrow k = 3, 1$$

For $k = 3$, the system equations becomes

$$4x + 8y = 12 \text{ and } 3x + 6y = 8$$

$$\Rightarrow \quad x + 2y = 3,\ x + 2y = \frac{8}{3}$$

Thus, in this case the system of equations has no solution.

For $k = 1$, the system of equations becomes

$$2x + 8y = 4 \text{ and } x + 4y = 2$$

$$\Rightarrow \quad x + 4y = 2,\ x + 4y = 2$$

This system has infinite number of solutions.

Example 108: Suppose $a, b, c \in \mathbf{R}$ and let

$$f(x) = \begin{vmatrix} 0 & a-x & b-x \\ -a-x & 0 & c-x \\ -b-x & -c-x & 0 \end{vmatrix}$$

Then coefficient of x^2 in $f(x)$ is

(a) $-(a+b+c)$ (b) $a+b+c$
(c) 0 (d) $ab+bc+ca$

Ans. (c)

Solution: Expanding along C_1

$$f(x) = (a+x)\begin{vmatrix} a-x & b-x \\ -c-x & 0 \end{vmatrix} - (b+x)\begin{vmatrix} a-x & b-x \\ 0 & c-x \end{vmatrix}$$

$$= (a+x)(b-x)(c+x) - (b+x)(a-x)(c-x)$$

$$= -[(x+a)(x-b)(x+c) + (x-a)(x+b)(x-c)]$$

$$= -[2x^3 + x^2(a-b+c-a+b-c) + \ldots]$$

$$= -2x^3 + 0x^2 + \ldots$$

$\therefore$ coefficient of x^2 in $f(x)$ is 0.

Example 109: If the system of equations

$$x - ky - z = 0, kx - y - z = 0, x + y - z = 0$$

has a non-zero solution, then possible values of k are

(a) $-1, 2$ (b) $1, 2$
(c) $0, 1$ (d) $-1, 1$

Ans. (d)

Solution: Subtracting the last equation from the first two equations, we get

$$-(k+1)y = 0$$

$$(k-1)x - 2y = 0.$$

If $k = -1$, we can choose y in infinite number of ways, corresponding to which we can choose x and z in infinite number of ways.
If $k = 1$, then $y = 0$ and x and hence z can be chosen in infinite number of ways.

Example 110: Let $a_2, a_3 \in \mathbf{R}$ be such that $|a_2 - a_3| = 6$. Let

$$f(x) = \begin{vmatrix} 1 & a_3 & a_2 \\ 1 & a_3 & 2a_2 - x \\ 1 & 2a_3 - x & a_2 \end{vmatrix}, x \in \mathbf{R}$$

The maximum value of $f(x)$ is

(a) 6 (b) 9
(c) 12 (d) 36

Ans. (b)

Solution: Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$f(x) = \begin{vmatrix} 1 & a_3 & a_2 \\ 0 & 0 & a_2 - x \\ 0 & a_3 - x & 0 \end{vmatrix}$$

$$= -(a_2 - x)(a_3 - x) = -[x^2 - (a_2 + a_3)x + a_2 a_3]$$

$$= \frac{1}{4}(a_2 - a_3)^2 - \left(x - \frac{a_2 + a_3}{2}\right)^2 \le 9$$

$f(x)$ attains maximum value 9 when

$$x = \frac{1}{2}(a_2 + a_3).$$

EXERCISES

Concept-based Straight Objective Type Questions

1. Suppose $A = (a_{ij})_{n \times n}$, where $a_{ij} \in \mathbf{R}$.
   If det $(\text{adj}(A)A^{-1}) = 3$, then det $(\text{adj}(A))$ equals:
   (a) $\sqrt{3}$ (b) 3
   (c) $3\sqrt{3}$ (d) 9

2. If a, b, c are in A.P. and p is a real number, and
   $$\Delta = \begin{vmatrix} p+c & p+2 & p+a \\ p+b & p+5 & p+b \\ p+a & p+8 & p+c \end{vmatrix}$$
   then Δ equals:
   (a) $-p^3$ (b) p^3
   (c) $p^3 - 2\,abc$ (d) 0

3. Suppose $x \neq 1$ and
   $$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$
   then $\Delta = 0$
   (a) for exactly two distinct complex numbers
   (b) for exactly four distinct complex numbers
   (c) for exactly two distinct real numbers
   (d) none of these

4. Let $D = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$, then $\dfrac{D}{(10!)^3} - 260$ equals:

(a) 1 (b) 2
(c) 3 (d) 4

5. Let

$$P(x) = \begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix},$$

the product of zeros of $P(x)$ is

(a) 0 (b) 6
(c) – 6 (d) 12

6. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of an equilateral triangle whose side is 4 units. Let

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

then Δ^2 is equal to

(a) 64 (b) 128
(c) 192 (d) 256

7. Suppose a, b, c are in A.P. If p, q, r are also in A.P., then value of

$$\Delta = \begin{vmatrix} x^2+a & x+p & c \\ x^2+b & x+q & b \\ x^2+c & x+r & a \end{vmatrix},$$

is dependent on

(a) x (b) a, b, c
(c) p, q, r (d) none of these

8. Suppose a, b are two non-zero numbers. Let

$$\Delta = \begin{vmatrix} 2 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ a^2+b^2 & a^3+b^3 & a^4+b^4 \end{vmatrix},$$

then Δ is equal to:

(a) 0 (b) ab
(c) $a^6 + b^6$ (d) $a^3b^5 + a^5b^3$

9. Suppose a, b, $c > 1$. Let

$$\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log(2007a) & \log(2007b) & \log(2007c) \\ \log(2017a) & \log(2017b) & \log(2017c) \end{vmatrix}$$

then Δ is equal to:

(a) 0 (b) $\log(4024\, abc)$
(c) $\log\left(\dfrac{2017}{2007}\right)$ (d) none of these

10. Suppose $a, b, c \in \mathbf{R}$. Let

$$\Delta = \begin{vmatrix} (a+2016)^2 & (b+2016)^2 & (a+2016)^2 \\ (a-2016)^2 & (b-2016)^2 & (c-2016)^2 \\ a^2 & b^2 & c^2 \end{vmatrix}$$

If $\Delta = k\,(2016)^3\,(a - b)\,(b - c)(c - a)$, then k is equal to:

(a) – 1 (b) – 4
(c) 4 (d) 1

11. Let $P(x) = \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x+3 \\ -3 & 2x & x+2 \end{vmatrix}$,

sum of the zeros of $P(x)$ is

(a) – 6 (b) – 7
(c) 13/5 (d) –12/5

12. Let n be an integer and $x, y, z > 1$. Suppose

$$\Delta = \begin{vmatrix} x^{n+1} & x^{n+2} & x^{n+3} \\ y^{n+1} & y^{n+2} & y^{n+3} \\ z^{n+1} & z^{n+2} & z^{n+3} \end{vmatrix}$$

If $\Delta = (x - y)\,(y - z)(z - x)\,x^2\,y^2\,z^2$, then n is equal to:

(a) –1 (b) 0
(c) 1 (d) 2

13. Suppose a, b, c are distinct real numbers. Let

$$P(x) = \begin{vmatrix} 0 & x^3-a & x^4-b \\ x^3+a & 0 & x^5+c \\ x^4+b & x^5-c & 0 \end{vmatrix}$$

A value of x satisfying $P(x) = 0$ is

(a) $-(a + b + c)$ (b) $a + b + c$
(c) $a + b - c$ (d) 0

14. Suppose $a, b, c \in \mathbf{R}$ and $abc \neq 0$. Let

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix}$$

If $\Delta = 0$, then $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$ is equal to:

(a) 0 (b) –1
(c) –2 (d) –3

15. Suppose n and m are natural numbers such that

$$\Delta = \begin{vmatrix} x^m & x^{m+2} & x^{2m} \\ 1 & x^n & 2^n \\ x^{m+5} & x^{n+6} & x^{2m+5} \end{vmatrix}$$

Then a possible relationship between n and m is

(a) $n = m + 2$ (b) $n = m + 1$
(c) $n = m$ (d) $n = m - 1$

LEVEL 1

Straight Objective Type Questions

16. Suppose a, b, $c \in \mathbf{R}$ and $a + b + c \neq 0$. Let

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

If $\Delta = 0$, then

(a) $a = b = c$ (b) $a^3 + b^3 - c^3 = 0$
(c) $a = b + c$ (d) $a = b = c = 0$

17. Distance of line

$$y = \begin{vmatrix} x+1 & x & x \\ x & x+2 & x \\ x & x & x+3 \end{vmatrix}$$

from the origin is

(a) $\frac{6}{11}$ (b) $\frac{7}{13}$
(c) $\frac{6}{\sqrt{122}}$ (d) $\frac{7}{\sqrt{122}}$

18. Suppose a, b, $c \in \mathbf{R}$. Let

$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Then Δ equals

(a) $3(a+b+c)(bc+ca+ab)$
(b) $a+b+c$
(c) $3(a+b+c)(a^2+b^2+c^2)$
(d) 0

19. Suppose a, b, c are in A.P. Let

$$\Delta = \begin{vmatrix} p^2+2^{n+1}+2a & q^2+2^{n+2}+3b & r^2+a \\ 2^n+a & 2^{n+1}+b & 2b \\ p^2+2^n+a & q^2+2^{n+1}+2b & r^2-c \end{vmatrix}$$

Then Δ equals:

(a) -1 (b) 0
(c) $p^2q^2r^2 - 3abc$ (d) $p^2q^2r^2 - 4(a+b+c)$

20. Suppose a, b, c, d, e and f are in G.P. with common ratio > 1. Let p, q, r be three real numbers. Let

$$\Delta = \begin{vmatrix} a^2 & d^2 & p \\ b^2 & e^2 & q \\ c^2 & f^2 & r \end{vmatrix}$$

Then Δ depends on

(a) a, b, c (b) d, e, f
(c) p, q, r (d) none of these

21. Suppose point (x, y, z) in space satisfies the equation

$$\begin{vmatrix} x^2+1 & xy & xz \\ yx & y^2+1 & yz \\ zx & zy & z^2+1 \end{vmatrix} = 5$$

Then (x, y, z) lies on a

(a) plane (b) straight line
(c) sphere (d) none of these

22. Suppose A, B, and C are angles of a triangle. Let

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

then Δ equals:

(a) 0 (b) -1
(c) -2 (d) -3

23. Let

$$\Delta = \begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix}$$

and $f(x) = (x - a)(x - b)(x - c)$

Determinant Δ is equal to:

(a) $f(x) - x^3$ (b) $f'(x)$
(c) $xf'(x) - f(x)$ (d) $f'(x) - xf''(x)$

24. Straight line

$$\begin{vmatrix} 2-x-y & 4 & 4 \\ 2x & x-y-2 & 2x \\ 2y & 2y & y-2-x \end{vmatrix} = 0$$

passes through the fixed point

(a) $(-2, -2)$ (b) $(-2, 0)$
(c) $(0, -2)$ (d) $(-1, -1)$

25. Suppose $a \in \mathbf{R}$. Let

$$f(x) = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$$

Then $f(2x) - f(x)$ is equal to

(a) $3xa^2$ (b) $3x^2a$
(c) xa^2 (d) a^2x

26. If α, β, γ are the roots of $x^3 + ax^2 + b = 0$, then the determinant

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \text{ equals}$$

(a) $-a^3$ (b) $a^3 - 3b$
(c) $a^2 - 3b$ (d) a^3

27. If α, β, γ are the roots of $x^3 + bx + c = 0$, then the determinant

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \text{ equals}$$

(a) $-b^3$ (b) $b^3 - 3c$
(c) $b^2 - 3c$ (d) 0

28. If a, b, c are distinct and different from zero and

$$\Delta = \begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix} = 0, \text{ then}$$

(a) $a^{-1} + b^{-1} + c^{-1} = 0$
(b) $a^{-1} + b^{-1} - c^{-1} = 0$
(c) $a^{-1} - b^{-1} + c^{-1} = 0$
(d) $a^{-1} - b^{-1} - c^{-1} = 0$

29. The determinant $\Delta = \begin{vmatrix} \lambda a & \lambda^2 + a^2 & 1 \\ \lambda b & \lambda^2 + b^2 & 1 \\ \lambda c & \lambda^2 + c^2 & 1 \end{vmatrix}$ equals

(a) $\lambda(a - b)(b - c)(c - a)$
(b) $\lambda(a^2 + b^2 + c^2)$
(c) $\lambda(a + b + c)$
(d) $\lambda^2(a - b)(b - c)(c - a)$

30. If α, β, γ are real numbers, then the determinant

$$\Delta = \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} \text{ equals}$$

(a) 0
(b) -1
(c) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$
(d) none of these

31. If $bc + ca + ab = 18$, and

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \lambda \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

the value of λ is
(a) -1 (b) 0
(c) 9 (d) 18

32. If $x \neq 0$, the determinant

$$\Delta = \begin{vmatrix} a_0 & a_1 & a_2 \\ -x & x & 0 \\ 0 & -x & x \end{vmatrix}$$

vanishes if
(a) $a_0 + a_1 + a_2 = 0$ (b) $a_0 + a_1 = 2a_2$
(c) $a_0 + a_2 = 2a_1$ (d) none of these

33. If $x \in \mathbf{R}$, the determinant

$$\Delta = \begin{vmatrix} 1 & \cos x & 0 \\ -1 & 1 - \cos x & \sin x + \cos x \\ 0 & -1 & 1 - \sqrt{2}\sin(x + \pi/4) \end{vmatrix} \text{ equals}$$

(a) 0 (b) -1
(c) 1 (d) none of these

34. The factors of $\Delta = \begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix}$ are

(a) $x - a$, $x - b$ and $x + a + b$
(b) $x - a$, $x - b$ and $x - a - b$
(c) $x + a$, $x + b$ and $x - a - b$
(d) none of these

35. If $\theta \,\varepsilon\, \mathbf{R}$, maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 & 1 & 1 + \cos\theta \end{vmatrix} \text{ is}$$

(a) 1/2 (b) $\sqrt{3}/2$
(c) $\sqrt{2}$ (d) $3\sqrt{2}/4$

36. If $\Delta = \begin{vmatrix} x + a & b & c \\ a & x + b & c \\ a & b & x + c \end{vmatrix} = 0$, then x equals

(a) $a + b + c$ (b) $-(a + b + c)$
(c) $0, a + b + c$ (d) $0, -(a + b + c)$

37. The determinant

$$\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 12 & 10 & 2 \end{vmatrix} \text{ equals}$$

(a) $2\sin^2\theta$
(b) $12\sec^2\theta - 10\tan^2\theta$
(c) $12\sec^2\theta - 10\tan^2\theta + 5$
(d) 0

38. If $\Delta = \begin{vmatrix} -a & 2b & 0 \\ 0 & -a & 2b \\ 2b & 0 & -a \end{vmatrix} = 0$, then

(a) $1/b$ is a cube root of unity
(b) a is one of the cube roots of unity

(c) b is one of the cube roots of 8

(d) a/b is a cube root of 8

39. The determinant

$$\Delta = \begin{vmatrix} 1 & 1+i & i \\ 1+i & i & 1 \\ i & 1 & 1+i \end{vmatrix} \text{ equals}$$

(a) $7 + 4i$ (b) $-7 + 4i$

(c) $-7 - 4i$ (d) $2(i - 1)$

40. If a, b, c are non-zero real numbers, then

$$\Delta = \begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix} \text{ equals}$$

(a) 0 (b) $bc + ca + ab$

(c) $a^{-1} + b^{-1} + c^{-1}$ (d) $abc - 1$

41. If $a, b, c > 1$, then $\Delta = \begin{vmatrix} \log_a (abc) & \log_a b & \log_a c \\ \log_b (abc) & 1 & \log_b c \\ \log_c (abc) & \log_c b & 1 \end{vmatrix}$ equals

(a) 0

(b) $\log_a b + \log_b c + \log_c a$

(c) $\log_{abc} (a + b + c)$

(d) none of these

42. If $\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = \lambda \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$

then λ equals

(a) 0 (b) 1

(c) x (d) $1 - x^2$

43. Let $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, $0 \le \theta \le 2\pi$.The

(a) $\Delta = 0$ (b) $\Delta \in (2, \infty)$

(c) $\Delta \in (2, 4)$ (d) $\Delta \in [2, 4]$

44. The determinant $\Delta = \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(a) $(b - c)(c - a)(a - b)$

(b) $abc(b - c)(c - a)(a - b)$

(c) $(a + b + c)(b - c)(c - a)(a - b)$

(d) 0

45. If $\Delta = \begin{vmatrix} 6i & -3i & w \\ 4 & 3i & -w \\ 20 & 3 & iw \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$

(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

46. If $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, the number of distinct real roots of $\Delta = 0$

$$\Delta = \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} \text{ is}$$

(a) 0 (b) 2

(c) 1 (d) 3

47. If $\omega \ne 1$ is a complex cube root of unity, and

$$x + iy = \begin{vmatrix} 1 & i & -\omega \\ -i & 1 & \omega^2 \\ \omega & -\omega^2 & 1 \end{vmatrix} \text{ then}$$

(a) $x = -1, y = 0$ (b) $x = 1, y = -1$

(c) $x = 1, y = 1$ (d) none of these

48. If $e^{ix} = \cos x + i\sin x$ and

$$x + iy = \begin{vmatrix} 1 & e^{\pi i/4} & e^{\pi i/3} \\ e^{-\pi i/4} & 1 & e^{2\pi i/3} \\ e^{-\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} \end{vmatrix}, \text{ then}$$

(a) $x = -1, y = \sqrt{2}$

(b) $x = 1, y = -\sqrt{2}$

(c) $x = -\sqrt{2}, y = \sqrt{2}$

(d) none of these

49. If $a, b, c \,\varepsilon\, \mathbf{R}$, the number of real roots of the equation

$$\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0, \text{ is}$$

(a) 0 (b) 1

(c) 2 (d) 3

50. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = -7$ and

$$\Delta_1 = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}, \text{ then}$$

(a) $\Delta = 7$ (b) $\Delta = 343$

(c) $\Delta = -49$ (d) $\Delta = 49$

51. If $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha+\cos\beta \\ \sin\beta & \cos\alpha & \sin\beta+\cos\beta \\ \sin\gamma & \cos\alpha & \sin\gamma+\cos\beta \end{vmatrix}$ then Δ equals
(a) $\sin\alpha\sin\beta\sin\gamma$
(b) $\cos\alpha\sec\beta\tan\gamma$
(c) $\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\beta\cos\gamma$
(d) 0

52. Suppose $a, b, c \in \mathbf{R}$ and $a, b, c > 0$,
Let $\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log(7a) & \log(49b) & \log(343c) \\ \log(3a) & \log(9b) & \log(27c) \end{vmatrix}$
then Δ is equals to
(a) 0 (b) –1
(c) 1 (d) 30

53. The values of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation
$$\Delta = \begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$
are given by
(a) $\pi/24, 5\pi/24$ (b) $7\pi/24, 11\pi/24$
(c) $5\pi/24, 7\pi/24$ (d) $11\pi/24, \pi/24$

54. The number of distinct roots of the equation
$$P(x) = \begin{vmatrix} x^2-1 & x^2+2x+1 & 2x^2+3x+1 \\ 2x^2+x-1 & 2x^2+5x-3 & 4x^2+4x-3 \\ 6x^2-x-2 & 6x^2-7x+2 & 12x^2-5x-2 \end{vmatrix} = 0$$
is
(a) 6 (b) 5
(c) 3 (d) 4

55. If $a_r = \cos\dfrac{2r\pi}{9} + i\sin\dfrac{2r\pi}{9}$, then value of the determinant
$$\Delta = \begin{vmatrix} 1 & a_8 & a_7 \\ a_3 & a_2 & a_1 \\ a_6 & a_5 & a_4 \end{vmatrix}$$ is
(a) – 1 (b) 1
(c) 0 (d) – 2

56. If $a \neq p$, $b \neq q$, $c \neq r$ and the system of equations
$px + by + cz = 0$
$ax + qy + cz = 0$
$ax + by + rz = 0$
has a non-zero solution, then value of
$\dfrac{p+a}{p-a} + \dfrac{q+b}{q-b} + \dfrac{r+c}{r-c}$ is
(a) 2 (b) – 2
(c) 1 (d) 1

57. For a fixed positive integer n, if
$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$
then $\dfrac{D}{n!(n+1)!(n+2)!}$ is equal to
(a) – 4 (b) – 2
(c) 2 (d) 4

58. If a, b, c are in A.P., and $\Delta = \begin{vmatrix} x+2 & x+7 & a \\ x+5 & x+11 & b \\ x+8 & x+15 & c \end{vmatrix}$
then Δ equals
(a) 0 (b) 1
(c) $-(a+b+c)$ (d) $a+b+c$.

59. If $\Delta = \begin{vmatrix} 1+y & 1-y & 1-y \\ 1-y & 1+y & 1-y \\ 1-y & 1-y & 1+y \end{vmatrix} = 0$, then value of y are
(a) 0, 3 (b) 2, – 1
(c) – 1, 3 (d) 0, 2

60. The determinant
$$\Delta = \begin{vmatrix} al+a'l' & am+a'm' & an+a'n' \\ bl+b'l' & bm+b'm' & bn+b'n' \\ cl+c'l' & cm+c'm' & cn+c'n' \end{vmatrix}$$
is equal to
(a) $(abc + a'b'c')(lmn + l'm'n')$
(b) $abc\,lmn + a'b'c'\,l'm'n'$
(c) $(a^2+b^2+c^2)(l^2+m^2+n^2) + (a'^2+b'^2+c'^2)(l'^2+m'^2+n'^2)$
(d) 0

61. If $a = i$, $b = \omega$, $c = \omega^2$ where ω is complex cube root of unity, then
$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9b+6c \end{vmatrix}$$ is equal to
(a) $-\omega$ (b) $-\omega^2$
(c) i (d) $-i$

62. If $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+3}C_1 & {}^{m+6}C_1 \\ {}^mC_2 & {}^{m+3}C_2 & {}^{m+6}C_2 \end{vmatrix} = 2^\alpha\, 3^\beta\, 5^\gamma$, then
$\alpha + \beta + \gamma$ is equal
(a) 3 (b) 5
(c) 7 (d) none of these

63. Suppose $a, b, c, x, y \in \mathbf{R}$. Let
$$\Delta = \begin{vmatrix} 1 & 2+ax & 3+ay \\ 1 & 2+bx & 3+by \\ 1 & 2+cx & 3+cy \end{vmatrix}$$

Then Δ is independent of

(a) a, b, c (b) x, y

(c) a, b, c, y (d) a, b, c, x, y

64. If A, B and C are the angles of a triangle, then the determinant

$$\Delta = \begin{vmatrix} 0 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

(a) $\sin^2 A$ (b) $\sin^2 B$

(c) $\sin^2 C$ (d) 0

65. Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim_{x \to 0} \frac{f(x)}{x^2}$ is given by

(a) 0 (b) – 1

(c) 2 (d) 3

66. If ω is a complex cube root of unity, then value of

$$\Delta = \begin{vmatrix} a_1 + b_1\omega & a_1\omega^2 + b_1 & c_1 + b_1\omega \\ a_2 + b_2\omega & a_2\omega^2 + b_2 & c_2 + b_2\omega \\ a_3 + b_3\omega & a_3\omega^2 + b_3 & c_3 + b_3\omega \end{vmatrix}$$

is

(a) 0 (b) – 1

(c) 2 (d) none of these

67. If $pqr \neq 0$ and the system of equations

$$(p + a)x + by + cz = 0$$
$$ax + (q + b)y + cz = 0$$
$$ax + by + (r + c)z = 0$$

has a non-trivial solution, then value of $\frac{a}{p} + \frac{b}{q} + \frac{c}{r}$ is

(a) – 1 (b) 0

(c) 1 (d) 2

68. The system of equations

$$ax + by + (a\alpha + b)z = 0$$
$$bx + cy + (b\alpha + c)z = 0$$
$$(a\alpha + b)x + (b\alpha + c)y = 0$$

has a non-zero solutions if a, b, c are in

(a) A.P. (b) G.P.

(c) H.P. (d) A.G.P.

69. If the system of equations

$$ax + ay - z = 0$$
$$bx - y + bz = 0$$
$$-x + cy + cz = 0$$

(where $a, b, c \neq -1$) has a non-trivial solution, then value of $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$ is

(a) 2 (b) – 1

(c) – 2 (d) 0

70. The values of λ for which the system of equations

$$(\lambda + 5)x + (\lambda - 4)y + z = 0$$
$$(\lambda - 2)x + (\lambda + 3)y + z = 0$$
$$\lambda x + \lambda y + z = 0$$

has a non-trivial solution is (are)

(a) – 1, 2 (b) 0, – 1

(c) 0 (d) none of these

71. Given the system of equations

$$(b + c)(y + z) - ax = b - c$$
$$(c + a)(z + x) - by = c - a$$
$$(a + b)(x + y) - cz = a - b$$

(where $a + b + c \neq 0$); then $x : y : z$ is given by

(a) $b - c : c - a : a - b$ (b) $b + c : c + a : a + b$

(c) $a : b : c$ (d) $\frac{a}{b} : \frac{b}{c} : \frac{c}{a}$

72. If $a, b, c \,\varepsilon\, \mathbf{R}$ and $a + b + c \neq 0$ and the system of equations

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0$$

has a non-zero solution, then $a : b : c$ is given by

(a) $1 : \alpha : \beta$
where α, β are roots of $ax^2 + bx + c = 0$

(b) $1 : r : r^2$ where r is some positive real number

(c) $1 : k : 2k$ where k is some positive real number

(d) none of these

73. If p is a constant and

$$f(x) = \begin{vmatrix} x^3 & x^3 & 3x^2 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}, \text{ then } f'''(x) \text{ is}$$

(a) proportional to x^3 (b) proportional to x^2

(c) proportional to x (d) a constant

74. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$$D = \begin{vmatrix} a_1 + pb_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 + pb_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 + pb_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix} \text{ then}$$

(a) $D = \Delta$ (b) $D = \Delta(1 - pqr)$

(c) $D = \Delta(1 + pqr)$ (d) $D = \Delta(1 + p + q + r)$

75. Number of real values of λ for which the system of equations

$$(\lambda + 3)x + (\lambda + 2)y + z = 0$$
$$3x + (\lambda + 3)y + z = 0$$
$$2x + 3y + z = 0$$

has a non-trivial solutions is

(a) 0 (b) 1

(c) 2 (d) infinite

Assertion-Reason Type Questions

76. **Statement-1:** If

$$\Delta(x) = \begin{vmatrix} \sin x - \cos x & \sin\left(x - \frac{\pi}{3}\right) & \cos\left(x - \frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3} - x\right) & \tan\left(x - \frac{\pi}{4}\right) & \sec\left(x - \frac{\pi}{4}\right) \\ \cos\left(\frac{2\pi}{3} + x\right) & \sec\left(\frac{2\pi}{3} + x\right) & \cot\left(x + \frac{\pi}{4}\right) \end{vmatrix}$$

then $\Delta\left(\frac{\pi}{4}\right) = 0$

Statement-2: If A is a skew symmetric matrix of odd order, then $|A| = 0$.

77. **Statement-1:** Let $p < 0$ and $\alpha_1, \alpha_2, \ldots \alpha_9$ be the nine roots of $x^9 = p$, then

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{vmatrix} = 0$$

Statement-2: If two rows of a determinant are identical, then determinant equals zero.

78. **Statement-1:**

$$w = \begin{vmatrix} \sqrt{6} & 2i & 3 + \sqrt{6}i \\ \sqrt{12} & \sqrt{3} + \sqrt{8}i & 3\sqrt{2} + \sqrt{6}i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}i & \sqrt{27} + 2i \end{vmatrix}$$

is a purely imaginary number.

Statement-2: $|z| = |\bar{z}|$ for each complex number z.

79. **Statement-1:** If $a_i, b_i \in \mathbf{N}$, for $i = 1, 2, 3$ and

$$\Delta(x) = \begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & (1+x)^{a_1 b_3} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix},$$

then coefficient of x in expansion of $\Delta(x)$ is 0.

Statement-2: If $P(x) = (1 + x)^n$, $n \in \mathbf{N}$ then coefficient of x in the expansion of $P(x)$ is $P'(0)$.

80. **Statement-1:** If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, then the system of homogenous equations

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0$$

has infinite number of solutions.

Statement-2: If $|A| = 0$, the system of equations $AX = B$ has infinite number of solutions.

81. Suppose $a, b, c \in \mathbf{R}$. Consider the system of linear equations:

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0$$

Statement-1: If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = bc + ca + ab$, then the system of equations has infinite number of solutions.

Statement-2: If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, then the system of equations has unique solutions.

82. Suppose $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$.

Let

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 \\ a_1b_2 + a_2b_1 & 2a_2b_2 & a_2b_3 + a_3b_2 \\ a_1b_3 + a_3b_1 & a_2b_3 + a_3b_2 & 2a_3b_3 \end{vmatrix}$$

Statement-1: $\Delta_2 \neq 0$ for some values of $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$.

Statement-2: If $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are non-collinear than $\Delta_1 \neq 0$.

83. Suppose a_1, b_1, c_1, d_1 a_2, b_2, c_2, d_2 are eight integers.

Statement-1: $\left(a_1^2 + b_1^2 + c_1^2 + d_1^2\right)\left(a_2^2 + b_2^2 + c_2^2 + d_2^2\right)$ can be written in the form $a^2 + b^2 + c^2 + d^2$ where a, b, c, d are some integers.

Statement-2: If $a, b, c, d \in \mathbf{Z}$, then

$$a^2 + b^2 + c^2 + d^2 = \begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$$

LEVEL 2

Straight Objective Type Questions

84. If $l_1^2 + m_1^2 + n_1^2 = 1$ etc., and $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ etc., and

$$\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

then

(a) $|\Delta| = 3$ (b) $|\Delta| = 2$

(c) $|\Delta| = 1$ (d) $\Delta = 0$

85. If a, b, c are non-zero real number and

$$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

then Δ equals

(a) abc (b) $a^2 b^2 c^2$

(c) $bc + ca + ab$ (d) 0

86. Let $\Delta(x) = \begin{vmatrix} x^2 - 5x + 3 & 2x - 5 & 3 \\ 3x^2 + x + 9 & 6x + 1 & 9 \\ 7x^2 - 6x + 9 & 14x - 6 & 21 \end{vmatrix}$

$= ax^3 + bx^2 + cx + d,$

then a equals

(a) – 1 (b) 0

(c) 2 (d) none of these

87. The number of distinct values of t for which the system

$(a + t)x + by + cz = 0$

$ax + (b + t)y + cz = 0$

$ax + by + (c + t)z = 0$

has a non-trivial solution is

(a) 1 (b) 2

(c) 3 (d) none of these

88. If $a^2 + b^2 + c^2 = 1$, then

$$\begin{vmatrix} a^2 + (b^2 + c^2)\cos\theta & ab(1-\cos\theta) & ac(1-\cos\theta) \\ ba(1-\cos\theta) & b^2 + (c^2 + a^2)\cos\theta & bc(1-\cos\theta) \\ ca(1-\cos\theta) & cb(1-\cos\theta) & c^2 + (a^2 + b^2)\cos\theta \end{vmatrix}$$ equals

(a) $\cos^2\theta$ (b) 0

(c) 1 (d) $\sin^2\theta$

89. Suppose $\alpha, \beta, \gamma, \theta \in \mathbf{R}$ and

$$A(\alpha, \beta, \gamma, \theta) = \begin{vmatrix} \cos(\alpha+\theta) & \sin(\alpha+\theta) & 1 \\ \cos(\beta+\theta) & \sin(\beta+\theta) & 1 \\ \cos(\gamma+\theta) & \sin(\gamma+\theta) & 1 \end{vmatrix}$$

Numerical value of $A\left(-\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{2\pi}{13}\right)$ is

(a) 0 (b) – 1

(c) 2 (d) none of these

90. If a, b, c are positive integers such that $a > b > c$ and

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -2$$

then $3a + 7b - 10c$ equals

(a) 10 (b) 11

(c) 12 (d) 13

91. If $A, B, C, P, Q, R \in \mathbf{R}$, and

$$\Delta = \begin{vmatrix} \cos(A+P) & \cos(A+Q) & \cos(A+R) \\ \cos(B+P) & \cos(B+Q) & \cos(B+R) \\ \cos(C+P) & \cos(C+Q) & \cos(C+R) \end{vmatrix}$$

(a) Δ depends on P, Q, R

(b) Δ depends on A, B, C

(c) Δ depends on A, B, C, P, Q, R

(d) none of these

92. Let $f(x) = \begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$ then

(a) $f\left(\frac{\pi}{3}\right) = 1$ (b) $f'\left(\frac{\pi}{3}\right) = -\sqrt{3}$

(c) $f\left(\frac{\pi}{2}\right) = -1$ (d) none of these

93. Consider the set A consisting of all determinants of order 3 with entries 0 and 1 only. Let B be the subset of A consisting of all the determinants with value 1. Let C be the subset of A consisting of all the determinants with value –1. Then

(a) $C = \phi$

(b) B has as many elements as C

(c) $A = B \cap C$

(d) $A = B \cup C$

94. let $\omega = e^{2\pi i/3}$ and consider the system of linear equations

$$x + y + z = a$$
$$x + \omega y + \omega^2 z = b$$
$$x + \omega^2 y + \omega z = c$$

If x, y, z is a solution of the above system of equations, then value of $\frac{|a|^2 + |b|^2 + |c|^2}{|x|^2 + |y|^2 + |z|^2}$ is

(a) 9 (b) 6
(c) 3 (d) 1

95. If a, b, c are distinct and $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$ then abc equals

(a) 0 (b) 1
(c) –1 (d) –2

96. If the adjoint of a 3×3 matrix P is $\begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$ then the possible value(s) of determinant P is (are)

(a) ± 2 (b) ± 3
(d) ± 1 (d) 0

97. If $\begin{vmatrix} 1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$, then x equals

(a) 17, 21 (b) 0,19
(c) 0, 35 (d) 21, 35

Previous Years' AIEEE/JEE Main Questions

1. If the equation $ax^2 + 2bx + c = 0$ has equal roots then the determinant

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

is

(a) positive (b) negative
(c) 0 (d) dependent on a. **[2002]**

2. If $l, m, n > 0$ and l, m, n are the pth, qth, rth terms of a G.P., then determinant

$$\Delta = \begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} \text{ equals}$$ **[2002]**

(a) 0 (b) – 1
(c) $p + q + r$ (d) none of these

3. If 1, ω, ω^2 are the cube roots of unity then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} \text{ equals}$$

(a) 1 (b) 2
(c) ω^2 (d) 0 **[2003]**

4. If the system of linear equations $x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$ has a non-zero solution, then a, b, c

(a) are in G.P
(b) are in H.P.
(c) satisfy $a + 2b + 3c = 0$
(d) are in A.P. **[2003]**

5. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in G.P., then the value of the determinant

$$\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+1} & \log a_{n+2} & \log a_{n+3} \\ \log a_{n+2} & \log a_{n+3} & \log a_{n+4} \end{vmatrix}$$

is

(a) 2 (b) 1
(c) 0 (d) – 2 **[2004, 2005]**

6. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

then $f(x)$ is a polynomial of degree

(a) 3 (b) 2
(1) 1 (d) 0 **[2005]**

7. The system of equations

$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$

has no solution, if α is

(a) not – 2
(b) 1
(c) – 2
(d) either – 2 or 1 **[2005]**

8. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0$, $y \neq 0$, then D is
   (a) divisible by neither x nor y
   (b) divisible by both x and y
   (c) divisible by x but not y
   (d) divisible by y but not x **[2007]**

9. Let a, b, c be such that $b(a + c) \neq 0$. If

$$\Delta = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$$

   then the value of n is
   (a) any odd integer
   (b) any integer
   (c) zero
   (d) any even integer
   [2009]

10. If a, b, c are sides of a scalene triangle, then the value of $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is:
   (a) non-negative
   (b) negative
   (c) positive
   (d) non-positive
   [2013 online]

11. **Statement-1:** The system of linear equations
   $x + (\sin \alpha)y + (\cos \alpha)z = 0$
   $x + (\cos \alpha)y + (\sin \alpha)z = 0$
   $x - (\sin \alpha)y - (\cos \alpha)z = 0$
   has a non-trivial solution for only one value of α lying in the interval $(0, \pi/2)$.
   Statement-2: The equation in α

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

   has only one solution lyining in the interval $(0, \pi/2)$. **[2013 online]**

12. If α, $\beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

   $= K(1 - \alpha)^2 (1- \beta)^2 (\alpha - \beta)^2$, then K is equal to:
   (a) $\dfrac{1}{\alpha\beta}$ (b) 1
   (c) –1 (d) $\alpha\beta$
   [2014]

13. If a, b, c are non-zero real numbers and if the system of equations
   $(a - 1)x = y + z$,
   $(b - 1)y = z + x$,
   $(c - 1)z = x + y$,
   has a non-trivial solution, then $ab + bc + ca$ equals:
   (a) $a + b + c$ (b) abc
   (c) 1 (d) –1 **[2014 online]**

14. Let for $i = 1, 2, 3$, $p_i(x)$ be a polynomial of degree 2 in x, $p_i'(x)$ and $p_i''(x)$ be the first and second order derivatives of $p_i(x)$ respectively. Let,

$$A(x) = \begin{bmatrix} p_1(x) & p_1'(x) & p_1''(x) \\ p_2(x) & p_2'(x) & p_2''(x) \\ p_3(x) & p_3'(x) & p_3''(x) \end{bmatrix}$$

   and $B(x) = [A(x)]^T A(x)$. Then determinant of $B(x)$.
   (a) is a polynomial of degree 6 in x
   (b) is a polynomial of degree 3 in x
   (c) is a polynomial of degree 2 in x
   (d) does not depend on x
   [2014 online]

15. If

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = \text{k}\lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}, \lambda \neq 0,$$

   then k is equal to:
   (a) $4\lambda abc$ (b) $-4\lambda abc$
   (c) $4\lambda^2$ (d) $-4\lambda^2$
   [2014 online]

16. If $\Delta_r = \begin{vmatrix} r & 2r-1 & 3r-2 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{1}{2}n(n-1) & (n-1)^2 & \dfrac{1}{2}(n-1)(3n+4) \end{vmatrix}$

   then the value of $\sum_{r=1}^{n-1} \Delta_r$:
   (a) depends only on a
   (b) depends only on n
   (c) depends both on a and n
   (d) is independent of both a and n.
   [2014 online]

17. The set of all values of λ for which the system of linear equations:

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

has a non-trivial solution,

(a) is an empty set
(b) is a singleton
(c) contains two elements
(d) contains more than two elements **[2015]**

18. If $\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = ax - 12,$

then a is equal to:

(a) 12 (b) 24
(c) – 12 (d) – 24 **[2015 online]**

19*. The least value of the product xyz for which the determinant $\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix}$ is non-negative, is:

(a) $-2\sqrt{2}$ (b) $-16\sqrt{2}$
(c) – 8 (d) – 1 **[2015 online]**

20. The system of linear equations

$x + \lambda y - z = 0$
$\lambda x - y - z = 0$
$x + y - \lambda z = 0$

has a non-trivial solution for

(a) infinitely many values of λ
(b) exactly one value of λ
(c) exactly two values of λ
(d) exactly three values of λ **[2016]**

21. The number of distinct real roots of the equation.

$$\Delta = \begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$

in the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is:

(a) 1 (b) 4
(c) 2 (d) 3 **[2016 online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If the system of equations

$x + y + z = 0$
$ax + by + z = 0$
$bx + y + z = 0$

has a non-trivial solution, then

(a) $b^2 = 2b + 1$ (b) $b^2 = 2b - 1$
(c) $b - a = 0$ (d) $b^2 = 2b$ **[2006]**

2. The system of linear equations

$(\lambda + 3)x + (\lambda + 2)y + z = 0$
$3x + (\lambda + 3)y + z = 0$
$2x + 3y + z = 0$

has a non-trivial solution

(a) if $\lambda = 1$ (b) if $\lambda = -1$
(c) for no real value of λ
(d) if $\lambda = 0$ **[2007]**

3. If

$$\Delta_k = \begin{vmatrix} 2(3^{k-1}) & 3(4^{k-1}) & 4(5^{k-1}) \\ \alpha & \beta & \gamma \\ 3^n - 1 & 4^n - 1 & 5^n - 1 \end{vmatrix}$$

then the value of $\sum_{k=1}^{n} \Delta_k$ depends

(a) only on α and β not on γ
(b) on all α, β and γ
(c) on none of α, β and γ
(d) only on α, not on β and γ **[2007]**

4. Let $A = (a_{ij})$ be a 3×3 matrix whose determinant is 5. The determinant of the matrix $B = (2^{i-j} a_{ij})$ is

(a) 5 (b) 10
(c) 20 (d) 40 **[2008]**

5. Let $\Delta_r = \begin{vmatrix} 2^{r-1} & 2(3^{r-1}) & 4(5^{r-1}) \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, for $r = 1, 2, \ldots, n$. Then $\sum_{r=1}^{n} \Delta_r$ is

(a) independent of α, β, γ and n
(b) independent of n only
(c) depends on α, β, γ and n
(d) independent of α, β, γ only **[2010]**

* Question is incorrect as xyz can take any real value.

6. If x_1, x_2, x_3,…, x_{13} are in A.P. then the value of $\begin{vmatrix} e^{x_1} & e^{x_4} & e^{x_7} \\ e^{x_4} & e^{x_7} & e^{x_{10}} \\ e^{x_7} & e^{x_{10}} & e^{x_{13}} \end{vmatrix}$ is:
(a) 27 (b) 0
(c) 1 (d) 9 **[2011]**

7. The value of the determinant
$$\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$$ is equal to:
(a) $5\sqrt{3}(\sqrt{6}-5)$ (b) $5\sqrt{3}(\sqrt{6}-\sqrt{5})$
(c) $5(\sqrt{6}-5)$ (d) $\sqrt{3}(\sqrt{6}-\sqrt{5})$ **[2012]**

8. If the system of linear equations, $x + 2ay + az = 0$, $x + 3by + bz = 0$ and $x + 4cy + cz = 0$ has a non-zero solution, then a, b, c satisfy:
(a) $2b = a + c$ (b) $b^2 = ac$
(c) $2ac = ab + bc$ (d) $2ab = ac + bc$ **[2013]**

9. In a ΔABC, if
$$\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0,$$
then $\sin^2 A + \sin^2 B + \sin^2 C$ is
(a) $\frac{3}{2}\sqrt{3}$ (b) $\frac{9}{4}$
(c) $\frac{5}{4}$ (d) 2 **[2014]**

10. The system of linear equations
$x - y + z = 1$
$x + y - z = 3$
$x - 4y + 4z = \alpha$ has:
(a) a unique solution when $\alpha = 2$
(b) a unique solution when $\alpha \neq -2$
(c) an infinite number of solutions, when $\alpha = 2$
(d) an infinite number of solutions, when $\alpha = -2$
[2015]

11. For all values of $\theta \in \left(0, \frac{\pi}{2}\right)$, the determinant of the matrix $\Delta = \begin{bmatrix} -2 & \tan\theta + \sec^2\theta & 3 \\ -\sin\theta & \cos\theta & \sin\theta \\ -3 & -4 & 3 \end{bmatrix}$ always lies in the interval:
(a) [3, 5] (b) (4, 6)
(c) $\left(\frac{5}{2}, \frac{19}{4}\right)$ (d) $\left[\frac{7}{2}, \frac{21}{4}\right]$ **[2016]**

Answers

Concept-based

1. (d)	**2.** (d)	**3.** (a)	**4.** (d)
5. (a)	**6.** (c)	**7.** (d)	**8.** (a)
9. (a)	**10.** (b)	**11.** (d)	**12.** (c)
13. (d)	**14.** (d)	**15.** (a)	

Level 1

16. (a)	**17.** (c)	**18.** (a)	**19.** (b)
20. (d)	**21.** (c)	**22.** (a)	**23.** (c)
24. (d)	**25.** (a)	**26.** (d)	**27.** (d)
28. (a)	**29.** (a)	**30.** (a)	**31.** (d)
32. (a)	**33.** (c)	**34.** (a)	**35.** (a)
36. (d)	**37.** (d)	**38.** (d)	**39.** (d)
40. (a)	**41.** (a)	**42.** (d)	**43.** (d)
44. (d)	**45.** (d)	**46.** (c)	**47.** (a)
48. (d)	**49.** (b)	**50.** (d)	**51.** (d)
52. (a)	**53.** (b)	**54.** (d)	**55.** (c)
56. (d)	**57.** (c)	**58.** (a)	**59.** (a)
60. (d)	**61.** (d)	**62.** (a)	**63.** (d)
64. (a)	**65.** (b)	**66.** (a)	**67.** (a)
68. (b)	**69.** (a)	**70.** (d)	**71.** (a)
72. (d)	**73.** (d)	**74.** (c)	**75.** (a)
76. (a)	**77.** (d)	**78.** (b)	**79.** (a)
80. (c)	**81.** (c)	**82.** (d)	**83.** (a)

Level 2

84. (c)	**85.** (d)	**86.** (b)	**87.** (b)
88. (a)	**89.** (c)	**90.** (d)	**91.** (d)
92. (b)	**93.** (b)	**94.** (c)	**95.** (b)
96. (a)	**97.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (c)	**2.** (a)	**3.** (d)	**4.** (b)
5. (c)	**6.** (b)	**7.** (c)	**8.** (b)
9. (a)	**10.** (b)	**11.** (b)	**12.** (b)
13. (b)	**14.** (d)	**15.** (c)	**16.** (d)
17. (c)	**18.** (b)	**19.** (?)	**20.** (d)
21. (c)			

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (c)	**3.** (c)	**4.** (a)
5. (a)	**6.** (b)	**7.** (c)	**8.** (c)
9. (b)	**10.** (d)	**11.** (a)	

Hints and Solutions

Concept-based

1. det (adj (A)) A^{-1}) = det (adj(A)) det(A^{-1})
$$= (\det(A))^2 \frac{1}{\det(A)} = \det(A)$$

2. Use $R_1 \to R_1 + R_3 - 2R_2$

3. Using $R_2 \to R_2 - x^2R_1$ and $R_3 \to R_3 - xR_1$, we get

$$\Delta = \begin{vmatrix} 1 & x & \\ 0 & 1-x^3 & x(1-x^3) \\ 0 & 0 & 1-x^3 \end{vmatrix}$$

$$= (1 - x^3)^2 = (1 - x)^2 (1 + x + x^2)^2$$

As $x \neq 1, \Delta = 0 \Rightarrow 1 + x + x^2 = 0 \Rightarrow x = \omega, \omega^2$

4. Taking 10!, 11! and 12! common from C_1, C_2, C_3 respectively, we get

$$D = (10!\ 11!\ 12!)\ D_1$$

where

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 11 & 12 & 13 \\ (11)(12) & (12)(13) & (13)(14) \end{vmatrix}$$

Using $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 11 & 1 & 1 \\ (11)(12) & 2(12) & 2(13) \end{vmatrix} = 2$$

Thus, $\dfrac{D}{(10!)^3} = 2(11)\ (12) = 264$

5. Product of zeros = $-P(0) = 0$

6. $\dfrac{1}{2}|\Delta|$ = area of triangle = $\dfrac{\sqrt{3}}{4}(4)^2 = 4\sqrt{3}$

$\Rightarrow \quad |\Delta| = 8\sqrt{3} \quad \Rightarrow \quad \Delta^2 = 192$

7. Use $R_1 \to R_1 + R_3 - 2R_2$.

8. Write

$$\Delta = \begin{vmatrix} 1+1 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ a^2+b^2 & a^3+b^3 & a^4+b^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 0 \end{vmatrix} = 0$$

9. Use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $\log x - \log y = \log (x/y)$ to obtain

$$\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log(2007) & \log(2007) & \log(2007) \\ \log(2017) & \log(2017) & \log(2017) \end{vmatrix} = 0$$

[$\because R_2$ and R_3 are proportional]

10. Using $R_2 \to R_2 - R_1$, we get

$$\Delta = -4(2016)\begin{vmatrix} (a+2016)^2 & (b+2016)^2 & (c+2016)^2 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 - 2\ (2016)\ R_2 - R_3$, we get

$$\Delta = -4(2016)^3 \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= -4\ (2016)^3\ (a - b)\ (b - c)\ (c - a)$$

11. Using $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, we get

$$P(x) = \begin{vmatrix} x & -6 & -1 \\ x+2 & -3(x+2) & x+2 \\ -3-x & 2(x+3) & x+3 \end{vmatrix}$$

$$= (x + 2)\ (x + 3) \begin{vmatrix} x & -6 & -1 \\ 1 & -3 & 1 \\ -1 & 2 & 1 \end{vmatrix}$$

Using $R_1 \to R_1 + R_3$, $R_2 \to R_2 - R_3$, we get

$$P(x) = (x + 2)\ (x + 3) \begin{vmatrix} x-1 & -4 & 0 \\ x & -5 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= (x + 2)\ (x + 3)\ (-5x + 13)$$

sum of zeros of $P(x)$ is $-2 - 3 + 13/5 = -12/5$.

12. Write Δ as

$$\Delta = x^{n+1}\ y^{n+1}\ z^{n+1} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (xyz)^{n+1}\ (x - y)\ (y - z)(z - x)$$

$$\therefore \quad n + 1 = 2 \quad \Rightarrow \quad n = 1.$$

13. When $x = 0$, $P(0)$ is a skew-symmetric determinant of odd order.

14. Write $\Delta = \Delta_1 + a\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix}$$

and $$\Delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix}$$

$$= (1 + 2b)\ (1 + 3c) - (1 + c)$$

$$= 2\ (b + c + 3c)$$

In Δ_1, apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to obtain

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1+b & b & -b \\ 1+c & 0 & 2c \end{vmatrix} = 2bc$$

Thus, $\Delta = 2(bc + ac + ab + 3abc)$

$\therefore \quad \Delta = 0 \quad \Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = -3$

15. Taking x^m common from R_1 and x^{m+5} from R_3, we get

$$\Delta = x^{2m+5}\begin{vmatrix} 1 & x^2 & x^m \\ 1 & x^n & 2^n \\ 1 & x^{n-m} & x^m \end{vmatrix}$$

If $n - m = 2$, then R_1 and R_3 are identical and hence $\Delta = 0$.

Level 1

16. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix}$$

$$= -2(a+b+c)\,[(b-c)^2 + (b-a)\,(c-a)]$$
$$= -(a+b+c)\,[(b-c)^2 + (c-a)^2 + (a-b)^2]$$

Now, use $\Delta = 0$, $a + b + c \neq 0$.

17. Write $y = \Delta_1 + \Delta_2$ where

$$\Delta_1 = \begin{vmatrix} 1 & x & x \\ 0 & x+2 & x \\ 0 & x & x+3 \end{vmatrix} = 5x+6$$

and $$\Delta_2 = x\begin{vmatrix} 1 & x & x \\ 1 & x+2 & x \\ 1 & x & x+3 \end{vmatrix} = x\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 6x$$

Thus, $y = 11x + 6$

Its distance from the origin $= \frac{6}{\sqrt{122}}$

18. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Now, use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & a+2c \end{vmatrix}$$

$$\Delta = (a+b+c)\,[(a+2b)\,(a+2c) - (a-c)\,(a-b)]$$
$$= 3(a+b+c)\,(bc+ca+ab)$$

19. Using $R_1 \to R_1 - R_3 - R_2$ we get R_1 consists of all zeros.

20. If R is the common ratio of the G.P., then $b = aR$, $c = aR^2$, $d = aR^3$, $e = aR^4$, $f = aR^5$, and

$$\Delta = (a^2)(a^2R^6)\begin{vmatrix} 1 & 1 & p \\ R^2 & R^2 & q \\ R^4 & R^4 & r \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are identical]

21. Multiply C_1 by x, C_2 by y and C_3 by z to obtain

$$5 = \frac{1}{xyz}\begin{vmatrix} (x^2+1)x & xy^2 & xz^2 \\ yx^2 & (y^2+1)y & yz^2 \\ zx^2 & zy^2 & z(z^2+1) \end{vmatrix}$$

Taking x common from R_1, y from R_2 and z from R_3, we get

$$5 = \frac{xyz}{xyz}\begin{vmatrix} x^2+1 & y^2 & z^2 \\ x^2 & y^2+1 & z^2 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$5 = (x^2 + y^2 + z^2 + 1)\,\Delta_1$$

where $$\Delta_1 = \begin{vmatrix} 1 & y^2 & z^2 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & y^2 & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Thus, $x^2 + y^2 + z^2 = 4$, which represents a sphere.

22. Apply $R_1 \to aR_1 + bR_2 + cR_3$ and use $a = b\cos C + c\cos B$ etc.

23. Write $\Delta = \Delta_1 + x\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} a-x & x & x \\ 0 & b & x \\ 0 & x & c \end{vmatrix} = (a-x)(bc-x^2)$$

and $$\Delta_2 = \begin{vmatrix} 1 & x & x \\ 1 & b & x \\ 1 & x & c \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ 0 & b-x & 0 \\ 0 & 0 & c-x \end{vmatrix} \qquad [\text{use } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$= (b-x)\,(c-x)$$

Thus, $\Delta = (a - x)(bc - x^2) + x(b - x)(c - x)$

$= 2x^3 - (a + b + c)x^2 + abc$

Also, $xf'(x) - f(x)$

$= x[(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)] - f(x)$

$= 2x^3 - (a + b + c)x^2 + abc$

24. Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$0 = (x+y+2)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-y-2 & 2x \\ 2y & 2y & y-2-x \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$0 = (x+y+2)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -(x+y+2) & 0 \\ 2y & 0 & -(x+y+2) \end{vmatrix}$$

$\Rightarrow \quad 0 = (x + y + 2)^3$

$\Rightarrow \quad x + y + 2 = 0$

It passes through (–1, –1).

25. Write $f(x) = \Delta_1 + x\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} a & x & x \\ 0 & x+a & x \\ 0 & x & x+a \end{vmatrix} = a[(x+a)^2 - x^2]$$

$= 2\,xa^2 + a^3$

and $$\Delta_2 = \begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a^2$$

Thus, $\quad f(x) = 2xa^2 + a^3 + xa^2$

$= 3xa^2 + a^3$

$\therefore \; f(2x) - f(x) = 3xa^2$

26. $\alpha + \beta + \gamma = -a$, $\beta\gamma + \gamma\alpha + \alpha\beta = 0$, $\alpha\beta\gamma = -b$.

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} \quad (1)$$

Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = -a\begin{vmatrix} 1 & \beta & \gamma \\ 0 & \gamma-\beta & \alpha-\gamma \\ 0 & \alpha-\gamma & \beta-\alpha \end{vmatrix}$$

$= -a[(\gamma - \beta)(\beta - \alpha) - (\alpha - \gamma)^2]$

$= -a[\beta\gamma + \gamma\alpha + \alpha\beta - (\alpha^2 + \beta^2 + \gamma^2)] \quad (2)$

$= -a[0 - a^2] = a^3$

27. In this case $\alpha + \beta + \gamma = 0$.
Thus, from (1) in solution to question 26, we get $\Delta = 0$.

28. We can write

$$\Delta = (abc)^2\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

where $\quad \alpha = \dfrac{1}{a}, \beta = \dfrac{1}{b}, \gamma = \dfrac{1}{c}$

From (2) in solution question 26, we have

$\Delta = (abc)^2(\alpha + \beta + \gamma)[(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2]$

$\Rightarrow \alpha + \beta + \gamma = 0 \quad \Rightarrow \quad a^{-1} + b^{-1} + c^{-1} = 0$

29. Using $C_2 \to C_2 - \lambda^2 C_3$, we can write

$$\Delta = \lambda\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = \lambda\begin{vmatrix} a & a^2 & 1 \\ b-a & b^2-a^2 & 0 \\ c-b & c^2-b^2 & 0 \end{vmatrix}$$

$$= \lambda(b-a)(c-b)\begin{vmatrix} 1 & b+a \\ 1 & c+b \end{vmatrix}$$

$= \lambda(b - a)(c - b)(c - a)$

$= \lambda(a - b)(b - c)(c - a)$

30. Use $C_2 \to C_2 - C_3 + C_1$ to show that C_2 consists of all zeros.

31. We know

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Let $$\Delta_1 = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-b^2 & c^3-b^3 \end{vmatrix}$$

$$= (b-a)(c-b)\begin{vmatrix} b+a & b^2+a^2+ba \\ c+b & c^2+b^2+cb \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1$, we get

$$\Delta_1 = (b-a)(c-b)\begin{vmatrix} b+a & a^2 \\ c+b & c^2 \end{vmatrix}$$

$$= (b-a)(c-b)\begin{vmatrix} b+a & a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

[using $R_2 \to R_2 - R_1$]

$$= (b-a)(c-b)(c-a)\begin{vmatrix} b+a & a^2 \\ 1 & c+a \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(bc+ca+ab)$$

Thus, $\lambda = 18$

32. Show that $\Delta = (a_0 + a_1 + a_2)x^2$
33. Using $R_2 \to R_2 + R_1$, we get

$$\Delta = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & 1 & \sin + \cos x \\ 0 & -1 & 1-(\sin x + \cos x) \end{vmatrix}$$

Using $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & 1 & \sin + \cos x \\ 0 & 0 & 1 \end{vmatrix} = 1$$

34. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (x+a+b)\begin{vmatrix} 1 & a & b \\ 1 & x & b \\ 1 & b & x \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = (x+a+b)\begin{vmatrix} 1 & a & b \\ 0 & x-a & 0 \\ 0 & b-a & x-b \end{vmatrix}$$

$$= (x+a+b)(x-a)(x-b)$$

35. Use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$to obtain

$$\Delta = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$$

36. Use $C_1 \to C_1 + C_2 + C_3$ to obtain

$$\Delta = (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

Now use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to obtain

$$\Delta = x^2(x+a+b+c)$$

Thus, $\Delta = 0 \Rightarrow x = 0$ or $x = -(a+b+c)$

37. Use $C_1 \to C_1 - C_2$.
38. Show $\Delta = -a^3 + 8b^3$

Thus, $\Delta = 0 \Rightarrow \left(\frac{a}{b}\right)^3 = 8$

$\Rightarrow a/b$ is a cube root of 8.

39. Using $C_1 \to C_1 + C_2 + C_3$, we can write $\Delta = 2(1+i)\Delta_1$ where

$$\Delta_1 = \begin{vmatrix} 1 & 1+i & i \\ 1 & i & 1 \\ 1 & 1 & 1+i \end{vmatrix} = \begin{vmatrix} 1 & 1+i & i \\ 0 & -1 & 1-i \\ 0 & -i & 1 \end{vmatrix} = i$$

[using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

Thus, $\Delta = 2(i-1)$

40. Write $\Delta = abc\begin{vmatrix} 1 & \frac{1}{c} & \frac{1}{a}+\frac{1}{b} \\ 1 & \frac{1}{a} & \frac{1}{b}+\frac{1}{c} \\ 1 & \frac{1}{b} & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

and use $C_2 \to C_2 + C_1$, to show that $\Delta = 0$

41. Applying $C_1 \to C_1 - C_2 - C_3$ and reduce Δ to determinant in Example 36.
42. Applying $R_1 \to R_1 - xR_2$ we get

$$\Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

Take $1 - x^2$ common from R_1 and apply $R_2 \to R_2 - xR_1$ to obtain

$$\Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

43. Applying $R_1 \to R_1 + R_3$ to obtain

$$\Delta = \begin{vmatrix} 0 & 0 & 2 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} = 2(\sin^2\theta + 1)$$

As $0 \le \sin^2\theta \le 1$, we get $\Delta \in [2, 4]$.

44. Write $\Delta = \Delta_1 - \Delta_2$ where

$$\Delta_1 = \begin{vmatrix} b^2-ab & b & bc-ac \\ ab-a^2 & a & b^2-ab \\ bc-ac & c & ab-a^2 \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} b^2-ab & c & bc-ac \\ ab-a^2 & b & b^2-ab \\ bc-ac & a & ab-a^2 \end{vmatrix}$$

In Δ_1 use $C_1 \to C_1 - (b-a)C_2$ to show that $\Delta_1 = 0$.

In Δ_2 use $C_3 \to C_3 - (b - a)C_2$ to show that $\Delta_2 = 0$.

Thus, $\Delta = 0$.

45. We can write

$$\Delta = -3iw \begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0$$

$\therefore \quad x = 0, y = 0$

46. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (\sin x + 2\cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (\sin x + 2\cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}$$

$$= (\sin x + 2\cos x)(\sin x - \cos x)^2$$

$\Delta = 0 \Rightarrow \tan x = -2, \tan x = 1.$

As $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$, $-1 \le \tan x \le 1$.

Thus, $\tan x = 1 \Rightarrow x = \pi/4$.

47. Applying $C_1 \to C_1 + iC_2$, we get

$$x + iy = \begin{vmatrix} 0 & i & -\omega \\ 0 & 1 & \omega^2 \\ \omega - i\omega^2 & -\omega^2 & 1 \end{vmatrix}$$

$$= (\omega - i\omega^2)(i\omega^2 + \omega)$$

$$= \omega^2 - i^2\omega^4 = \omega^2 + \omega = -1$$

$\Rightarrow x = -1, y = 0.$

48. Taking conjugate, we get

$$x - iy = \begin{vmatrix} 1 & e^{-\pi i/4} & e^{-\pi i/3} \\ e^{\pi i/4} & 1 & e^{-2\pi i/3} \\ e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & e^{\pi i/4} & e^{\pi i/3} \\ e^{-\pi i/4} & 1 & e^{2\pi i/3} \\ e^{-\pi i/3} & e^{-2\pi i/3} & e^{2\pi i/3} \end{vmatrix}$$

Now,

$$2iy = (x + iy) - (x - iy)$$

$$= \begin{vmatrix} 1 & e^{\pi i/4} & 0 \\ e^{-\pi i/4} & 1 & 0 \\ e^{-\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} - e^{2\pi i/3} \end{vmatrix}$$

$$= (e^{-2\pi i/3} - e^{2\pi i/3})[1 - 1] = 0$$

$\Rightarrow y = 0.$

49. Show that the determinant equals $x^3 + (a^2 + b^2 + c^2)x = 0$.

$\Rightarrow x = 0$ as $x \in \mathbf{R}$

$\therefore$ Number of real roots of the equation is one.

50. Note that

$$\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$$

where C_{ij} cofactor of (i, j)th element of

$$\begin{bmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{bmatrix}$$

$\therefore \quad \Delta_1 = \Delta^2 = 49$

51. Applying $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\beta \\ \sin\beta & \cos\alpha & \cos\beta \\ \sin\gamma & \cos\alpha & \cos\beta \end{vmatrix} = 0$$

52. Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, and $\log x - \log y = \log\left(\dfrac{x}{y}\right)$, we get

$$\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log 7 & \log(7^2) & \log(7^3) \\ \log(3) & \log(3^2) & \log(3^3) \end{vmatrix}$$

$$= (\log 7)(\log 3)\Delta_1$$

$$\text{where } \Delta_1 = \begin{vmatrix} \log a & \log b & \log c \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$\therefore \quad \Delta = 0$

53. $C_3 \to C_3 + C_1 + C_2$, gives

$$\Delta = (2 + 4\sin 4\theta) \begin{vmatrix} 1 + \cos^2\theta & \sin^2\theta & 1 \\ \cos^2\theta & 1 + \sin^2\theta & 1 \\ \cos^2\theta & \sin^2\theta & 1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = 2(1 + 2\sin 4\theta) \begin{vmatrix} 1 + \cos^2\theta & \sin^2\theta & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= 2(1 + 2\sin 4\theta)$$

$\Delta = 0 \Rightarrow \sin 4\theta = -1/2$

Now, $0 \le \theta \le \dfrac{\pi}{2} \Rightarrow 0 \le 4\theta \le 2\pi$

$\therefore \quad 4\theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6} \quad \Rightarrow \quad \theta = \dfrac{7\pi}{24}, \dfrac{11\pi}{24}$

54.

$$P(x) = \begin{vmatrix} (x-1)(x+1) & (x+1)^2 & (2x+1)(x+1) \\ (2x-1)(x+1) & (2x-1)(x+3) & (2x+3)(2x-1) \\ (2x+1)(3x-2) & (2x-1)(3x-2) & (4x+1)(3x-2) \end{vmatrix}$$

$$= (x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & x+1 & 2x+1 \\ x+1 & x+3 & 2x+3 \\ 2x+1 & 2x-1 & 4x+1 \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$, we get

$$P(x) = (x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & 2 & 3 \\ x+1 & 2 & 1 \\ 2x+1 & -2 & -1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, we get

$$P(x) = (x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & 2 & 3 \\ 2 & 0 & -2 \\ 3x & 0 & 2 \end{vmatrix}$$

$$P(x) = -4(x+1)(2x-1)(3x-2)(3x+2)$$

$\therefore$ $P(x) = 0$ has four distinct roots.

55. Put $a_r = w^r$ where $w = \cos \dfrac{2\pi}{9} + i \sin \dfrac{2\pi}{9}$.

Note that $w^9 = 1 \Rightarrow \dfrac{1}{w} = w^8$. Now,

$$\Delta = \begin{vmatrix} 1 & w^8 & w^7 \\ w^3 & w^2 & w \\ w^6 & w^5 & w^4 \end{vmatrix} = 0$$

as C_2 and C_3 are proportional.

56. As the system has a non-zero solution

$$\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0$$

Expanding along R_3, we get

$-(a-p)(q-b)c + (r-c)[p(q-b) - b(a-p)] = 0.$

Dividing $(p-a)(q-b)(r-c)$, we get

$$\frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0.$$

$$\Rightarrow \quad \frac{c-r}{r-c} + \frac{r}{r-c} + \frac{p}{p-a} + \frac{b-q}{q-b} + \frac{q}{q-b} = 0$$

$$\Rightarrow \quad \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2.$$

Now use

$$\frac{p+a}{p-a} = \frac{2p-(p-a)}{p-a} = \frac{2p}{p-a} - 1 \text{ etc.}$$

57. We can write D as

$$D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_3$, $R_2 \to R_2 - R_1$, we get

$$D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix}$$

$$\Rightarrow \quad \frac{D}{n!(n+1)!(n+2)!} = 2$$

58. Use $R_1 \to R_1 + R_3 - 2R_2$

59. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (3-y)\begin{vmatrix} 1 & 1-y & 1-y \\ 1 & 1+y & 1-y \\ 1 & 1-y & 1+y \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = (3-y)\begin{vmatrix} 1 & 1-y & 1-y \\ 0 & 2y & 0 \\ 0 & -2y & 2y \end{vmatrix} = (3-y)(4y^2)$$

$\therefore \quad \Delta = 0 \Rightarrow y = 0$ or $y = 3$.

60. Write $\Delta = l\Delta_1 + l'\Delta_2$, where

$$\Delta_1 = \begin{vmatrix} a & am+a'm' & an+a'n' \\ b & bm+b'm' & bn+b'n' \\ c & cm+c'm' & cn+c'n' \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} a' & am+a'm' & an+a'n' \\ b' & bm+b'm' & bn+b'n' \\ c' & cm+c'm' & cn+c'n' \end{vmatrix}$$

In Δ_1 apply $C_2 \to C_2 - mC_1$, $C_3 \to C_3 - nC_1$ to obtain

$$\Delta_1 = \begin{vmatrix} a & a'm' & a'n' \\ b & b'm' & b'n' \\ c & c'm' & c'n' \end{vmatrix} = 0$$

Similarly $\Delta_2 = 0$. Thus, $\Delta = 0$

61. Imitate Example 35.

62. We have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ m & m+3 & m+6 \\ \frac{1}{2}m(m-1) & \frac{1}{2}(m+3)(m+2) & \frac{1}{2}(m+6)(m+5) \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ m & 3 & 3 \\ \frac{1}{2}m(m-1) & 3(m+1) & 3(m+4) \end{vmatrix}$$

$= 3^2(m + 4 - m - 1) = 3^3$

$\therefore \quad \alpha + \beta + \gamma = 3$

63. Use $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$ to obtain C_2 and C_3 are proportional.

64. Write $\Delta = \dfrac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ a\cos C & -1 & \cos A \\ a\cos B & \cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 + bC_2 + cC_3$, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} a & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = \sin^2 A$$

65. For $x \neq 0$,

$$\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & 1 & 1 \\ 2\frac{\sin x}{x} & 1 & 2 \\ \tan x & 1 & 1 \end{vmatrix}$$

$$\lim_{x\to 0}\frac{f(x)}{x^2} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

[using $R_1 \to R_1 - R_3$]

66. Applying $C_2 \to C_2 - \omega^2 C_1$ we find the second column of Δ becomes 0.

67. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$$

Write

$$\Delta = pqr\begin{vmatrix} 1+\frac{a}{p} & \frac{b}{q} & \frac{c}{r} \\ \frac{a}{p} & 1+\frac{b}{q} & \frac{c}{r} \\ \frac{a}{p} & \frac{b}{q} & 1+\frac{c}{r} \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = pqr\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)\begin{vmatrix} 1 & \frac{b}{q} & \frac{c}{r} \\ 1 & 1+\frac{b}{q} & \frac{c}{r} \\ 1 & \frac{b}{q} & 1+\frac{c}{r} \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = pqr\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right) \qquad (1)$$

$$\therefore \quad \Delta = 0, pqr \neq 0 \Rightarrow \frac{a}{p}+\frac{b}{q}+\frac{c}{r} = -1.$$

68. The system will have a non-zero solution if

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - \alpha R_1 - R_2$ to obtain

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+2b\alpha+c) \end{vmatrix}$$

$$= -(a\alpha^2 + 2b\alpha + c)(ac - b^2)$$

Note that $\Delta = 0$ if a, b, c are in G.P.

69. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a & a & -1 \\ b & -1 & b \\ -1 & c & c \end{vmatrix} = 0.$$

Using $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} a & 0 & -(1+a) \\ b & -(1+b) & (b+1) \\ -1 & c+1 & 0 \end{vmatrix}$$

$= -a(b + 1)(c + 1) - (1 + a)[b(c + 1) - (1 + b)]$

$$\therefore \quad \Delta = 0 \Rightarrow -\frac{a}{1+a} - \frac{b}{1+b} + \frac{1}{1+c} = 0$$

$$\Rightarrow \quad \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2.$$

70. The system will have a non-trivial solution if $\Delta = 0$ where

$$\Delta = \begin{vmatrix} \lambda+5 & \lambda-4 & 1 \\ \lambda-2 & \lambda+3 & 1 \\ \lambda & \lambda & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - \lambda C_3$, $C_1 \to C_1 - \lambda C_3$, we get

$$\Delta = \begin{vmatrix} 5 & -4 & 1 \\ -2 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 7 \neq 0$$

Thus, the system cannot have a non-trivial solution.

71. Adding above the system of equations, we get $(a + b + c)(x + y + z) = 0 \Rightarrow x + y + z = 0$
$\therefore \quad (b + c)(-x) - ax = b - c$
$\Rightarrow \quad x = -\dfrac{b-c}{a+b+c}$ etc.

72. The above system of equations will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

But $\Delta = -\dfrac{1}{2}(a + b + c)[(b - c)^2 + (c - a)^2 + (a - b)^2]$

$\therefore \quad \Delta = 0, a + b + c \neq 0$
$\Rightarrow \quad a = b = c.$
Thus, $a : b : c = 1 : 1 : 1$

73. $f'''(x) = \begin{vmatrix} 6 & 6 & 0 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}$

$\therefore \quad f'''(x)$ is a constant.

74. Write
$D = \Delta_1 + p\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} b_1 & b_1 + qc_1 & c_1 + ra_1 \\ b_2 & b_2 + qc_2 & c_2 + ra_2 \\ b_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix}$$

Now show that $\Delta_1 = \Delta$ and $\Delta_2 = pqr\,\Delta$.

75. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} \lambda+3 & \lambda+2 & 1 \\ 3 & \lambda+3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0.$$

Using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} \lambda & -1 & 0 \\ 1 & \lambda & 0 \\ 2 & 3 & 1 \end{vmatrix} = \lambda^2 + 1$$

Note that there is no real value of λ for which $\Delta = 0$.

76. Let A be a skew symmetric matrix of order n, then $|A'| = (-1)^n |A| \Rightarrow |A| = (-1)^n |A|$
If n is odd, then $|A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0.$
We have

$$\Delta\left(\frac{\pi}{4}\right) = \begin{vmatrix} 0 & \sin(\pi/12) & \cos(\pi/12) \\ -\sin(\pi/12) & 0 & \sec(\pi/12) \\ -\cos(\pi/12) & -\sec(\pi/12) & 0 \end{vmatrix} = 0$$

[using statement-2]

77. Statement-2 is true, see theory.
The roots of $x^9 = p$ are $p^{1/9}\,\omega^r$
where $\omega = \cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9}.$
value of determinant Δ depends on $\alpha_1, \alpha_2, \ldots \alpha_9$.
If we put $\alpha_k = p^{1/9}\,\omega^k$, then

$$\Delta = (p^{1/9})^3 \begin{vmatrix} 1 & \omega^8 & \omega^7 \\ 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{vmatrix} = 0$$

However, if $\alpha_1 = p^{1/9}$, $\alpha_2 = p^{1/9}\,\omega^8$,
$a_3 = p^{1/9}\,\omega^7$, $\alpha_4 = p^{1/9}\,\omega$, $\alpha_5 = p^{1/9}\,\omega^5$
$\alpha_6 = p^{1/9}\,\omega^4$, $\alpha_7 = p^{1/9}\,\omega^2$, $\alpha_8 = p^{1/9}\,\omega^3$,
$\alpha_9 = p^{1/9}\,\omega^6$, then

$$\Delta = (p^{1/9})^3 \begin{vmatrix} 1 & \omega^8 & \omega^7 \\ \omega & \omega^5 & \omega^4 \\ \omega^2 & \omega^3 & \omega^6 \end{vmatrix}$$

$$= p^{1/3}(2\omega^2 - \omega^7 - \omega^6) \neq 0.$$

78. Statement-2 is true.
Using $R_2 \to R_2 - \sqrt{2}\,R_1$ and $R_3 \to R_3 - \sqrt{3}\,R_1$, we get

$$w = \begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6}i \\ 0 & \sqrt{3} & (\sqrt{6}-2\sqrt{3})i \\ 0 & \sqrt{2} & (2-3\sqrt{2})i \end{vmatrix}$$

$$= \sqrt{6}\left[2\sqrt{3} - 3\sqrt{6} - 2\sqrt{3} + 2\sqrt{6}\right]i = -6i$$

79. We first show statement-2 is true.

$$P(x) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$$\Rightarrow \quad P'(x) = \binom{n}{1} + 2\binom{n}{2}x + \ldots + n\binom{n}{n}x^{n-1}$$

$\Rightarrow \quad P'(0) = \dbinom{n}{1}$ = coefficient of x in the expansion of $P(x)$

$\therefore \quad$ Statement-2 is true.
Note that $\Delta(x)$ consists of 6 terms of the form $(1 + x)^n$.
Thus, coefficients of x in $\Delta(x) = \Delta'(0)$

$$\text{But } \Delta'(0) = \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix} = 0$$

80. Statement-2 is not always true. For instance, the system of equations

$$x + 2y + 3z = 1$$
$$2x + 3y + 4z = 2$$
$$3x + 4y + 5z = 4$$

has no solution but $|A| = 0$

For statement-1, let

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we obtain

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

Also, note that $x = y = z$ satisfies each of the three equations.

Thus, the system of equations has infinite number of solutions.

81. Let

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= -\frac{1}{2}(a+b+c)\left[(b-c)^2+(c-a)^2+(a-b)^2\right]$$

If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = bc + ca + ab$, then $(b - c)^2 + (c - a)^2 + (a - b)^2 = 0$

$\Rightarrow \quad a = b = c \neq 0 \qquad [\because a + b + c \neq 0]$

Thus, the system of equation reduces to $x + y + z = 0$ which is satisfied by infinite number of solutions.

$\therefore$ Statement-1 is true.

If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, then at least one following is true:

$a^2 \neq bc, \quad b^2 \neq ca, \quad c^2 \neq ab.$

Suppose $b^2 - ac \neq 0$

Write first two equations as:

$$ax + by = (a + b)z$$
$$bx + cy = (b + c)z$$

Eliminating y, we get

$$(ac - b^2)x = (ac - b^2)z$$

$\Rightarrow \quad x = z$

Thus, from (1) equation, we get

$$by = bz \quad \Rightarrow \quad y = z$$

$\therefore \quad x = y = z$

Therefore, statement-2 is false.

82. If (a_1, b_1), (a_2, b_2) (a_3, b_3) are non-collinear, then $\Delta_1 \neq 0$. Therefore statement-2 is true.

We can write Δ_2 as

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

for all values of $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$.

83. As $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$$= (a + ib)(a - ib) + (c - id)(c + id)$$
$$= a^2 + b^2 + c^2 + d^2$$

we get statement-2 is true.

Now,

$$\left(a_1^2+b_1^2+c_1^2+d_1^2\right)\left(a_2^2+b_2^2+c_2^2+d_2^2\right)$$

$$= \begin{vmatrix} a_1+ib_1 & c_1+id_1 \\ -c_1+id_1 & a_1-ib_1 \end{vmatrix}\begin{vmatrix} a_2+ib_2 & c_2+id_2 \\ -c_2+id_2 & a_2-ib_2 \end{vmatrix}$$

$$= \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = a^2+b^2+c^2+d^2$$

where $a = a_1a_2 - b_1b_2 + c_1c_2 - d_1d_2$,

$b = a_2b_1 + a_1b_2 + c_1d_2 + c_2d_1$,

$c = -a_2c_1 + b_2d_1 - b_2c_1 + a_2d_1$,

$d = -b_2c_1 + a_2d_1 - b_1c_2 - b_1d_2$

Level 2

84. We have

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_{21} & b_{31} \\ b_{12} & a_2 & b_{32} \\ b_{13} & b_{23} & a_3 \end{vmatrix}$$

where $a_k = l_k^2 + m_k^2 + n_k^2 = 1$ for $k = 1, 2, 3$

and $b_{ij} = l_i l_j + m_i m_j + n_i n_j = 0 \ \forall \ i \neq j$

Thus, $\Delta^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow |\Delta| = 1.$

85. Using $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$

$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

using $C_3 \to C_3 + C_1$, we get C_2 and C_3 are propotional. Thus, $\Delta = 0$

86. Using $C_1 \to C_1 - (x^2/3)C_3$, $C_2 \to C_2 - (2x/3)\ C_3$, we get

$$\Delta(x) = \begin{vmatrix} -5x+3 & -5 & 3 \\ x+9 & 1 & 9 \\ -6x+9 & -6 & 21 \end{vmatrix}$$

Using $C_1 \to C_1 - xC_2$, we get

$$\Delta(x) = \begin{vmatrix} 3 & -5 & 3 \\ 9 & 1 & 9 \\ 9 & -6 & 21 \end{vmatrix} = a \text{ constant.}$$

Thus, $a = 0$.

Alternate Solution

Replacing x by $1/x$, we get

$$\begin{vmatrix} \frac{1}{x^2} & -5\left(\frac{1}{x}\right)+3 & 2\left(\frac{1}{x}\right)-5 & 3 \\ 3\left(\frac{1}{x^2}\right) & +\frac{1}{x}+9 & 6\left(\frac{1}{x}\right)+1 & 9 \\ 7\left(\frac{1}{x^2}\right) & -6\left(\frac{1}{x}\right)+9 & 14\left(\frac{1}{x}\right)-6 & 21 \end{vmatrix}$$

$$= a\left(\frac{1}{x^3}\right)+b\left(\frac{1}{x^2}\right)+c\left(\frac{1}{x}\right)+d$$

$$\Rightarrow \begin{vmatrix} 1-5x+3x^2 & 2-5x & 3 \\ 3+x+9x^2 & 6+x & 9 \\ 7-6x+9x^2 & 14-6x & 21 \end{vmatrix} = a + bx + cx^2 + dx^3$$

Putting $x = 0$, we get

$$a = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 7 & 14 & 21 \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are proportional]

87. The given system of equations will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a+t & b & c \\ a & b+t & c \\ a & b & c+t \end{vmatrix} = 0$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (a + b + c + t) \begin{vmatrix} 1 & b & c \\ 1 & b+t & c \\ 1 & b & c+t \end{vmatrix} = 0$$

Using $C_2 \to C_2 - bC_1$, $C_3 \to C_3 - cC_1$, we get

$\Delta = (a + b + c + t)t^2 = 0 \Rightarrow t = 0, -(a + b + c)$

Thus, there are just two distinct values of t.

88. First multiply C_1 by a, C_2 by b, C_3 by c, followed by multiplying R_1 by $1/a$, R_2 by $1/b$ and R_3 by $1/c$, we get

$$\Delta = \begin{vmatrix} a^2+(b^2+c^2)\cos\theta & b^2(1-\cos\theta) & c^2(1-\cos\theta) \\ a^2(1-\cos\theta) & b^2+(c^2+a^2)\cos\theta & c^2(1-\cos\theta) \\ a^2(1-\cos\theta) & b^2(1-\cos\theta) & c^2+(a^2+b^2)\cos\theta \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$ and $a^2 + b^2 + c^2 = 1$, we get

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\theta) & c^2(1-\cos\theta) \\ 1 & b^2+(1-b^2)\cos\theta & c^2(1-\cos\theta) \\ 1 & b^2(1-\cos\theta) & c^2+(1-c^2)\cos\theta \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\theta) & c^2(1-\cos\theta) \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix} = \cos^2\theta$$

89. We have

$$\frac{dA}{d\theta} = \begin{vmatrix} -\sin(\alpha+\theta) & \sin(\alpha+\theta) & 1 \\ -\sin(\beta+\theta) & \sin(\beta+\theta) & 1 \\ -\sin(\gamma+\theta) & \sin(\gamma+\theta) & 1 \end{vmatrix}$$

$$+\begin{vmatrix} \cos(\alpha+\theta) & \cos(\alpha+\theta) & 1 \\ \cos(\beta+\theta) & \cos(\beta+\theta) & 1 \\ \cos(\gamma+\theta) & \cos(\gamma+\theta) & 1 \end{vmatrix}$$

$= 0 + 0 = 0$

$\Rightarrow A$ is independent θ. Thus, $A(\alpha, \beta, \gamma, \theta) = A(\alpha, \beta, \gamma, 0)$

$$\Rightarrow A\left(-\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{2\pi}{13}\right) = A\left(-\frac{\pi}{2}, 0, \frac{\pi}{2}, 0\right)$$

$$= \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2$$

90. We have $\Delta = (a - b)(b - c)(c - a) = -2$.

As $a > b > c$, $a - b$, $b - c$ are positive integers and $c - a$ is a negative integers.

Only possibilities are

$a - b = 2,\ b - c = 1,\ c - a = -1$ (1)

or $a - b = 1,\ b - c = 2,\ c - a = -1$ (2)

or $a - b = 1,\ b - c = 1,\ c - a = -2$ (3)

(1) and (2) lead us to $0 = 2$.

$\therefore\ a - b = 1,\ b - c = 1,\ c - a = -2$.

Now, $3a + 7b - 10c = 3(a - c) + 7(b - c) = 13$

91. Write

$$\Delta = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \begin{vmatrix} \cos P & -\sin P & 0 \\ \cos Q & -\sin Q & 0 \\ \cos R & -\sin R & 0 \end{vmatrix}$$

$= 0$

Alternate Solution

Write $\Delta = \cos P\ \Delta_1 - \sin P\ \Delta_2$

$$\text{where } \Delta_1 = \begin{vmatrix} \cos A & \cos(A+Q) & \cos(A+R) \\ \cos B & \cos(B+Q) & \cos(B+R) \\ \cos C & \cos(C+Q) & \cos(C+R) \end{vmatrix}$$

$$\text{and } \Delta_2 = \begin{vmatrix} \sin A & \cos(A+Q) & \cos(A+R) \\ \sin B & \cos(B+Q) & \cos(B+R) \\ \sin C & \cos(C+Q) & \cos(C+R) \end{vmatrix}$$

In Δ_1, use $C_2 \to C_2 - \cos Q\ C_1$

$C_3 \to C_3 - \cos R\ C_1$

$$\Delta_1 = \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix}$$

$= 0$ [$\because$ C_2 and C_3 are propotional]

Similarly, $\Delta_2 = 0$

Thus, $\Delta = 0$

92. Expanding along C_1, we get

$$f(x) = 2\cos x \begin{vmatrix} 2\cos x & 1 \\ 1 & 2\cos x \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 2\cos x \end{vmatrix}$$

$= 8\cos^3 x - 4\cos x = 4\cos x \cos 2x$

$$\Rightarrow f\left(\frac{\pi}{3}\right) = -1,\ f\left(\frac{\pi}{2}\right) = 0.$$

$f'(x) = -4[\sin x \cos 2x + 2\cos x \sin 2x]$

$= -4[\sin 3x + \cos x \sin 2x]$

$$\Rightarrow f'\left(\frac{\pi}{3}\right) = -4\left[\sin \pi + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\right]$$

$= -\sqrt{3}$

93. For each $\Delta \in B$, there exist $\Delta_1 \in C$ where Δ_1 is obtained by interchanging 1st and 2nd row of Δ, similarly, for each $\Delta \in C$ there exists $\Delta_1 \in B$.

$\therefore$ B and C has same number of elements.

94. $|a|^2 = |x|^2 + |y|^2 + |z|^2 + \bar{x}y + x\bar{y} + x\bar{z} + \bar{x}z + y\bar{z} + \bar{y}z$

$\therefore\ |a|^2 + |b|^2 + |c|^2 = 3(|x|^2 + |y|^2 + |z|^2) +$

$(\bar{x}y + x\bar{y} + x\bar{z} + \bar{x}z + y\bar{z} + \bar{y}z)(1 + \omega + \omega^2)$

$$\Rightarrow \frac{|a|^2 + |b|^2 + |c|^2}{|x|^2 + |y|^2 + |z|^2} = 3$$

95. Write the determinant as $= abc\ \Delta_1 - \Delta_2$

$$\text{where } \Delta_1 = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

$$\text{and } \Delta_2 = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \Delta_1$$

$\therefore\ (a - b)(b - c)(c - a)(abc - 1) = 0$

Since, a, b, c are distinct, we get

$abc - 1 = 0$ or $abc = 1$

96. $$|P|^2 = |\text{adj } P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 4$$

$\Rightarrow |P| = \pm 2$

97. Using $R_2 \to R_2 + 5R_1$, $R_3 \to R_3 - 4R_1$, we get

$$\begin{vmatrix} 1 & -3 & 4 \\ 0 & x-13 & 22 \\ 0 & 13 & x-22 \end{vmatrix} = 0$$

$\Rightarrow (x - 13)(x - 22) - (13)(22) = 0$

$\Rightarrow x(x - 35) = 0 \Rightarrow x = 0, 35$

Previous Years' AIEEE/JEE Main Questions

1. As $ax^2 + 2bx + c = 0$ has equal roots,

$b^2 - ac = 0$

Using $R_3 \to R_3 - xR_1 - R_2$, we get

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ 0 & 0 & \alpha \end{vmatrix}$$

where $\alpha = 0 - x(ax + b) - (bx + c)$

$= -(ax^2 + 2bx + c)$

$\therefore \quad \Delta = \alpha\,(ac - b^2) = 0$

2. As l, m, n are pth, qth, rth terms of a G.P., log l, log m, log n are pth, qth, rth terms of an A.P. Let first term of the A.P. be a and its common difference be d.

Now,

$$\Delta = \begin{vmatrix} a+(p-1)d & p & 1 \\ a+(q-1)d & q & 1 \\ a+(r-1)d & r & 1 \end{vmatrix}$$

Using $C_1 \to C_1 - dC_2 - (a - d)C_3$, we get

$$\Delta = \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = 0$$

3. If n is a multiple of 3, we get each element of Δ becomes 1.

$\therefore \quad \Delta = 0$

If $n = 3k + 1$, then

$$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

If $n = 3k + 2$, then

$$\Delta = \begin{vmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

4. As the system has a non-zero solution

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$$\begin{vmatrix} 1/a & 2 & 1 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

Using $R_1 \to R_1 - 2R_2 + R_3$, we get

$$\Rightarrow \begin{vmatrix} 1/a - 2/b + 1/c & 0 & 0 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

$$\frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

$\Rightarrow \quad a, c, b$ are in H.P.

5. Let R be the common ratio of the G.P. Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get that the given determinant

$$= \begin{vmatrix} \log a_n & \log R & \log R \\ \log a_{n+1} & \log R & \log R \\ \log a_{n+2} & \log R & \log R \end{vmatrix} = 0$$

6. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & 1+b^2x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix}$$

$= (1 - x)^2$

which is a polynomial of degree 2.

7. If A denotes the coefficient matrix, then

$$|A| = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

$$= (\alpha+2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-1 \end{vmatrix}$$

$= (\alpha + 2)(\alpha - 1)^2$

If $|A| \neq 0$, the system has a unique solution.

If $|A| = 0$, then $\alpha = -2$ or $\alpha = 1$. For $\alpha = 1$, the system of equations becomes $x + y + z = 0$ which has infinite number of solutions.

For $\alpha = -2$, the system of equation becomes

$$-2x + y + z = -3$$
$$x - 2y + z = -3$$

$$x + y - 2z = -3$$

which on adding becomes $0 = -9$

Thus, the system has no solution if $\alpha = -2$.

8. $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix} = xy$$

which is divisible by both x and y.

9. Let $\Delta_2 = \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix}$

Using the reflection property, we get

$$\Delta_2 = \begin{vmatrix} a+1 & a-1 & (-1)^{n+2}a \\ b+1 & b-1 & (-1)^{n+1}b \\ c-1 & c+1 & (-1)^n c \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^n c & c-1 & c+1 \end{vmatrix}$$

Thus,

$$\Delta = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^n c & c-1 & c+1 \end{vmatrix}$$

$$= \begin{vmatrix} a+(-1)^{n+2}a & a+1 & a-1 \\ -b+(-1)^{n+1}b & b+1 & b-1 \\ c+(-1)^n c & c-1 & c+1 \end{vmatrix}$$

The first column consists of all 0's if n is any odd integer.

$\therefore \quad \Delta = 0$ if n is any odd integer.

10. Using $C_1 \to C_1 + C_2 + C_3$, we get

$\Delta = (a + b + c)\,\Delta_1$

where

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$= -(b - c)^2 - (a - b)(a - c)$

$= -(a^2 + b^2 + c^2 - bc - ca - ab)$

$\Rightarrow \quad \Delta_1 = -\frac{1}{2}[(b-c)^2 + (c-a)^2 + (a-b)^2] < 0$

As $a + b + c > 0$, we get

$\Delta = (a + b + c)\,\Delta_1 < 0$

11. Using $C_1 \to C_1 - C_3$ in Δ, we get

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ 0 & \cos\alpha & \sin\alpha \\ 2\cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix}$$

$= 2\cos\alpha\,(\sin^2\alpha - \cos^2\alpha)$

$= -2\cos\alpha\cos 2\alpha$

$\Delta = 0$ for $\alpha = \pi/4 \in (0, \pi/2)$.

This is the only value of α lying in $(0, \pi/2)$ for which $\Delta = 0$.

The system of linear equation will have a non-trivial solution if and only if

$$\Delta_1 = \begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Using $R_2 \to R_2 + R_1$, we get

$$\Delta_1 = \begin{vmatrix} 2 & 0 & 0 \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-\cos^2\alpha + \sin^2\alpha) = 0$

$\Rightarrow \quad -2\cos 2\alpha = 0$

This is true for only one value of $\alpha \in (0, \pi/2)$ viz, $\alpha = \pi/4$.

Thus, statement-1 is also true. However statement-2 is not a correct reason for statement-1.

12. $\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1^2$$

where $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha - 1 & \beta - \alpha \\ 1 & \alpha^2 - 1 & \beta^2 - \alpha^2 \end{vmatrix}$$

$$= (\alpha - 1)(\beta - \alpha) \begin{vmatrix} 1 & 1 \\ \alpha + 1 & \beta + \alpha \end{vmatrix}$$

$$= (\alpha - 1)\ (\beta - 1)\ (\beta - \alpha)$$

Thus, $\Delta = (\alpha - 1)^2\ (\beta - 1)^2 (\alpha - \beta)^2$

$\therefore$ $K = 1$

13. As the given system of equations has a non-trivial solutions.

$$\Delta = \begin{vmatrix} a-1 & -1 & -1 \\ -1 & b-1 & -1 \\ -1 & -1 & c-1 \end{vmatrix} = 0$$

Write

$$\Delta = \begin{vmatrix} a & -1 & -1 \\ 0 & b-1 & -1 \\ 0 & -1 & c-1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & -1 \\ 1 & b-1 & -1 \\ 1 & -1 & c-1 \end{vmatrix}$$

$$= a[(b-1)(c-1)-1] - \begin{vmatrix} 1 & 0 & 0 \\ 1 & b & 0 \\ 1 & 0 & c \end{vmatrix}$$

[use $C_2 \to C_2 + C_1$, C_3 — $C_3 + C_1$]

$= a\ (bc - b - c) - bc$

As $\Delta = 0$, we get

$ab + bc + ca = abc$

14. We have $|B(x)| = |(A(x))^T A(x)|$

$= |A(x)^T|\ |A(x)|$

$= |A(x)|\ |A(x)| = |A(x)|^2$

Let $p_i(x) = a_i x^2 + b_i x + c_i$ $(i = 1, 2, 3)$

Applying $C_1 \to C_1 - \frac{1}{2}x^2 C_3, C_2 \to C_2 - xC_3$.

we get

$$|A(x)| = 2 \begin{vmatrix} b_1 x + c_1 & b_1 & a_1 \\ b_2 x + c_2 & b_2 & a_2 \\ b_3 + c_3 & b_3 & a_3 \end{vmatrix}$$

Applying $C_1 \to C_1 - xC_2$, we get

$$|A(x)| = 2 \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

$\therefore$ $|B(x)|$ is independent of x.

15. Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a\lambda + \lambda^2 & 2b\lambda + \lambda^2 & 2c\lambda + \lambda^2 \\ -4a\lambda & -4b\lambda & -4c\lambda \end{vmatrix}$$

Take -4λ common from R_3, and applying $R_2 \to R_2 - 2\lambda R_3$, we get

$$\Delta = -4\lambda^3 \begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$= \lambda(4\lambda^2) \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore$ $k = 4\lambda^2$

16. As $\sum_{r=1}^{n-1} r = \frac{1}{2}n(n-1), \sum_{r=1}^{n-1}(2r-1) = (n-1)^2$

and $\sum_{r=1}^{n-1}(3r-2) = \frac{3}{2}n(n-1) - 2(n-1)$

$$= \frac{1}{2}(n-1)(3n-4)$$

Thus,

$$\sum_{r=1}^{n-1} \Delta_r = \begin{vmatrix} n(n-1)/2 & (n-1)^2 & (n-1)(3n-4)/2 \\ n/2 & n-1 & a \\ n(n-1)/2 & (n-1)^2 & (n-1)(3n+4)/2 \end{vmatrix}$$

Using $R_3 \to R_3 - R_1$, we get

$$\sum_{r=1}^{n-1} \Delta_r = \begin{vmatrix} n(n-1)/2 & (n-1)^2 & (n-1)(3n-4)/2 \\ n/2 & n-1 & a \\ 0 & 0 & 4(n-1) \end{vmatrix}$$

Expanding along R_3, we get

$$\sum_{r=1}^{n-1} \Delta_r = 4(n-1)\begin{vmatrix} n(n-1)/2 & (n-1)^2 \\ n/2 & n-1 \end{vmatrix}$$

$$= 4(n-1)\left(\frac{n}{2}\right)(n-1)\begin{vmatrix} n-1 & n-1 \\ 1 & 1 \end{vmatrix} = 0$$

17. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

Using $R_1 \to R_1 + R_3$, $R_2 \to R_2 + 2R_3$,

$$\Delta = \begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 0 & 1-\lambda & 2-2\lambda \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

Using $C_3 \to C_3 - C_1$, we get

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & -\lambda+1 \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)^2 [- \lambda + 1 - 4] = 0 \Rightarrow \lambda = 1, - 3$

Thus, the set contains two elements

18. Using $R_2 \to R_2 - R_1 - R_3$, we get

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Expanding along R_2, we get

$$\Delta = -(-4)\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, we get

$$\Delta = 4\begin{vmatrix} x+1 & -3 \\ 2x-1 & 0 \end{vmatrix}$$

$= 4(3)\ (2x - 1)$

$= 24x - 12$

$\therefore\ a = 24$

TIP : Put $x = 1$ to obtain

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 3 & 0 \\ 6 & 1 & 1 \end{vmatrix} = a - 12$$

$\Rightarrow 12 = a - 12$

$\Rightarrow a = 24$

19. Let x be any real number and $y = 1$, $z = 1$, then

$$\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \geq 0$$

and $xyz = x$ can take any real value, thus, minimum value of xyz does not exist.

20. The given system of equations has a non-trivial solution if

$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda - \lambda - \lambda - 1 + 1 + \lambda^3 = 0$

$\Rightarrow \lambda\ (\lambda - 1)\ (\lambda + 1) = 0$

$\Rightarrow \lambda = 0, -1, 1$

When $\lambda = 0$, $x = -y = z = k \neq 0$

When $\lambda = -1$, $x = 0$, $y = -z = k \neq 0$

When $\lambda = 1$, $y = 0$, $x = -z = k \neq 0$

Thus, there are three values of λ, for which the system of equations has a non-trivial solutions.

21. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (2 \sin x + \cos x) \begin{vmatrix} 1 & \sin x & \sin x \\ 1 & \cos x & \sin x \\ 1 & \sin x & \cos x \end{vmatrix}$$

Using $C_2 \to C_2 - (\sin x)\ C_1$

and $C_3 \to C_3 - (\sin x)\ C_1$,

we get

$$\Delta = (2 \sin x + \cos x) \begin{vmatrix} 1 & 0 & 0 \\ 1 & \cos x - \sin x & 0 \\ 1 & 0 & \cos x - \sin x \end{vmatrix}$$

$= (2 \sin x + \cos x)\ (\cos x - \sin x)^2$

$\therefore\ \Delta = 0 \Rightarrow \tan x = -\dfrac{1}{2}, 1$

As $-\pi/4 \leq x \leq \pi/4$, $-1 \leq \tan x \leq 1$

and $\tan x$ is one-to-one in the interval $[-\pi/4, \pi/4]$.

Thus, there are two values of x.

Previous Years' B-Architecture Entrance Examination Questions

1. From first and third equation

$$(b - 1)x = 0$$

If $b \neq 1$, $x = 0$.

In this case $y + z = 0$, $by + z = 0$

$\Rightarrow \quad (b - 1)y = 0 \quad \Rightarrow y = 0$

Thus, $z = 0$

That is, if $b \neq 1$, then the system has only trivial solution.

$\therefore \quad b = 1$

From first and second equation,

$(a - 1)\, x = 0.$

If $a \neq 1$, $x = 0$ and $y + z = 0$

In this case the system has infinite number of solution.

If $a = 1$, all the three equations become identical to $x + y + z = 0$ and the system has infinite number of solutions.

$b = 1 \Rightarrow (b - 1)^2 = 0 \quad \Rightarrow b^2 = 2b - 1$

2. As the system has a non-trivial solution,

$$\Delta = \begin{vmatrix} \lambda+3 & \lambda+2 & 1 \\ 3 & \lambda+3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

Using $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} \lambda+1 & \lambda-1 & 0 \\ 1 & \lambda & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$\Rightarrow \Delta = \lambda(\lambda + 1) - (\lambda - 1) = 0$

$\Rightarrow (\lambda - 1)^2 = 0 \quad \Rightarrow \lambda = 1$

For $\lambda = 1$, the system becomes

$$4x + 3y + z = 0 \qquad \text{(i)}$$

$$3x + 4y + z = 0 \qquad \text{(ii)}$$

$$2x + 3y + z = 0 \qquad \text{(iii)}$$

From (i) and (iii) $x = 0$

From (i) and (ii) $3y + z = 0$

and $\quad 4y + z = 0$

$\Rightarrow \quad y = 0$, $z = 0$

Thus, the system has non-trivial solution for no value of λ.

3. $$\sum_{k=1}^{2} \Delta_k = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ 3^n - 1 & 4^n - 1 & 5^n - 1 \end{vmatrix}$$

where $a = \sum_{k=1}^{n} 2(3^{k-1})$

$$= \frac{2(3^n - 1)}{3 - 1} = 3^n - 1,$$

$$b = \sum_{k=1}^{n} 3(4^{k-1}) = \frac{3(4^n - 1)}{4 - 1} = 4^n - 1$$

and $c = \sum_{k=1}^{n} 4(5^{k-1}) = \frac{4(5^n - 1)}{5 - 1} = 5^n - 1$

Thus, $$\sum_{k=1}^{n} \Delta_k = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix} = 0$$

4. $$\det(B) = \begin{vmatrix} a_{11} & 2^{-1}a_{12} & 2^{-2}a_{13} \\ 2a_{21} & a_{22} & 2^{-1}a_{23} \\ 2^2 a_{31} & 2a_{32} & a_{33} \end{vmatrix}$$

$$= (2^{-2})(2^{-1}) \begin{vmatrix} 2^2 a_{11} & 2a_{12} & a_{13} \\ 2^2 a_{21} & 2a_{22} & a_{23} \\ 2^2 a_{31} & 2a_{32} & a_{33} \end{vmatrix}$$

$$= (2^{-3})(2^2)(2) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= \det(A) = 5$

5. $$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

where $\qquad a = \sum_{r=1}^{n} 2^{r-1} = 2^n - 1$

$$b = \sum_{r=1}^{n} 2(3^{r-1}) = \frac{2(3^n - 1)}{3 - 1}$$

$= 3^n - 1$

and $\qquad c = \sum_{r=1}^{n} 4(5^{r-1}) = \frac{4(5^n - 1)}{5 - 1}$

$= 5^n - 1$

Thus, $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix} = 0$

6. Taking e^{x_1} common from R_1, e^{x_4} common from R_2 and e^{x_7} from R_3, we get

   $\Delta = e^{x_1 + x_4 + x_7}\Delta_1$

   where

   $$\Delta_1 = \begin{vmatrix} 1 & e^{3d} & e^{6d} \\ 1 & e^{3d} & e^{6d} \\ 1 & e^{3d} & e^{6d} \end{vmatrix} = 0$$

   where d is common difference of A.P.

   $\therefore\ \Delta = 0$

7. Taking $\sqrt{5}$ common from C_2 and C_3, we get

   $\Delta = 5\Delta_1$

   where $\Delta_1 = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+\sqrt{26} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix}$

   Applying, $C_1 \to C_1 - \sqrt{3}C_2 - \sqrt{13}C_3$, we get

   $$\Delta_1 = \begin{vmatrix} -\sqrt{3} & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

   $= -\sqrt{3}(5-\sqrt{6})$

   Thus, $\Delta = 5\sqrt{3}(\sqrt{6}-5)$

8. As the given system has a non-zero solution

   $$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

   $$\Rightarrow abc\begin{vmatrix} 1/a & 2 & 1 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

   Using $R_1 \to R_1 - 2R_2 + R_3$, we get

   $$abc\begin{vmatrix} \frac{1}{a}-\frac{2}{b}+\frac{1}{c} & 0 & 0 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

   $\Rightarrow\ abc\left(\frac{1}{a}-\frac{2}{b}+\frac{1}{c}\right)(-1) = 0$

   $\Rightarrow\ 2ac = ab + bc$

9. Using $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$, we get

   $$\begin{vmatrix} 0 & a-c & b-a \\ 1 & c & a \\ 0 & b-c & c-a \end{vmatrix} = 0$$

   $\Rightarrow\ (a-c)^2 + (b-a)(b-c) = 0$

   $\Rightarrow\ a^2 - 2ac + c^2 + b^2 - ab - bc + ac = 0$

   $\Rightarrow\ a^2 + b^2 + c^2 - bc - ca - ab = 0$

   $\Rightarrow\ (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

   $\Rightarrow\ a = b = c \quad \Rightarrow A = B = C$

   $\Rightarrow\ \sin A = \sin B = \sin C = \sqrt{3}/2$

   Thus, $\sin^2 A + \sin^2 B + \sin^2 C = 9/4$

10. From the first two equations $x = 2$.

    $\therefore\ y - z = 1$ and $-4y + 4z = \alpha - 2$

    $\Rightarrow\ 4y - 4z + (-4y + 4z) = 4 + (\alpha - 2)$

    $\Rightarrow\ 0 = \alpha + 2$

    Thus, the system has no solution for $\alpha \neq -2$.

    If $\alpha = -2$, the system is consistent and has infinite number of solutions, since $y - z = 1$ has infinite number of solutions.

11. Using $C_1 \to C_1 + C_3$, we get

    $$\Delta = \begin{vmatrix} 1 & \tan\theta + \sec^2\theta & 3 \\ 0 & \cos\theta & \sin\theta \\ 0 & -4 & 3 \end{vmatrix}$$

    $= 3\cos\theta + 4\sin\theta$

    $\frac{d\Delta}{d\theta} = -3\cos\theta + 4\sin\theta$

    $\frac{d\Delta}{d\theta} = 0 \Rightarrow \frac{\sin\theta}{4} = \frac{\cos\theta}{3} = \frac{1}{5}$

    $\Rightarrow \sin\theta = 4/5,\ \cos\theta = 3/5$

    Max $\Delta = \max\ \{\Delta(0), \Delta(\sin^{-1}(4/5)), \Delta(1)\}$

    $= \max\left\{3, \frac{24}{5}, 4\right\} = \frac{24}{5}$

    min $\Delta = 3$.

    $\therefore\ \Delta \in [3,\ 5]$