3. SEQUENCES AND SERIES

1. SEQUENCE

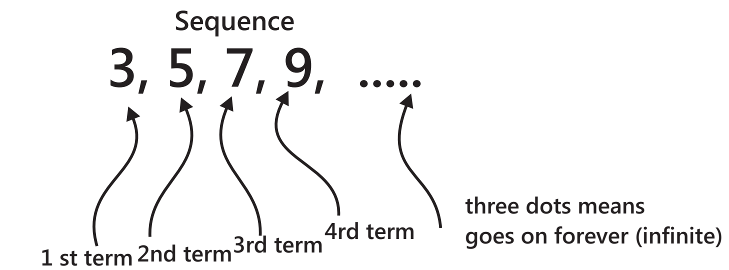
1.1 Introduction

A sequence can be defined as an ordered collection of things (usually numbers) or a set of numbers arranged one after another. Sometimes, sequence is also referred as progression. The numbers a_1, a_2, a_3 .....a_n are known as terms or elements of the sequence. The subscript is the set of positive integers 1, 2, 3..... that indicates the position of the term in the sequence. T_n is used to denote the n^{th} term.

Some examples of a sequence are as follows:

0, 7, 26......................., 1, 4, 7, 10......................., 2, 4, 6, 8.......................

Note: The minimum number terms in a sequence should be 3.



("term"", element" or ‘‘ member"" mean the same thing)

Figure 3.1

1.2 Finite and Infinite Sequences

A sequence containing a finite number of terms is called a finite sequence. If the sequence contains a infinite number of terms, it is known as an infinite sequence. It is infinite in the sense that it never ends. Examples of infinite and finite sequences are as follows:

{1, 2, 3, 4......} is an infinite sequence

{20, 25, 30, 35....} is an infinite sequence

{1, 3, 5, 7} is the sequence of the first 4 odd numbers, which is a finite sequence

1.3 Rule

A sequence usually has a rule, on the basis of which the terms in the sequence are built up. With the help of this rule, we can find any term involved in the sequence. For example, the sequence {3, 5, 7, 9} starts at the number 3 and jumps 2 every time.

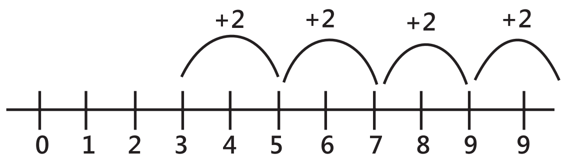


Figure 3.2

As a Formula:

Saying 'start at the number 3 and jump 2 every time' is fine, but it does not help to calculate the 10^{th} term or 100^{th} term or n^{th} term. Hence, we want a formula for the sequence with "n" in it (where n is any term number). What would the rule for {3, 5, 7, 9.......} be? First, we can see the sequence goes up 2 every time; hence, we can guess that the rule will be something like '2 times n' (where 'n' is the term number). Let us test it out.

n	**Test Rule**	**Term**
1	$2n = 2 \times 1 = 2$	3
2	$2n = 2 \times 2 = 4$	5
3	$2n = 2 \times 3 = 6$	7

That nearly worked! But it is less by 1 every time. Let us try changing it to 2n+1.

n	**Test Rule**	**Term**
1	$2n + 1 = 2 \times 1 + 1 = 3$	3
2	$2n + 1 = 2 \times 2 + 1 = 5$	5
3	$2n + 1 = 2 \times 3 + 1 = 7$	7

That Works: Therefore, instead of saying 'starts at the number 3 and jumps 2 every time,' we write the expression 2n + 1. We can now calculate, e.g. the 100^{th} term as $2 \times 100 + 1 = 201$.

1.4 Notation

The notation T_n is used to represent the general term of the sequence. Here, the position of the term in the sequence is represented by n. To mention for the '$5'^{th}$ term, just write T_5.

Thus, the rule for {3, 5, 7, 9...} can be written as the following equation: $T_n = 2n + 1$.

To calculate the 10^{th} term, we can write $T_{10} = 2n + 1 = 2 \times 10 + 1 = 21$

Illustration 1: Find out the first 4 terms of the sequence, $\{T_n\} = \{-1/n\}^n$. **(JEE MAIN)**

Sol: By substituting n = 1, 2, 3 and 4 in$\{T_n\} = \{-1/n\}^n$, we will get the first 4 terms of given sequence.

$T_1 = (-1/1)^1 = -1$

$T_2 = (-1/2)^2 = 1/4$

$T_3 = (-1/3)^3 = -1/27$

$T_4 = (-1/4)^4 = 1/256$

$\Rightarrow \quad \{T_n\} = \{-1, ¼, -1/27, 1/256 ...\}$

Illustration 2: Write the sequence whose n^{th} term is (i) 2^n and (ii) log(nx). **(JEE MAIN)**

Sol: By substituting n = 1, 2, 3........, we will get the sequence.

(i) n^{th} term = 2^n

$a_1 = 2^1, a_2 = 2^2$.......

Sequence $\Rightarrow$ $2^1, 2^2, \ldots\ldots 2^n$

(ii) n^{th} term = $a_n = \log(nx)$

$a_1 = \log(x)$

$a_2 = \log(2x)$, $a_n = \log(nx)$

Sequence $\Rightarrow$ log(x), log(2x),................log(nx)

2. SERIES

Series is something that we get from a given sequence by adding all the terms. If we have a sequence as $T_1, T_2, \ldots, T_n$, then the series that we get from this sequence is $T_1 + T_2 + \ldots + T_n$. S_n is used to represent the sum of n terms. Hence, $S_n = T_1 + T_2 + \ldots + T_n$

3. SIGMA AND PI NOTATIONS

3.1 Sigma Notation

The meaning of the symbol Σ (sigma) is summation. To find the sum of any sequence, the symbol Σ (sigma) is used before its n^{th} term. For example:

(i) $\sum_{n=1}^{9} n = 1+2+3+\ldots\ldots+9$

(ii) $\sum_{r=1}^{n} r^a$ or $\sum n^a = 1^a+2^a+3^a+\ldots\ldots+n^a$

(iii) $\sum_{i=1}^{5} \frac{i+1}{2i+4} = \frac{1+1}{2\times1+4} + \frac{2+1}{2\times2+4} + \frac{3+1}{2\times3+4} + \frac{4+1}{2\times4+4} + \frac{5+1}{2\times5+4}$

Properties of Σ (Sigma)

(i) $\sum_{i=1}^{k} a = a + a + a\ldots$ (k times) = ka, where a is a constant.

(ii) $\sum_{i=1}^{k} ai = a\sum_{i=1}^{k} i$, where a is a constant.

(iii) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(iv) $\sum_{i=i_0}^{i_n} \sum_{j=j_0}^{j_n} a_i a_j = \sum_{j=j_0}^{j_n} \sum_{i=i_0}^{i_n} a_i a_j$

3.2 Pi Notation

The symbol ∏ denotes the product of similar terms. For example:

(i) $\prod_{n=1}^{6} n = 1 \times 2 \times 3 \times 4 \times 5 \times 6$

(ii) $\prod_{n=1}^{k} n^m = 1^m \times 2^m \times 3^m \times 4^m \times \ldots\ldots\ldots\ldots \times k^m$

(iii) $\prod_{n=1}^{k} n = 1 \times 2 \times 3 \times \ldots\ldots\ldots \times k = k!$

4. ARITHMETIC PROGRESSION

The sequence in which the successive terms maintain a constant difference is known as an arithmetic progression (AP). Consider the following sequences:

$$\begin{array}{cccc} a, & a+d, & a+2d, & a+3d \\ T_1, & T_2, & T_3, & T_4 \end{array}$$

$\because T_2 - T_1 = T_3 - T_2 = T_4 - T_3$ = constant (common difference)

The given sequence is an example of AP. The set of natural numbers is also an example of AP.

4.1 General Term

General term (n^th term) of an AP is given by $T_n = a + (n-1)d$, where a is the first term of the sequence and d is the common difference of the sequence.

Note:

(i) General term is also denoted by ℓ (last term).

(ii) n (number of terms) always belongs to the set of natural numbers.

(iii) Common difference can be zero, + ve or – ve.

If $d > 0 \Rightarrow$ increasing AP and the sequence tends to $+\infty$

If $d < 0 \Rightarrow$ decreasing AP and the sequence tends to $-\infty$

If $d = 0 \Rightarrow$ constant AP (all the terms remain same)

(iv) The n^{th} term from end is $(m - n + 1)$ term from the beginning, where m is the total number of terms and is given by the following expression:

$T_{m-n+1} = T_m - (n-1)d$

PLANCESS CONCEPTS

- If the m^{th} term is n and the n^{th} term is m, then the $(m+n)^{th}$ term is 0.
- If m times the m^{th} term is equal to n times the n^{th} term, then the $(m+n)^{th}$ term is 0.

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Illustration 3: If the 5^{th} term of an AP is 17 and its 7th term is 15, then find the 22^{th} term. **(JEE MAIN)**

Sol: Using the formula $T_n = a + (n-1)d$, we can solve above problem.

Given $a + 4d = 17$ and $a + 6d = 15$

$\Rightarrow \quad 2d = -2 \Rightarrow d = -1, a = 21$

$\therefore \quad T_{22} = 21 - 21 = 0$

Illustration 4: If 11 times the 11^{th} term of an AP is equal to 9 times the 9^{th} term, then find the 20^{th} term. **(JEE MAIN)**

Sol: By solving $11(a + 10d) = 9(a + 8d)$, we will get the value of a and d.

$\therefore 2a = -38d \Rightarrow a = -19d$

$\therefore 20^{th}$ term $= a + 19d = 0$

Illustration 5: Check whether the sequences given below are AP or not. **(JEE MAIN)**

(i) $T_n = n^2$ (ii) $T_n = an + b$

Sol: By taking the difference of two consecutive terms, we can check whether the sequences are in AP or not.

(i) $T_n = n^2$; $T_{n-1} = (n-1)^2$

Difference = $T_n - T_{n-1} = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1$

This difference varies with respect to the term. Hence, the sequence is not an AP.

(ii) $T_n = an + b$; $T_{n-1} = a(n-1) + b$

Difference = $(an + b) - (a(n-1) + b) = a$ (constant)

Hence, the sequence is an AP.

Illustration 6: The 2nd, 31st and the last term of an AP are given as $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$, respectively. Find the first term and the number of terms. **(JEE MAIN)**

Sol: Using $T_n = a + (n-1)d$, we can get the first term and common difference. Suppose a be the first term and d be the common difference of the AP.

Given, $T_2 = 7\frac{3}{4} \Rightarrow a + d = \frac{31}{4}$ .... (i)

$T_{31} = \frac{1}{2} \Rightarrow a + 30d = \frac{1}{2}$ .... (ii)

Subtracting (i) from (ii), we get $29d = \frac{1}{2} - \frac{31}{4} = -\frac{29}{4} \Rightarrow d = \frac{-1}{4}$

Putting the value of d in (i), we get $a - \frac{1}{4} = \frac{31}{4} \Rightarrow a = \frac{31}{4} + \frac{1}{4} = \frac{32}{4} = 8$

Suppose the number of terms be n, so that $T_n = -\frac{13}{2}$

i.e. $a + (n-1)d = -\frac{13}{2} \Rightarrow 8 + (n-1)\left(-\frac{1}{4}\right) = -\frac{13}{2}$

$\Rightarrow 32 - n + 1 = -26 \Rightarrow n = 59$

Hence, the first term = 8 and the number of terms = 59.

Illustration 7: Prove that the square roots of three unequal prime numbers cannot be three terms of an AP. **(JEE ADVANCED)**

Sol: Here by Considering $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ to be the λ^{th}, μ^{th} and v^{th} terms of an AP and solving them using $T_n = a + (n-1)$ d, we prove the problem.

If possible let $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ be the three terms of an AP. a, a + d, a + 2d........., where $p \neq q \neq r$ and they are prime numbers. Let them be the λ^{th}, μ^{th} and v^{th} terms, respectively.

$\therefore\ \sqrt{p} = a + (\lambda - 1)d$

$\sqrt{q} = a + (\mu - 1)d$

$\sqrt{r} = a + (\nu - 1)d$

$\therefore\ \sqrt{p} - \sqrt{r} = (\lambda - \mu)d$

Also, $\sqrt{q} - \sqrt{r} = (\mu - v)d$

$$\therefore\ \frac{\sqrt{p} - \sqrt{q}}{\sqrt{q} - \sqrt{r}} = \frac{\lambda - \mu}{\mu - v} \text{ or } \frac{\left(\sqrt{p} - \sqrt{q}\right)\left(\sqrt{q} + \sqrt{r}\right)}{\left(\sqrt{q} - \sqrt{r}\right)\left(\sqrt{q} + \sqrt{r}\right)} = \frac{\lambda - \mu}{\mu - v}$$

or $\sqrt{pq} + \sqrt{pr} - q - \sqrt{qr} = \frac{\lambda - \mu}{\mu - v}(q - r)$ or $\sqrt{pq} + \sqrt{pr} - \sqrt{qr} = q + \frac{\lambda - \mu}{\mu - v}(q - r)$ = rational number

Since p, q, r are unequal primes, $\sqrt{pq}, \sqrt{pr}$ and $\sqrt{qr}$ are unequal pure irrational numbers. Thus, LHS is irrational, but irrational $\neq$ rational.

Hence, the problem is proved.

Illustration 8: If x, y and z are real numbers satisfying the equation $25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$, then prove that x, y and z are in AP. **(JEE ADVANCED)**

Sol: By solving the equation $25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$, we can prove that x, y and z are in AP.

We have

$$(15x)^2 + (5y)^2 + (3z)^2 - (15x)(5y) - (5y)(3z) - (3z)(15x) = 0$$

$$\Rightarrow \quad (15x - 5y)^2 + (5y - 3z)^2 + (3z - 15x)^2 = 0$$

$$\Rightarrow \quad 15x - 5y = 0,\ 5y - 3z = 0,\ 3z - 15x = 0$$

$$\Rightarrow \quad 15x = 5y = 3z \Rightarrow \frac{x}{1} = \frac{y}{3} = \frac{y}{5}\ (= k \text{ say})$$

$$\therefore \quad x = k,\ y = 3k,\ z = 5k$$

Thus, x, y and z are in AP.

Illustration 9: Let $a_1, a_2, a_3, \ldots, a_n$ be in AP, where $a_1 = 0$ and the common difference $\neq 0$. Show that

$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + \ldots\ldots \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$$ **(JEE ADVANCED)**

Sol: Given $a_1 = 0$ and $d = a_2 - a_1 = a_2 - 0 = a_2$

By solving LHS and RHS separately, we can solve the problem.

$$\text{LHS} = \frac{a_3 - a_2}{a_2} + \frac{a_4 - a_2}{a_3} + \frac{a_5 - a_2}{a_4} + \ldots + \frac{a_{n-1} - a_2}{a_{n-2}} + \frac{a_n}{a_{n-1}}$$

$$= \frac{a_2}{a_2} + \frac{a_3}{a_3} + \frac{a_4}{a_4} + \ldots\ldots + \frac{a_{n-2}}{a_{n-2}} + \frac{a_n}{a_{n-1}}$$

$$= (n-3) + \frac{a_n}{a_{n-1}} = (n-3) + \frac{a_1 + (n-1)d}{a_1 + (n-2)d} = (n-3) + \frac{n-1}{n-2} \quad \{\because a_1 = 0\}$$

$$\text{RHS} = \frac{a_1 + (n-2)d}{a_2} + \frac{a_2}{a_1 + (n-2)d} = (n-2) + \frac{1}{n-2} = (n-3) + 1 + \frac{1}{n-2} = (n-3) + \frac{n-1}{n-2}$$

$\therefore$ LHS = RHS

PLANCESS CONCEPTS

A sequence obtained by multiplication or division of corresponding terms of two APs may not be in AP

For example, let the first AP be 2, 4, 6, 8,................ and the second AP be 1, 2, 3, 4, 5.......

Multiplying these two, we get 2,8,18,32, ..., which is clearly not an AP

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4.2 Series of an AP

Series of an AP can be obtained as

$S_n = a + (a + d) + (a + 2d)$ ...... $[a+(n-1)d]$

$S_n = [a + (n-1)d] + [a+(n-2)d]$ ...... $+ a$ (writing in the reverse order)

$\therefore\ 2S_n = n(2a + (n-1)\,d)$

$\therefore$ Sum to n terms, $S_n = \frac{n}{2}(2a+(n-1)d) = \frac{n}{2}\left(T_1 + T_n\right)$

Illustration 10: Find the sum of the first 19 terms of an AP when $a_4 + a_8 + a_{12} + a_{16} = 224$. **(JEE MAIN)**

Sol: We need to find out the sum of the first 19 terms of an AP, i.e. $\frac{19}{2}$ (2a + 18d), and we can represent the given equation as (a + 3d) + (a + 7d) + (a + 11d) + (a + 15d) = 224.

Given (a + 3d) + (a + 7d) + (a + 11d) + (a +15d) = 224

$\Rightarrow\ 4a + 36d = 224 \Rightarrow a + 9d = 56$

Sum of the first 19 terms $\Rightarrow S = \frac{19}{2}\ (2a + 18d) = \frac{19}{2}\times 2\times 56 = 1064$

Illustration 11: The sum of n terms of two arithmetic progressions is in the ratio of $\frac{7n+1}{4n+27}$. Find ratio of the 11th terms? **(JEE MAIN)**

Sol: Since we know the sum of n terms, i.e. $S_n = \frac{n}{2}\left[2a+\left(n-1\right)d\right]$, we can write the equation as

$\frac{7n+1}{4n+27} = \frac{\frac{n}{2}(2a_1+(n-1)d_1)}{\frac{n}{2}(2a_2+(n-1)d_2)}$. Hence, by putting n = 11 in this equation, we can obtain the ratio of the 11th terms.

$$\frac{7n+1}{4n+27} = \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_1+\left(\frac{n-1}{2}\right)d_2}$$

We want the ratio of $\frac{a_1+10d_1}{a_2+10d_2}$. Hence, $\frac{n-1}{2} = 10 \Rightarrow n = 21$

$$\Rightarrow \frac{a_1+10d_2}{a_2+10d_2} = \frac{148}{111}$$

Illustration 12: In an AP of n terms, prove that the sum of the kth term from the beginning and the kth term from the end is independent of k and equal to the sum of the first and last terms. **(JEE MAIN)**

Sol: Using the formula $T_k = a+(k-1)d$ and $T_{n-k+1} = [a+(n-k)d]$, we can obtain the kth term from the beginning and end, respectively, and after that by adding these values we can prove the given problem.

Suppose a be the first term and d be the common difference of the AP.

$\therefore$ kth term from the beginning = $T_k = a+(k-1)d$ ... (i)

Let l be the last term of the AP and l = a + (n – 1) d

The k^{th} term from the end of the given AP is the $(n - k + 1)^{th}$ term from the beginning.

$\therefore \quad T_{n-k+1} = [a + (n - k)d]$ ...(ii)

Adding (i) and (ii), we get

$\therefore$ The required sum = $T_k + T_{n-k+1} = [a + (k-1)d] + \left[a + (n-k)d\right]$ 2a + (k – 1 + n – k) d

= 2a + (n – 1) d .... (iii), which is independent of k

Moreover, the sum of the first and last terms = a + l = a + [a + (n – 1) d] = 2a + (n – 1) d. ...(iv)

Thus, the sum of the first and last terms is independent of k and (3) = (4). Hence proved.

Illustration 13: If $a_1, a_2, a_3, \ldots, a_n$ is an AP of non-zero terms , then prove that **(JEE ADVANCED)**

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots.. + \frac{1}{a_{n-1}a_n} = \frac{n-1}{a_1a_n}$$

Sol: By considering a as the first term and d as the common difference, we can write a_n as a + (n – 1) d, where n = 1, 2, 3,… n.

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots.. + \frac{1}{a_{n-1}a_n} = \frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \ldots\ldots \frac{1}{[a+(n-2)d][a+(n-1)d]}$$

$$= \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right) + \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+2d}\right) + \ldots + \frac{1}{d}\left(\frac{1}{a+(n-2)d} - \frac{1}{a+(n-1)d}\right)$$

$$= \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+(n-1)d}\right) = \frac{a+(n-1)d-a}{ad(a+(n-1)d)} = \frac{(n-1)d}{ad(a+(n-1)d)} = \frac{n-1}{a(a+(n-1)d)} = \frac{n-1}{a_1a_n}$$

PLANCESS CONCEPTS

Facts:

- If each term of an AP is increased, decreased, multiplied or divided by the same non-zero number, the resulting sequence is also an AP.
- The sum of the two terms of an AP equidistant from the beginning and end is constant and is equal to the sum of the first and last terms.

  $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots.$

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Illustration 14: Split 69 into three parts such that they are in AP and the product of the two smaller parts is 483. **(JEE MAIN)**

Sol: By considering the three parts as a – d, a, and a + d and using the given conditions, we can solve the given problem.

Sum of the three terms = 69 $\Rightarrow$ (a – d) + a + (a + d) = 69

$\Rightarrow$ 3a = 69 $\Rightarrow$ a = 23 ..... (i)

Product of the two smaller parts = 483 $\Rightarrow$ a (a – d) = 483

$\Rightarrow$ 23(23 – d) = 483 $\Rightarrow$ 23 – d = 21 $\Rightarrow$ d = 23 – 21 = 2

Hence, the three parts are 21, 23 and 25.

Illustration 15: Divide 32 into four parts that are in AP such that the ratio of the product of extremes to the product of mean is 7: 15. **(JEE MAIN)**

Sol: We can consider the four parts as (a – 3d), (a – d), (a + d) and (a + 3d).

Sum of the four parts = 32

$\Rightarrow$ (a – 3d) + (a – d) + (a + d) + (a + 3d) = 32 $\Rightarrow$ 4a = 32 $\Rightarrow$ a = 8

And $\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15} \Rightarrow \frac{a^2-9d^2}{a^2-d^2} = \frac{7}{15} \Rightarrow \frac{64-9d^2}{64-d^2} = \frac{7}{15}$

$\Rightarrow$ 128 d^2 = 512 $\Rightarrow$ d^2 = 4 $\Rightarrow$ d = ± 2

Thus, the required parts are 2, 6, 10 and 14.

Illustration 16: If a + b + c $\neq$ 0 and $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in AP, then prove that $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are also in AP. **(JEE ADVANCED)**

Sol: Here $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in AP; therefore, by adding 1 to each term and then by dividing each term by a + b +c, we will get the required result $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in AP

Adding 1 to each term, find that $\left(\frac{b+c}{a}+1\right),\left(\frac{c+a}{b}+1\right),\left(\frac{a+b}{c}+1\right)$ are in AP

i.e. $\frac{a+b+c}{a},\frac{c+a+b}{b},\frac{a+b+c}{c}$ are in AP

Dividing each term by a + b + c, we find that $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP

Illustration 17: If a $\left(\frac{1}{b}+\frac{1}{c}\right)$,b$\left(\frac{1}{a}+\frac{1}{c}\right)$,c$\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP, then prove that a, b, c are in AP. **(JEE ADVANCED)**

Sol: By adding 1 and then multiplying by $\left(\frac{abc}{ab+bc+ac}\right)$ to each term, we will get the result.

$$a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{a}+\frac{1}{c}\right), c\left(\frac{1}{a}+\frac{1}{b}\right) \text{ are in AP} \Rightarrow a\left(\frac{b+c}{bc}\right), b\left(\frac{a+c}{ac}\right), c\left(\frac{b+a}{ab}\right) \text{ are in AP}$$

Adding 1, we find that $\frac{ab+ac}{bc}+1,\frac{ab+bc}{ac}+1,\frac{bc+ac}{ab}+1$ are in AP

$\Rightarrow$ $\frac{ab+ac+bc}{bc},\frac{ab+bc+ac}{ac},\frac{bc+ac+ab}{ab}$ are in AP

Multiplying by $\left(\frac{abc}{ab+bc+ac}\right)$ to all the terms, we find that a, b, c are in AP

4.3 Arithmetic Mean

The arithmetic mean (AM) A of any two numbers a and b is given by the equation (a + b)/2. Please note that the sequence a, A, b is in AP. If $a_1, a_2, ..., a_n$ are n numbers, the (AM) A, of these numbers is given by:

$$A = \frac{1}{n}(a_1 + a_2 + ... + a_n)$$

Inserting 'n' AMs between 'a' and 'b'

Suppose $A_1, A_2, A_3, ..., A_n$ be the n means between a and b. Thus, a, $A_1, A_2, ..., A_n$, b is an AP and b is the $(n + 2)^{th}$ term.

Thus, b = a + (n + 1)d $\Rightarrow$ $d = \frac{b-a}{n+1}$

Now,

$A_1 = a + d$

$A_2 = a + 2d$

:

$A_n = a + nd$

$$\sum_{i=1}^{n} A_i = na + (1+2+3+.....+n)d = na + \left(\frac{n(n+1)}{2}\right)d = na + \left(\frac{n(n+1)}{2}\right)\left(\frac{b-a}{n+1}\right)$$

$$= \frac{n}{2}[2a + b - a] = nA \text{ where, } A = \frac{a+b}{2}$$

Note: The sum of the n AMs inserted between a and b is equal to n times A.M. between them.

Illustration 18: Insert 20 AMs between the numbers 4 and 67. **(JEE MAIN)**

Sol: Given, a = 4 and b = 67; therefore by using the formula $d = \frac{b-a}{n+1}$, we can solve it.

$$d = \frac{67-4}{20+1} = 3$$

$A_1 = a + d \Rightarrow A_1 = 7$

$A_2 = a + 2d \Rightarrow A_2 = 10$

$A_3 = a + 3d \Rightarrow A_3 = 13$

$A_{20} = a + 20\,d \Rightarrow A_{20} = 63$

Thus, between 4 and 67, 20 AMs are 7, 10, 13, 16, ..., 63.

Illustration 19: If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b, find the value of n. **(JEE ADVANCED)**

Sol: Since the A.M. between a and b = $\frac{a+b}{2}$, we can obtain the value of n by equating this to $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$.

$\Rightarrow \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$ [Given] $\Rightarrow 2a^n + 2b^n = a^n + ab^{n-1} + a^{n-1}b + b^n$

$\Rightarrow a^n - a^{n-1}b = ab^{n-1} - b^n$ $\Rightarrow a^{n-1}(a - b) = b^{n-1}(a - b)$

$\Rightarrow a^{n-1} = b^{n-1}$ [$\because a \neq b$] $\Rightarrow \left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0 \left(\because \left(\frac{a}{b}\right)^0 = 1\right)$

$\Rightarrow n - 1 = 0$ $\Rightarrow n = 1$

Illustration 20: Between 1 and 31, m arithmetic means are inserted in such a way that the ratio of the 7th and (m – 1)th means is 5: 9. Calculate the value of m. **(JEE MAIN)**

Sol: AMs inserted between 1 and 31 are in AP. Thus, by considering d to be the common difference of AP and obtaining the 7th and (m – 1)th means we can solve the problem.

Suppose $A_1, A_2, A_3, A_4, \ldots\ldots A_m$ be the m AMs between 1 and 31.

Thus, 1, $A_1, A_2, \ldots. A_m$, 31 are in AP

The total number of terms is m + 2 and $T_{m+2} = 31$

$1 + (m + 2 - 1)\, d = 31 \quad \Rightarrow (m + 1)\, d = 30 \Rightarrow d = \dfrac{30}{m+1}$

$$A_7 = T_8 = a + 7d = 1 + 7 \times \frac{30}{m+1} = \frac{m+1+210}{m+1} = \frac{m+211}{m+1}$$

$$A_{m-1} = T_m = 1 + (m-1)\, d = 1+(m-1)\times\frac{30}{m+1} = \frac{m+1+30m-30}{m+1} = \frac{31m-29}{m+1}$$

$$\frac{A_7}{A_{m-1}} = \frac{(m+211)/(m+1)}{(31m-29)/(m+1)} = \frac{m+211}{31m-29} = \frac{5}{9} \qquad \text{[Given]}$$

$$\Rightarrow \quad \frac{m+211}{31m-29} = \frac{5}{9} \quad \Rightarrow \quad 9m + 1899 = 155m - 145$$

$$\Rightarrow \quad 146m = 2044 \quad \Rightarrow \quad m = \frac{2044}{146} = 14 \ \text{;Thus, } m = 14$$

Illustration 21: Gate receipts at the show of "Baghbaan" amounted to Rs 9500 on the first night and showed a drop of Rs 250 every succeeding night. If the operational expenses of the show are Rs 2000 a day, find out on which night the show ceases to be profitable? **(JEE MAIN)**

Sol: Here, a = 9500 and d = –250. The show ceases to be profitable on the night when the receipts are just Rs 2000. Thus, by considering that it will happen at nth night and using $T_n = a + (n - 1)\, d$, we can solve this problem.

We have the cost of gate receipt on the first night (a) = 9500

Common difference (d) = –250

Suppose, it happens on the nth night, then

$$2000 = 9500 + (n - 1)(-250) \quad \Rightarrow \quad 2000 - 9500 = -250n + 250$$

$$\Rightarrow \quad -7500 - 250 = -250n \qquad \Rightarrow \quad -7750 = -250n \qquad \Rightarrow \quad n = \frac{7750}{250} = 31$$

PLANCESS CONCEPTS

(a) If the sum of n terms S_n is given, then the general term $T_n = S_n - S_{n-1}$, where S_{n-1} is sum of (n – 1) terms of AP.

(b) In a series, if S_n is a quadratic function of n or T_n is a linear function of n, the series is an AP.

(i) If $T_n = an + b$, the series so formed is an AP and its common difference is a.

(ii) If $S_n = an^2 + bn + c$, the series so formed is an AP and its common difference is 2a.

PLANCESS CONCEPTS

(c) If in a finite AP, the number of terms is odd, then its middle term is the A.M. between the first term and last term and its sum is equal to the product of the middle term and number of terms.

(d) It is found that the sum of infinite terms of an AP is ∞, if d > 0 and –∞, if d < 0.

(e) If for an AP, the pth term is q and the qth term is p, then the mth term is = p + q – m.

(f) If for an AP, the sum of p terms is q and sum of q terms is p, then the sum of (p + q) terms is – (p + q).

(g) If for an AP, the sum of p terms is equal to the sum of q terms, then the sum of (p +q) terms is zero.

(h) If for different APs, $\frac{S_n}{S'_n}=\frac{f_n}{\phi_n}$, then $\frac{T_n}{T'_n}=\frac{f(2n-1)}{\phi(2n-1)}$.

(i) If for two APs, $\frac{T_n}{T'_n}=\frac{An+B}{Cn+D}$, then we find that $\frac{S_n}{S'_n}=\frac{A\left(\frac{n+1}{2}\right)+B}{C\left(\frac{n+1}{2}\right)+D}$.

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An Important Property of AP: A sequence is said to be an AP if the sum of its n terms is of the form $An^2 + Bn$, where A and B are constants. Thus, the common difference of the AP is 2A.

Proof: Suppose, a and d be the first term and common difference of AP, respectively, and S_n be the sum of n terms.

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$$\Rightarrow \quad S_n=an+\frac{n^2}{2}d-\frac{n}{2}d=\left(\frac{d}{2}\right)n^2+\left(a-\frac{d}{2}\right)n$$

$\Rightarrow$ $S_n = An^2 + Bn$, where $A = \frac{d}{2}$ and $B=a-\frac{d}{2}$

Hence, the sum of n terms of an AP is of the form $An^2 + Bn$.

Conversely, suppose the sum S_n of n terms of a sequence a_1, a_2, a_3, ...a_n..... is of the form $An^2 + Bn$. Then, we have to prove that the sequence is an AP.

We have $S_n = An^2 – Bn$

$\Rightarrow$ $S_{n-1} = A(n-1)^2 + B(n-1)$ [On replacing n by n = 1]

Now, $A_n = S_n - S_{n-1}$

$\Rightarrow$ $A_n = \{An^2 + Bn\} - \{A(n-1)^2 - B(n-1)\} = 2An + (B - A)$

$\Rightarrow$ $A_{n+1} = 2A(n+1) + (B - A)$ [On replacing n by (n + 1)]

$\therefore$ $A_{n+1} - A_n = \{2A(n+1) + B - A\} - \{2An + (B - A)\} = 2A$

Since $A_{n+1} - A_n = 2A$ for all $n \in N$, the sequence is an AP with a common difference 2A.

For example, if $S_n = 3n^2 + 2n$, we can say that it is the sum of the n terms of an AP with a common difference of 6.

5. GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (GP) if the ratio of successive terms is constant.

In general, G.P. is written in the following form: a, ar, ar^2, ..., ar^{n-1},....

where a is the first term and r is the common ratio

5.1 General Term

If a is the first term and r is the common ratio, then $T_n = ar^{n-1}$.

Illustration 22: The 5th, 8th and 11th terms of a G.P. are given as p, q and, s respectively. Prove that $q^2 = ps$. **(JEE MAIN)**

Sol: By using $T_n = ar^{n-1}$ and solving it, we can prove the problem.

Given, $T_5 = p, T_8 = q, T_{11} = s$ ....(i)

Now, $T_5 = ar^{5-1} = ar^4 \Rightarrow ar^4 = p$ .... (ii) [Using (i)]

$T_8 = ar^{8-1} = ar^7 \Rightarrow ar^7 = q$ ... (iii) [Using (i)]

$T_{11} = ar^{11-1} = ar^{10} \Rightarrow ar^{10} = s$ ..... (iv)

On squaring (iii), we get

$q^2 = a^2 r^{14} = a\cdot a.r^4\cdot r^{10} = (ar^4)(ar^{10})$

$\Rightarrow q^2 = ps$ [Using (ii) and (iv)] proved.

5.2 Series of GP

Let us suppose $S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ ... (i)

Multiplying 'r' on both the sides of (i) and shifting the RHS terms by one place, we get

$S_n r = 0 + ar + ar^2 + \ldots\ldots + ar^n$ ... (ii)

By subtracting (ii) from (i), we get

$S_n(1-r) = a - ar^n = a(1-r^n)$

$S_n = \frac{a(1-r^n)}{1-r}$, where $r \neq 1$

Thus, the sum of the first n terms of a G.P. is given by $\Rightarrow S_n = \left(\frac{a(r^n-1)}{r-1}\right) = \left(\frac{T_{n+1}-a}{r-1}\right)$

And $S_n = na$, when r = 1

Note: If r = 1, then the sequence is of both AP and GP, and its sum is equal to na, i.e. $S_n = na$.

If $|r| < 1$, the nth term of G.P. converges to zero and the sum becomes finite.

The sum to infinite terms of G.P. = $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{a(r^n-1)}{r-1}$

As $|r| < 1$ $r^n \to 0$ as $n \to \infty$ $\therefore$ $S_\infty = \frac{a}{1-r}$

5.3 Geometric Mean

If a, b and c are three positive numbers in GP, then b is called the geometrical mean (GM) between a and c, and $b^2 = ac$. If a and b are two real numbers of the same sign and G is the G.M. between them, $G^2 = ab$.

Note: If a and b are two number of opposite signs, then the G.M. between them does not exist.

To Insert 'n' GMs Between a and b: If a and b are two positive numbers and we have to insert n GMs, G_1, G_2, ..., G_n, between the two numbers 'a' and 'b' then a, G_1, G_2, ..., G_n, b will be in GP. The series consists of (n + 2) terms and the last term is b and the first term is a.

Thus, $b = ar^{n+2-1} \Rightarrow b = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$\Rightarrow \quad G_1 = ar,\ G_2 = ar^2 \ ....G_n = ar^n$ or $G_n = a^{yn+1} \cdot b^{yn+1} = (ab)^{yn+1}$

Note: The product of n GMs inserted between 'a' and 'b' is equal to the n^{th} power of the single G.M. between 'a' and 'b,' i.e.

$\prod_{r=1}^{n} G_r = (G)^n$, where $G = \sqrt{ab}$ (GM between a and b)

5.4 Relation between A.M. and GM

For any two non-negative number A.M. ≥ G.M.

Proof: Let two non-negative numbers be $\sqrt{a}$ and $\sqrt{b}$.

Now, we can write $\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0 \Rightarrow a - 2\sqrt{ab} + b \ge 0 \quad \Rightarrow a + b \ge 2\sqrt{ab} \Rightarrow \frac{a+b}{2} \ge \sqrt{ab} \qquad \Rightarrow$ A.M. ≥ GM

Note: (i) Equality for AM, G.M. (i.e. A.M. = GM) exists when a = b.

(ii) Since A.M. ≥ GM; $(AM)_{min}$ = GM; $(GM)_{max}$ = AM

Illustration 23: If x, y and z have the same sign, then prove that $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \ge 3$. **(JEE ADVANCED)**

Sol: As we know that A.M. ≥ G.M., therefore by obtaining A.M. and G.M. of $\frac{x}{y}$, $\frac{y}{z}$ and $\frac{z}{x}$ we can prove the problem.

Let $\frac{x}{y} = x_1$; $\frac{y}{z} = x_2$; $\frac{z}{x} = x_3$

$\therefore \ \frac{x_1+x_2+x_3}{3} \ge (x_1x_2x_3)^{1/3} \Rightarrow \frac{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}{3} \ge 1$

Hence proved.

Illustration 24: Calculate the values of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the G.M. between a and b. **(JEE ADVANCED)**

Sol: We know that the G.M. between a and b = $\sqrt{ab}$, but here G.M. between a and b is $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$.

$\Rightarrow \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab} \qquad \Rightarrow \quad a^{n+1} + b^{n+1} = (a^n + b^n)(ab)^{1/2}$

$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{n+\frac{1}{2}} \cdot b^{\frac{1}{2}} + a^{\frac{1}{2}} \cdot b^{n+\frac{1}{2}} \qquad \Rightarrow a^{n+1} - a^{n+\frac{1}{2}} \cdot b^{\frac{1}{2}} = a^{\frac{1}{2}} \cdot b^{n+\frac{1}{2}} - b^{n+1}$

$\Rightarrow \quad a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \qquad \Rightarrow a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = 1 = \left(\frac{a}{b}\right)^0 \qquad \Rightarrow \quad n+\frac{1}{2} = 0 \Rightarrow \quad n = -\frac{1}{2}$

Illustration 25: Find the sum to n terms for the series 9 + 99 + 999 ...... n. **(JEE ADVANCED)**

Sol: The given series can be written as S = $(10 - 1) + (10^2 - 1) + (10^3 - 1) \ldots\ldots + (10^n - 1)$.

Thus, by using $S_n = \frac{a(1-r^n)}{1-r}$, we can find out the required sum.

$\therefore \quad S = (10 + 10^2 + 10^3 + \ldots + 10^n) - n; \quad S = \frac{10(1-10^n)}{1-10} - n$

Illustration 26: If a_1, a_2 and a_3 are in G.P. with a common ratio r (r > 0 and a > 0), then values of r for which inequality $9a_1 + 5a_3 > 14a_2$ hold good are? **(JEE ADVANCED)**

Sol: Since $a_1 = \frac{a}{r}$ $a_2 = a$, $a_3 = ar$, by substituting these values to the given inequality we will get the result.

$a_1 = \frac{a}{r}, a_2 = a, a_3 = ar$ Now, $\frac{9a}{r} + 5ar > 14a$

$\Rightarrow \quad 5ar^2 - 14ar + 9a > 0 \qquad \Rightarrow \quad 5r^2 - 14r + 9 > 0$

$\Rightarrow \quad 5r^2 - 5r - 9r + 9 > 0 \qquad \Rightarrow \quad 5r(r-1) - 9(r-1) > 0$

$\Rightarrow \quad (5r-9)(r-1) > 0 \qquad \Rightarrow \quad r \in R - \left(1, \frac{5}{9}\right)$

PLANCESS CONCEPTS

- The product of n geometric means between a and (1/a) is 1.
- Let the first term of a G.P. be negative; if r > 1, then it is a decreasing G.P. and if 0< r < 1, then it is an increasing GP.
- If $a_1, a_2, a_3, \ldots, a_n$ are in AP, $a^{a_1}, a^{a_2}, a^{a_3}, \ldots, a^{a_n}$ will be in G.P. whose common ratio is a^d.

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Illustration 27: On a certain date, the height of a plant is 1.6 m. If the height increases by 5 cm in the following year and if the increase in each year is half of that in the preceding years, show that the height of the plant will never be 1.7 m. **(JEE MAIN)**

Sol: Here, the sum of the increases in the height of the plant in the first, second, third, ... year is equal to (1.7 – 1.6) m = 0.1 m = 10 cm.

According to the question, increases in the height of the plant in the first, second, third, ... year are 5, $\frac{5}{2}, \frac{5}{4}$, ... cm, respectively.

Let it reach the height of 1.7 m (i.e. increases [1.7 – 1.6] m = 0.1 m = 10 cm).

Therefore, the sum of 5, $\frac{5}{2}, \frac{5}{4}$, ...to n terms = 10

$\Rightarrow \quad \frac{5\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 10 \qquad \left[\because a = 5, r = ½, S_n = \frac{a(1-r^n)}{1-r}\right]$

$\Rightarrow \quad 10\left(1-\frac{1}{2^n}\right) = 10 \Rightarrow 1 - \frac{1}{2^n} = 1 \Rightarrow \frac{1}{2^n} = 0$, which does not hold for any n. Thus, the plant will never reach the height of 1.7 m.

Illustration 28: A manufacturer reckons that the value of a machine (price = Rs 15,625) will depreciate each year by 20%. Calculate the estimated value at the end of 5 years. **(JEE MAIN)**

Sol: Here the value of the machine after 5 years= ar^5, where a= 15,625. We will obtain the value of r using the given condition.

The present value of the machine = Rs 15,625

The value of the machine in the next year = Rs 15,625 × $\frac{80}{100}$

The value of the machine after 2 years = Rs 15,625 × $\frac{80}{100}\times\frac{80}{100}$

The values of the machine in the present year, after 1 year and after 2 years are

Rs 15,625, Rs 15,625 × $\frac{80}{100}$ and Rs 15,625 × $\frac{80}{100}\times\frac{80}{100}$, respectively

These values form a GP.

Here, the first term is Rs 15,625 and the common ratio is $\frac{80}{100}$,i.e. $\frac{4}{5}$.

Thus, thee value of the machine after 5 years = ar^5 = Rs 15,625 $\times\left(\frac{4}{5}\right)^5 = \frac{15625\times1024}{625\times5} = 1024\times5$ = Rs 5120

5.5 Properties of GP

(a) If each term of a G.P. is multiplied or divided by the same non-zero quantity, then the resulting sequence is also a GP.

(b) If in a finite GP, the number of terms is odd, then its middle term is the G.M. of the first and last terms.

(c) If a, b and c are in GP, then $\frac{b}{a}=\frac{c}{b}\Rightarrow b^2=ac$ (which is the condition of GP).

(d) The reciprocals of the terms of a given G.P. also give a G.P. with a common ratio of $\frac{1}{r}$.

Proof: Let $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ be the terms of a G.P. with the common ratio r.

Then, $\frac{a_{n+1}}{a_n}=r$ for all $n\in N$ ... (i)

The sequence formed by the reciprocals of the terms of the above G.P. is given by

$$\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3},\ldots,\frac{1}{a_n},\ldots$$

Now, $\frac{1/a_{n+1}}{1/a_n}=\frac{a_n}{a_{n+1}}=\frac{1}{r}$ [Using (i)]

Hence, the new sequence is also a G.P. with the common ratio 1/r.

(e) If each term of a G.P. is raised to the same power (say k), then the resulting sequence also forms a G.P. with the common ratio as r^k.

Proof: Let $a_1, a_2, a_3, a_4, \ldots, a_n \ldots$ be the terms of a G.P. with the common ratio r.

Then, $\frac{a_{n+1}}{a_n}=r$ for all $n\in N$ (i)

Let k be a non-zero real number. Consider the sequence. $a_1^k, a_2^k, a_3^k, \ldots, a_n^k, \ldots$

Here, $\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k = r^k$ for all $n \in d$ [Using (i)]

Thus, $a_1^k, a_2^k, a_3^k, ..., a_n^k, ...$ is a G.P. with a common ratio r^k.

(f) In a GP, the product of the terms equidistant from the beginning and the end is always the same and it is equal to the product of the first and last terms (only for finite GP).

Proof: Let $a_1, a_2, a_3, ..., a_n$ be a finite G.P. with the common ratio r. Then,

k^{th} term from the beginning = $a_k = a_1 r^{k-1}$

k^{th} term from the end = $(n - k + 1)^{th}$ term from the beginning

$= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ (k^{th} term from the beginning) (k^{th} term from the end) $= a_k a_{n-k+1}$

$= a_1 r^{k-1} a_1 r^{n-k} = a_1^2 r^{n-1} = a_1 . a_1 r^{n-1} = a_1 a_n$ for all k = 2,3,....,n –1

Thus, the product of the terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last terms.

(g) If the terms of a G.P. are chosen at regular intervals, the new sequence so formed also forms a G.P. with the common ratio as r^p , where p is the size of interval.

For example:

2, 4, 8, 16, 32, 64, 128,..... (GP, where r = 2)

4, 16, 64 ..... (also a GP, where r = 4)

(h) If $a_1, a_2, a_3, ..., a_r.....$ is a G.P. of non-zero, non-negative terms, then $\log a_1, \log a_2, ..., \log a_n, ...$ is an AP and vice versa.

(i) If $T_1, T_2, T_3...$ and t_1, t_2, t_3 are two GPs, $T_1t_1, T_2t_2, T_3t_3....$ is also in GP.

Proof: Let the two GPs be $T_1, T_2, ..., T_n, ...$ with the common ratio R

$$\Rightarrow \frac{T_{n+1}}{T_n} = R \qquad ... (i)$$

and $t_1, t_2, ..., t_n, ...$ with common the ratio r

$$\Rightarrow \frac{t_{n+1}}{t_n} = r \qquad .... (ii)$$

Multiplying each term of the sequence (i) by the corresponding term of (ii), we get $\left(\frac{T_{n+1}}{T_n}\right)\left(\frac{t_{n-1}}{t_n}\right) = Rr$

Thus, the resulting sequence is also in G.P. with the common ratio Rr.

(j) The resulting sequence thus formed by dividing the terms of a G.P. by the corresponding terms of another G.P. is also a GP.

Proof: Let the two GPs be $T_1, T_2, ..., T_n, ...$ with the common ratio R

$$\Rightarrow \frac{T_{n+1}}{T_n} = R \qquad ... (i)$$

and $t_1, t_2, ..., t_n....$ with the common ratio r

$$\Rightarrow \frac{t_{n+1}}{t_n} = r \qquad ...(ii)$$

Dividing each term of the sequence (i) by the corresponding term of (ii), we get

$$\frac{\frac{T_{n+1}}{t_{n+1}}}{\frac{T_n}{t_n}} = \left(\frac{R}{r}\right)$$ [Using (i) and (ii)]

Thus, the resulting sequence is also in G.P. with the common ratio $\left(\frac{R}{r}\right)$.

Illustration 29: If sum of infinite terms of G.P. is 15 and sum of squares of infinite terms of G.P. is 45, then find GP.

(JEE MAIN)

Sol: As the sum of infinite terms $S_\infty = \frac{a}{1-r}$, therefore by using this formula we can obtain the value of the common ratio.

$$\frac{a}{1-r} = 15$$

Now, $a^2, a^2r^2, a^2r^4, ...;$ $\quad \frac{a^2}{1-r^2} = 45 \quad \therefore \quad \frac{225(1-r)(1-r)}{(1-r)(1+r)} = 45$

$225 - 225r = 45 + 45r; \qquad 180 = 270\,r \qquad \therefore \qquad r = 2/3$

Illustration 30: If $x = 1 + a + a^2 + ... \infty$, $y = 1 + b + b^2 + ... \infty$ and $|a| < 1$, $|b| < 1$, then prove that $1 + ab + a^2b^2 + ... = \frac{xy}{x+y-1}$

(JEE MAIN)

Sol: By using the formula $S_\infty = \frac{a}{1-r}$, we can solve problem.

$x = 1 + a + a^2 + ...\text{ to } \infty = \frac{1}{1-a} \qquad (\because |a| < 1)$

$\Rightarrow \quad 1 - a = \frac{1}{x} \Rightarrow a = 1 - \frac{1}{x} \quad \Rightarrow a = \frac{x-1}{x}$ ... (i)

Also, $y = 1 + b + b^2 + ...\text{ to } \infty = \frac{1}{1-b} \qquad (\because |b| < 1)$

$\Rightarrow \quad 1 - b = \frac{1}{y} \Rightarrow b = 1 - \frac{1}{y} \quad \Rightarrow b = \frac{y-1}{y}$ ... (ii)

$\therefore \quad 1 + ab + a^2b^2 + ...\text{ to } \infty = \frac{1}{1-ab} \qquad (\because |a| < 1, |b| < 1 \Rightarrow |ab| < 1)$

$= \frac{1}{1 - \frac{x-1}{x}\cdot\frac{y-1}{y}}$ [Using (i) and (ii)] $= \frac{xy}{xy - xy + x + y - 1} = \frac{xy}{x+y-1}$

Hence proved.

Illustration 31: If $S_1, S_2, S_3, ..., S_p$ denote the sum of an infinite G.P. whose first terms are 1, 2, 3, ... , p, respectively and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{(p+1)}$, respectively, show that $S_1 + S_2 + S_3 + ... + S_p = \frac{p(p+3)}{2}$.

(JEE ADVANCED)

Sol: By using $S_\infty = \frac{a}{1-r}$ we can obtain $S_1, S_2, S_3, ..., S_p$ and after that by adding them we can prove the given equation.

For S_1, we have $a = 1$, $r = \frac{1}{2}$ $\therefore$ $S_1 = \frac{1}{1-\frac{1}{2}} = 2$

For S_2, we have $a = 2$, $r = \frac{1}{3}$ $\therefore$ $S_2 = \frac{2}{1-\frac{1}{3}} = 3$

For S_3, we have $a = 3$, $r = \frac{1}{4}$ $\therefore$ $S_3 = \frac{3}{1-\frac{1}{4}} = 4$

For S_p, we have $a = p$, $r = \frac{1}{p+1}$ $\therefore$ $S_p = \frac{p}{1-\frac{1}{p+1}} = p + 1$

Adding all these, we get $S_1 + S_2 + S_3 + \text{.........} + S_p = 2 + 3 + 4 + \text{........} + (p + 1)$

$$= \frac{p}{2}[2 + (p + 1)] = \frac{p}{2}[p + 3] = \frac{p(p+3)}{2}$$

Hence proved.

6. ARITHMETIC GEOMETRIC PROGRESSION

A series formed by multiplying the corresponding terms of AP and G.P. is called arithmetic geometric progression (AGP).

Let a = first term of AP, b = first term of GP, d = common difference and r = common ratio of GP, then

AP: $a, a + d, a + 2d, a+ 3d, \ldots, a + (n - 1) d$

GP: $b, br, br^2, br^3, \ldots, br^{n-1}$

AGP: $ab, (a + d) br, (a + 2d) br^2 \ldots (a + (n - 1) d) br^{n-1}$ (Standard appearance of AGP)

The general term (n^th term) of an AGP is given as $T_n = [a + (n - 1)d]\, br^{n-1}$.

6.1 Series of AGP

To find the sum of n terms of an AGP, we suppose that its sum as S_n and then multiply both the sides by the common ratio of the corresponding G.P. and then subtract as in the following way. Thus, we get a G.P. whose sum can be easily obtained.

$S_n = ab + (a + d) br + (a + 2d) br^2 + \ldots + (a + (n-1)d) br^{n-1}$ ...(i)

$rS_n = 0 + abr + (a + d) br^2 + \ldots + (a + (n - 1)d) br^n$ ...(ii)

After subtraction, we get

$S_n(1 - r) = ab + [dbr + dbr^2 + \ldots + \text{ up to } (n - 1) \text{ terms}] - [(a + (n - 1)d)br^n]$

$$S_n(1 - r) = ab + \frac{dbr(1-r^{n-1})}{1-r} - (a+(n-1)d)br^n$$

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$$. This is the sum of n terms of AGP

For an infinite AGP, as $n \to \infty$, then $r^n \to 0$ ($\because |r| < 1$)

$$\Rightarrow \quad S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

Illustration 32: If $|x| < 1$, then find the sum $S = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots + \infty$. **(JEE MAIN)**

Sol: The sum can be found out by calculating the value of Sx – S.

$Sx = x + 2x^2 + 3x^3 + 4x^3 + \ldots\ldots\infty$

$S(1 - x) = 1 + x + x^2 + x^3 + \ldots\ldots + \infty$; $S(1 - x) = \dfrac{1}{(1-x)} \Rightarrow S = \dfrac{1}{(1-x)^2}$

Illustration 33: If $|x| < 1$, then find the sum $S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\infty$. **(JEE ADVANCED)**

Sol: Similar to above illustration.

$S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\infty$

$Sx = x + 3x^2 + 6x^3 \ldots\ldots\ldots\infty$

$S(1-x) = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots\ldots\ \infty$

$S(x)(1-x) = x + 2x^2 + 3x^3 \ldots\ldots\ \infty$

$S(1-x)^2 = 1 + x + x^2 + \ldots\ldots\ \infty$

$S(1 - x)^2 = \dfrac{1}{1-x}$ $S = \dfrac{1}{(1-x)^3}$

7. MISCELLANEOUS SEQUENCES

Type 1: Some Standard Results

(i) Sum of the first n natural numbers $= \displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(ii) Sum of the first n odd natural numbers $= \displaystyle\sum_{r=1}^{n} (2r-1) = n^2$

(iii) Sum of the first n even natural numbers $= \displaystyle\sum_{r=1}^{n} 2r = n(n+1)$

(iv) Sum of the squares of the first n natural numbers $= \displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Proof: $\displaystyle\sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$

Consider $(x + 1)^3 = x^3 + 1 + 3x^2 + 3x$

$(x + 1)^3 - x^3 = 3x^2 + 3x + 1$

Put $x = 1, 2, 3 \ldots.\ n$

$2^3 - 1^3 = 3.1^2 + 3.1 + 1$

$3^3 - 2^3 = 3.2^2 + 3.2 + 1$

$(n + 1)^3 - n^3 = 3n^2 + 3.n + 1$

Adding all, we get

$\Rightarrow$ $(n + 1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + \ldots.+ n^2) + 3(1 + 2 + \ldots. + n) + n$

$\Rightarrow$ $(n + 1)^3 - 1 = 3\Sigma n^2 + 3\dfrac{3n(n+1)}{2} + n \Rightarrow$ $3\Sigma n^2 = (n + 1)^3 - 1 - \dfrac{3n(n+1)}{2} - n$

$\Rightarrow \quad 3\Sigma n^2 = \quad n^3 + 1 + 3n^2 + 3n - 1 - \dfrac{3n(n+1)}{2} - n$

$\Rightarrow \quad 3\Sigma n^2 = n^3 + 3n^2 + 2n - \dfrac{3n(n+1)}{2}$

$\Rightarrow \quad 3\Sigma n^2 = \dfrac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{2} \Rightarrow 3\Sigma n^2 = \dfrac{2n^3 + 3n^2 + n}{2}$

$\Rightarrow \quad 3\Sigma n^2 = \dfrac{2n^3 + 3n^2 + n^2 + n}{2} \Rightarrow 3\Sigma n^2 = \dfrac{2n^2(n+1) + n(n+1)}{2}$

$\Rightarrow \quad 3\Sigma n^2 = \dfrac{n(n+1) \times (2n+1)}{2} \Rightarrow \Sigma n^2 = \dfrac{n(n+1) \times (2n+1)}{6}$

(v) Sum of the cubes of first n natural numbers $\sum_{r=1}^{n} r^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Proof: Consider $(x + 1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Put $x = 1, 2, 3 \ldots\ldots n$

$2^4 - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$

$3^4 - 2^4 = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$

$4^4 - 3^4 = 4 \cdot 3^2 + 6 \cdot 2^2 + 4 \cdot 3 + 1$

:

$(n + 1)^4 - n^4 = 4 \cdot n^3 + 6 \cdot n^2 + 4 \cdot n + 1$

Adding all, we get

$(n + 1)^4 - 1^4 = 4(1^3 + 2^3 + \ldots\ldots + n^3) + 6(1^2 + 2^2 + \ldots. + n^2) + 4(1 + 2 + 3 \ldots.. + n) + n$

$= 4\Sigma n^3 + 6\left(\dfrac{n(n-1)(2n+1)}{6}\right) + 4\left(n\left(\dfrac{n+1}{2}\right)\right) + n$

On simplification, we get

$\Sigma n^3 = \left(n\left(\dfrac{n+1}{2}\right)\right)^2$

(vi) Sum of the fourth powers of the first n natural numbers (Σn^4)

$\Sigma n^4 = 1^4 + 2^4 + \ldots.. + n^4 \quad ; \quad \Sigma n^4 = \dfrac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$

[The result can be proved in the same manner as done for Σn^3]

Illustration 34: Find the value of $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} (1)$. **(JEE MAIN)**

Sol: Using the formula $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ and $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$, we can solve the problem.

Let $S = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} (1)$

$S = \sum_{i=1}^{n} \sum_{j=1}^{i} (j) = \sum_{i=1}^{n} \dfrac{i(i+1)}{2} = \dfrac{1}{2}[\Sigma n^2 + \Sigma n] = \dfrac{1}{2}\left[\dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2}\right] = \dfrac{n(n+1)(2n+4)}{12} = \dfrac{n(n+1)(n+2)}{6}$

Illustration 35: Find the sum of 1.2.3 + 2.3.4 + 3.4.5...... n terms. **(JEE MAIN)**

Sol: The given series is in the form of $T_n = n(n + 1)(n + 2) = n^3 + 3n^2 + 2n$.

Therefore, by using $\sum_{r=1}^{n} r^3 = n^2\left(\frac{n+1}{2}\right)^2$ $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, we can solve the problem.

$T_n = n(n + 1)(n + 2) = n(n^2 + 3n + 2)$

$T_n = n^3 + 3n^2 + 2n$

$\Sigma T_n = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n = n^2\left(\frac{n+1}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + n(n+1)$

Type 2: Using Method of Difference: If $T_1, T_2, T_3, T_2, T_4\ T_5$.... is a sequence whose terms are sometimes in AP and sometimes in GP, then for such series we first compute their nth term and then compute the sum to n terms using sigma notation.

Illustration 36: Find $S_n = 6 + 13 + 22 \ldots\ldots\ldots + T_n$. **(JEE ADVANCED)**

Sol: By calculating $[S_n + (-S_n)]$, we will get T_n. After that we will obtain $\Sigma\ T_n$ and thus we will get the result.

$\Sigma_n = 6 + 13 + 22 \ldots\ldots\ldots T_n$

$-\Sigma_n = -6 - 13 \ldots\ldots\ldots - T_{n-1} - T_n$

$\Rightarrow$ $0 = 6 + (7 + 9 + 11 \ldots\ldots (T_n - T_{n-1})) - T_n$

$\Rightarrow$ $T_n = 6 + (7 + 9 + 11 \ldots. (T_n - T_{n-1})) = 6 + (n - 1)(7 + n - 2) = 6 + (n - 1)(n + 5)$

$\Rightarrow$ $T_n = 6 + n^2 + 4n - 5 = n^2 + 4n + 1$

$\Sigma_n = \Sigma n^2 + 4\Sigma n + n = \frac{n(n+1)(n+1)}{6} + 2n(n + 1) + n$

Illustration 37: Find $S = 1+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{3}+\frac{1}{3^2}\right)+\ldots\ldots$ n terms. **(JEE ADVANCED)**

Sol: Given, $T_n = 1+\frac{1}{3}+\frac{1}{3^2}+\ldots..+\frac{1}{3^{n-1}}$; therefore by obtaining ΣT_n, we will get the result.

$S = 1+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{3}+\frac{1}{9}\right)\ldots\ldots.$

$T_n = 1+\frac{1}{3}+\frac{1}{3^2}+\ldots..+\frac{1}{3^{n-1}} = \frac{3\left(1-\frac{1}{3^n}\right)}{2}$

$\Sigma T_n = \frac{3n}{2}-\frac{3}{2}\Sigma\frac{1}{3^n} = \frac{3n}{2}-\frac{3}{2}\left(\frac{1}{3}+\frac{1}{3^2}\ldots\ldots\frac{1}{3^n}\right) = \frac{3}{2}\left(n-\frac{3}{2}\left(1-\frac{1}{3^n}\right)\right)$

Type 3: Splitting the nth term as a difference of two: Here, S is a series in which each term is composed of the reciprocal of the product of r factors in an AP.

Illustration 38: Find the sum of n terms of the series $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+.....$ **(JEE ADVANCED)**

Sol: Here n^{th} term of the series will be $T_n = \frac{1}{n(n+1)(n+2)(n+3)}$.

By considering $S_n = c - \lambda$, where $\lambda = \frac{1}{3(n+1)(n+2)(n+3)}$, we will get the result.

First calculate the n^{th} term, $T_n = \frac{1}{n(n+1)(n+2)(n+3)}$

Now, let the sum of the above series be given by:

$S_n = c - \lambda$, where λ is obtained by replacing the first factor by (last factor – first factor) ... (i)

Hence,

$$\lambda = \frac{1}{3(n+1)(n+2)(n+3)} \quad \begin{bmatrix}\text{First factor} = n \\ \text{Last factor} = n+3\end{bmatrix} \qquad \text{... (ii)}$$

Using (ii) $\Rightarrow$ $S_n = c - \frac{1}{3(n+1)(n+2)(n+3)}$ ... (iii)

To calculate 'c', put n = 1 in (iii)

$$S_1 = c - \frac{1}{1\cdot2\cdot3\cdot4} \Rightarrow \frac{1}{1\cdot2\cdot3\cdot4} = c - \frac{1}{3\cdot2\cdot3\cdot4} \quad \Rightarrow \quad c = 1/18$$

Put the value of 'c' in (iii)

$$S_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)},\ S_n = \frac{1}{3}\left\{\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right\}$$

Remark $\Rightarrow$ If we want to calculate S_∞, then $n \to \infty$, $\frac{1}{(n+1)(n+2)(n+3)} \to 0 \Rightarrow S_\infty = \frac{1}{18}$

Note: The above method is applicable only when the series looks like as follows:

$$\frac{1}{a(a+d)(a+2d)} + \frac{1}{(a+d)(a+2d)(a+3d)} + \frac{1}{(a+2d)(a+3d)(a+4d)} + ...$$

Type 4: Vn Method: This is method of resolving the nth term into partial fraction and summation by telescopic cancellation. First, find the n^{th} term of the series and try to create a denominator part in the numerator by using partial fraction whenever the series is in the form of fraction or T_n is in the form of fraction.

For example, let us suppose a summation where the n^{th} term is like the following:

$$T_n = \frac{2}{n^2 - 1}$$

Using the partial fraction, we can write the nth term as $T_n = \frac{1}{n-1} - \frac{1}{n+1}$

Now, when we find the summation, there will be telescopic cancellation and thus we will get the sum of the given series.

Type 5: Dealing with Sn^4: This technique is valid for Σn^2 and Σn^3. In this type, there is a series in which each term is composed of factors in an AP, i.e. factors of several terms being in AP.

$$T_n = \frac{1}{5}[(n+1)(n+2)(n+3)(n+4)[n-(n-1)]] = \frac{1}{5}(n(n+1)(n+2)(n+3)(n+4) - (n-1)(n+1)(n+2)(n+3)(n+4))$$

$$T_1 = \frac{1}{5}(1\cdot2\cdot3\cdot4\cdot5-0)$$

$$T_2 = \frac{1}{5}(2\cdot3\cdot4\cdot5\cdot6-1\cdot2\cdot3\cdot4\cdot5)$$

$$T_3 = \frac{1}{5}(3\cdot4\cdot5\cdot6\cdot7-2\cdot3\cdot4\cdot5\cdot6)$$

$$T_n = = \frac{1}{5}(n(n+1)(n+2)(n+3)(n+4) - (n-1)(n+1)(n+2)(n+3)(n+4))$$

Adding all, we have

$$S_n = \frac{1}{5}\left(n(n+1)(n+2)(n+3)(n+4)\right)$$

Note: This method will be applicable only when the series looks like the following:

a(a + d) (a+ 2d) + (a + d) (a + 2d) (a + 3d) + (a + 2d) (a + 3d) (a + 4d)+....+ up to n term, where a = first term and d = common difference

PLANCESS CONCEPTS

- $$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

  $$\frac{1}{1.2.3}+\frac{1}{2.3.4}+....+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

  $$\frac{1}{a_1a_2....a_r}+\frac{1}{a_2a_3....a_{r+1}}+.....+\frac{1}{a_na_{n+1}....a_{n+r-1}}=\frac{1}{(r-1)(a_2-a_1)}\left[\frac{1}{a_1a_2....a_{r-1}}-\frac{1}{a_{n+1}a_{n+2}....a_{n+r-1}}\right]$$

- $$a_1a_2....a_r + a_2a_3....a_{r+1} +.....+ a_na_{n+1}....a_{n+r-1} = \frac{1}{(r+1)(a_2-a_1)}[a_na_{n+1}....a_{n+r} - a_0a_1a_2....a_n]$$

  Where $a_1a_2....a_n$ are in AP and $a_0 = a_1 - d$

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8. HARMONIC PROGRESSION

A sequence will be in harmonic progression (HP) if the reciprocals of its terms are in AP, e.g. if a_1, a_2, a_3, ... are in HP, then $\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3}$ .... are in AP. For every AP, there will be a corresponding HP, and the standard H.P. is $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d}+...........+\frac{1}{a+(n-1)d}$.

The terms of a harmonic series are the outcomes of an AP.

Note:

(i) 0 cannot be a term of H.P. because ∞ is not a term of AP, but ∞ can be a term of HP.

(ii) There is no general formula for finding the sum to n terms of HP.

(iii) If a, b and c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad \Rightarrow \quad b = \frac{2ac}{a+c}$$

$\Rightarrow$ a, b and c are in HP

Moreover, $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$ i.e. $\frac{a-b}{ab} = \frac{b-c}{bc}$; i.e. $\frac{a}{c} = \frac{a-b}{b-c}$

Illustration 39: If the 3rd, 6th and last terms of a H.P. are $\frac{1}{3}, \frac{1}{5}, \frac{3}{203}$, then find the number of terms. **(JEE MAIN)**

Sol: If n^{th} term of a H.P. is $\frac{1}{a}$, then the n^{th} term of the corresponding AP will be a. Thus, by using

$T_n = a + (n-1)d$, we will get the result.

Let a be the first term and d be the common difference of the corresponding AP.

If the 3rd term of H.P. = $\frac{1}{3}$; then the 3rd term of the corresponding AP = 3

$\Rightarrow \quad a + 2d = 3$ ..... (i)

If the 6th term of H.P. = $\frac{1}{5}$; then the 6th term of the corresponding AP = 5

$\Rightarrow \quad a + 5d = 5$ ..... (ii)

From (i) and (ii), we get $d = \frac{2}{3} \Rightarrow a = \frac{5}{3}$

If the n^{th} term of H.P. = $\frac{3}{203}$; then n^{th} term of AP = $\frac{203}{3}$

$$a + (n-1)d = \frac{203}{3}; \qquad \frac{5}{3} + (n-1)\frac{2}{3} = \frac{203}{3}$$

$5 + 2n - 2 = 203; \quad n = 100$

Illustration 40: If $a_1, a_2, \ldots\ldots\ldots a_n$ are in H.P. then the expression $a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n$ is equal to.

(JEE ADVANCED)

Sol: As $\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots \frac{1}{a_n}$ are in AP, taking $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$, we can obtain the values of a_1a_2, a_2a_3 and so on.

$a_1, a_2, \ldots\ldots a_n$ are in HP

$\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots \frac{1}{a_n}$ are in AP

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d \text{ (say)}$$

$$\Rightarrow \quad a_1a_2 = \frac{1}{d}(a_1 - a_2),\ a_2a_3 = \frac{1}{d}(a_2 - a_3), \ldots\ldots, a_{n-1}a_n = \frac{1}{d}(a_{n-1} - a_n)$$

Hence, $a_1a_2 + a_2a_3 + ...+a_{n-1}a_n = \frac{1}{d}[a_1 - a_2 + a_2 - a_3 + ...+a_{n-1} - a_n] = \frac{1}{d}(a_1 - a_n)$

But $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)\,d \Rightarrow \frac{a_1 - a_n}{a_n a_1} = (n-1)\,d$

$\therefore \quad a_1a_2 + a_2a_3 + ... + a_{n-1}a_n = (n-1)a_1a_n$

8.1 Harmonic Mean

If a, b and c are in HP, then the middle term is called the harmonic mean (HM) between them. If H is the HM between a and b, then a, H, c are in H.P. and $H = \frac{2ac}{a+c}$.

To Insert n HMs Between a and b

Let $H_1, H_2, ..., H_n$ be the n HMs between a and b.

Thus, a, $H_1, H_2, ..., H_n$ b are in HP.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} .... \frac{1}{H_n}, \frac{1}{b}$ are in AP.

$\Rightarrow \frac{1}{b} = \frac{1}{a} + (n+1)d; \frac{1}{b} - \frac{1}{a} = (n+1)\,d;\ d = \frac{a-b}{ab(n+1)}$

$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + d \qquad \Rightarrow \frac{1}{H_2} = \frac{1}{a} + 2d$

$\Rightarrow \frac{1}{H_3} = \frac{1}{a} + 3d \qquad \Rightarrow \frac{1}{H_n} = \frac{1}{a} + nd$

Adding all, we get

$$\sum_{i=1}^{n} \frac{1}{H_i} = \frac{n}{a} + \frac{d(n)(n+1)}{2} = \frac{n}{a} + \frac{n(n+1)}{2}\frac{(a-b)}{ab(n+1)} = n\left[\frac{1}{a} + \frac{a-b}{2ab}\right] = \frac{n}{2ab}[2b + a - b] = \frac{n(a+b)}{2ab} = n\frac{1}{H}$$

Note: The sum of the reciprocals of all the n HMs between a and b is equal to n times the reciprocal of the single HM between a and b.

For example, between 1 and $\frac{1}{100}$ if 100 HMs are inserted, then $\sum_{i=1}^{100} \frac{1}{H_i} = 5050$.

8.2 Sum of the Reciprocal of 'n' Harmonic Means

The sum of reciprocal of n harmonic means $= \frac{n(a+b)}{2ab}$

To Insert n Harmonic Means Between a and b

a, $H_1, H_2, H_3 ....H_n$, b $\rightarrow$ H.P.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3} .... \frac{1}{H_n}, \frac{1}{b} \rightarrow$ A.P.

$\frac{1}{b} = \frac{1}{a} + (n+1)\,d \Rightarrow (n+1)\,d = \frac{a-b}{ab}$

$d = \dfrac{a-b}{(n+1)ab}$

Illustration 41: Find the sum of $\dfrac{1}{H_1}+\dfrac{1}{H_2}+\dfrac{1}{H_3}........\dfrac{1}{H_n}$.

Sol: Using $\dfrac{1}{H_n}=\dfrac{1}{a}+nd$, we can obtain the values of $\dfrac{1}{H_1}$, $\dfrac{1}{H_2}$ and so on. Then, by obtaining the value of $\sum\limits_{n=1}^{n}\dfrac{1}{H_n}$, we will get the result.

$$\frac{1}{H_1}=\frac{1}{a}+d \qquad \frac{1}{H_2}=\frac{1}{a}+2d$$

$$\frac{1}{H_n}=\frac{1}{a}+nd \quad\Rightarrow\quad \sum_{n=1}^{n}\frac{1}{H_i}=\frac{n}{a}+\frac{n(n+1)}{2}d$$

$$=\frac{n}{a}+\frac{n(n+1)\times(a-b)}{2(n+1)ab}=\frac{n}{a}\frac{(2b+a-b)}{2b}=\frac{n}{2ab}(a+b)$$

(i) For 3 numbers a, b and c, HM is defined as the reciprocals of the mean of the reciprocals of a, b and c, i.e. means of reciprocal = $\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$; HM = $\dfrac{3}{\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)}$

(ii) If a_1, a_2, a_3, ..., a_n are n numbers, then

$$AM = \left(\left|\frac{a_1+a_2+a_3+a_3+........+a_n}{n}\right|\right)$$

$$GM = (a_1\, a_2\, a_3\, ...a_n)^{1/n}$$

$$HM = \left(\frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_2}+.......\dfrac{1}{a_n}}\right)$$

Illustration 42: If a^2, b^2, c^2 are in AP, then show that b + c, c + a, a + b are in HP.

Sol: Given that a^2, b^2 and c^2 are in AP. Thus, by adding ab + ac + bc to each term and then dividing each term by (a + b)(b + c)(c + a), we will get the result.

By adding ab + ac + bc to each term, we find that $a^2 + ab + ac + bc$, $b^2 + ba + bc + ac$, $c^2 + ca + cb + ab$ are in AP, i.e.

(a + b)(a + c), (b + c)(b + a), (c + a)(c + b) are in AP

$\therefore$ Dividing each terms by (a + b)(b + c)(c + a), we find that

$\dfrac{1}{b+c},\dfrac{1}{c+a},\dfrac{1}{a+b}$ are in AP, i.e.

b + c, c + a, a + b are in HP

Illustration 43: If H_1, H_2, ..., H_n are n harmonic means between a and b ($\neq$ a), then find the value of $\dfrac{H_1+a}{H_1-a}+\dfrac{H_n+b}{H_n-b}$.

Sol: As a, H_1, H_2, ..., H_n, b are in HP, $\dfrac{1}{a},\dfrac{1}{H_1},\dfrac{1}{H_2}.......\dfrac{1}{H_n},\dfrac{1}{b}$ are in AP. By considering d as the common difference of this

AP and using $T_n = a + (n - 1)\,d$ we can solve this problem.

$\frac{1}{b} = \frac{1}{a} + (n + 1)d$ and $\frac{1}{H_n} - \frac{1}{H_1} = (n - 1)d$

Now, $$\frac{H_1 + a}{H_1 - a} = \frac{1/a + 1/H_1}{1/a - 1/H_1} = \frac{1/a + 1/H_1}{-d}$$

and $$\frac{H_n + b}{H_n - b} = \frac{1/b + 1/H_n}{1/b - 1/H_n} = \frac{1/b + 1/H_n}{d}$$

$$\therefore \quad \frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = \frac{1/a + 1/H_1}{-d} + \frac{1/b + 1/H_n}{d} = \frac{1}{d}\left[\left(\frac{1}{b} - \frac{1}{a}\right) + \left(\frac{1}{H_n} - \frac{1}{H_1}\right)\right] = 2n$$

9. RELATION BETWEEN AM, G.M. AND HM

If a and b are two positive numbers, then it can be shown that $A \geq G \geq H$ and A, G, H are in GP, i.e. $G^2 = AH$.

Proof: Given that, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab}$

$\Rightarrow \quad A - G = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0$

$\Rightarrow \quad A \geq G$ ......(i)

$G - H = \sqrt{ab} - \frac{2ab}{a+b}$

$\Rightarrow \quad G - H = \sqrt{ab}\left(\frac{a + b - 2\sqrt{ab}}{a+b}\right) \quad \Rightarrow \quad G - H = \frac{\sqrt{ab}}{a+b}(\sqrt{a} - \sqrt{b})^2 \geq 0$

$\Rightarrow \quad G \geq H$ ......(ii)

Using (i) and (ii), we find that

$A \geq G \geq H$

Please note that the equality holds only when a = b.

Proof of G^2 = AH

Proof: $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

Now, $AH = ab = G^2 \quad \Rightarrow \quad$ A, G & H are in G.P.

Moreover, $\frac{A}{G} = \frac{G}{H}$; $\quad \therefore \quad A \geq G \quad \Rightarrow \quad G \geq H$

Therefore, $A \geq G \geq H$; in fact, RMS ≥ A.M. ≥ G.M. ≥ HM (where RMS is root mean square).

PLANCESS CONCEPTS

- If a and b are two positive quantities, then AM, G.M. and HM are always in GP, i.e. only for two numbers.
- If there are three numbers. then AM, G.M. and HM are in G.P. only when the three numbers. are in GP.

  For example, 2, 4, 8 $\rightarrow$ GP

  GM = 4; A.M. = $\frac{14}{3}$; HM = $\frac{24}{7}$
- For two positive numbers, it has been shown that $A \geq G \geq H$, equality holding for equal numbers.
- For n non-zero positive numbers, it has been shown that $A \geq G \geq H$, equality holding when all the numbers are equal.

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Illustration 44: If a, b and c are unequal positive numbers in HP, then prove that

$$\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4.$$ **(JEE ADVANCED)**

Sol: As a, b and c are in HP, therefore $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$. Thus, by substituting this to LHS, we can prove the given problem.

$$\text{LHS} = \frac{\frac{1}{b}+\frac{1}{a}}{\frac{2}{b}-\frac{1}{a}}+\frac{\frac{1}{b}+\frac{1}{c}}{\frac{2}{b}-\frac{1}{c}}=\frac{\frac{1}{b}+\frac{1}{a}}{\frac{1}{c}}+\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}},\ \text{using (i)}$$

$$=\frac{c}{b}+\frac{c}{a}+\frac{a}{b}+\frac{a}{c}=\frac{a+c}{b}+\frac{a}{c}+\frac{c}{a}.$$

Now, A.M. > G.M. $\Rightarrow \frac{\frac{a}{c}+\frac{c}{a}}{2}>\sqrt{\frac{a}{c}.\frac{c}{a}}$ or $\frac{a}{c}+\frac{c}{a}>2.$

$$\frac{a+c}{b}=\frac{a+c}{\frac{2ac}{a+c}}=\frac{\left(a+c\right)^2}{2ac}=\frac{\left(a-c\right)^2}{2ac}+2$$

$$\therefore\ \text{LHS} = \frac{a+c}{b}\left(\frac{a}{c}+\frac{c}{a}\right)>2+2=4$$

10. PROPERTIES OF AM, G.M. AND HM

(i) The equation with a and b as its roots is $x^2 - 2Ax + G^2 = 0$

Proof: The equation with a and b as its roots is $x^2 - (a + b)\,x + ab = 0$

$\Rightarrow \quad x^2 - 2Ax + G^2 = 0 \qquad (\therefore A = \frac{a+b}{2}, G = \sqrt{ab})$

(ii) If A, G and H are the arithmetic, geometric and harmonic means, respectively, between three given numbers a, b and c, then the equation having a, b, c as its roots is $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

Proof: As given, A = $\frac{a+b+c}{3}$, G = $(abc)^{1/3}$ and $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$

$\Rightarrow$ a + b + c = 3A, abc = G^3 and $\frac{3G^3}{H}$ = ab + bc + ca

The equation having a, b and c as its roots is $x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc = 0$

$\Rightarrow \quad x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

Illustration 45: The harmonic means between two numbers is given as 4, their A.M. is A, and G.M. is G, satisfy the relation $2A + G^2 = 27$. Determine the two numbers. **(JEE ADVANCED)**

Sol: Let a and b be the two numbers and H = 4 be the harmonic mean between them. Therefore, by using A.M. = $\frac{a+b}{2}$ and G.M. = $\sqrt{ab}$, we can obtain the values of a and b.

H = 4 (given)

As A, G and H are in GP, therefore $G^2 = AH \Rightarrow \quad G^2 = 4A$

Also, $2A + G^2 = 27$ (given; $\therefore G^2 = 4A$)

$\therefore \quad 6A = 27$

$\Rightarrow \quad A = \frac{9}{2} \quad \Rightarrow \quad \frac{a+b}{2} = \frac{9}{2} \Rightarrow \quad a + b = 9$

We have, $G^2 = 4A$ and $A = 9/2 \Rightarrow \quad G^2 = 18 \quad \Rightarrow \quad ab = 18$

The quadratic equation having a and b as its roots is $x^2 - (a + b)x + ab = 0$ or, $x^2 - 9x + 18 = 0$

$\Rightarrow x = 3, 6$

Thus, the two numbers are 3 and 6.

Illustration 46: If 2a + b + 3c = 1 and a > 0, b > 0, c > 0, then find the greatest value of $a^4b^2c^2$ and obtain the corresponding values of a, b and c. **(JEE ADVANCED)**

Sol: Since there is a^4,. take four equal parts of 2a; as there is b^2, take two equal parts of b; as there is c^2, take two equal parts of 3c. Since A.M. ≥ G.M., obtaining A.M. and G.M. of these numbers will help in solving this illustration.

Let us consider the positive numbers $\frac{2a}{4}, \frac{2a}{4}, \frac{2a}{4}, \frac{2a}{4}, \frac{b}{2}, \frac{b}{2}, \frac{3c}{2}, \frac{3c}{2}$.

For the numbers, A = $\frac{\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{b}{2}+\frac{b}{2}+\frac{3c}{2}+\frac{3c}{2}}{4+2+2} = \frac{2a+b+3c}{8} = \frac{1}{8}$

($\therefore$ 2a + b + 3c = 1)

$$G = \left(\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{b}{2}.\frac{b}{2}.\frac{3c}{2}.\frac{3c}{2}\right)^{\frac{1}{8}} = \left(\frac{1}{2^4}.\frac{1}{2^2}.\frac{1}{2^2}.3^2a^4b^2c^2\right)^{\frac{1}{8}}$$

$$\therefore \quad A \ge G \quad \Rightarrow \quad \frac{1}{8} \ge \left(\frac{3^2}{2^8}a^4b^2c^2\right)^{\frac{1}{8}}$$

$$\text{or} \quad \frac{1}{8^8} \ge \frac{3^2}{2^8}a^4b^2c^2 \quad \text{or} \quad \frac{2^8}{3^2.8^8} \ge a^4b^2c^2 \quad \text{or} \quad \frac{1}{9.4^8} \ge a^4b^2c^2.$$

Hence, the greatest value of $a^4b^2c^2 = \dfrac{1}{9.4^8}$

It has been found that when the equality holds, the greatest value takes place.

We know that A = G when all the numbers are equal, i.e.

$$\frac{2a}{4} = \frac{b}{2} = \frac{3c}{2} \qquad \Rightarrow \qquad a = b = 3c$$

$$\therefore \quad \frac{a}{3} = \frac{b}{3} = \frac{c}{1} = k \qquad \therefore \qquad a = 3^k,\ b = 3k,\ c = k$$

$$\therefore \quad 2a + b + 3c = 1 \qquad \Rightarrow \qquad 6k + 3k + 3k = 1$$

$$\therefore \quad k = \frac{1}{12} \qquad \therefore \qquad a = \frac{3}{12}, b = \frac{3}{12}, c = \frac{1}{12}, \text{ i.e. } a = \frac{1}{4}, b = \frac{1}{4}, c = \frac{1}{12}$$

Arithmetic Mean of the m^{th} power

Suppose $a_1, a_2, \ldots, a_n$ be n positive real numbers (not all equal) and let m be a real number, then

$$\frac{a_1^m + a_2^m + \ldots a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots a_n}{n}\right)^m, \text{ if } m \in R - [0, 1]$$

If $m \in (0, 1)$, then $\dfrac{a_1^m + a_2^m + \ldots a_n^m}{n} < \left(\dfrac{a_1 + a_2 + \ldots a_n}{n}\right)^m$

Thus, if $m \in \{0, 1\}$, then $\dfrac{a_1^m + a_2^m + \ldots a_n^m}{n} = \left(\dfrac{a_1 + a_2 + \ldots a_n}{n}\right)^m$

PROBLEM-SOLVING TACTICS

(a) When looking for a pattern in a sequence or series, writing out several terms will help you see the pattern, do not simplify directly. If you do this way, it is often easier to spot the pattern (if you leave terms as products, sums, etc.).

(b) If each term of an AP is multiplied by (or divided by a non-zero) fixed constant C, the resulting sequence is also an AP, with a common difference C times $\left(\text{or } \frac{1}{c}\text{times}\right)$ the previous.

(c) **Tips for AP problems**

(i) When the number of terms are three, then we take the terms as a – d, a, a + d;

Five terms as a - 2d, a - d, a, a + d, a + 2d

Here, we take the middle term as 'a' and common difference as 'd'.

(ii) When the number of terms is even, then we take:

Four terms as a – 3d, a – d, a + d, a + 3d;

Six terms as a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d

Here, we take 'a – d' and 'a + d' as the middle terms and common difference as '2d'.

(iii) If the number of terms in an AP is even, then take the number of terms as 2n and if odd then take it as (2n + 1).

(d) Tips for G.P. problems

(i) When the number of terms is odd, then we take three terms as a/r, a, ar; five terms as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.
Here, we take the middle term as 'a' and common ratio as 'r.'

(ii) When the number of terms is even, then we take four terms as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$; six terms as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$.
Here, we take $\frac{'a'}{r}$ and 'ar' as the middle terms and common ratio as 'r^2.'

(e) Tips for H.P. problems

For three terms, we take as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

For four terms, we take as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

For five terms, we take as $\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$

FORMULAE SHEET

Arithmetic Progression: Here, a, d, A and S_n represent the first term, common difference, A.M. and sum of the numbers, respectively, and T_n stands for the nth term.

1.	$T_n = a + (n - 1)d$	4.	$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$
2.	$T_n = \frac{T_{n-1} + T_{n+1}}{2}$	5.	$A = \frac{(a_1 + a_2 + \ldots + a_n)}{n}$
3.	$S_n = \frac{n}{2}(a + T_n)$	6.	Insertion of n arithmetic means between a and b is $A_n = a + \frac{n(b-a)}{n+1}$

Geometric Progression: Here, a, r, S_n and G represent the first term, common ratio, sum of the terms and G.M., respectively, and T_n stands for the nth term.

1.	$T_n = a.r^{n-1}$	4.	$S_n = \frac{a(r^n - 1)}{r-1}$
2.	$T_n = \sqrt{T_{n-1} \cdot T_{n+1}}$	5.	$S_\infty = \frac{a}{1-r}$ (for $-1 < r < 1$)
3.	$S_n = \frac{T_{n+1} - a}{r-1}$	6.	Insertion of n geometric means between a and b is $G_1 = ar$, $G_2 = ar^2$ ....$G_n = ar^n$ or $G_n = b/r$, where $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Arithmetic Geometric Progression: Here, a = the first term of AP, b = the first term of GP, d = common difference and r = common ratio of GP.

1.	$S_n = ab + (a + d)br + (a + 2d)br^2 + (a + 3d)br^3 + ...$
2.	$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}$
3.	$S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$ (for $-1 < r < 1$)

Harmonic Progression

1. $a_n = \frac{1}{a+(n-1)d}$, where $a = \frac{1}{a_1}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$

2. $\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right)$

3. Insertion of n harmonic means between a and b

$$\frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab}$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab} \text{ and so on } \Rightarrow \left[\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}\right]$$

1.	The sum of n natural numbers	$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$
2.	The sum of n odd natural numbers	$\sum_{r=1}^{n} (2r-1) = n^2$
3.	The sum of n even natural numbers	$\sum_{r=1}^{n} 2r = n(n+1)$
4.	The sum of squares of n natural numbers	$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$
5.	The sum of cubes of n natural numbers	$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

Solved Examples

JEE Main/Boards

Example 1: Find the r^{th} term if the p^{th} term of an AP is q and the q^{th} term is p .

Sol: Using $T_n = a + (n - 1) d$, we can obtain the p^{th}, q^{th} and r^{th} terms.

Let the initial term and common difference of the given AP be a and d, respectively.

As given,

$q = a + (p - 1) d$ ... (i)

$p = a + (q - 1) d$ ... (ii)

Subtracting (i) by (ii), we find that

$q - p = (p - q) d$

$\therefore d = -1$

Putting $d = -1$ in (i), we get

$a = q + p - 1$

$\therefore t_r = a + (r - 1)d$

$= (q + p - 1) - r + 1 = p + q - r$

Example 2: Find out the number of terms in a given AP 20, 25, 30, 35, ... 100.

Sol: We know that $T_n = a + (n - 1) d$.

Given, $a = 20$, $d = 5$ and $T_n = 100$. Therefore, by solving the equation, we will get the number of terms.

Let the number of terms be n.

Given, $T_n = 100$, $a = 20$, $d = 5$

$T_n = a + (n - 1) d$

$\Rightarrow 100 = 20 + (n - 1) 5 \Rightarrow 80 = (n - 1) 5$

$\Rightarrow 16 = (n - 1) \Rightarrow n = 17$

Example 3: Solve the following series:

99 + 95 + 91 +87 + ... to 20 terms

Sol: Using $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, we can solve the given problem.

We know that the terms of the given series are in AP. Given,

$D = -4$, $a = 99$ and $n = 20$

$$\because S_n = \frac{n\left(2a + (n-1)d\right)}{2}$$

$$S_{20} = \frac{20}{2}[2 \times 99 + (20 - 1)(-4)]$$

$= 10[198 + 19 \times (-4)] = 10(198 - 76) = 1220$

Example 4: Find out the G.P. if the fifth and second terms of a G.P. are 81 and 24, respectively.

Sol: We know that in GP, the n^{th} term is given by $T_n = a.r^{n-1}$. Thus, by using this formula, we can find the GP.

Given, $T_5 = 81$ and $T_2 = 24$

$\therefore \quad 81 = ar^4$ ... (i)

and $24 = ar$ ... (ii)

Dividing (i) by (ii), we get

$$\frac{81}{24} = r^3 \Rightarrow r^3 = \frac{27}{8} \Rightarrow r^3 = \left(\frac{3}{2}\right)^3 \Rightarrow r = \frac{3}{2}$$

Substituting the value of r in (ii), we get, a = 16

Thus, the required G.P. is 16, 24, 36, 54 ,....

Example 5: If the sum of four numbers in AP is 50 and the greatest of them is four times the least, then find the numbers.

Sol: Let the four numbers in AP be a, a +d, a + 2d, a +3d with d > 0.

As given, sum of the numbers is 50.

$a + (a + d) + (a + 2d) + (a + 3d) = 50$

$\therefore 4a + 6d = 50$

$\Rightarrow 2a + 3d = 25$ ...(i) and

$a + 3d = 4a$

$\Rightarrow 3d = 3a$

$\therefore d = a$

$\therefore$ Equation (i) becomes $5a = 25$

Thus, $a = 5 = d$

Therefore, the four number are 5, 10, 15 and 20.

Example 6: If $S_1, S_2, S_3, \ldots, S_p$ are the sum of p infinite geometric progression whose first terms are 1, 2, 3,...., p and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{p+1}$, respectively, then prove that $S_1 + S_2 + ... + S_p = \frac{p(p+3)}{2}$.

Sol: As we know $S_\infty = \frac{a}{1-r}$, therefore by using this formula we can obtain the value of $S_1, S_2, ...S_p$.

We know that $S_\infty = \frac{a}{1-r}$

$$\therefore \quad S_1 = \frac{1}{1-\frac{1}{2}} = 2;\ S_2 = \frac{2}{1-\frac{1}{3}} = 3$$

$$S_p = \frac{p}{1-\frac{1}{p+1}} = p+1$$

$$S_1 + S_2 + ... + S_p = \frac{p}{2}[2\times 2 + (p-1)1] = \frac{p}{2}[p+3]$$

Example 7: Solve the series $1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + ... + 100\cdot 2^{99}$.

Sol: Let $S = 1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + ... + 100\cdot 2^{99}$. Therefore, by multiplying 2 on both the sides and then taking the difference, we can solve the given problem.

Given,

$$S = 1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + ... + 100\cdot 2^{99} \quad ... (i)$$

Multiplying 2 on both the sides,

$$2S = 1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + ... + 99\cdot 2^{99} + 100\cdot 2^{100} \quad ... (ii)$$

Subtracting (ii) from (i), we get

$$-S = 1 + 1\cdot 2 + 1\cdot 2^2 + 1\cdot 2^3 + ... + 1\cdot 2^{99} - 100\cdot 2^{100}$$

$$-S = \frac{1-2^{100}}{1-2} - 100\cdot 2^{100};$$

$$\Rightarrow S = 99\cdot 2^{100} + 1$$

Example 8: If $(5n + 4) : (9n + 6)$ is the ratio of the sums of the n^{th} terms of two APs, then find out the ratio of their 13^{th} terms.

Sol: Let a_1 and a_2 be the first terms of the two APs and d_1 and d_2 be their respective common difference.

Applying $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, we can solve the given problem.

Given,

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{a_1 + \frac{n-1}{2}d_1}{a_2 + \frac{n-1}{2}d_2} = \frac{5n+4}{9n+6} \quad ... (i)$$

The ratio of the 13^{th} terms is $\frac{a_1 + 12d_1}{a_2 + 12d_2}$ [which is obtained from (i) with n = 25]

$$\therefore \quad \frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{129}{231}$$

Example 9: If the 7^{th} and 8^{th} terms of an H.P. are 8 and 7, respectively, then find its 15^{th} term.

Sol: We know that $t_n = \frac{1}{a + (n-1)d}$. Therefore, by using this formula we can solve the given problem.

Given, $T_7 = 8$ =and $T_8 = 7$

$$\therefore \frac{1}{a+6d} = 8 \Rightarrow 8a + 48d - 1 = 0 \quad ..(i)$$

$$\frac{1}{a+7d} = 7 \Rightarrow 7a + 49d - 1 = 0 \quad ..(ii)$$

By solving these two equations, we find that d = a

$$\therefore \text{From eq.(i), we get } \frac{1}{7a} = 8$$

$$\Rightarrow a = d = \frac{1}{56}$$

$$\therefore \quad T_{15} = \frac{1}{a+14d} = \frac{56}{15}$$

Example 10: Suppose x y and z are positive real numbers, which are different from 1.

If $x^{18} = y^{21} = z^{28}$, then show that 3, $3\log_y(x)$. $3\log_z(y)$ and $7\log_x(z)$ are in AP.

Sol: By applying log on $x^{18} = y^{21} = z^{28}$, we can find the values of $\log_y x$, $\log_z y$ and $\log_x z$.

Given, $x^{18} = y^{21} = z^{28}$

Taking log, we find that

$$18 \log x = 21 \log y = 28 \log z$$

$$\log_y x = \frac{\log x}{\log y} = \frac{7}{6}$$

$$\Rightarrow 3\log_y x = \frac{7}{2} \quad ... (i)$$

$$\log_z y = \frac{\log y}{\log z} = \frac{4}{3}; \quad 3\log_z y = 4 \quad ... (ii)$$

$\log_x z = \dfrac{\log z}{\log x} = \dfrac{9}{14}$

$\Rightarrow 7\log_x z = \dfrac{9}{2}$ ... (ii)

The numbers 3, $\dfrac{7}{2}$, 4 and $\dfrac{9}{2}$ are in AP with common difference = $\dfrac{1}{2}$.

$\therefore$ 3, $\log_y x$, $3\log_z y$ and $7\log_x z$ are in AP.

Example 11: If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$ and if a, b and c are positive and in GP, then prove that x, y and z are in AP.

Sol: This problem can be solved by taking log on $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$.

$a^{1/x} = b^{1/y} = c^{1/z}$

$\Rightarrow \quad \dfrac{\log a}{x} = \dfrac{\log b}{y} = \dfrac{\log c}{z} = k$

$\Rightarrow \quad \log a = kx, \log b = ky, \log c = kz$ ... (i)

a, b and c are in GP

$\Rightarrow \quad b^2 = ac$

$\therefore \quad 2\log b = \log a + \log c$

$\Rightarrow \quad 2ky = kx + kz$ by (i)

$\Rightarrow \quad 2y = x + z$

$\Rightarrow$ x, y and z are in AP.

Example 12: Determine the relation between x, y and z if 1, $\log_y x$, $\log_z y$, $-15\log_x z$ are in AP.

Sol: By considering the common difference as d and obtaining its value by $\log_y x = 1 + d$ and $\log_z y = 1 + 2d$, we can determine the required relation.

Suppose d be the common difference of the given AP, then

$\log_y x = 1 + d \Rightarrow x = y^{1+d}$ ... (i)

$\log_z y = 1 + 2d \Rightarrow y = z^{1+2d}$ ... (ii)

$15\log_x z = -(1 + 3d)$

$\Rightarrow \quad z = x^{\frac{1+3d}{-15}}$ ... (iii)

Elimination y and z from equations (i), (ii) and (iii), we get

$x = x^{\frac{(1-d)(1+2d)(1+3d)}{-15}}$

$\therefore \quad 1 = \dfrac{(1+d)(1+2d)(1+3d)}{-15}$

or (1 + d) (1 + 2d) (1 + 3d) + 15 = 0

or $(d + 2)(6d^2 + 5d + 8) = 0$

$\Rightarrow \quad d = -2$

The other factors do not give any real solution.

$\therefore\ x = y^{-1}, y = z^{-3}, z = x^{1/3}$ or $\quad x = y^{-1} = z^3$

Example 13: There are four numbers of which the first three are in G.P. and the last there are in AP, with a common difference of 6. If the first number and the last number are equal, then find the numbers.

Sol: Let the four numbers be a, a – 2d, a – d, a, where d = 6

a, a – 12, a – 6 are in GP.

$\Rightarrow \quad a(a - 6) = (a - 12)^2$

$\Rightarrow \quad a^2 - 6a = a^2 - 24a + 144$

$\Rightarrow \quad 18a = 144$

$\Rightarrow \quad a = 8$

The numbers are 8, – 4, 2 and 8.

Example 14: a, b and c are the p^{th}, q^{th} and r^{th} terms of both an AP and a GP, respectively, then prove that $a^{b-c}.\ b^{c-a}.\ c^{a-b} = 1$ (both progressions have the same first term.)

Sol: By using formula $T_n = a + (n - 1)\, d$ and $T_n = a.r^{n-1}$, we can obtain the p^{th}, q^{th} and r^{th} terms of both an AP and a GP.

$T_p = a = a_1 + (p - 1)\, d_1 = a_1(r_1)^{p-1}$ ... (i)

$T_q = b = a_1 + (q - 1)d_1 = a_1(r_1)^{q-1}$ ... (ii)

$T_r = c = a_1 + (r - 1)d_1 = a_1(r_1)^{r-1}$ ... (iii)

From (i), (ii), (iii)

$a - b = (p - q)\, d_1$

$b - c = (q - r)\, d_1$

$c - a = (r - p)\, d_1$

Therefore, $a^{b-c}.\ b^{c-a}.\ c^{a-b}$

$= (a_1 r_1^{p-1})^{b-c}\ (a_1 r_1^{q-1})^{c-a}\ (a_1 r_1^{r-1})^{a-b}$

$= a_1^{b-c+c-a+a-b}.r_1^{(p-1)(b-c)+(q-1)(c-a)+(r-1)(a-b)}$

$= a_1^0.\ r_1^{(p-1)(q-r)d_1+(q-1)(r-p)d_1+(r-1)(p-q)d_1}$

$= a_1^0.r_1^0 = 1$

JEE Advanced/Boards

Example 1: If the sum of first n terms of three arithmetic progressions are S_1, S_2 and S_3, the first term of each being 1 and the common differences being 1, 2 and 3, respectively, then prove that $S_1 + S_3 = 2S_2$

Sol: Using $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$, we can get the values of S_1, S_2 and S_3.

Given, a = 1, $d_1 = 1$, $d_2 = 2$, $d_3 = 3$

$S_1 = \frac{n}{2}[2a + (n - 1) d_1]$

$= \frac{n}{2}[2 \times 1 + (n - 1) 1] \qquad = \frac{n}{2}[1 + n]$

$S_2 = \frac{n}{2}[2a + (n - 1)d_2] = \frac{n}{2}[2 \times 1 + (n - 1) 2] = n^2$

$S_3 = \frac{n}{2}[2a + (n - 1) d_3] = \frac{n}{2}[2 \times 1 + (n - 1) 3]$

$= \frac{n}{2}[3n - 1]$

$S_1 + S_3 = \frac{n}{2}[1 + n + 3n - 1] = 2n^2 = 2S$

Example 2: Calculate the sum to n terms of the series: 8 +88 + 888 + ...

Sol: We can solve this problem by taking 8 as common from given series and applying various operations.

Let S_n = 8 + 88 + 888 +....... to n terms = 8 [1 + 11 + 111 + ...]

$= \frac{8}{9}$ [9 + 99 + 999 + ...] = $\frac{8}{9}$ [(10 – 1) + (100 – 1) +(1000 – 1) + ... to n terms]

$= \frac{8}{9}$ [10 + 100 + 1000 + ... + to n terms] – $\frac{8}{9}$ n = $\frac{8}{9}$

$\frac{(10^n-1)}{9} - \frac{8n}{9}$

$= \frac{8}{81}[10^{n+1} - 9n - 10]$

Example 3: If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in AP, where $a_i > 0$ for all I, then show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Sol: We can write x – y = p as

$\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right) = p$.

Thus, by following this method we can represent difference of $a_1, a_2, a_3, \ldots\ldots, a_n$.

$a_1, a_2, a_3, \ldots\ldots, a_n$ are in AP.

$a_2 - a_1 = a_3 - a_2 = \ldots\ldots\ldots = a_n - a_{n-1} = d$ (say)

$\Rightarrow \quad \left(\sqrt{a_2}+\sqrt{a_1}\right)\left(\sqrt{a_2}-\sqrt{a_1}\right)$

$= \left(\sqrt{a_3}+\sqrt{a_2}\right)\left(\sqrt{a_3}-\sqrt{a_2}\right)\ldots\ldots\ldots$

$= \left(\sqrt{a_n}+\sqrt{a_{n-1}}\right)\left(\sqrt{a_n}-\sqrt{a_{n-1}}\right) = d$

$\Rightarrow \frac{1}{\sqrt{a_2}+\sqrt{a_1}} = \frac{1}{d}\left(\sqrt{a_2}-\sqrt{a_1}\right), \frac{1}{\sqrt{a_3}+\sqrt{a_2}}$

$= \frac{\sqrt{a_3}-\sqrt{a_2}}{d}, \ldots\ldots\ldots$

$\frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}} = \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d}$

$\text{LHS} = \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots\ldots\ldots +$

$= \frac{1}{d}\left(\sqrt{a_n}-\sqrt{a_1}\right)$

$\because d = \frac{a_n - a_1}{n-1}$

$\Rightarrow \text{LHS} = \frac{\sqrt{a_n}-\sqrt{a_1}}{a_n - a_1}(n-1) = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}} = \text{RHS}$

Example 4: A series of natural numbers is divided into groups: (1); (2, 3, 4); (5, 6, 7, 8, 9) and so on. Prove that the sum of the numbers in the n^{th} group is $(n - 1)^3 + n^3$.

Sol: In this problem, the last term of each group is the square of the corresponding number of the group. Thus, the first term of the n^{th} group is $(n - 1)^2 + 1 = n^2 - 2n + 2$. Hence, by using $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$, we can solve the problem.

The number of terms in the first group = 1

The number of terms in the second group = 3

The number of terms in the third group = 5

$\therefore$ The number of terms in the n^{th} group = 2n–1

The common difference of the numbers in the n^{th} group = 1

The required sum = $\frac{2n-1}{2}$ $[2(n^2 - 2n + 2) + (2n - 2) 1]$

$= \frac{2n-1}{2}$ $[2n^2 - 2n + 2] = (2n - 1)[n^2 - n + 1]$

$= 2n^3 - 3n^2 + 3n - 1 = n^3 + (n - 1)^3$

Example 5: Find the sum of the series

$1 + \frac{1}{5}+\frac{3}{5^2}+\frac{5}{5^3}$ + ... to ∞.

Sol: We can write given series as $S_\infty = 1 + S_1$, where $S_1 = \frac{1}{5}+\frac{3}{5^2}+\frac{5}{5^3}$ + ... to ∞ Thus, by multiplying $\frac{1}{5}$ on both the sides and subtracting, we can obtain the required sum.

Let $S_\infty = 1 + \frac{1}{5}+\frac{3}{5^2}+\frac{5}{5^3}$ + ... to ∞

$\therefore$ $S_\infty = 1 + S_1$ ....(i)

Where

$\therefore$ $\frac{1}{5}S_1 = \frac{1}{5^2}+\frac{3}{5^3}+\frac{5}{5^4}$ + ... to ∞ ....(ii)

Subtracting, (ii) from (i)

$\therefore$ $\frac{4}{5}S_1 = \frac{1}{5} + 2\left(\frac{1}{5^2}+\frac{1}{5^3}+\text{.......to } \infty\right)$

$= \frac{1}{5}+2\frac{\frac{1}{5^2}}{1-\frac{1}{5}} = \frac{1}{5}+\frac{2}{4}\left(\frac{1}{5}\right) = \frac{3}{10}$,

$S_1 = \frac{3}{8}$

From (i), $S_\infty = 1+\frac{3}{8} = \frac{11}{8}$

Example 6: If $n \in N$ and $n > 1$, then prove that

(a) $n^n \geq 1.3.5$ ....$(2n - 1)$ (b) $2^n \geq 1 + n\,2^{\frac{n-1}{2}}$

Sol: (a) Use the inequality A.M. ≥ GM.

(b) By solving $\frac{1+2+2^2+....+2^{n-1}}{n} \geq (1.2.2^2 ....2_{n-1})^{1/n}$,

we can prove the given equation.

(a) $\therefore \frac{1+3+5+......+(2n-1)}{n} \geq \left[1.3.5\ ....(2n-1)\right]^{1/n}$

$\Rightarrow \frac{\frac{n}{2}\left[1+(2n-1)\right]}{n} \geq \left[1.3.5\ ....(2n-1)\right]^{1/n}$

$\Rightarrow n = \left[1.3.5\ ....(2n-1)\right]^{1/n}$

$\therefore n^n = 1.3.5\ ....(2n-1)$

(b) $\frac{1+2+2^2+....+2^{n-1}}{n} \geq (1.2.2^2 ....2_{n-1})^{1/n}$

$\Rightarrow \frac{2^n-1}{2-1}\times\frac{1}{n} \geq \left(2^{\frac{n(n-1)}{2}}\right)^{1/n}$

$\Rightarrow \frac{2^n-1}{n} \geq 2^{n-1/2}$

$\Rightarrow 2^n \geq n\sqrt{2^{n-1}}+1$

JEE Main/Boards

Exercise 1

Q.1 In a G.P. sum of n terms is 364. First term is 1 and common ratio is 3. Find n.

Q.2 The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of this infinite series in 24. Then find the series.

Q.3 Sum of n terms of the series,

(i) 0.7 + 0.77 + 0.777 + ...(ii) 6 + 66 + 666 + ...

Q.4 If a, b, c are in A.P., prove that

(i) b + c, c + a, a + b are also in A.P.

(ii) $\frac{1}{bc},\frac{1}{ca},\frac{1}{ab}$ are also in A.P.

(iii) $a^2(b + c), b^2(c + a), c^2(a + b)$ are also in A.P.

(iv) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are also in A.P.

Q.5 The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers.

Q.6 Find the sum of the integers between 1 and 200 which are

(i) Multiple of 3 (ii) Multiple of 7

(iii) Multiple of 3 and 7

Q.7 The sum of first n terms of two A.P.'s are in the ratio (3n – 3): (5n + 21). Find the ratio of their 24^{th} terms.

Q.8 If the p^{th} term of an A.P. is x and q^{th} term is y, show that the sum first (p + q) terms is $\frac{p+q}{2}\left\{x+y+\frac{x-y}{p-q}\right\}$

Q.9 If a, b, c are in H.P. prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

Q.10 Find the sum of n terms of the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\ldots$.

Q.11 Let a, b, c, d, e be five real numbers such that a, b, c are in A.P.; b, c, d are in G.P.; c, d, e are in H.P. If a = 2 and e = 18, find all possible values of b, c and d.

Q.12 Find the sum of first n terms of the series: $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots\ldots$

Q.13 Find the sum of first 2n terms of the series: $1^2 + 2 + 3^2 + 4 + 5^2 + 6 + \ldots.$

Q.14 The H.M of two numbers is 4 and their A.M. (A) and G.M. (G) satisfy the relation $2A + G^2 = 27$. Find the numbers.

Q.15 Find the sum of first 10 terms of the series: $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots..$

Q.16 Find the sum of first 20 terms of the series: $1\cdot3^2 + 2\cdot5^2 + 3\cdot7^2 + \ldots.$

Q.17 Find three numbers a, b, c between 2 and 18 such that:

(i) Their sum is 25.

(ii) The numbers 2, a, b are consecutive terms of an A.P. and

(iii) The numbers b, c, 18 are consecutive terms of a G.P.

Q.18 If a > 0, b > 0 and c > 0, prove that:

$(a + b + c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9$

Q.19 If A_1, A_2, G_1, G_2 ; and H_1, H_2 be two A.M.'s, G.M.'s and H.M's between two numbers, then prove that:

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Q.20 Find the coefficient of x^{99} and x^{98} in the polynomial: $(x - 1)(x - 2)(x - 3) \ldots (x - 100)$.

Q.21 The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon

Q.22 A number consists of three digits in G.P.. The sum of the digits at units and hundreds place exceeds twice the digit at tens place by 1 and the sum of the digits at tens and hundreds place is two third of the sum of the digits at tens and units place. Find the number.

Q.23 25 trees are planted in a straight line at intervals of 5 meters. To water them the gardener must bring water for each tree separately from a well 10 meters from the first tree in line with the trees. How far he will have to cover in order to water all the tree beginning with the first if he starts from the well.

Q.24 Natural numbers have been grouped in the following way 1 ; (2, 3) ; (4, 5, 6); (7, 8, 9, 10) ; ...... Show that the sum of the numbers in the nth group is $\frac{n(n^2+1)}{2}$.

Q.25 In three series of GP's, the corresponding numbers in G.P. are subtracted and the difference of the numbers are also found to be in G.P. Prove that the three sequences have the same common ratio.

Q.26 If a_1, a_2, a_3 ,.... Are in A.P such that $a_i \neq 0$, show that

$$S = \frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots..+\frac{1}{a_na_{n+1}} = \frac{n}{a_1a_{n+1}}$$

Also evaluate $\lim\limits_{a\to\infty} S$.

Q.27 If 9 arithmetic means and 9 harmonic means be inserted between 2 and 3, prove that $A+\frac{6}{H}=5$, where A is any arithmetic mean and H the corresponding harmonic mean.

Q.28 If x + y + z = 1 and x, y, z are positive numbers, show that $(1-x)(1-y)(1-z) \geq 8xyz$.

Q.29 Show that any positive integral power (greater than 1) of a positive integer m, is the sum of m consecutive odd positive integers. Find the first odd integer for $m^r (r > 1)$.

Exercise 2

Single Correct Choice Type

Q.1 If a, b, c are distinct positive real in H.P., then the value of the expression, $\frac{b+a}{b-a}+\frac{b+c}{b-c}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Q.2 The sum of infinity of the series

$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+........$ is equal to

(A) 2 (B) 5/2 (C) 3 (D) None

Q.3 Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried out the job starting with the stone in the middle, carrying stones in succession, thereby covering a distance of 4.8 km. Then the number of stones is

(A) 15 (B) 29 (C) 31 (D) 35

Q.4 If $S = 1^2 + 3^2 + 5^2 + ... + (99)^2$ then the value of the sum $2^2 + 4^2 + 6^2 + ... + (100)^2$ is

(A) S + 2550 (B) 2S (C) 4S (D) S +5050

Q.5 In an A.P. with first term and the common difference d (a, d ≠ 0), the ratio 'S' of the sum of the first n terms to sum of n terms succeeding them does not depend on n. Then the ratio "a/d" and the ratio 'ρ', respectively are

(A) $\frac{1}{2},\frac{1}{4}$ (B) $2,\frac{1}{3}$ (C) $\frac{1}{2},\frac{1}{3}$ (D) $\frac{1}{2},2$

Q.6 If x ∈ R, the numbers $(5^{1+x} + 5^{1-x})$, $a/2\ (25^x + 25^{-x})$ form an A.P. then 'a' must lie in the interval

(A) [1, 5] (B) [2, 5] (C) [5, 12] (D) [12, ∞]

Q.7 If the sum of the first 11 terms of an arithmetical progression equals that of the first 19 terms, then the sum of its first 30 terms, is

(A) Equal to 0 (B) Equal to – 1

(C) Equal to 1 (D) Non unique

Q.8 Let s_1, s_2, s_3 ..... and t_1, t_2, t_3 .... are two arithmetic sequence such that $s_1 = t_1 \neq 0$; $s_2 = 2t_2$ and $\sum_{i=1}^{10} s_i = \sum_{i=1}^{15} t_i$. Then the value of $\frac{s_2-s_1}{t_2-t_1}$ is

(A) 8/3 (B) 3/2 (C) 19/8 (D) 2

Q.9 Let a_n, n ∈ I be the n^{th} term an A.P. with common difference 'd' and all whose terms are non-zero. If n approaches infinity, then the sum

$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+.....+\frac{1}{a_na_{n+1}}$ will approach

(A) $\frac{1}{a_1d}$ (B) $\frac{2}{a_1d}$ (C) $\frac{1}{2a_1d}$ (D) a_1d

Q.10 The sum of the first three terms of an increasing G.P. is 21 and the sum of their squares is 189. Then the sum of its first n term is

(A) $3(2^n - 1)$ (B) $12\left(1-\frac{1}{2^n}\right)$

(C) $6\left(1-\frac{1}{2^n}\right)$ (D) $6(2^n - 1)$

Q.11 The sum $\sum_{n=1}^{\infty}\left(\frac{n}{n^4+4}\right)$ is equal to

(A) 1/4 (B) 1/3 (C) 3/8 (D) 1/2

Q.12 If a ≠ 1 and $(\ln a^2) + (\ln a^2)^2 + (\ln a^2)^3 + ... = 3[\ln a + (\ln a)^2 + (\ln a)^3 + (\ln a)^4 + ...]$, then 'a' is equal to

(A) $e^{1/5}$ (B) $e^{1/2}$ (C) $3e^{1/2}$ (D) $e^{1/4}$

Previous Years' Questions

Q.1 If a, b, c d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$, then a, b, c, d ***(1987)***

(A) Are in A.P. (B) Are in G.P.

(C) Are in H.P. (D) Satisfy ab = cd

Q.2 Sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+....$ is equal to

(A) $2^n - n - 1$ (B) $1 - 2^{-n}$

(C) $n + 2^{-n} - 1$ (D) $2^n + 1$

Q.3 If x > 1, y > 1, z > 1 are in G.P. then $\frac{1}{1+\ln x},\frac{1}{1+\ln y},\frac{1}{1+\ln z}$ are in ***(1998)***

(A) AP (B) H.Π. (C) G.Π. (D) None

Q.4 If a, b, c, d are positive real number such that a + b + c + d = 2, then M = (a + b) (c +d) satisfies the relation ***(2000)***

(A) $0 < M \le 1$ (B) $1 \le M \le 2$

(C) $2 \le M \le 3$ (D) $3 \le M \le 4$

Q.5 Let the positive numbers a, b, c, d be in A.P. then abc, abd, acd, bcd are ***(2001)***

(A) not in AP/GP/HP (B) in AP

(C) in GP (D) in HP

Q.6 Suppose a, b, c are in AP and a^2, b^2, c^2 are in G.P. If a < b < c and a + b + c = $\frac{3}{2}$, then the value of a is ***(2002)***

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2\sqrt{3}}$

(C) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (D) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

Q.7 An infinite G.P. has first term x and sum 5, then x belongs to ***(2004)***

(A) x < – 10 (B) – 10 < x < 0

(C) 0 < x < 10 (D) x > 10

Q.8 If the sum of first n terms of an AP is cn^2, then the sum of squares of these n terms is ***(2009)***

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$

(C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

Q.9 The first two terms of a geometric progression add up to 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is ***(2008)***

(A) –4 (B) –12 (C) 12 (D) 4

Q.10 The sum to the infinity of the series is

$1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+.......$ ***(2009)***

(A) 2 (B) 3 (C) 4 (D) 6

Q.11 If m is the A. M. of two distinct real numbers l and n(l, n > 1) and G1, G2 and G3 are three geometric means between l and n, then $G_1^4+2G_2^4+G_3^4$ equals: ***(2015)***

(A) $4\, l^2\, mm$ (B) $4\, lm^2 n$ (C) $4\, lmn^2$ (D) $4\, l^2m^2n^2$

Q.12 The sum of first 9 terms of the series is

$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+.......$ ***(2015)***

(A) 71 (B) 96 (C) 142 (D) 192

Q.13 If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is: ***(2016)***

(A) $\frac{4}{3}$ (B) 1 (C) $\frac{7}{3}$ (D) $\frac{8}{5}$

Q.14 If the sum of the first ten terms of the series is $\frac{16}{5}m$ then m is equal to $\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+.......$ ***(2016)***

(A) 101 (B) 100 (C) 99 (D) 102

Q.15 Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is ***(2014)***

(A) $2-\sqrt{3}$ (B) $2+\sqrt{3}$ (C) $\sqrt{2}+\sqrt{3}$ (D) 3 + 2

Assertion Reasoning Type

Q.16 Statement-I: The sum of the series 1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + ... + (361 + 380 + 400) is 8000.

Statement-II: $\sum_{k=1}^{n}(k^3-(k-1)^3)=n^3$ for any natural number n. ***(2012)***

(A) Statement-I is false, statement-II is true

(B) Statement-I is true, statement-II is true; statement-II is a correct explanation for statement-I

(C) Statement-I is true, statement-II is true; statement-II is not a correct explanation for statement-I

(D) Statement-I is true, statement-II is false

Q.17 Statement-I: The sum of the series 1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + ... + (361 + 380 + 400) is 8000.

Statement-II: $\sum_{k=1}^{n}(k^3-(k-1)^3)=n^3$ for any natural number n.

Q.18 If 100 times the 100th term of an AP with non zero common difference equals the 50 times its 50th term, then the 150th term of this AP is ***(2012)***

(A) –150 (B) 150 times its 50th term

(C) 150 (D) Zero

Q.19 The real number k for which the equation, 2x3 + 3x + k = 0 has two distinct real roots in [0, 1] ***(2013)***

(A) Lies between 1 and 2

(B) Lies between 2 and 3

(C) Lies between -1 and 0

(D) Does not exist.

Q.20 If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a,b,c \in R$, have a common root, then a : b : c is ***(2013)***

(A) 1 : 2 : 3 (B) 3 : 2 : 1 (C) 1 : 3 : 2 (D) 3 : 1 : 2

Q.21 The sum of first 20 terms of the sequence 0.7, 0.77, 0.777,....., is ***(2013)***

(A) $\frac{7}{81}(179-10^{-20})$ (B) $\frac{7}{9}(99-10^{-20})$

(C) $\frac{7}{81}(179+10^{-20})$ (D) $\frac{7}{9}(99+10^{-20})$

Q.22 If x, y, z are in A.P. and tan-1x, tan-1y and tan-1z are also in A.P., then ***(2013)***

(A) x = y = z (B) 2x = 3y = 6z

(C) 6x = 3y = 2z (D) 6x = 4y = 3z

JEE Advanced/Boards

Exercise 1

Q.1 (i) The harmonic mean of two numbers is 4. The arithmetic mean A & the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers.

(ii) The A.M. of two numbers exceeds their G.M. by 15 and HM by 27. Find the numbers.

Q.2 If the 10^{th} term of an H.P. is 21 and 21^{st} term of the same H.P. is 10, then find the 210^{th} term.

Q.3 If sinx, $\sin^2 2x$ and cosx.sin4x form an increasing geometric sequence, then find the numerical value of cos2x. Also find the common ratio of geometric sequence.

Q.4 If a, b, c, d, e be 5 numbers such that a, b, c are in AP; b, c, d are in G.P. & c, d, e are in H.P. then,

(i) Prove that a, c, e are in GP

(ii) Prove that $e = (2b - a)^2/a$

(iii) If a = 2 & e = 18, find all possible values of b, c, d.

Q.5 Let a_1 and a_2 be two real values of α for which the numbers $2\alpha^2$, α^4, 24 taken in that order form an arithmetic progression. If β_1 and β_2 are two real values of β for which the numbers 1, β^2, $6 - \beta^2$ taken in that order form a geometric progression, then find the value of $(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2)$.

Q.6 Two distinct, real infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is 18. If the second term of both the series can be written in the form $\frac{m-n}{p}$, where m, n and p are positive integers, and m is not divisible by the square of any prime, find the value of 100m + 10n+p.

Q.7 Let $S = \sum_{n=1}^{99} \frac{5^{100}}{(25)^n + 5^{100}}$. Find [s].

Where [y] denotes largest integer less than or equal to y.

Q.8 Given that the cubic $ax^3 - ax^2 + 9bx - b = 0$ $(a \neq 0)$ has all three positive roots. Find the harmonic mean of the roots independent of a and b, hence deduce that the root are all equal. Find also the minimum value of (a + b), if a and b $\in$ N.

Q.9 A computer solved several problems in succession. The time it took the computer to solve each successive problem was the same number of times smaller than the time it took to solve the preceding problem. How many problems were suggested to the computer if it spent 63.5 min to solve all the problems except for the first, 127 min to solve all the problems except for the last one, an 31.5 min to solve all the problems except for the first two?

Q.10 The sequence $a_1, a_2, a_3, \ldots\ldots a_{98}$ satisfies the relation $a_{n+1} = a_n + 1$ for 1,2,3,.... 97 and has the sum equal to 4949. Evaluate $\sum_{k=1}^{49} a_{2k}$.

Q.11 Let a and b be positive integers. The value of xyz is 55 or $\frac{343}{55}$, according as a, x, y, z, b are in arithmetic progression or harmonic progression resp.. Find the value of $(a^2 + b^2)$.

Q.12 If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, then find c and all the roots.

Q.13 If a, b, c be in G.P. & $\log_c a$, $\log_b c$, $\log_a b$ be in AP, then find the common difference of the AP if $\log_a c = 4$.

Q.14 The first term of a geometric progression is equal to b – 2, then third term is b + 6, and the arithmetic mean of the first and third term to the second term is in the ratio 5: 3. Find the positive integral value of b.

Q.15 In a G.P. the ratio of the sum of the first eleven terms to the sum of the last eleven terms is 1/8 and the ratio of the sum of all the terms without the first nine to the sum of all the terms with out the last nine is 2. Find the number of terms in the GP.

Q.16 If sum of first n terms of an AP (having positive terms) is given by $S_n = (1+2T_n)(1 - T_n)$ where T_n is the n^{th} term of series then $T_2^2 = \frac{\sqrt{a} - \sqrt{b}}{4}$ (a, b $\in$ N). Find the value of (a + b).

Q.17 Given a three digit number whose digits are three successive terms of a G.P. If we subtract 792 form it, we get a number written by the same digits in the reverse order. Now if we subtract four from the hundred's digit of the initial number and leave the other digits unchanged, we get a number whose digits are successive terms of an A.P. Find the number.

Q.18 For $0 < \theta < \frac{\pi}{4}$, let $S(\theta) = 1 + (1 + \sin\theta)\cos\theta + (1 + \sin\theta + \sin^2\theta)\cos^2\theta + \ldots\ldots \infty$.

Then find the value of $\left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right) S\left(\frac{\pi}{4}\right)$.

Q.19 If $\tan\left(\frac{\pi}{12} - x\right), \tan\frac{\pi}{12}, \tan\left(\frac{x}{12} + x\right)$ in order are three consecutive terms of a G.P., then sum of all the solutions in [0, 314] is $k\pi$. Find the value of k.

Exercise 2

Single Correct Choice Type

Q.1 The arithmetic mean of the nine numbers in the give set {9, 99,999,......999999999} is a 9 digit number N, all whose digits are distinct. The number N does not contain the digit

(A) 0 (B) 2 (C) 5 (D) 9

Q.2 $\sum_{k=1}^{360} \left(\frac{1}{k\sqrt{k+1} + (k+1\sqrt{k})}\right)$ is the ratio of two relative prime positive integers m and n. The value of (m + n) is equal to

(A) 43 (B) 41 (C) 39 (D) 37

Q.3 The sum $\sum_{k=1}^{100} \frac{k}{k^4 + k^2 + 1}$ is equal to

(A) $\frac{4950}{10101}$ (B) $\frac{5050}{10101}$

(C) $\frac{5151}{10101}$ (D) None of these

Q.4 A circle of radius r is inscribed in a square. The mid point of sides of the square have been connected by line segment and a new square resulted. The sides of these square were also connected by segments so that a new square was obtained and so on, then the radius of the circle inscribed in the nth square is

(A) $\left[2^{\frac{1-n}{2}}\right]r$ (B) $\left[2^{\frac{3-3n}{2}}\right]r$

(C) $\left[2^{\frac{n}{2}}\right]r$ (D) $\left[2^{\frac{5-3n}{2}}\right]r$

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statment-II is correct explanation for statment-I.

(B) Statement-I is true, statement-II is true and statment-II is NOT the correct explanation for statment-I.

(C) Statement-I is true, statement -II is false.

(D) Statement-I is false, statement-II is true

Q.5 Statement-I: If 27 abc ≥ (a + b + c)^3 and 3a + 4b + 5c = 12 then $\frac{1}{a^2}+\frac{1}{b^3}+\frac{1}{c^5}=10$, where a, b, c are positive real numbers.

Statement-II: For positive real numbers A.M. ≥ G.M.

Multiple Correct Choice Type

Q.6 Let a_1, a_2, a_3 .... and b_1, b_2, b_3 .... be arithmetic progressions such that a_1 = 25, b = 75 and a_{100} + b_{100} = 100. Then,

(A) The difference between successive terms in progression 'a' is opposite of the difference in progression 'b'.

(B) a_n + b_n = 100, for any n.

(C) (a_1 + b_1), (a_2 + b_2), (a_3 + b_3),... are in AP

(D) $\sum_{r=1}^{100}(a_r+b_r)=10000$

Q.7 If sin (x – y), sin x and sin (x + y) are in H.P. then the value of sin x.sec $\frac{y}{2}$ =

(A) 2 (B) $2^{1/2}$ (C) – 2 (D) $-2^{1/2}$

Q.8 The sum of the first three terms of the G.P. in which the difference between the second and the first term is 6 and the difference between the fourth and the third term 54, is

(A) 39 (B) –10.5 (C) 27 (D) –27

Q.9 If the roots of the equation $x^3 + px^2 + qx - 1 = 0$ form an increasing GP, where p and q are real, then

(A) p + q = 0

(B) p ∈ (–3, ∞)

(C) One of the roots is unity

(D) One root is smaller than 1

Q.10 If the triplets log a, log b, log c and (log a – log 2b), (log 2b – log3c), (log 3c – loga) are in arithmetic progression then

(A) 18 (a + b + c)^2 = 18(a^2 + b^2 + c^2) + ab

(B) a, b, c are in GP

(C) a, 2b, 2c are in HP

(D) a, b, c can be the lengths of the sides of a triangle.

(Assume all logarithmic terms to be defined)

Q.11 x_1, x_2 are the roots of the equation $x^2 - 3x + A = 0$; x_3, x_4 are roots of the equation $x^2 - 12x + B = 0$, such that x_1, x_2, x_3, x_4 form an increasing G.P. then

(A) A = 2 (B) B = 32

(C) $x_1 + x_3 = 5$ (D) $x_2 + x_4 = 10$

Previous Years' Question

Q.1 If the first and the (2n – 1)th term of an AP, G.P. and H.P. are equal and their nth terms are a, b, and c respectively, then ***(1988)***

(A) a = b = c (B) a ≥ b ≥ c

(C) a + c =b (D) ac – b^2 = 0

Q.2 Let S_1, S_2.... be squares such that for each n ≥ 1 the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm ? ***(1999)***

(A) 7 (B) 8 (C) 9 (D) 10

Q.3 Let S_k, k = 1,2,....., 100, denotes the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!}+\sum_{k=1}^{100}|(k^2-3k+1)S_k|$ is ***(2010)***

Q.4 Let a_1, a_2, a_3,....., a_{11} be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{(k-2)}$, for k = 3, 4,...., 11. If $\frac{a_1^2 + a_2^2 + ... + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ...a_{11}}{11}$ is equal to ***(2010)***

Q.5 Le a_1, a_2, a_3,.... a_{100} be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i$ $, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let m = 5n. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is .... ***(2011)***

Paragraph 1: Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For n > 2, let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

Q.6 Which one of the following statements is correct?

(A) $G_1 > G_2 > G_3 >$......

(B) $G_1 < G_2 < G_3 <$ ......

(C) $G_1 = G_2 = G_3 =$......

(D) $G_1 < G_3 < G_5 <$...... and $G_2 > G_4 > G_6 >$......

Q.7 Which one of the following statements is correct?

(A) $A_1 > A_2 > A_3 >$ ......

(B) $A_1 < A_2 < A_3 <$ ......

(C) $A_1 > A_3 > A_5 >$...... and $A_2 < A_4 < A_6 <$ ......

(D) $A_1 < A_3 < A_5 <$...... and $A_2 > A_4 > A_6 >$......

Q.8 Which one of the following statements is correct?

(A) $H_1 > H_2 > H_3 >$ ......

(B) $H_1 < H_2 < H_3 <$ ......

(C) $H_1 > H_3 > H_5 >$...... and $H_2 < H_4 < H_6 <$......

(D) $H_1 < H_3 < H_5 <$...... and $H_2 > H_4 > H_6 >$......

Paragraph 2: Let V_r denote the sum of the first 'r' terms of an arithmetic progression (A.P.), whose first term is 'r' and the common difference is (2r – 1). Let $T_r = V_{r+1} - V_{r-2}$ and $Q_r = T_{r+1} - T_r$ for r = 1, 2, ... ***(2007)***

Q.9 The sum $V_1 + V_2 + ... + V_n$ is

(A) $\frac{1}{12}$ $n(n + 1)(3n^2 - n + 1)$

(B) $\frac{1}{12}$$n(n + 1)(3n^2 + n + 2)$

(C) $\frac{1}{2}$ $n(2n^2 - n + 1)$

(D) $\frac{1}{3}$ $(2n^3 - 2n + 3)$

Q.10 T_r is always

(A) An odd number (B) An even number

(C) A prime number (D) A composite number

Q.11 Which one of the following is a correct statement?

(A) Q_1, Q_2, Q_3....... are in A.P. with common difference 5.

(B) Q_1, Q_2, Q_3....... are in A.P. with common difference 6.

(C) Q_1, Q_2, Q_3....... are in A.P. with common difference 11.

(D) $Q_1 = Q_2 = Q_3 =$.......

Q.12 Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$ for n = 1, 2, 3, ... Then, ***(2008)***

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$

(C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

Q.13 Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$. ***(2008)***

Statement-I: The numbers b_1, b_2, b_3, b_4 are neither in A.P. Nor in G.P.

Statement-II: The numbers b_1, b_2, b_3, b_4 are in H.P.

(A) Statement-I is true, statement-II is true; statement-II is a correct explanation for statement-I

(B) Statement-I is true, statement-II is true; statement-II is not a correct explanation for statement-I.

(C) Statement-I is true, statement-II is false

(D) Statement-I is false, statement-II is true

Q.14 If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is ***(2009)***

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$

(C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

Q.15 Let S_k, k = 1, 2,...., 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!}+\sum_{k=1}^{100}\left|(k^2-3k+1)S_k\right|$ is ***(2010)***

Q.16 Let a_1, a_2, a_3, ..., a11 be real numbers satisfying a1 = 15, 27 − $2a_2$ > 0 and ak = $2a_k$−1 − a_k−2 for k = 3, 4, ...,11. If $\frac{a_1^2+a_2^2+....+a_{11}^2}{11}=90$, then the value of $\frac{a_1+a_2+....+a_{11}}{11}$ is equal to ***(2010)***

Q.17 Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is ***(2011)***

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

Q.18 Let a_1, a_2, a_3, ... be in harmonic progression with a_1 = 5 and a_{20} = 25. The least positive integer n for which an< 0 is ***(2012)***

(A) 22 (B) 23 (C) 24 (D) 25

Q.19 Let $S_n=\sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s) ***(2013)***

(A) 1056 (B) 1088 (C) 1120 (D) 1332

Q.20 Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is b + 2, then the value of $\frac{a^2+a-14}{a+1}$ is _______ ***(2014)***

Q.21 Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is ***(2015)***

Q.22 Let b_i > 1 for i = 1, 2, ..., 101. Suppose loge b1 loge b_2, ..., loge b_{101} are in Arithmetic Progression (A.P.) with the common difference loge 2. Suppose a_1, a_2, ..., a_{101} are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ...+ b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$, then ***(2016)***

(A) s > t and $a_{101} > b_{101}$ (B) s > t and $a_{101} < b_{101}$

(C) s < t and $a_{101} > b_{101}$ (D) s < t and $a_{101} < b_{101}$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.3 Q.11 Q.14
Q.17 Q.21 Q.25
Q.27

Exercise 2

Q.2 Q.4 Q.10
Q.13

Previous Years' Questions

Q.2 Q.5 Q.8

JEE Advanced/Boards

Exercise 1

Q.6 Q.9 Q.12
Q.15 Q.17

Exercise 2

Q.1 Q.4 Q.5 Q.12

Previous Years' Questions

Q.1 Q.3 Q.4
Q.6 Q.7 Q.8

Answer Key

JEE Main/Boards

Exercise 1

Q.1 n = 6

Q.2 $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$ is the series.

Q.3 (a) $\frac{7n}{9} - \frac{7}{81}\left(1 - \left(\frac{1}{10}\right)^n\right)$; (b) $\frac{2}{27}[10^{n+1} - 9n - 10]$

Q.5 (2, 4, 6) or (6, 4, 2)

Q.6 (i) 6633 (ii) 2842 (iii) 945

Q.7 69: 128

Q.10 $\frac{35}{16} - \frac{3}{16(5^{n-2})} - \frac{(3n-2)}{4(5^{n-1})}$

Q.11 c = 6, b = 4, d = 9; b = – 2, c = – 6, d = – 18

Q.12 $\frac{1}{4}n(n+1)(n+2)(n+3)$

Q.13 $\frac{1}{3}n(4n^2 + 3n + 2)$

Q.14 6,3

Q.15 4960

Q.16 188090

Q.17 a = 5, b = 8, c = 12.

Q.20 –5050, $\frac{1}{2}[(5050)^2 - 338350]$

Q.21 9

Q.22 469

Q.23 3370 m

Q.26 $\frac{1}{a_1(a_2 - a_1)}$

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** A **Q.3** C **Q.4** D **Q.5** C **Q.6** D
Q.7 A **Q.8** C **Q.9** A **Q.10** A **Q.11** C **Q.12** D

Previous Years' Questions

Q.1 B **Q.2** C **Q.3** B **Q.4** A **Q.5** D **Q.6** D
Q.7 C **Q.8** C **Q.9** B **Q.10** B **Q.11** B **Q.12** B
Q.13 A **Q.14** A **Q.15** B **Q.16** B **Q.17** D **Q.18** D
Q.19 A **Q.20** C **Q.21** A

JEE Advanced/Boards

Exercise 1

Q.1 (i) 6, 3 ; (ii) 120, 30 **Q.2** 1 **Q.3** $\frac{\sqrt{5}-1}{2};\sqrt{2}$
Q.4 (iii) b = 4, c = 6, d = 9 or b = – 2, c = – 6, d = – 18 **Q.5** 12 **Q.6** 518 **Q.7** 49
Q.8 28 **Q.9** 8 problems, 127.5 minutes **Q.10** 2499 **Q.11** 50
Q.12 C = 9; (3, –3/2, – 3/5) **Q.13** 13/4 **Q.14** 3 **Q.15** n = 38
Q.16 6 **Q.17** 931 **Q.18** 2 **Q.19** 4950

Exercise 2

Single Correct Choice Type

Q.1 A **Q.2** D **Q.3** B **Q.4** A

Assertion Reasoning Type

Q.5 D

Multiple Correct Choice Type

Q.6 A, B, C, D **Q.7** B, C **Q.8** A, B **Q.9** A, C, D **Q.10** B, D **Q.11** A, B, C, D

Previous Years' Questions

Q.1 A, B, D **Q.2** B, C, D **Q.3** 4 **Q.4** 0 **Q.5** 3 or 9 **Q.6** C
Q.7 A **Q.8** B **Q.9** B **Q.10** D **Q.11** B **Q.12** A, D
Q.13 C **Q.14** C **Q.15** 3 **Q.16** 0 **Q.17** B **Q.18** D
Q.19 A, D **Q.20** 6 **Q.21** 9 **Q.22** B

Solutions

JEE Main/Boards

Exercise 1

Sol 1: Sum of n terms is 364

$a + ar + ar^2$ .... $ar^{n-1} = 364$

$\frac{a(r^n - 1)}{r - 1} = 364$

given r = 3, a = 1

$\Rightarrow \frac{(3^n - 1)}{(3 - 1)} = 364 \Rightarrow 3^n - 1 = 728$

$\Rightarrow 3^n = 729$

$\Rightarrow \boxed{n = 6}$

Sol 2: Sum of infinite G.P. is 2 $\Rightarrow \frac{a}{-r+1} = 2$

$\Rightarrow a = -2(r - 1)$

Series is a, ar, ar^2 .... (|r| < 1)

$\Rightarrow a^3, (ar)^3, (ar^2)^3$ ....(2)

First term of this infinite series is a^3 and ratio is r^3

Hence sum of this infinite series is $\frac{a^3}{-r^3 + 1}$

Given $\frac{a^3}{-r^3 + 1} = 24$

$\frac{8(r-1)^3}{\left(-r^3+1\right)} = 24 \Rightarrow \frac{(r-1)^2}{(r^2 + r + 1)} = 3$

$\Rightarrow r^2 + 1 - 2r = 3r^2 + 3r + 3$

$\Rightarrow 2r^2 + 5r + 2 = 0 \Rightarrow 2r^2 + 4r + r + 2 = 0$

$\Rightarrow (2r + 1)(r + 2) = 0$

$\Rightarrow r = -2, r = -\frac{1}{2}$

$|r| < 1 \Rightarrow r = -\frac{1}{2} \Rightarrow a = +3$

Series is $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, ...$

Sol 3: (a) Sum upto n terms

$S_n = 0.7 + 0.77 + 0.777$ .... n terms

$= 7[0.1 + 0.11 + 0.111 + ...] = \frac{7}{9}[0.9 + 0.99 + 0.999 ...]$

$= \frac{7}{9}[1 - 0.1 + 1 - 0.01 + 1 - 0.001 ...]$

$= \frac{7}{9}[n - (0.1 + 0.01 + 0.001 ...)]$

$= \frac{7}{9}\left[n - \frac{0.1(1-(0.1)^n)}{1 - 0.1}\right] = \frac{7}{9}\left[n - \frac{1}{9}(1 - (0.1)^n)\right]$

$= \frac{7n}{9} - \frac{7}{81}\left(1 - \left(\frac{1}{10}\right)^n\right)$

(b) $6 + 66 + 666 = 6[1 + 11 + 111 ...]$

$= \frac{6}{9}[9 + 99 + 999 + ...] = \frac{2}{3}[10 - 1 + 100 - 1 + 1000 - 1 ...]$

$= \frac{2}{3}\left[\frac{10(10^n - 1)}{10 - 1} - n\right] = \frac{2}{3} \times \frac{10}{9}(10^n - 1) - \frac{2n}{3}$

$= \frac{2}{27}\left(10^{n+1} - 10\right) - \frac{2n \times 9}{3 \times 9} = \frac{2}{27}\left[10^{n+1} - 9n - 10\right]$

Sol 4: a, b, c are in AP

(i) b + c, c + a, a + b are also in AP

a, b, c are in AP $\Rightarrow$ 2b = a + c

$\Rightarrow b - a = c - b$

$\Rightarrow \boxed{a - b = b - c}$

Difference between term of given AP = a – b, b – c which are equal by equation (i)

Hence b + c, c + a, a + b is an AP

(ii) $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are also in AP

Common difference = $\frac{(b-a)}{cab}, \frac{(c-b)}{abc}$

By equation (i) b – a = c – b ie difference between terms is same

Hence the given series is in AP

(iii) $a^2(b + c), b^2(c + a), c^2(a + b)$

Difference = $\underbrace{b^2c + b^2a - a^2b - a^2c}_{d_1}, \underbrace{c^2a + c^2b - b^2c - b^2a}_{d_2}$

$d_1 = c(b^2 - a^2) + ab(b - a) = (ca + ab + cb)(b - a)$

$= (ca + ab + bc)(c - b)$ [from eq.(i)

$d_2 = a(c^2 - b^2) + bc(c - b) = (ac + ab + bc)(c - b)$

$d_1 = d_2$

Hence given series is an AP

(iv) $a\left(\frac{1}{b}+\frac{1}{c}\right),\ b\left(\frac{1}{c}+\frac{1}{a}\right),\ c\left(\frac{1}{a}+\frac{1}{b}\right)$

$\Rightarrow d_1 = \frac{1}{c}(b-a) + \frac{b}{a} - \frac{a}{b} \Rightarrow d_2 = \frac{1}{a}(c-b) + \frac{c}{b} - \frac{b}{c}$

$d_1 = \frac{b-a}{c} + \frac{(b-a)(b+a)}{ab}$

$d_2 = \frac{c-b}{a} + \frac{(c-b)(c+b)}{bc} = (b-a)\left[\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right]$

$= (b-a)\left[\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right] = (c-b)\left[\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right]$

From eq^n (i) $\Rightarrow d_1 = d_2$

Hence given series is also an AP

Sol 5: Sum of first 3 numbers in AP is 12

Let $a - r, a, a + r$ be the first 3 numbers

$3a = 12 \Rightarrow a = 4$

$\Rightarrow (a - r)^3 + a^3 + (a + r)^3 = 288$

$\Rightarrow 2a^3 + 6ar^2 + a^3 = 288 \Rightarrow 3a^3 + 6ar^2 = 288$

$\Rightarrow 6 \times 4 \times r^2 = 288 - 3(4^3) \Rightarrow 24r^2 = 96$

$\Rightarrow r = \pm 2$

So numbers are, $4 - 2, 4, 4 + 2 = (2, 4, 6)$ (for $r = 2$)

$(4 + 2, 4, 4 - 2)$ for ($r = -2$)

$\Rightarrow (6, 4, 2)$

Sol 6: (i) Sum of integers between 1 & 200 which are multiple of 3

$3, 6, 9, \ldots 198 \Rightarrow n = 66$

Hence sum $= \frac{n}{2}[a + l]$

$= \frac{(66)}{2}[3 + 198] = 33[201] = 6633$

(ii) Multiple of 7

$7, 14, 21 \ldots 196 \Rightarrow n = 28$

Now sum $= \frac{n}{2}[a+l] = \frac{28}{2}[7 + 196] = 14[203] = 2842$

(iii) Multiple of 3 and 7

$21, 42, 63 \ldots 189 \Rightarrow n = 9$

Sum $= \frac{9}{2}[21 + 189] = \frac{9}{2} . 210 = 945$

Sol 7: Sum of first n terms of 2 AP's are in ratio

$= \frac{3n-3}{5n+21}$

$\Rightarrow$ Let the AP be $a_1, a_1 + d_1 \ldots$

2nd AP be $a_2, a_2 + d_2 \ldots$

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_1]} = \frac{3n-3}{5n+21}$$

$$\Rightarrow \frac{a_1 + \frac{(n-1)}{2}d_1}{a_2 + \frac{(n-1)d_2}{2}} = \frac{3n-3}{5n+21} \quad .....(i)$$

Ratio of 24th term well be $\frac{a_1+23d_1}{a_2+23d_2}$

Putting n = 47 in equation (i), we well get desired ratio

$$\frac{a_1+23d_1}{a_2+23d_2} = \frac{3(47)-3}{5(47)+21} = \frac{138}{256} = \frac{69}{128}$$

Sol 8: given

$a + (p - 1)d = x$ ...(i)

$a + (q - 1)d = y$ ...(ii)

sum of first $(p + q)$ terms

$= \frac{p+q}{2}[2a+(p+q-1)d]$ ...(iii)

Subtracting (ii) from (i)

$(p - q)d = x - y$

$\Rightarrow d = \frac{x-y}{p-q}$ and putting this value in equation (i)

$a + \frac{(p-1)(x-y)}{p-q} = x$

$\Rightarrow a = x - \frac{(px-py-x+y)}{p-q} = \frac{-qx+py+x-y}{p-q}$

Putting values of a and d in equation (iii)

$$S_{p+q} = \frac{p+q}{2}\left[\frac{2x-2y-2qx+2py+(p+q-1)(x-y)}{p-q}\right]$$

$$= \frac{p+q}{2}\left[\frac{x-y-qx+py+px-qy}{p-q}\right]$$

$$= \frac{p+q}{2}\left[(x+y)\frac{(p-q)}{(p-q)}+\frac{(x-y)}{p-q}\right] = \frac{p+q}{2}\left[x+y+\frac{x-y}{p-q}\right]$$

Sol 9: a, b, c are in HP

i.e $\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$ ...(i)

$\frac{2a_1c_1}{a_1+c_1}$ [where a_1 & c_1 are 1st & 3rd terms of given series]

$$= \frac{\frac{2ac}{(a+b)(b+c)}}{\frac{a}{b+c}+\frac{c}{a+b}} = \frac{2ac}{a^2+ab+c^2+bc} = \frac{2ac}{a^2+c^2+b(a+c)}$$

$= \frac{2ac}{a^2+c^2+(2ac)}$ (from equation (i)

$$= \frac{2ac}{(a+c)(a+c)} = \frac{b}{a+c} = b_1$$

Middle term of given series, hence $\frac{2a_1c_1}{a_1+c_1} = b_1$ ie given Series is H.P

Sol 10: $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3}$

Sum of first n terms

$S_n = \frac{1}{5^0} + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} \ldots \frac{1+(n-1)3}{5^{n-1}}$ ...(i)

$\frac{S_n}{5} = \frac{1}{5} + \frac{4}{5.5} + \frac{7}{5.5^2} + \frac{10}{5.5^3} \ldots \frac{1+(n-1)3}{5^n}$ ...(ii)

Subtracting (ii) from (i)

$$= 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} \ldots - \left(\frac{3n-2}{5^n}\right)$$

$$= 1 - \frac{(3n-2)}{5^n} + 3\left[\frac{1}{5}+\frac{1}{5^2}\ldots\frac{1}{5^{n-2}}\right]$$

$$= 1 - \frac{(3n-2)}{5^n} + 3.\frac{1}{5}\left[\frac{\left(1-\frac{1}{5^{n-1}}\right)}{\left(1-\frac{1}{5}\right)}\right]$$

$$= 1 - \left(\frac{3n-2}{5^n}\right) + \frac{3}{5}\frac{(5^{n-1}-1)\times 5}{5^{n-1}\times 4}$$

$$\frac{4s_n}{5} = 1 - \left(\frac{3n-2}{5^n}\right) + \frac{15}{4}\frac{(5^{n-1}-1)}{5^n}$$

$$S_n = \frac{5}{4} - \frac{(3n-2)}{4.5^{n-1}} + \frac{75}{16}\left(\frac{1}{5}-\frac{1}{5^n}\right)$$

$$= \frac{5\times 4}{4\times 4} + \frac{15}{16} - \frac{3}{16\times 5^{n-2}} \quad \frac{-(3n-2)}{4\times 5^{n-1}}$$

$$= \frac{35}{16} - \frac{3}{16(5^{n-2})} \quad \frac{-(3n-2)}{4(5^{n-1})}$$

Sol 11: a, b, c are in AP

$\Rightarrow 2b = a + c$ ...(i)

b, c, d in GP

$c^2 = bd$ ...(ii)

c, d, e are in HP

$d = \frac{2ce}{c+e}$ ...(ii)

Given that a = 2, e = 18

We have $(2b - 2) = c$ from (i) and $\frac{(2b-2)^2}{b}$ = d from (ii)

and also $\frac{(2b-2)^2}{b} = \frac{2\times(2b-2)18}{(2b-2)+18}$ from (iii)

$\Rightarrow (2b - 2) = \frac{36b}{2b+16} \Rightarrow (b - 1)(b + 8) = 9b$

$\Rightarrow b^2 + 7b - 8 = 9b \Rightarrow b^2 - 2b - 8=0$

$\Rightarrow b = 4, -2$

$\Rightarrow c = 6, -6$

$\Rightarrow d = 9, -18$

b, c, d = [4, 6, 9] and [–2, –6, – 18]

Sol 12: $S_n = 1.2.3 + 2.3.4 + 3.4.5 \ldots$

$T_n = n(n + 1)(n + 2) = n(n^2 + 3n + 2)$

$= n^3 + 3n^2 + 2n = n^3 + 3n^2 + 2n$

$\Sigma_n = \Sigma T_n = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 3\frac{(n)(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{3(2n+1)}{3}+2\right]$$

$= \frac{n(n+1)}{2}\left[\frac{n^2+n+4n+2+4}{2}\right]$

$= \frac{n(n+1)}{4}(n^2 + 5n + 6) = \frac{n(n+1)(n+2)(n+3)}{4}$

Sol 13: $S_n = 1^2 + 2 + 3^2 + 4 + 5^2 + 6$

(first 2n numbers)

$1^2 + 3^2 + 5^2 \ldots$ n terms + 2 + 4 + 6... n terms

$= 2(1 + 2 + 3 \ldots + n) + 1^2 + 3^2 + 5^2 \ldots$

$= 2n\frac{(n+1)}{2} + 1^2 + 2^2 + 3^2 \ldots (2n-1)^2 - 2^2$

$- 4^2 - 6^2 \ldots (2n-2)^2$

$= n(n+1) + \frac{(2n-1)(2n-1+1)(4n-2+1)}{6}$

$- 2^2\{1^2 + 2^2 + \ldots (n-1)^2\}$

$= n(n+1) + \frac{(2n-1)n(4n-1)}{3}$

$- 2^2\frac{(n-1)n(2n-1)}{6}$

$= n(n+1) + \frac{(2n-1)(n)(4n-1)}{3}$

$- \frac{2^2(n-1)n(2n-1)}{6}$

$= n(n+1) + \frac{n}{3}[(2n-1)(4n-1) - 2(n-1)(2n-1)]$

$= n(n+1) + \frac{n}{3}(2n-1)[2n+1]$

$= n(n+1) + \frac{\left(4n^2-1\right)n}{3} = \frac{n}{3}[4n^2 + 3n + 2]$

Sol 14: HM of 2 numbers is 4 ie $\frac{2ab}{a+b} = 4$

$\boxed{ab = 2a+2b}$ ...(i)

AM $= \frac{a+b}{2}$ and G.M. $= \sqrt{ab}$

We have $2A + G^2 = 27 \Rightarrow a + b + ab = 27$

$a + b + 2a + 2b = 27$ [from (i)]

$\Rightarrow a + b = 9$ and $ab = 18$

[a = 6, b = 3]; [a = 3, b = 6]

Sol 15: $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$

$= 3^3 + 5^3 + 7^3 + \ldots + 21^3$

$- 2^3(1 + 2^3 + 3^3 \ldots 10^3) + 1 - 1$

$= 1^3 + 2^3 + 3^3 + 4^3 \ldots 21^3 - 2^4(1 + 2^3 \ldots 10^3) - 1$

$= \left[\frac{21\times(21+1)}{2}\right]^2 - 2^4\left[\frac{(10)(10+1)}{2}\right]^2 - 1$

$= (21 . 11)^2 - 16(5 . 11)^2 - 1$

$= 11^2(21 . 21 - 16 . 25) - 1$

$= 121 \times 41 - 1 = 4961 - 1 = 4960$

Sol 16: $S_n = 1.3^2 + 2.5^2 + 3.7^2 + \ldots$

$T_n = n(2n+1)^2 = 4n^3 + n + 4n^2$

$\Sigma_n = \Sigma T_n = 4\Sigma n^3 + \Sigma n + 4\Sigma n^2$

$= 4\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)}{2} + 4\frac{n(n+1)(2n+1)}{6}$

$= n^2(n+1)^2 + \frac{n(n+1)}{2} + \frac{2}{3}n(n+1)(2n+1)$

$= n(n+1)\left[n^2 + n + \frac{1}{2} + \frac{4n}{3} + \frac{2}{3}\right]$

$= n(n+1)\left[n^2 + \frac{7n}{3} + \frac{7}{6}\right]$

$S_{20} = 20 \times 21\left[20^2 + \frac{7}{3}\times 20 + \frac{7}{6}\right]$

$= \frac{420}{6}\left[2400 + 280 + 7\right] = 70(2687) = 188090$

Sol 17: $a, b, c \in (2, 18)$

$a + b + c = 25$ ...(i)

$2a = 2 + b$ ...(ii)

$c^2 = 18b$ ...(iii)

$b = 2a - 2$

$c = 25 - a - 2a + 2 = 27 - 3a$

$\Rightarrow (27 - 3a)^2 = 18(2a - 2)$

$\Rightarrow (9 - a)^2 = 4(a - 1) \Rightarrow a^2 + 81 - 18a = 4a - 4$

$\Rightarrow a^2 - 22a + 85 = 0$

a = 17, 5

b = 32, 8

c = 24, 12

Numbers are (5, 8, 12)

Sol 18: a > 0, b > 0, c > 0

To prove $(a + b + c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

We know that A.M. ≥ G.M. ≥ HM

Therefore, A.M. ≥ HM

For 3 numbers a, b, c

$AM = \frac{a+b+c}{3}$, $HM = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$

$\Rightarrow \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$

$\Rightarrow (a + b + c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

Sol 19: Let the 2 number be a, b

a, A_1, A_2, b are in AP, (a, G_1, G_2, b) are in GP,

(a, H_1, H_2, b) are in HP

$A_1 + A_2 = a + b$ ...(i)

$G_1 G_2 = ab$ ...(ii)

By properties of respective series

$$\frac{H_1H_2}{H_1+H_2} = \frac{ab}{a+b}$$

$$\frac{H_1H_2}{H_1+H_2} = \frac{G_1G_2}{A_1+A_2}$$

$\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}$ Hence proved

Sol 20: (x –1) (x – 2) (x – 3) ... (x – 100)

Coefficient of $x^{99} = \frac{-b}{a}$

We can see that a = 1

and $b = 1 + 2 + 3 \ldots 100 = \frac{100\times101}{2} = 5050$

$\therefore$ Coefficient of $x^{99} = -5050$

Coefficient of x^{98}

$= 1\times2+1\times3+....+1\times100+2\times3+2\times4+....+99\times100$

$= \frac{1}{2}\left\{\left(1+2+....+100\right)^2-\left(1^2+2^2+100^2\right)\right\}$

$= \frac{1}{2}\left\{\left(5050\right)^2-338350\right\}$

Sol 21: Given that,for a polygon of "n" sides, we have

$\alpha = 120^o$; d = 5

Sum of interior angle

$(n – 2)\, 180^o = a + (n – 1)d$

$= \frac{n}{2}\left[2(120)+(n-1)5\right] = \frac{n}{2}\left[5n+235\right]$

$5n^2 + 235n = 360n – 720$

$\Rightarrow 5n^2 – 125n + 720 = 0$

$\Rightarrow n^2 – 25n + 144 = 0 \Rightarrow n = 16, 9$

If n = 16, then interior angle will be greater than 180^o.

Hence the answer is 9.

Sol 22: Let the number be a, b, c

$b^2 = ac$, $a + c = 2b + 1$, $b + a = \frac{2}{3}(b + c)$

$\Rightarrow a = 2b + 1 – c$

$\Rightarrow 3b + 1 – c = \frac{2}{3}(b+c)$

$\Rightarrow 2b + 2c = 9b + 3 – 3c$

$\Rightarrow 7b = 5c – 3$

$c = \frac{7b+3}{5} \Rightarrow a = 2b + 1 - \frac{(7b+3)}{5}$

$a = \frac{3b+2}{5} \Rightarrow b^2 = \frac{(3b+2)(7b+3)}{25}$

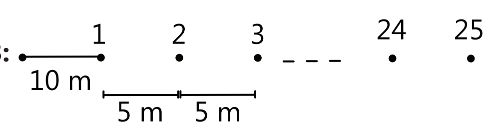
$\Rightarrow 25b^2 = 21b^2 + 23b + 6 \Rightarrow 4b^2 – 23b – 6 = 0$

$\Rightarrow 4b^2 – 24b + b – 6 = 0$

$\Rightarrow b = 6, c = 9$

$\Rightarrow a = 4$ Number is 469

Sol 23:



Sn = 10 + 10 + 15 + 15 + 20 + 20

... 85 + 85 + ... + 125 + 125 + 130

= 2[10 + 15 + 20 ... 130] – 130

= 10[2 + 3 + 4 ... 26] – 130

$= 10\left[\frac{26\times27}{2}-1\right] - 130 = 10 \times 350 – 130$

= 3500 – 130 = 3370 m

Sol 24: Number of elements in nth group = n

First number in the group will be $\frac{n(n-1)+2}{2}$

$S_n = \frac{n}{2}$ [n(n – 1) + 2 + (n – 1)1]

$= \frac{n}{2}[n^2 - n + 2 + n - 1] = \frac{n}{2}[n^2 + 1]$

Sol 25: Let the 3 number in G.P. be a, ar, ar^2 & other 3 numbers be a_1, a_1r_1, $a_1r_1^2$

$(a_1r_1 - ar)^2 = (a_1r_1^2 - ar^2)(a_1 - a)$

$a_1^2r_1^2 + a^2r^2 - 2aa_1rr_1 = a_1^2r_1^2 - aa_1r^2 - aa_1r^2 + a^2r^2$

$2rr_1 = r^2 + r_1^2$

$\Rightarrow (r_1 - r)^2 = 0$

$\Rightarrow r = r_1$

Ratio for third G.P. = $\frac{a_1r_1 - ar}{a_1 - a} = \frac{(a_1 - a)r}{(a_1 - a)} = r$

Hence ratio of the three G.P. is same

Sol 26: $S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} \ldots \frac{1}{a_na_{n+1}}$

$= \frac{1}{a_2 - a_1}\left[\frac{a_2 - a_1}{a_1a_2} + \frac{a_3 - a_2}{a_2a_3} \ldots\right]$

As $a_2 - a_1 = a_3 - a_2 = \ldots a_{n+1} - a_n$

$= \frac{1}{a_2 - a_1}\left[\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} \ldots + \frac{1}{a_n} - \frac{1}{a_{n+1}}\right]$

$= \frac{1}{a_2 - a_1}\left[\frac{1}{a_1} - \frac{1}{a_{n+1}}\right] = \frac{a_{n+1} - a_1}{(a_2 - a_1)a_1a_{n+1}}$

$= \frac{nd}{d(a_1a_{n+1})} = \frac{n}{a_1a_{n+1}}$

$S = \frac{n}{a_1a_{n+1}} = \frac{n}{a_1(a_1 + nd)}$

$\lim_{n\to\infty} S = \lim_{n\to\infty} \frac{n}{a_1(a_1 + nd)} = \lim_{n\to\infty} \frac{1}{a_1\left(d + \frac{a_1}{n}\right)} = \frac{1}{a_1d}$

Sol 27: 2, AM_1, AM_2 … AM_9, 3

2, HM_1, HM_2 … HM_9, 3

Suppose we take AM_n and HM_n

$\Rightarrow 2 + 10d = 3 \Rightarrow d = \frac{1}{10}$

$AM_n = 2 + nd = 2 + \frac{n}{10}$

$\frac{1}{2}, \frac{1}{HM_1} \ldots \frac{1}{HM_9}, \frac{1}{3}$ in AP

$\Rightarrow \frac{1}{2} + 10d = \frac{1}{3} \Rightarrow d = \frac{-1}{60}$

$HM_n = \frac{60}{30 - n}$

$A + \frac{6}{H} = 2 + \frac{n}{10} + \frac{6}{60}(30 - n)$

$= 2 + \frac{n}{10} + 3 - \frac{n}{10} = 5$

Hence proved

Sol 28: x + y + z = 1

For x, y, z. A.M. ≥ HM

$\Rightarrow \frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$

Therefore, $xy + yz + zx - 9xyz \geq 0$

Hence proved.

Sol 29: We must prove that for some m and p;

$M^p = \frac{m}{2}[2a + (m - 1)2]$, for some odd a

= m[a + m – 1]

Let us prove this by induction

Taking, P = 2

$m^2 = m[a + m - 1] \Rightarrow a = 1$ is the required a.

$m^{p+1} = m^p.m$

$= m[a + m - 1]\, m = m[ma + m^2 - m]$

$= m[ma + m^2 - 2m + 1 + m - 1]$

$= m[ma + (m - 1)^2 + m - 1]$

= m [A + m – 1]

We must prove that A is odd.

A is odd

For even m, ma is even and $(m - 1)^2$ is odd ⇒ A is odd
for odd m, ma is odd and $(m - 1)^2$ is even ⇒ A is odd

∴ By induction hypothesis,

$M^p = \frac{m}{2}[2a + (m - 1)2]$, with odd a.

Hence proved

Exercise 2

Single Correct Choice Type

Sol 1: (B) $b = \frac{2ac}{a+c}$ [a, b, c are in HP] ...(i)

$= \frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{\frac{2ac}{a+c}+a}{\frac{2ac}{a+c}-c} + \frac{\frac{2ac}{a+c}+c}{\frac{2ac}{a+c}-c}$

$= \frac{3ac+a^2}{ac-a^2} + \frac{3ac+c^2}{ac-c^2} = \frac{3c+a}{c-a} + \frac{(3a+c)}{a-c}$

$= \frac{3c+a}{c-a} - \frac{(3a+c)}{c-a} = \frac{2c-2a}{c-a} = 2$

Sol 2: (A) Given summation is, $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3}$

... $T_n = \frac{2}{n(n+1)} \Rightarrow T_n = 2\left[\frac{1}{n}-\frac{1}{n+1}\right]$

$S_n = ST_n = 2\left[\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}...\frac{1}{\infty}\right] = 2$

Sol 3: (C)

• • • • • • •

←10m→ ←middle→ n stones

n stones

$\Rightarrow$ 2[20 + 40 + 60 ...n terms] = 4800

$\Rightarrow$ 120 = 1 + 2 ... n terms

$\Rightarrow 120 = \frac{n(n+1)}{2} \Rightarrow n^2 + n = 240$

$\Rightarrow$ n = 15

Total no of stone = 2n+1 = 31 [C]

Sol 4: (D) $S = 1^2 + 3^2 + 5^2 ... 99^2$

$S_1 = 2^2 + 4^2 + 6^2 ... (100)^2$

$S_1 - S = 2^2 - 1^2 + 4^2 - 3^2 + 6^2 - 5^2 ...$

= 2+1+4+3+6+5+ ... +100+99

$= \frac{100}{2}[1+100] = 50\,[101] = 5050$

Sol 5: (C) a, a + d, ... a + (n – 1)d, a + nd, ... a + (2n – 1) d

$S_1 = \frac{n}{2}[2a + (n-1)d]$

$S_2 = \frac{n}{2}[2a + 2nd + (n-1)d]$

$\frac{s_1}{s_2} = \frac{2a+(n-1)d}{2a+2nd+(n-1)d}$

$\Rightarrow \frac{2+(n-1)\frac{d}{a}}{2+(3n-1)\frac{d}{a}} = s$

$\Rightarrow 2 + (n-1)\frac{d}{a} = 2s + (3n-1)\frac{d}{a}s$

$\Rightarrow 2s - 2 = \frac{d}{a}[n - 3ns + s - 1]$

$(2s-2)\frac{a}{d} = n(1-3s) + s - 1$

This is independent of n ie coefficient of n will be zero

$\Rightarrow s = \frac{1}{3}; \Rightarrow \frac{a}{d} = \frac{1}{2}$

Sol 6: (D) $a = 5^{1+x} + 5^{1-x} + 25^x + 25^{-x}$

$\Rightarrow a = 5.5^x + \frac{5}{5^x} + (5^x)^2 + (5^{-x})^2$

Let $5^x = t \Rightarrow a = 5t + \frac{5}{t} + t^2 + \frac{1}{t^2}$

$\frac{5t+\frac{5}{t}}{2} \ge 5; \frac{t^2+\frac{1}{t^2}}{2} \ge 1$ hence $a \ge 10 + 2$

$\therefore \quad a \ge 12$

Sol 7: (A) $S_{11} = S_{19}$

$\frac{11}{2}[2a + 10d] = \frac{19}{2}[2a + 18d]$

$16a = -232\,d \Rightarrow \frac{a}{d} = \frac{-29}{2}$

$S_{30} = \frac{30}{2}[2a + 29d] = 30\left[a+\frac{29d}{2}\right] = 0$

Sol 8: (C) $S_2 = 2t_2$

$S_1 + d_s = 2(t_1 + d_t)$

$d_s = t_1 + 2d_t$

$\Rightarrow \frac{10}{2}[2s_1 + 9d_s] = \frac{15}{2}[2t_1 + 14d_t]$

$\Rightarrow 18d_s = 2t_1 + 42d_t \Rightarrow 18d_s = 2d_s - 4d_t + 42d_t$

$\Rightarrow 16d_s = 38d_t$

$\frac{d_s}{d_t} = \frac{19}{8}$

Sol 9: (A) The given expression is equal to

$$\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\ldots\right) = \frac{1}{a_1 d}$$

Sol 10: (A) $a + ar + ar^2 = 21$ ...(i)

$a^2 + a^2r^2 + a^2r^4 = 189$ ...(ii)

Squaring equation (i) & then dividing by (ii)

$$\Rightarrow \frac{a^2(1+r+r^2)^2}{a^2(1+r^2+r^4)} = \frac{441}{189}$$

$$\Rightarrow \frac{(1+r+r^2)(1+r+r^2)}{(1+r+r^2)(1-r+r^2)} = \frac{441}{189}$$

$\Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow r = 2, \frac{1}{2} \Rightarrow a = 3, 12$

GP is 3, 6, 12 ... $S_n = 3\frac{(2^n-1)}{2-1} = 3(2^n - 1)$ Hence, (A) is the correct choice

Sol 11: (C) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+4} = \frac{1}{n\left(n^2+\frac{4}{n^2}+4-4\right)}$

$$= \sum\frac{1}{n\left[\left(n+\frac{2}{n}\right)^2-4\right]} = \sum\frac{1}{n\left[n+\frac{2}{n}-2\right]\left[n+\frac{2}{n}+2\right]}$$

$$= \sum\frac{n}{(n^2+2-2n)(n^2+2n+2)}$$

$$= \frac{1}{4}\sum\left[\frac{1}{n^2-2n+2}-\frac{1}{n^2+2n+2}\right]$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{(n-1)^2+1}-\frac{1}{(n+1)^2+1}\right]$$

$$= \frac{1}{4}\left(1+\frac{1}{2}+\sum_{n=1}^{\infty}\left[\frac{1}{(n+1)^2+1}-\frac{1}{(n+1)^2+1}\right]\right) = \frac{3}{8}$$

Sol 12: (D) $\ln a^2 + (\ln a^2)^2 + (\ln a^2)^3$ .....

$= 3\{\ln a + (\ln a)^2 + (\ln a)^3 + \ldots.\}$

$$\Rightarrow \frac{2\ln a}{1-2\ln a} = \frac{3\ln a}{1-\ln a}$$

$\Rightarrow 2 - 2\ln a = 3 - 6\ln a$

$\Rightarrow 1 = 4\ln a$

$\Rightarrow a = e^{1/4}$

Previous Years' Questions

Sol 1: (B) Here, $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$

$\Rightarrow (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \le 0$

$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$

(Since, sum of squares is never less than zero)

$\Rightarrow$ Each of the squares is zero

$\therefore (ap - b)^2 = (bp - c)^2 = (cp - d)^2 = 0$

$\Rightarrow p = \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

$\therefore$ a, b, c are in G.P.

Sol 2: (C) Sum of the n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots.$ upto n terms can be written as

$\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)\ldots.$ upto n terms

$$= n-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.+n\text{terms}\right)$$

$$= n-\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = n+2^{-n}-1$$

Sol 3: (B) Let the common ratio of the G.P. be r. Then,

$Y = xr$ and $z = xr^2$

$\Rightarrow \ln y = \ln x + \ln r$ and $\ln z = \ln x + 2\ln r$

Let $A = 1 + \ln x$, $D = \ln r$

Then, $\frac{1}{1+\ln x} = \frac{1}{A}$,

$$\frac{1}{1+\ln y} = \frac{1}{1+\ln x+\ln r} = \frac{1}{A+D}$$

and $\frac{1}{1+\ln z} = \frac{1}{1+\ln x+2\ln r} = \frac{1}{A+2D}$

Therefore, $\frac{1}{1+\ln x},\frac{1}{1+\ln y},\frac{1}{1+\ln z}$ are in H.P.

Sol 4: (A) Since A.M. $\ge$ G.M., then

$$\frac{(a+b)+(c+d)}{2} \ge \sqrt{(a+b)(c+d)}$$

$\Rightarrow M \le 1$

Also,(a + b) + (c + d) > 0($\because$ a, b, c, d > 0)

$\therefore 0 < M \le 1$

Sol 5: (D) Since a, b, c, d are in A.P.

$\Rightarrow \frac{a}{abcd}, \frac{b}{abcd}, \frac{c}{abcd}, \frac{d}{abcd}$ are in AP

$\Rightarrow \frac{1}{bcd}, \frac{1}{cda}, \frac{1}{abd}, \frac{1}{abc}$ are in A.P.

$\Rightarrow$ bcd, cda, abd, abc are in HP.

$\Rightarrow$ abc, abd, cda, bcd are in HP.

Sol 6: (D) Since a, b, c are in AP.

Leta = A – D, b = A, c = A + D

Given, a + b + c = $\frac{3}{2}$

$\Rightarrow (A - D) + A + (A + D) = \frac{3}{2} \Rightarrow 3A = \frac{3}{2} \Rightarrow A = \frac{1}{2}$

$\therefore$ The numbers are $\frac{1}{2} - D, \frac{1}{2}, \frac{1}{2} + D$

Also, $\left(\frac{1}{2} - D\right)^2, \frac{1}{4}, \left(\frac{1}{2} + D\right)^2$ are in GP.

$$\Rightarrow \left(\frac{1}{4}\right)^2 = \left(\frac{1}{2} - D\right)^2 \left(\frac{1}{2} + D\right)^2 \Rightarrow \frac{1}{16} = \left(\frac{1}{4} - D^2\right)^2$$

$$\Rightarrow \frac{1}{4} - D^2 = \pm\frac{1}{4} \Rightarrow D^2 = \frac{1}{2} \Rightarrow D = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow a = \frac{1}{2} + \frac{1}{\sqrt{2}} \text{ or } \frac{1}{2} - \frac{1}{\sqrt{2}}$$

So, out of the given values, a = $\frac{1}{2} - \frac{1}{\sqrt{2}}$ is the right choice

Sol 7: (C) We know that, the sum of infinite term of G.P. is

$$S_\infty = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \infty, & |r| \le 1 \end{cases}$$

$\therefore S_\infty = \frac{x}{1-r} = 5$ (thus |r| < 1)

or $1 - r = \frac{x}{5} \Rightarrow r = \frac{5-x}{5}$ exists only when |r| < 1

i.e., $-1 < \frac{5-x}{5} < 1$

or –10 < – x < 0$\Rightarrow$ 0 < x < 10

Sol 8: (C) Let $S_n = cn^2$

$S_{n-1} = c(n-1)^2 = cn^2 + c - 2cn$

$\therefore T_n = 2cn - c$ ($\because$ $T_n = S_n - S_{n-1}$)

$T^2_n = (2cn - c)^2 = 4c^2n^2 + c^2 - 4c^2n$

$\therefore$ Sum = ST_n^2

$$= \frac{4c^2n(n+1)(2n+1)}{6} + nc^2 - 2c^2n(n+1)$$

$$= \frac{2c^2n(n+1)(2n+1) + 3nc^2 - 6c^2n(n+1)}{3}$$

$$= \frac{nc^2(4n^2 + 6n + 2 + 3 - 6n - 6)}{3} = \frac{nc^2(4n^2 - 1)}{3}$$

Sol 9: (B) Let a, ar, ar^2, ...

$a + ar = 12$ ... (i)

$ar^2 + ar^3 = 48$ ... (ii)

Dividing (ii) by (i), we have

$$\frac{ar^2(1+r)}{a(r+1)} = 4$$

$\Rightarrow r^2 = 4$ if $r \ne -1$

$\therefore r = -2$

Also, a = – 12 (using (i)).

Sol 10: (B) Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...$ ... (i)

$\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + ...$ ... (ii)

Dividing (i) and (ii)

$$S\left(1 - \frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + ...$$

$$\frac{2}{3}S = \frac{4}{3} + \frac{4}{3^2}\left(1 + \frac{1}{3} + \frac{1}{3^2} + ...\right)$$

$$\Rightarrow \frac{2}{3}S = \frac{4}{3} + \frac{4}{3^2}\left(\frac{1}{1 - \frac{1}{3}}\right) = \frac{4}{3} + \frac{4}{3^2}\frac{3}{2} = \frac{4}{3} + \frac{2}{3} = \frac{6}{2}$$

$$\Rightarrow \frac{2}{3}S = \frac{6}{3} \Rightarrow S = 3$$

Sol 11: (B) $M = \frac{\ell + n}{2}$

ℓ, G1 , G2 , G3 , n are in G.P.

$$r = \left(\frac{n}{\ell}\right)^{\frac{1}{4}}$$

$G_1 = \ell\left(\frac{n}{\ell}\right)^{\frac{1}{4}}$ $\quad G_2 = \ell\left(\frac{n}{\ell}\right)^{\frac{1}{2}}$ $\quad G_3 = \ell\left(\frac{n}{\ell}\right)^{\frac{3}{4}}$

$G_1^4 + 2G_2^4 + G_3^4$

$= \ell^4 \times \frac{n}{\ell} + 2\ell^4 \times \frac{n^2}{\ell^2} + \ell^4 \times \frac{n^3}{\ell^3}$

$= \ell^3 n + 2\ell^2 n^2 + \ell n^3$

$= n\ell(\ell^2 + 2n\ell + n^2)$

$= n\ell(\ell + n)^2$

$= 4m^2 n\ell$

Sol 12: (B) $T_n = \dfrac{\frac{n^2(n+1)^2}{4}}{n^2}$

$T_n = \frac{1}{4}(n+1)^2$

$T_n = \frac{1}{4}[n^2 + 2n + 1]$

$S_n = \sum_{n=1}^{n} T_n$

$S_n = \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + n(n+1) + n\right]$

$n = 9$

$S_9 = \frac{1}{4}\left[\frac{9\times10\times19}{6} + 9\times10 + 9\right] = \frac{1}{4}[285 + 90 + 9] = \frac{384}{4} = 96$

Sol 13: (A) a + d, a + 4d, a + 8d → G.P

∴ $(a + 4d)^2 = a^2 + 9ad + 8d2$

⇒ $8d^2 = ad$ ⇒ a = 8d

∴ 9d, 12d, 16d → G.P.

Common ratio $r = \frac{12}{9} = \frac{4}{3}$

Sol 14: (A)

$\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 + \left(\frac{24}{5}\right)^2 + \ldots\ldots.$

$\frac{8^2}{5^2} + \frac{12^2}{5^2} + \frac{16^2}{5^2} + \frac{20^2}{5^2} + \frac{24^2}{5^2} + \ldots\ldots$

$T_n = \frac{(4n+4)^2}{5^2}; S_n = \frac{1}{5^2}\sum_{n=1}^{10} 16(n+1)^2 = \frac{16}{25}\sum_{n=1}^{10}(n^2 + 2n + 1)$

$= \frac{16}{25}\left[\frac{10\times11\times21}{6} + \frac{2\times10\times11}{2} + 10\right] = \frac{16}{25}\times505 = \frac{16}{5}m$

⇒ m = 101

Sol 15: (B) a, ar, ar^2 → G.P.

a, 2ar, ar^2 → A.P.

2 × 2ar = a + ar^2

$4r = 1 + r^2$

⇒ $r^2 - 4r + 1 = 0$

$r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\boxed{r = 2 + \sqrt{3}}$

$r = 2 - \sqrt{3}$ is rejected

∴ (r > 1)

G.P. is increasing.

Sol 16: (B) Statement-I has 20 terms whose sum is 8000 and statement-II is true and supporting statement-I.

∴ k^{th} bracket is $(k-1)^2 + k(k-1) + k^2 = 3k^2 - 3k + 1$.

Sol 17: (D) $100(T_{100}) = 50(T_{50})$ ⇒ 2[a + 99d] = a + 49d ⇒ a + 149d = 0 ⇒ $T_{150} = 0$

Sol 18: (D) $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0 \quad \forall x \in R$

⇒ f(x) is strictly increasing function

⇒ f(x) = 0 has only one real root, so two roots are not possible

Sol 19: (A) $x^2 + 2x + 3 = 0$ ... (i)

$ax^2 + bx + c = 0$ ... (ii)

Since equation (i) has imaginary roots

So equation (ii) will also have both roots same as (i). Thus

$\frac{a}{1} = \frac{b}{2} = \frac{c}{3} \Rightarrow a = \lambda, b = 2\lambda, c = 3\lambda$

Hence 1 : 2 : 3

Sol 20: (C) $\frac{7}{10} + \frac{77}{100} + \frac{777}{10^3} + \ldots\ldots +$ up to 20 terms

$= 7\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{10^3} + \ldots\ldots + \text{up to 20 terms}\right]$

$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots\ldots + \text{up to 20 terms}\right]$

$$=\frac{7}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots\ldots+\text{up to 20 terms}\right]$$

$$=\frac{7}{9}\left[20-\frac{\frac{1}{10}\left(1-\left(\frac{1}{10}\right)^{20}\right)}{1-\frac{1}{10}}\right]==\frac{7}{9}\left[20-\frac{1}{9}\left(1-\left(\frac{1}{10}\right)^{20}\right)\right]$$

$$=\frac{7}{9}\left[\frac{179}{9}+\frac{1}{9}\left(\frac{1}{10}\right)^{20}\right]=\frac{7}{81}\left[179+(10)^{-20}\right]$$

Sol 21: (A) 2y = x + z

2 tan-1 y = tan-1 x + tan-1 (z)

$$\tan^{-1}\left(\frac{2y}{1-y^2}\right)=\tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\frac{x+z}{1-y^2}=\frac{x+z}{1-xz}$$

$\Rightarrow$ y2 = xz or x + z = 0 $\Rightarrow$ x = y = z

JEE Advanced/Boards

Exercise 1

Sol 1: (i) Let 2 numbers be a, b

Given H.M. = $\frac{2ab}{a+b}$ = 4 $\Rightarrow$ a + b = $\frac{ab}{2}$

We have A.M. = $\frac{a+b}{2}$ and G.M. = $\sqrt{ab}$

(G.M.)^2 = ab ... (i)

2A.M. = a + b ... (ii)

2A + G^2 = 27

a + b + ab = 27 using (i) and (ii)

$\frac{3ab}{2}$ = 27 $\Rightarrow$ ab = 18

a + b = 9

$\Rightarrow$ a, b = 3, 6

(ii) A.M. = G.M. + 15 = H.M. + 27

$\frac{a+b}{2}$ = $\frac{2ab}{a+b}$ + 27 = $\sqrt{ab}$ + 15

HM = $\frac{G^2}{AM}$ (we know this)

$\Rightarrow$ (A – 27) (A) = (A – 15)^2

$\Rightarrow$ 27A = 225 – 30 A

A = 75 ...(i)

G.M. = 60 ...(ii)

H.M. = 48

a + b = 150 using (i) and (ii)

ab = 3600

a = 120

b = 30

Sol 2: H_{10} = 21, H_{21} = 10

$\frac{1}{H_{10}} = \frac{1}{H_1} + 9d = \frac{1}{21}$

$\frac{1}{H_{10}} = \frac{1}{H_1} + 20d = \frac{1}{10}$

$\Rightarrow$ 11d = $\frac{11}{210}$ $\Rightarrow$ d = $\frac{1}{210}$

$\frac{1}{H_1} + \frac{9}{210} = \frac{10}{210}$

$\frac{1}{H_1} = \frac{1}{210}$

$\frac{1}{H_{210}} = \frac{1}{H_1} + 209d = \frac{1}{210} + \frac{209}{210} = \frac{210}{210} = 1$

Sol 3: sin x, sin^{2}2x, cos x sin 4x are in GP.

sin^{4}2x = sin x cos x sin 4x

(2 sin x cos x)^4= sin x cos x 2 sin 2x cos 2x

16 sin^4 x cos^4 x = 4 sin^2 x cos^2 x cos 2x

4 sin^{2}x cos^{2}x = cos 2x

Sin^{2}2x = cos 2x

1 – cos^2 2x = cos 2x

cos^2 2x + cos 2x – 1 = 0

cos 2x = $\frac{-1\pm\sqrt{5}}{2}$

$\therefore$ cos θ can never be equal to $\frac{-1-\sqrt{5}}{2}$ i.e

$\therefore$ cos 2x = $\frac{-1+\sqrt{5}}{2}$

Common ratio = $\frac{\sin^2 2x}{\sin x} = \frac{4\sin^2 x\cos^2 x}{\sin x}$

$= 4\cos^2 x \sin x = 2\cos x \sin 2x$

$= 2\sqrt{\dfrac{1+\cos 2x}{2}}\sqrt{1-\cos^2 2x}$

$= \sqrt{2}\sqrt{1-\dfrac{(6-2\sqrt{5})}{4}}\sqrt{1+\left(\dfrac{\sqrt{5}-1}{2}\right)}$

$= \sqrt{2}\sqrt{\dfrac{\sqrt{5}-1}{2}}\cdot\sqrt{\dfrac{\sqrt{5}+1}{2}} = \sqrt{2}\cdot\sqrt{\dfrac{4}{4}} = \sqrt{2}$

Sol 4: a, b, c, d, e be 5 numbers

a b c in AP, b c d in GP, c d e in HP

$2b = a + c$, $c^2 = bd$, $d = \dfrac{2ce}{c+e}$, Let b be b

c be br, d be br^2

$br^2 = \dfrac{2bre}{br+e} \Rightarrow br^2 + er = 2e \Rightarrow e = \dfrac{br^2}{2-r}$

$a = 2b - br = b(2-r)$

$ae = b^2r^2 = c^2$ hence a,c,e are in GP ...(i)

(ii) $\Rightarrow \dfrac{(2b-a)^2}{a} = \dfrac{c^2}{a} = \dfrac{b^2r^2}{b(2-r)} = \dfrac{br^2}{2-r} = e$

Hence proved.

(iii) $a = 2$ $e = 18 \Rightarrow c = \pm 6$

bc, d, e = (4, 6, 9); (–2, –6, –18)

$\Rightarrow b = 4, -2 \Rightarrow d = 9, -18$

Sol 5: $2\alpha^2, \alpha^4, 24$ form A.P.

$\alpha^4 = \alpha^2 + 12$ ...(i)

$\alpha^4 - \alpha^2 = 12$

$\Rightarrow \alpha = 2, -2$ $(\alpha_1, \alpha_2 = 2, -2$

$(\beta^2)^2 = 1(6-\beta^2)$

$\beta^4 + \beta^2 = 6$

$\beta^4 + \beta^2 - 6 = 0$

$\beta^4 + 3\beta^2 - 2\beta^2 - 6 = 0$

$\beta^2 = 2$

$\beta_1, \beta_2 = \sqrt{2}, -\sqrt{2}$

$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 4 + 4 + 2 + 2 = 12$

Sol 6: 2G.P.s

$\dfrac{a_1}{1-r_1} = 1 \Rightarrow a_1 = 1 - r_1$

$\dfrac{a_2}{1-r_2} = 1 \Rightarrow a_2 = 1 - r_2$

$a_1r_1 = a_2r_2 \Rightarrow (1-r_1)r_1 = (1-r_2)r_2$

$\Rightarrow r_1 - r_2 = (r_1 - r_2)(r_1 + r_2)$

$\Rightarrow r_1 + r_2 = 1$

$a_1r_1^2 = 1/8 \Rightarrow (1-r_1)r_1^2 = 1/8$

$\Rightarrow (2r_1 - 1)(4r_1^2 - 2r_1 - 1) = 0$

If $r_1 = ½$ then $r_1 = r_2$

$\Rightarrow 4r_1^2 - 2r_1 - 1 = 0$

$\Rightarrow r_1 = \dfrac{1\pm\sqrt{5}}{4}$

If $r_1 = \dfrac{1-\sqrt{5}}{4}$ then, $r_2 > 1$.

$\Rightarrow r_1 = \dfrac{1+\sqrt{5}}{4}$

$\therefore a_1r_1 = (1-r_1)r_1 = \left(1-\left(\dfrac{1+\sqrt{5}}{4}\right)\right)\left(\dfrac{1+\sqrt{5}}{4}\right)$

$= \left(\dfrac{3-\sqrt{5}}{4}\right)\left(\dfrac{1+\sqrt{5}}{4}\right) = \dfrac{\sqrt{5}-1}{8} = \dfrac{\sqrt{m}-n}{p}$

$\therefore 100m + 10n + p = 500 + 10 + 8 = 518$

Sol 7: $S = \displaystyle\sum_{n=1}^{99}\dfrac{5^{100}}{25^n + 5^{100}}$

$T_1 = \dfrac{5^{100}}{5^2 + 5^{100}}$

$T_{99} = \dfrac{5^{100}}{5^{2\times 99} + 5^{100}} = \dfrac{5^{100}}{\dfrac{5^{200}}{5^2} + 5^{100}} = \dfrac{5^2}{5^2 + 5^{100}}$

$T_1 + T_n = 1$

$S = T_1 + T_2 + ... + T_{99} = 1 + 1 ... T_{50}$

$= 49 + T_{50} = 49 + \dfrac{5^{100}}{5^{100} + 5^{100}} = 49 + {}^1/2$

$[S] = 49$

Sol 8: $ax^3 - ax^2 + 9bx - b = 0$

HM roots $= \dfrac{3}{\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}} = \dfrac{3\alpha\beta\gamma}{\alpha\beta+\beta\gamma+\gamma\alpha}$

$\alpha+\beta+\gamma = 1$

$\alpha\beta+\beta\gamma+\gamma\alpha = \dfrac{9b_1}{a_1}$

$\alpha\beta\gamma = \frac{b_1}{a_1}$

H.M .= $\frac{1}{3}$

$\frac{\alpha+\beta+\gamma}{3} \geq$ H.M.$_{(abc)}$

$\frac{\alpha+\beta+\gamma}{3} \geq \frac{1}{3}$

$\alpha+\beta+\gamma \geq 1$ ...(i)

Its given $\alpha+\beta+\gamma = 1$

Equality of equation (i) holds only if $\alpha = \beta = \gamma$

i.e all the roots are $\frac{1}{3}$

$\frac{b}{a} = \alpha^3 = \left(\frac{1}{3}\right)^3$

b = 27a

b + a = 28a

$\therefore$ a is an integer, min (a + b) = 28

Sol 9: Let time taken to solve 1st problem be S time to solve second problem will be $\frac{S}{r}$

$\frac{S}{r} + \ldots + \frac{S}{r^{n-1}} = 63.5$ ...(i)

$S_n = 127 = S + \frac{s}{r} \ldots \frac{s}{r^{n-2}}$ ...(ii)

$31.5 = \frac{s}{r^2} + \ldots \frac{s}{r^{n-1}}$ ...(iii)

$\frac{S}{r} = 32$ ...(iv)

$127 + \frac{S}{r^{n-1}} = 63.5 + S$

$63.5 + \frac{S}{r^{n-1}} = S$

$\frac{S}{r} - \frac{S}{r^n} = 63.5\left(1-\frac{1}{r}\right)$

$32 - \frac{s}{r^n} = 63.5 - \frac{63.5}{r}$

$\frac{63.5}{r} \quad \frac{S}{r^n} = 31.5$ ...(v)

$127 = \frac{S\left(1-\frac{1}{r^{n-1}}\right)}{\left(1-\frac{1}{r}\right)}$

$127 - \frac{127}{r} = S - \frac{S}{r^{n-1}}$

$127\left(1-\frac{1}{r}\right) = 32r - r\left(\frac{63.5}{r} - 31.5\right)$

from (v)

$127 - \frac{127}{r} = 63.5\,r - 63.5$

$\Rightarrow r = 2$

$\therefore$ s = 64

$\frac{s}{r^{n-1}} = \frac{1}{2}$

$2^{n-2} = 64$

$\Rightarrow n = 8$

Sol 10: $a_n+1 = a_n + 1$ for n = 1 ... 97

$\Rightarrow a_2 = a_1 + 1$

$\Rightarrow a_3 = a_2 + 1 = a_1 + 2$

$\Rightarrow a_4 = a_1 + 3$

$a_n = a_1 + (n-1)$

$\Rightarrow a_1 + a_2 \ldots a_{98} = 4949 = \frac{98}{2}[2a_1 + 97.1]$

$101 = 2a_1 + 97 \Rightarrow a_1 = 2$

Now, we can write here Σa_{2k}

$= a_2 + a_4 + a_6 \ldots a_{98} = a_1 + 1 + a_1 + 3 \ldots a_1 + 97$

$= \frac{49}{2}[2a_1 + 2 + 48\times 2] = 49[a_1 + 49] = 49 \times 51$

= 2499

Sol 11: xyz = 55 or $\frac{343}{55}$ acc to a, x, y, z, b in AP/HP

For a, x, y, z, b in AP

x = a + d; y = a + 2d z = a + 3d

$b = a + 4d \Rightarrow d = \frac{b-a}{4}$

(a + d) (a + 2d) (a + 3d) = 55 ...(i)

For a, x, y, z, b in HP

$\frac{1}{x} = \frac{1}{a} + d_H;\ \frac{1}{b} = \frac{1}{a} + 4d_H \Rightarrow d_H = \frac{1}{4}\left[\frac{1}{b} - \frac{1}{a}\right]$

$\frac{1}{y} = \frac{1}{a} + 2d_H$

$\frac{1}{z} = \frac{1}{a} + 3d_H$

$$\frac{1}{xyz} = \left(\frac{1}{a}+d_H\right)\left(\frac{1}{a}+2d_H\right)\left(\frac{1}{a}+3d_H\right) = \frac{55}{343} \quad ...(ii)$$

Equation (i) can be written as

$$\left(a+\frac{b-a}{4}\right)\left(a+\frac{(b-a)2}{4}\right)\left(a+\frac{(b-a)3}{4}\right)=55$$

$$\frac{(3a+b)(2a+2b)(a+3b)}{64} = 55$$

Equation (ii) can be written as

$$\left(\frac{3}{4a}+\frac{1}{4b}\right)\left(\frac{1}{2a}+\frac{1}{2b}\right)\left(\frac{1}{4a}+\frac{3}{4b}\right) = \frac{55}{343}$$

$$\frac{(3b+a)(2b+2a)(b+3a)}{64a^2b^2ab} = \frac{55}{343}$$

$\Rightarrow a^3b^3 = 343$

$\Rightarrow ab = 7$

a & b are integers

i.e a = 1 , b = 7or a = 7, b = 1

i.e $a^2 + b^2 = 50$

Sol 12: $10x^3 - cx^2 - 54x - 27 = 0$

Let α, β, γ be the roots

$$\alpha + \beta + \gamma = \frac{c}{10} \quad ...(i)$$

$$a\beta + by + \gamma\alpha = -\frac{54}{10} \quad ...(ii)$$

$$ab\gamma = \frac{27}{10} \quad ...(iii)$$

α β & γ are in harmonic progression

i.e $\beta = \dfrac{2\alpha\gamma}{\alpha+\gamma}$

$$b\alpha + b\gamma = 2\alpha\gamma \quad ....(iv)$$

Putting this in equation (iii)

$\beta = -3/2$ this in equation (iv)

$$(\alpha + \gamma)\left(\frac{-3}{2}\right) = \frac{-3.6}{10}$$

$$\alpha + \gamma = \frac{12}{5}$$

$$\Rightarrow \alpha = 3;\ \gamma = \frac{-3}{5}$$

The 3 roots are 3, $\dfrac{-3}{2}$, $\dfrac{-3}{5}$

$$\frac{C}{10} = \alpha + \beta + y = \frac{12}{5} - \frac{3}{2} = \frac{9}{10} \Rightarrow C = 9$$

Sol 13: We have $b^2 = ac$

Also, $\log_c a$, $\log_b c$ and $\log_a b$ are in AP

We can write $\log_a b = \log_c a + (3 - 1)d$

$$\Rightarrow d = \frac{\log_a b - \log_c a}{2}$$

$$= \frac{\log_a \sqrt{ac} - \log_c a}{2}$$

$$= \frac{1+3\log_a c}{4}$$

Given that $\log_a c = 4$

$$\therefore d = \frac{1+3\times 4}{4} = \frac{13}{4}$$

Sol 14: $a = b - 2$

$ar^2 = b + 6$

$$\frac{a+ar^2}{2ar} = \frac{5}{3}$$

$$\frac{2b+4}{2ar} = \frac{5}{3}$$

$$\frac{3}{5}(b + 2) = ar$$

$$\frac{9}{25}(b + 2)^2 = (b + 6)(b - 2)$$

$\Rightarrow 9(b^2 + 4 + 4b) = 25(b^2 + 4b - 12)$

$16b^2 + 64b - 336 = 0$

$b^2 + 4b - 21 = 0$

$b^2 + 7b - 3b - 21 = 0$

b = 7, 3 $\Rightarrow$ +ve integral value of b is 3.

Sol 15: $\dfrac{S_{1-11}}{S_{n-10-n}} = \dfrac{1}{8}$ ... (i)

$$\frac{S_{10-n}}{S_{(n-8)-n}} = 2 \quad ... (ii)$$

$$S_{1-11} = a\frac{\left(r^{11}-1\right)}{r-1} \quad S_{10-n} = ar^9\frac{\left(r^{n-9}-1\right)}{r-1}$$

$$S_{(n-10)-n} = ar^{n-11}\frac{\left(r^{11}-1\right)}{r-1} \quad S_{(n-8)-n} = a\frac{\left(r^{n-9}-1\right)}{r-1}$$

Putting these values in equation (i) and equation (ii)

$$\frac{1}{r^{n-11}} = \frac{1}{8}$$

$r^9 = 2 \Rightarrow r = 2^{\frac{1}{9}}$

$\Rightarrow r^{n-11} = 2^3$

$\Rightarrow 2^{\frac{n-11}{9}} = 2^3$

$\Rightarrow n = 11 + 27 = 38$

Sol 16: $S_n = (1 + 2T_n)(1 - T_n)$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$= [1 + 2a + (n-1)2d][1 - a - (n-1)d]$

$T_n = a + (n-1)d$

$S_n = 1 + T_n - 2T_n^2$

$S_1 = 1 + T_1 - 2T_1^2 = T_1$

$T_1 = \frac{1}{\sqrt{2}}$

$S_2 = 1 + T_2 - 2T_2^2 = T_1 + T_2$

$1 - \frac{1}{\sqrt{2}} = 2T_2^2$

$\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right) = T_2^2 = \frac{2-\sqrt{2}}{4} = \frac{\sqrt{4}-\sqrt{2}}{4}$

$\Rightarrow a = 4$ and $b = 2$

$a + b = 6$

Sol 17: Let number be abc $(a > b > c)$

$a = a,\ b = \frac{a}{r},\ c = \frac{a}{r^2}$

$\Rightarrow 100a + \frac{10a}{r} + \frac{a}{r^2} - 100\frac{a}{r^2} - \frac{10a-a}{r} = 792$

$\Rightarrow 99a - \frac{99a}{r^2} = 792 \Rightarrow a - \frac{a}{r^2} = 8$

$\therefore$ New number $= 100(a-4) + 10b + c$

$\Rightarrow 2b = a - 4 + c \Rightarrow \frac{2a}{r} = a - 4 + \frac{a}{r^2}$

$\Rightarrow \frac{2a}{r} = a - 4 + a - 8$

$\Rightarrow 2a - 12 = \frac{2a}{r} \Rightarrow a = \frac{a}{r} + 6 = \frac{a}{r^2} + 8$

$\Rightarrow r = \frac{a}{a-6}$

$\Rightarrow \left(\frac{a}{a-6}\right)^2 = \frac{a}{a-8}$

$\Rightarrow a(a-8) = (a-6)^2 \Rightarrow a = 9,\ r = 3$

So the number is 931

Sol 18: $S(\theta) = 1 + (1 + \sin\theta)\cos\theta$

$+ (1 + \sin\theta + \sin^2\theta)\cos^2\theta \ldots \infty$

$= 1 + \cos\theta + \cos^2\theta \ldots$

$+ \sin\theta(\cos\theta + \cos^2\theta \ldots) + \sin^2\theta$

$= \frac{1}{1-\cos\theta} + \frac{\sin\theta}{1-\cos\theta}\cos\theta + \frac{\sin^2\theta\cos^2\theta}{1-\cos\theta}$

$= \frac{1}{1-\cos\theta}[1 + \sin\theta\cos\theta + \sin^2\theta\cos^2\theta \ldots]$

$S(\theta) = \frac{1}{(1-\sin\theta\cos\theta)(1-\cos\theta)}$

$S_{\left(\frac{\pi}{4}\right)} = \frac{1}{\left(1-\frac{1}{2}\right)}\left(1-\frac{1}{\sqrt{2}}\right) = \frac{2\sqrt{2}}{\left(\sqrt{2}-1\right)}$

Sol 19: $\tan\left(\frac{\pi}{12}-x\right), \tan\frac{\pi}{12}, \tan\left(\frac{\pi}{12}+x\right)$ are in GP

$\tan^2\frac{\pi}{12} = \tan\left(\frac{\pi}{12}-x\right)\ \tan\left(\frac{\pi}{12}+x\right)$

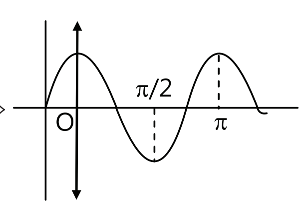
$= \frac{\sin\left(\frac{\pi}{12}+x\right)\sin\left(\frac{\pi}{12}-x\right)}{\cos\left(\frac{\pi}{12}+x\right)\cos\left(\frac{\pi}{12}-x\right)} \Rightarrow \frac{\cos 2x - \cos\frac{\pi}{6}}{\cos 2x + \cos\frac{\pi}{6}} = \tan^2\frac{\pi}{12}$

$\cos 2x = \frac{\cos\frac{\pi}{6}\tan^2\frac{\pi}{12} + \cos\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{12}} = \frac{\cos\frac{\pi}{6}\left[\tan^2\frac{\pi}{12} + 1\right]}{1 - \tan^2\frac{\pi}{12}}$

$= \cos\frac{\pi}{6}\left[\frac{\sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right)}{\cos^2\left(\frac{\pi}{12}\right) - \sin^2\left(\frac{\pi}{12}\right)}\right] = \frac{\sqrt{3}}{2}\left[\frac{1}{\cos\left(\frac{\pi}{6}\right)}\right] = 1$

$= \frac{\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} = 1$

$\therefore \cos 2x = 1$

$\Rightarrow$ 

Solutions are O, π, 2π, 3π ... 99π

$$\frac{99}{2}[2\pi+98\pi] = 50\ \pi\ .\ 99 = 4950\ \pi$$

K = 4950

Exercise 2

Single Correct Choice Type

Sol 1: (A) A.M. = 9 + 99 ... 999999999/9

$\Rightarrow$ 9[1 + 11 + 111 + ... 111111111]/9

= 123456789

This does not contain 0

Sol 2: (D) Given $\sum_{k=1}^{360} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} = \frac{m}{n}$

$$\sum_{k=1}^{360} = \frac{1}{\sqrt{k+1}\sqrt{k}}\left[\frac{1}{\sqrt{k}+\sqrt{k+1}}\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}\right]$$

$$\sum_{k=1}^{360} = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}\sqrt{k}}$$

$$\sum_{k=1}^{360} = \sum\left[\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right]$$

$$\therefore \sum_{k=1}^{360} T_k = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \ldots - \frac{1}{\sqrt{361}}$$

$$= 1-\frac{1}{19} = \frac{18}{19} = \frac{m}{n} \qquad \text{(given)}$$

$\Rightarrow$ m + n = 18 + 19 = 37

Sol 3: (B) $\sum_{k=1}^{100}\frac{k}{k^4+k^2+1} = \sum\frac{k}{(k^2+1)^2-k^2}$

$$= \sum\frac{k}{(k^2+1+k)(k^2+1-k)}$$

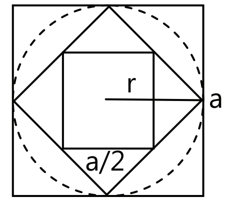
$$= \sum\frac{k}{2k}\left[\frac{1}{k^2+k+1}+\frac{1}{k^2-k+1}\right]$$

$$= \sum\frac{1}{2}\left[\frac{1}{k^2-k+1}-\frac{1}{k^2+k+1}\right]$$

$$= \sum\frac{1}{2}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{7}\ldots\frac{1}{100^2+100+1}\right]$$

$$= \frac{1}{2}\left[\frac{100^2+100}{10101}\right] = \frac{5050}{10101}$$

Sol 4: (A)



Circle inscribed in 1st circle = r = a/2

In 2nd circle = $r_1 = \frac{a}{2\sqrt{2}}$

In 3rd circle = $\frac{a}{2\times 2}$ this is G.P. with common ratio $\frac{1}{\sqrt{2}}$

$$a_r = \frac{a}{2}\left(\frac{1}{\sqrt{2}}\right)^{n-1} = r\left(\frac{1}{2}\right)^{\frac{n-1}{2}} = r2^{\left(\frac{1-n}{2}\right)}\ \text{[A]}$$

Assertion Reasoning Type

Sol 5: (D) Statement-I: If $(a + b + c)^3 \le 27$ abc

3a + 4b + 5c = 12

Statement-II: $\Rightarrow$ A.M. $\ge$ G.M. (True)

We Know A.M. $\ge$ G.M.

For three numbers a + b + c

$$\Rightarrow \frac{a+b+c}{3} \ge (abc)^{1/3}$$

$\Rightarrow (a + b + c)^3 \ge 27$ abc

Given $(a + b + c)^3 \le 27$ abc

$\Rightarrow$ a = b = c & $(a + b + c)^3 = 27$ abc

3a + 4a + 5a = 12 $\Rightarrow$ a = b = c = 1

$$\frac{1}{a^2} + \frac{1}{b^3} + \frac{1}{b^5} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} = 3 \ne 10$$

So statement-II is false.

Multiple Correct Choice Type

Sol 6: (A, B, C, D) $a_1 = 25$, $b_1 = 75$, $a_{100} + b_{100} = 100$

$a_1 + 99d_a + b_1 + 99d_b = 100$

$\Rightarrow d_a + d_b = 0$

$\Rightarrow d_a = -\,d_B$

(a) Hence a is correct

(b) $a_n + b_n = a_1 + (n-1)d_a + b_1 + (n-1)d_B$

$= a_1 + b_1 + (n-1)(d_a + d_B) = a_1 + b_1 = 100$

Hence, correct

(c) $(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \ldots$

= 100, 100, 100 is in AP

(d) $\sum_{r=1}^{100}(a_r + b_r) = \sum_{r=1}^{100}(100) = 10000$

Sol 7: (B, C) sin (x – y), sin x, sin (x + y) are in HP

$\sin x = \dfrac{2\sin(x-y)\sin(x+y)}{\sin(x-y)+\sin(x+y)}$

$\Rightarrow 2\sin^2 x \cos y = 2\cos^2 y - 2\cos^2 x$

$\Rightarrow \sin^2 x(\cos y - 1) = \cos^2 y - 1$

$\Rightarrow \sin^2 x = 1 + \cos y = 2\cos^2 y/2$

$\Rightarrow \sin x \sec\left(\dfrac{y}{2}\right) = \sqrt{2}$

Sol 8: (A, B) Given series is a ar ar^2 ar^3...

Given that $ar - a = 6$ and $ar^3 - ar^2 = 54$

$\Rightarrow ar^2(r-1) = 54 \Rightarrow a(r-1) = 6$

Dividing these (2)

$r^2 = 9$

$r = \pm 3$

a = 3, for r = 3

ar = 9

$ar^2 = 27$

a = 3

sum = 39 [A]

for $r = -3$; $a = \dfrac{-3}{2}$

$ar = \dfrac{9}{2}$; $ar^2 = \dfrac{-27}{2}$

$\therefore$ sum $= \dfrac{-3}{2} + \dfrac{9}{2} - \dfrac{27}{2} = 3 - \dfrac{27}{2} = \dfrac{-21}{2}$

Sol 9: (A, C, D) $x^3 + px^2 + qx - 1$

Roots form increasing GP

Roots be $\dfrac{a}{r}$, a, ar

$\dfrac{a}{r} + a + ar = -p$

$\dfrac{a^2}{r} + a^2 r + a^2 = q$

$a^3 = 1 \Rightarrow a = 1$

$\left(r + 1 + \dfrac{1}{r}\right) = -p$ ... (i)

$\left(\dfrac{1}{r} + r + 1\right) = q \Rightarrow q = -p \Rightarrow p + q = 0$

$\dfrac{r + 1 + \frac{1}{r}}{3} \geq 1$ [AM ≥ GM]

$r + 1 + \dfrac{1}{r} \geq 3$

$p \in (-\infty, 3)$ [B is incorrect]

one root (a) is unity

one root is $\dfrac{1}{r}$ & other is r, so if 1 root is greater than 1 and other less than [ACD]

Sol 10: (B, D) log a, log b, log c, $\log\dfrac{a}{2b}$, $\log\dfrac{2b}{3c}$, $\log\dfrac{3c}{a}$ are in AP

$a_1, a_2, a_3, a_4, a_5, a_6$

$2a_5 = a_4 + a_6$

$\dfrac{4b^2}{9c^2} = \dfrac{3c}{2b}$

$\dfrac{b}{c} = \dfrac{3}{2} \Rightarrow b = \dfrac{3c}{2}$

$a = \dfrac{3b}{2} = \dfrac{9c}{4}$

$a + b = \dfrac{15c}{4} > c$

$a + c = \dfrac{13c}{4} > \dfrac{3c}{2}$ (b)

$b + c = \dfrac{5c}{2} > \dfrac{9c}{4}$ (a)

Hence a, b, c can form Δ

log b – log a = log c – log b

2log b = log a + log c

$b^2 = ac$ ie a, b, c are in GP[B]

$a = a, b = ar, c = ar^2$

$18(a + b + c)^2 - 18a^2 - 18b^2 - 18c^2$

$= 18(2ab + 2bc + 2ac)$

$= 36(ab + bc + ac) > ab$ so A is incorrect

Sol 11: (A, B, C, D) $x^2 - 3x + A = 0$

$x_1 + x_2 = 3$

$x_1x_2 = A$

$x^2 - 12x + B = 0$

$x_3\, x_4 = B$

$x_3 + x_4 = 12$

$x_1 = a\, ;\, x_2 = ar\, ;\, x_3 = ar^2\, ;\, x_4 = ar^3$

$a^2r = A$

$a(1 + r) = 3$

$a^2r^5 = B$

$ar^2(1 + r) = 12$

$r = \pm 2$

$a = 1, -3$

$A = a^2r = 1 \times 2 = 2$

$a^2r^5 = 2^5 = 32 = B$

$x_1 + x_3 = a(1 + r^2) = 5$

$x_2 + x_4 = ar\,(1 + r^2) = 2.5 = 10$

Previous Years' Questions

Sol 1: (A, B, D) Since, first and (2n – 1)the terms are equal. Let first term be x and $(2n - 1)^{th}$ term by y. whose middle term is t_n.

Thus in arithmetic progression ; $t_n = \dfrac{x+y}{2} = a$

In geometric progression : $\sqrt{xy} = b$

In harmonic progression ; $t_n = \dfrac{2xy}{x+y} = c$

$\Rightarrow b^2 = ac$ and $a \ge b \ge c$ (using A.M. ≥ G.M. ≥ HM)

Here, equality holds (ie, a = b = c) only if all terms are same.

Sol 2: (B, C, D) Let a_n denotes the length of side of the square S_n.

We are given a_n = length of diagonal of S_{n+1}.

$\Rightarrow a_n = \sqrt{2}a_{n+1} \Rightarrow a_{n+1} = \dfrac{a_n}{\sqrt{2}}$

This show that a_1, a_2, a_3.... Form a G.P. with common ratio $1/\sqrt{2}$

Therefore, $a_n = a_1\left(\dfrac{1}{\sqrt{2}}\right)^{n-1}$

$\Rightarrow a_n = 10\left(\dfrac{1}{\sqrt{2}}\right)^{n-1}$ ($Qa_1 = 10$ given)

$\Rightarrow a_n^2 = 100\left(\dfrac{1}{\sqrt{2}}\right)^{2(n-1)}$

$\Rightarrow \dfrac{100}{2^{n-1}} \le 1\ (\because\ a_n^2 \le 1 \text{ given})$

$\Rightarrow 100 \le 2^{n-1}$

This is possible for $n \ge 8$. So (b), (c), (d) are the answer.

Sol 3: (4) We have $S_k = \dfrac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \dfrac{1}{(k-1)!}$

Now, $(k^2 - 3k + 1)\, S_k = \{(k - 2)\,(k - 1) - 1\} \times S_k$

$= \dfrac{1}{(k-3)!} - \dfrac{1}{(k-1)!} \Rightarrow \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k|$

$= 1 + 1 + 2 - \left(\dfrac{1}{99!} + \dfrac{1}{98!}\right) = 4 - \dfrac{100^2}{100!}$

$\Rightarrow \dfrac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = 4$

Sol 4: (0) $a_k = 2a_{k-1} - a_{k-2}$

$\Rightarrow a_1, a_2, \ldots, a_{11}$ are in AP

$\therefore \dfrac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = \dfrac{11a^2 + 35 \times 11d^2 + 10ad}{11} = 90$

$\Rightarrow 225 + 35d^2 + 150d = 90$

$\Rightarrow 35d^2 + 150d + 135 = 0$

$\Rightarrow d = -3, -\dfrac{9}{7}$

Given $a_2 < \dfrac{27}{2}$ $\quad\therefore d = -3$ and $d \ne -\dfrac{9}{7}$

$\Rightarrow \dfrac{a_1 + a_2 + \ldots + a_{11}}{11} = \dfrac{11}{2}[30 - 10 \times 3] = 0$

Sol 5: $a_1, a_2, a_3, \ldots A_{100}$ is an A.P.

$a_1 = 3,\ Sp = \sum_{i=1}^{p} a_1,\ 1 \le p \le 100$

$\dfrac{S_m}{S_n} = \dfrac{S_{5n}}{S_n} = \dfrac{\frac{5_n}{2}\left(6 + (5n-1)d\right)}{\frac{n}{2}\left(6 - d + nd\right)}$

Sm is independent of n of $6 - d = 0 \Rightarrow d = 6$

$a_1 = 3$

$a_2 = 3 + 6 = 9$

$a_2 = 9$

Sol 6: (C) Let a and b are two numbers. Then,

$$A_1 = \frac{a+b}{3}; G_1 = \sqrt{ab}; H_1 = \frac{2ab}{a+b}$$

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}, G_n = \sqrt{A_{n-1}H_{n-1}},$$

$$H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1} + H_{n-1}}$$

Clearly, $G_1 = G_2 = G_3 = ... = \sqrt{ab}$

Sol 7: (A) A_2 is A.M. of A_1 and H_1 and $A_1 > H_1$

$\Rightarrow A_1 > A_2 > H_1$

A_3 is A.M. of A_2 and H_2 and $A_2 > H_2$

$\Rightarrow A_2 > A_3 > A_4$

: : :

$\therefore A_1 > A_2 > A_3 > ....$

Sol 8: (B) As above $A_1 > H_2 > H_1$, $A_2 > H_3 > H_2$

$\therefore H_1 < H_2 < H_2 < ......$

Sol 9: (B) Here, $V_r = \frac{r}{2}[2r + (r-1)(2r-1)] = \frac{1}{2}(2r^3 - r^2 + r)$

$$\therefore \Sigma V_r = \frac{1}{2}[2\Sigma r^3 - \Sigma r^2 + \Sigma r]$$

$$= \frac{1}{2}\left[2\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[3n(n+1) - (2n+1) + 3]$$

$$= \frac{1}{12}n(n+1)(3n^2 + n + 2)$$

Sol 10: (D) $V_{r+1} - V_r = (r+1)^3 - r^3 - \frac{1}{2}[(r+1)^2 - r^2] + \frac{1}{2}(1)$

$= 3r^2 + 2r - 1$

$\therefore T_r = 3r^2 + 2r - 1 = (r + 1)(3r - 1)$

Which is a composite number.

Sol 11: (B) Since,

$T_r = 3r^2 + 2r - 1$

$\therefore T_{r+1} = 3(r + 1)^2 + 2(r + 1) - 1$

$\therefore Q_r = T_{r+1} - T_r = 3[2r + 1] + 2[1]$

$Q_r = 6r + 5$

Sol 12: (A, D) $S_n < \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1 + k/n + (k/n)^2}$

$$= \int_0^1 \frac{dx}{1 + x + x^2} = \frac{\pi}{3\sqrt{3}}$$

Now, $T_n > \frac{\pi}{3\sqrt{3}}$ as $h\sum\limits_{k=0}^{n-1} f(kh) > \int_0^1 f(x)dx > h\sum\limits_{k=1}^{n} f(kh)$

Sol 13: (C) $b_1 = a_1$, $b_2 = a_1 + a_2$, $b_3 = a_1 + a_2 + a_3$, $b_4 = a_1 + a_2 + a_3 + a_4$

Hence b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P. nor in H.P.

Sol 14: (C) $t_n = c\{n^2 - (n - 1)^2\} = c(2n - 1)$

$$\Rightarrow t_n^2 = c^2(4n^2 - 4n + 1)$$

$$\Rightarrow \sum_{n=1}^{n} t_n^2 = c^2\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$$

$$= \frac{c^2 n}{6}\left\{4(n+1)(2n+1) - 12(n+1) + 6\right\}$$

$$= \frac{c^2 n}{3}\left\{4n^2 + 6n + 2 - 6n - 6 + 3\right\} = \frac{c^2}{3}n(4n^2 - 1)$$

Sol 15: (3) $\sum\limits_{k=2}^{100}\left|(k^2 - 3k + 1)S_k\right|$

for $k = 2$ $|k^2 - 3k + 1)S_k| = 1$

$$\sum_{k=3}^{100}\left|\frac{k-1}{(k-2)!} - \frac{k-1+1}{(k-1)!}\right|$$

$$\sum_{k=3}^{100}\frac{1}{(k-3)!} + \frac{1}{(k-2)!} - \frac{1}{(k-2)!} - \frac{1}{(k-1)!}$$

$$\sum_{k=3}^{100}\left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right)$$

$$S = 1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{1!} - \frac{1}{3!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right) + \left(\frac{1}{3!} - \frac{1}{5!}\right) + \left(\frac{1}{4!} - \frac{1}{6!}\right) + \dots \left(\frac{1}{94!} - \frac{1}{96!}\right) + \left(\frac{1}{95!} - \frac{1}{97!}\right) + \left(\frac{1}{96!} - \frac{1}{98!}\right) + \left(\frac{1}{97!} - \frac{1}{99!}\right)$$

$$= 2 - \frac{1}{98!} - \frac{1}{99!}$$

$$\therefore E = \frac{100^2}{100!} + 3 - \frac{1}{98!} - \frac{1}{99.98!}$$

$$= \frac{100^2}{100!} + 3 - \frac{100}{99!} = \frac{100^2}{100.99!} + 3 - \frac{100}{99!} = 3$$

Sol 16: (0) $a_k = 2a_{k-1} - a_{k-2} \Rightarrow a_1, a_2, \dots, a_{11}$ are in A.P.

$$\therefore \frac{a_1^2 + a_2^2 + \dots + a_{11}^2}{11} = \frac{11a^2 + 35 \times 11d^2 + 10ad}{11} = 90$$

$\Rightarrow 225 + 35d2 + 150d = 90$

$35d2 + 150d + 135 = 0 \Rightarrow d = -3, -9/7$

Given $a^2 < \frac{27}{2} \therefore d = -3$ and $d \neq -9/7 \Rightarrow$

$$\frac{a_1 + a_2 + \dots + a_{11}}{11} = \frac{11}{2}\left[30 - 10 \times 3\right] = 0$$

Sol 17: (B) $ax^2 + bx + c = 0 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow \alpha = 1, \beta = -7$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{1} - \frac{1}{7}\right)^n = 7$$

Sol 18: (D) Corresponding A.P.

$\frac{1}{5}, \dots\dots\dots \frac{1}{25}$ (20^{th} term)

$$\frac{1}{25} = \frac{1}{5} + 19d \qquad \Rightarrow d = \frac{1}{19}\left(\frac{-4}{25}\right) = -\frac{4}{19 \times 25}$$

$$a_n < 0$$

$$\frac{1}{5} - \frac{4}{19 \times 25} \times (n-1) < 0$$

$$\frac{19 \times 5}{4} < n - 1$$

$$n > 24.75$$

Sol 19: (A, D) $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$

$$= \sum_{r=0}^{(n-1)} ((4r+4)^2 + (4r+3)^2 - (4r+2)^2 - (4r+1)^2)$$

$$= \sum_{r=0}^{(n-1)} (2(8r+6) + 2(8r+4)) = \sum_{r=0}^{(n-1)} (32r + 20)$$

$$= 16(n-1)n + 20n = 4n(4n+1) = \begin{cases} 1056 \text{ for } n = 8 \\ 1332 \text{ for } n = 9 \end{cases}$$

Sol 20: (6) $\frac{b}{a} = \frac{c}{b} = (\text{integer})$

$$b^2 = ac \Rightarrow c = \frac{b^2}{a}$$

$$\frac{a+b+c}{3} = b + 2$$

$$a + b + c = 3b + 6 \Rightarrow a - 2b + c = 6$$

$$a - 2b + \frac{b^2}{a} = 6 \Rightarrow 1 - \frac{2b}{a} + \frac{b^2}{a^2} = \frac{6}{a}$$

$$\left(\frac{b}{a} - 1\right)^2 = \frac{6}{a} \Rightarrow a = 6 \text{ only}$$

Sol 21: (9) Let seventh term be 'a' and common difference be 'd'

Given, $\frac{S_7}{S_{11}} = \frac{6}{11} \Rightarrow a = 15d$

Hence, $130 < 15d < 140$

$\Rightarrow d = 9$

Sol 22: (B) $\log(b_2) - \log(b_1) = \log(2)$

$\Rightarrow \frac{b_2}{b_1} = 2 \Rightarrow b_1, b_2, \dots$ are in G.P. with common ratio 2

$\therefore t = b_1 + 2b_1 + \dots\dots + 250\, b_1 = b_1\,(251 - 1)$

$S = a_1 + a_2 + \dots + a_{51}$

$$= \frac{51}{2}(a_1 + a_{51}) \qquad = \frac{51}{2}(b_1 + b_2) \qquad = \frac{51}{2}b_1(1 + 2^{50})$$

$$S - t = b_1\left(\frac{51}{2} + 51 \times 2^{49} - 2^{51} + 1\right) = b_1\left(\frac{53}{2} + 2^{49} \times 47\right)$$

$\Rightarrow S > t$

$b_{101} = 2_{100}\, b_1$

$a_{101} = a_1 + 100\,d = 2\,(a_1 + 50d) - a_1$

$= 2a_{51} - a_1 = 2b_{51} - b_1 = (2 \times 2_{51} - 1)\, b_1 = (2_{51} - 1)\, b_1$

$\therefore b_{101} > a_{101}$