6 Continuity and Differentiability

BASIC CONCEPTS

1. **Continuity and Discontinuity of Function:** A function $y = f(x)$ is said to be continuous in an interval if for every value of x in that interval y exist. If we plot the points, the graph is drawn without lifting the pencil.

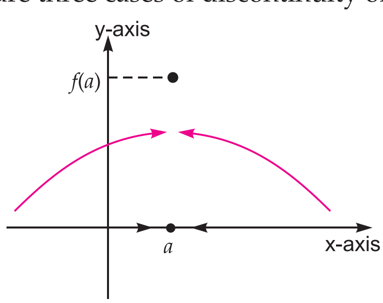
   If we have to lift the pencil on drawing the curve, then the function is said to be a discontinuous function.

2. **Continuity and Discontinuity of a Function at a Point:** A function $f(x)$ is said to be continuous at a point a of its domain if

$$\lim_{x \to a^-} f(x),\ \lim_{x \to a^+} f(x),\ f(a) \text{ exist and } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

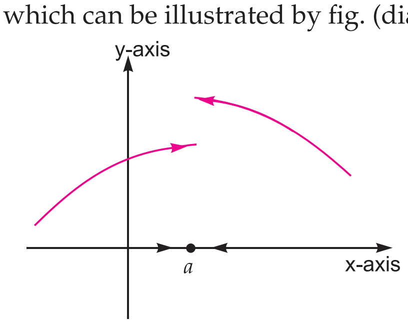
   A function $f(x)$ is said to be discontinuous at $x = a$ if it is not continuous at $x = a$.

   There are three cases of discontinuity of a function which can be illustrated by fig. (diagram) as.



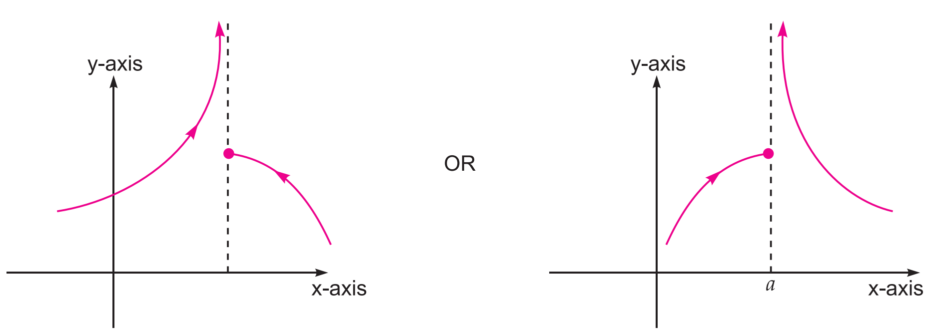
(*i*) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$

(Removable discontinuity)



(*ii*) $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

(Ist type discontinuity)



(*iii*) $\lim_{x \to a^-} f(x)$ does not exist or $\lim_{x \to a^+} f(x)$ does not exist (2nd type of discontinuity)

3. Properties of Continuous Function:

If f and g are two continuous functions at a point a, then

(*i*) $f+g$ is continuous at a.

(*ii*) $f-g$ is continuous at a.

(*iii*) $f \cdot g$ is continuous at a.

(*iv*) $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.

(*v*) $c.f$ is continuous at a, where c is a constant.

(*vi*) $|f|$ is continuous function at a.

- Every constant function is continuous function.
- Every polynomial function is continuous function.
- Identity function is continuous function.
- Every logarithmic and exponential function is a continuous function.

4. Important Series which are Frequently Used in Limits:

(*i*) $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$

(*ii*) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\infty$ and $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

(*iii*) $a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots$ and $\log|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3}\ldots$

(*iv*) $\log|1-x| = -x - \frac{x^2}{2} - \frac{x^3}{3}\ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(*v*) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ and $\tan x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

5. Differentiation from First Principle or Ab-initio Method or by Delta Method:

Given a function $f(x)$ and if there is a small increment h in x, let their corresponding increment is $f(x+h)$ in $f(x)$ *i.e.*,

$f(x) \to f(x+h)$, then, $\lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$ is called the differential coefficient of $f(x)$ with respect to x.

List of Useful Formulae:

6. (*i*) $\frac{d}{dx}(x^n) = nx^{n-1}$

(*ii*) $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}.a$

(*iii*) $\frac{d}{dx}(e^x) = e^x$

(*iv*) $\frac{d}{dx}e^{ax} = a.e^{ax}$

(*v*) $\frac{d}{dx}a^x = a^x.\log_e a$

(*vi*) $\frac{d}{dx}a^{bx} = ba^{bx}\log_e a$

(*vii*) $\frac{d}{dx}\log_e x = \frac{1}{x}$ and $\frac{d}{dx}\log_e ax = \frac{1}{x}$

(*viii*) $\frac{d}{dx}\log_a x = \frac{1}{x.\log_e a}$ and $\frac{d}{dx}\log_a bx = \frac{1}{x.\log_e a}$

7. (*i*) $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\sin ax = a\cos ax$

(*ii*) $\frac{d}{dx}\cos x = -\sin x$ and $\frac{d}{dx}\cos ax = -a\sin ax$

(*iii*) $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\tan ax = a\sec^2 ax$

(*iv*) $\frac{d}{dx}\cot x = -\text{cosec}^2 x$ and $\frac{d}{dx}\cot ax = -a\,\text{cosec}^2 ax$

(v) $\frac{d}{dx}\sec x = \sec x \tan x$ and $\frac{d}{dx}\sec ax = a\sec ax.\tan ax$

(vi) $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$ and $\frac{d}{dx}\text{cosec}\, ax = -a\,\text{cosec}\, ax.\cot ax$

8. (i) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\sin^{-1}ax = \frac{a}{\sqrt{1-a^2x^2}}$

(ii) $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\cos^{-1}ax = \frac{-a}{\sqrt{1-a^2x^2}}$

(iii) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ and $\frac{d}{dx}\tan^{-1}ax = \frac{a}{1+a^2x^2}$

(iv) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$ and $\frac{d}{dx}\cot^{-1}ax = \frac{-a}{1+a^2x^2}$

(v) $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$ and $\frac{d}{dx}\sec^{-1}ax = \frac{1}{x\sqrt{a^2x^2-1}}$

(vi) $\frac{d}{dx}\text{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$ and $\frac{d}{dx}\text{cosec}^{-1}ax = \frac{-1}{x\sqrt{a^2x^2-1}}$

9. Product Rule: Let u and v be two functions of x, then $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$.

i.e., $\frac{d}{dx}$ (Product of two functions)

$$= \text{First function } \frac{d}{dx}(\text{Second function}) + \text{Second function } \frac{d}{dx}(\text{First function})$$

10. Quotient Rule: If u and v are functions of x then,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad i.e., \qquad \frac{d}{dx}\left(\frac{N^r}{D^r}\right) = \frac{D^r\frac{d(N^r)}{dx} - N^r\frac{d(D^r)}{dx}}{(D^r)^2}$$

11. Chain Rule: Chain rule is applied when the given function is the function of function *i.e.*,

if y is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ or $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

12. Logarithmic Differentiation: *Logarithmic differentiations are used for differentiation of functions which consists of the product or quotients of a number of functions and/or the given function is of type $[f(x)]^{g(x)}$, where $f(x)$ and $g(x)$ both are differentiable functions of x.*

Therefore, in this method, we take the logarithm on both the sides of the function and then differentiate it with respect to 'x'. So, this process is called ***logarithmic differentiation****.*

General method: If $y = [f(x)]^{g(x)}$ then

$$\frac{dy}{dx} = y\left[\log f(x)g'(x) + g(x).\frac{1}{f(x)}.f'(x)\right]$$

13. Parametric Form: *Sometimes we come across the function when both x and y are expressed in terms of another variable say t i.e., $x = \phi(t)$ and $y = \psi(t)$. This form of a function is called parametric form and t is called the parameter.*

To obtain $\frac{dy}{dx}$ in parametric type of functions we follow any one of the following two steps:

(i) Try to obtain a relationship between x and y by eliminating the parameter and then proceed to get $\frac{dy}{dx}$ which is already discussed.

(*ii*) If it is not convenient to obtain such a relation between x and y, then differentiate x and y both with respect to parameter t to get $\frac{dx}{dt}$ and $\frac{dy}{dt}$ (treating t as independent variable and x and y as dependent variables). Finally, divide $\frac{dy}{dt}$ by $\frac{dx}{dt}$ to get $\frac{dy}{dx}$ *i.e.*, $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

or sometimes $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$, where θ is an independent variable.

14. Rolle's Theorem: *If $f(x)$ be a real valued function, defined in a closed interval $[a, b]$ such that:*

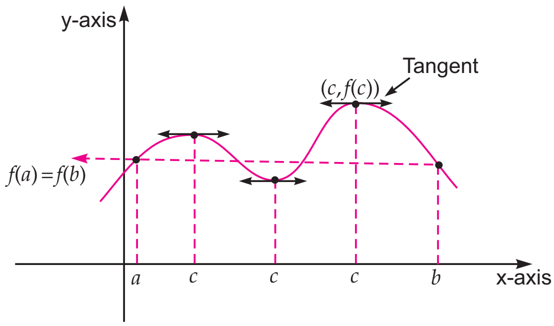
(*i*) it is continuous in closed interval $[a, b]$.

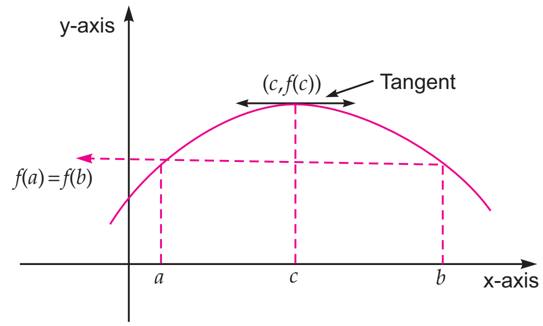
(*ii*) it is differentiable in open interval (a, b).

(*iii*) $f(a) = f(b)$. Then there exists at least one value $c \in (a, b)$ such that $f'(c) = 0$.

It is illustrated by diagram as

[**Note:** $f'(c) = 0$ means tangent at c is parallel to x-axis.]





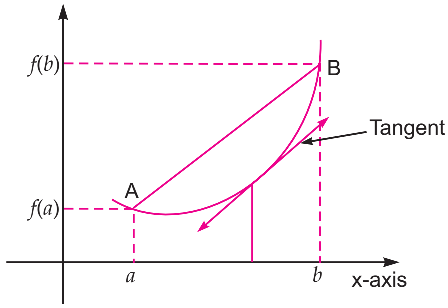
15. Lagrange's Mean Value Theorem:

If $f(x)$ is a real valued function defined in the closed interval $[a, b]$ such that:

(*i*) it is continuous in the closed interval $[a, b]$.

(*ii*) it is differentiable in the open interval (a, b).

Then there exists at least one real value $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. It is illustrated by diagram.



[**Note:** $f'(c) = \frac{f(b) - f(a)}{b - a}$ means tangent at c is parallel to chord AB.

i.e., Slope of tangent at c = Slope of chord AB

16. Limits: Let $f(x)$ be a function of x. Let a and l be two constants such that as $x \to a$, we have $f(x) \to l$, *i.e.*, the numerical difference between $f(x)$ and l can be made as small as we wish by taking x sufficiently close to a. In such a case, we say that the limit of function $f(x)$ as x approaches a is l. We write this as $\lim_{x \to a} f(x) = l$.

17. Procedure to Find $\lim_{x \to a} f(x)$:

(*i*) Putting $x = a$ in the given function. If $f(a)$ is a finite value, then $\lim_{x \to a} f(x) = f(a)$.

(*ii*) To find LHL of $f(x)$ at $x = a$ we put $x = a - h$, $h \to 0$ and find $\lim_{h \to 0} f(a - h)$ after simplification.

(*iii*) To find RHL of $f(x)$ at $x = a$ we put $x = a + h$, $h \to 0$ and find $\lim_{h \to 0} f(a + h)$ after simplification.

(*iv*) If LHL = RHL = k (say), then $\lim_{x \to a} f(x) = k$.

18. Fundamental Theorems on Limits: Some important theorems are given below which are frequently used in limits:

(*i*) $\lim_{x \to a} c = c$, *i.e.*, the limit of a constant quantity is constant itself.

(*ii*) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

i.e., the limit of sum of two functions is equal to the sum of their limits.

(*iii*) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

i.e., the limit of difference of two functions is equal to the difference of their limits.

(*iv*) $\lim_{x \to a} [f(x) \,.\, g(x)] = \lim_{x \to a} f(x) \,.\, \lim_{x \to a} g(x)$

i.e., the limit of the product of two functions is equal to the product of their limits.

(*v*) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$

i.e., the limit of quotient of two functions is equal to quotient of their limits.
provided $\lim_{x \to a} g(x)$ finite value not equal to zero.

(*vi*) $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$, where c is a constant.

i.e., the limit of the product of a constant and the function is equal to the product of the constant and the limit of the function.

(*vii*) $\lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)}$

(*viii*) $\lim_{x \to 0} f(-x) = \lim_{x \to 0^-} f(x)$

19. Evaluation of Limits:

(*i*) Direct substituting method: We substitute the value of the point in the given expression and if we get a finite number, then this number is the limit of the given function.

(*ii*) Factorisation method: On substituting $x = a$ in the given expression, if we get $\frac{0}{0}, \frac{\infty}{\infty}, \ldots$, etc. form, then we factorize the numerator and denominator and take $(x - a)$ as a common factor from numerator and denominator. After cancelling out $(x - a)$, we put $x = a$. If we get a finite number, then it is the required value otherwise repeat the step till we get a finite number.

(*iii*) Rationalisation method: Rationalisation method is applicable when

(*a*) numerator, denominator or both in square root or

(*b*) after substituting the value of limit if we get the negative number in square root. Hence, after simplifying in both the cases, we get the required value.

(*iv*) L' HOSPITAL Rule: With the help of this rule, if we have to evaluate $\lim_{x \to c} \frac{f(x)}{g(x)}$ such that it takes indeterminate form, *i.e.*, $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we differentiate numerator and denominator to get $\lim_{x \to c} \frac{f(x)}{g(x)}$, if it is determinate form then it is required value, otherwise repeat the step till we get a determinate form and thus required value.

[**Note:** According to L′ HOSPITAL rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}. \text{ where } g'(x) \neq 0 \ \forall\ x \in Z \text{ with } x \neq c]$$

20. Some Standard Results:

(*i*) (*a*) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0, n \in Q$ (*b*) $\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}, m, n \in Q$

(*c*) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (*d*) $\lim_{x \to 0} \cos x = 1$ (*e*) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(*ii*) Evaluation of limits of inverse trigonometric functions:

(*a*) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$ (*b*) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(*iii*) Evaluation of limits of exponential and logarithmic functions:

(*a*) $\lim_{x \to 0} e^x = 1$ (*b*) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(*c*) $\lim_{x \to 0} \frac{\log|1 + x|}{x} = 1$ (*d*) $\lim_{x \to 0} \frac{(1 + x)^n - 1}{x} = n$

(*e*) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(*iv*) Limits at infinity: This method is applied when $x \to \infty$.

Procedure to solve the infinite limits:

(*a*) Write the given expression in the form of rational function.

(*b*) Divide the numerator and denominator by highest power of x.

(*c*) Use the result $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

(*d*) Simplify and get the required result.

Selected NCERT Questions

1. Show that the function *f* defined by

$$f(x) = |1 - x + |x||,$$

where x is any real number, is a continuous function.

Sol. Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x. Then

$$(hog)(x) = h(g(x)) = h(1 - x + |x|)$$
$$= |1 - x + |x|| = f(x)$$

We know that h is a continuous function. Hence g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

2. Find all points of discontinuity of *f*, where *f* is defined by the following function:

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

[*CBSE Delhi 2010*]

Sol. We know that $|x|$ is a continuous function, so f is continuous $\forall\ x < -3$

$-2x$ is continuous everywhere, so f is continuous $\forall\ x, -3 < x < 3$

and $6x + 2$ is continuous everywhere, so f is continuous $\forall\ x > 3$

Now we have to check the continuity of the function at $x = -3$ and $x = 3$

At $x = -3$, $f(x) = |x| + 3$

$f(-3) = |-3| + 3 = 3 + 3 = 6$

LHL = $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} |x| + 3 = |-3| + 3 = 3 + 3 = 6$

RHL = $\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} -2x = -2 \times -3 = 6$

LHL = RHL = $f(-3)$, so it is continuous at $x = -3$.

Again at $x = 3$, $f(x) = 6x + 2$

Here $f(3) = 6 \times 3 + 2 = 18 + 2 = 20$

LHL = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -2x = -2 \times 3 = -6$

RHL = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 6x + 2 = 6 \times 3 + 2 = 20$

LHL $\neq$ RHL

So, it is discontinuous at $x = 3$.

3. Find all points of discontinuity of f where f is defined by the following function:

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Sol. For $x < 0$, $f(x) = \dfrac{x}{|x|}$ is continuous and for $x > 0$, $f(x)$ is a constant function so it is continuous.

At $x = 0$,

LHL $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\dfrac{x}{|x|}\right) = \lim_{x \to 0^-} \dfrac{x}{-x} = -1$, $\quad (\because |x| = -x$, when $x < 0)$

RHL $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-1) = -1$

and $f(0) = -1$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \Rightarrow f(x)$ is continuous at $x = 0$.

Hence, $f(x)$ has no points of discontinuity.

4. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1 & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$? **[*CBSE (F) 2011*]**

Sol. $\because$ $f(x)$ to be continuous at $x = 0$

$\therefore$ LHL = RHL = $f(0)$

i.e., $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ ...(*i*)

Now LHL $= \lim_{x \to 0^-} \lambda(x^2 - 2x) = \lambda(0 - 0) = 0$ ...(*ii*)

RHL $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1) = 4 \times 0 + 1 = 1$ ...(*iii*)

and $f(0) = 0$ ...(*iv*)

from (*ii*) and (*iii*), LHL $\neq$ RHL

So, $f(x)$ is not continuous for any value of λ.

For continuity at $x = 1$,

$$f(1) = 4 \times 1 + 1 = 4 + 1 = 5$$

and $\lim_{x \to 1} f(x) = \lim_{x \to 1} (4x + 1) = 4 \times 1 + 1 = 5$

As $\lim_{x \to 1} f(x) = f(1)$

So, it is continuous at $x = 1$.

5. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Sol. Let $g(x) = x - [x]$ be the greatest integer function. Let n be any integer. Then

$$\lim_{x \to n^-} g(x) = \lim_{h \to 0} g(n-h) = \lim_{h \to 0} (n-h) - [n-h] = \lim_{h \to 0} (n-h) - (n-1) = n - (n-1) = 1$$

$$\lim_{x \to n^+} g(x) = \lim_{h \to 0} g(n+h) = \lim_{h \to 0} (n+h) - [n+h] = \lim_{h \to 0} (n+h) - n = n - n = 0$$

$$\therefore \lim_{x \to n^-} g(x) \neq \lim_{x \to n^+} g(x)$$

LHL $\neq$ RHL so, f is discontinuous at all integral points.

6. Find the value of a and b such that the function $f(x)$ defined by:

$$f(x) = \begin{cases} 5; & \text{if } x \le 2 \\ ax + b; & \text{if } 2 < x < 10 \\ 21; & \text{if } x \ge 10 \end{cases}$$ **is a continuous function** **[*CBSE Delhi 2011*]**

Sol. Since, $f(x)$ is continuous.

$\Rightarrow$ $f(x)$ is continuous at $x = 2$ and $x = 10$

$\Rightarrow$ (LHL of $f(x)$ at $x = 2$) = (RHL of $f(x)$ at $x = 2$) = $f(x)$

$\Rightarrow$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$ ...(*i*)

Similarly, $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$ ...(*ii*)

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} 5 = 5$ $[\because f(x) = 5 \text{ if } x \le 2]$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (ax + b) = 2a + b$ $[\because f(x) = ax + b \text{ if } x > 2]$

$f(2) = 5$

Putting these values in (*i*), we get

$2a + b = 5$ ...(*iii*)

Again $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^{-1}} (ax + b) = 10a + b$ $[\because f(x) = ax + b \text{ if } x < 10]$

$\lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 21 = 21$ $[\because f(x) = 21 \text{ if } x > 10]$

$f(10) = 21$

Putting these values in (*ii*), we get

$10a + b = 21$ ...(*iv*)

Subtracting (*iii*) from (*iv*), we get

$$\begin{array}{r} 10a + b = 21 \\ \underline{-2a \pm b = -5} \\ 8a \quad = 16 \end{array} \Rightarrow a = 2$$

$\therefore$ $b = 5 - 2 \times 2 = 1$

Hence, the value of $a = 2$ and $b = 1$.

7. Find the derivative of the function $f(x) = \sin(x^2)$.

Sol. $\because \quad f(x) = \sin(x^2)$

$\therefore \quad f'(x) = \cos x^2 \,.\, 2x = 2x \cos x^2$

8. Prove that the function f given by $f(x) = |x-1|,\ x \in \mathbb{R}$ is not differentiable at $x = 1$.

Sol. Here $f(x) = |x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$

LHD at $x = 1$

$$\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h \to 0} \frac{f(1-h)-f(1)}{(1-h-1)} = \lim_{h \to 0} \frac{1-(1-h)-(1-1)}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

RHD at $x = 1$

$$\lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h \to 0} \frac{f(1+h)-f(1)}{(1+h-1)} = \lim_{h \to 0} \frac{(1+h)-1-(1-1)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$\therefore \quad$ LHD $\neq$ RHD

Thus, f is not differentiable at $x = 1$.

9. Find $\frac{dy}{dx}$, if $x - y = \pi$.

Sol. $\because \quad x - y = \pi \ \Rightarrow\ y = x - \pi \qquad \Rightarrow \qquad \frac{dy}{dx} = 1$

10. Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

Sol. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, putting $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right] = \tan^{-1}[\tan 3\theta] = 3\theta = 3\tan^{-1} x$$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = 3 \times \frac{1}{1+x^2} = \frac{3}{1+x^2}$$

11. Find $\frac{dy}{dx}$ if $y = \sin^{-1}(2x\sqrt{1-x^2});\ -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

Sol. Here $y = \sin^{-1}(2x\sqrt{1-x^2})$, putting $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$.

$$y = \sin^{-1}[2\sin\theta\sqrt{1-\sin^2\theta}] = \sin^{-1}[2\sin\theta\cos\theta]$$

$$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} x$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2.\frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

12. Differentiate the following with respect to x:

$\sqrt{e^{\sqrt{x}}},\ x > 0$

Sol. Let $y = \sqrt{e^{\sqrt{x}}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{de^{\sqrt{x}}}(\sqrt{e^{\sqrt{x}}}) \times \frac{d}{d\sqrt{x}} e^{\sqrt{x}} \times \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$$

Aliter : Let $y = \sqrt{e^{\sqrt{x}}}$ put $u = \sqrt{x}$ and $v = e^u$

$$y = \sqrt{v} \quad \Rightarrow \quad \frac{dy}{dv} = \frac{1}{2\sqrt{v}} = \frac{1}{2\sqrt{e^{\sqrt{x}}}}$$

$$v = e^u \quad \Rightarrow \quad \frac{dv}{du} = e^u = e^{\sqrt{x}}$$

$$u = \sqrt{x} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{e^{\sqrt{x}}}}.e^{\sqrt{x}}.\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}}.\frac{e^{\sqrt{x}}}{\sqrt{e^{\sqrt{x}}}} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$$

13. Differentiate with respect to x: $y = \left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$.

Sol. Let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(x+\frac{1}{x}\right)}$

then $y = u + v \quad \Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(*i*)

Now, $u = \left(x+\frac{1}{x}\right)^x$

$$\log u = \log\left(x+\frac{1}{x}\right)^x = x\log\left(x+\frac{1}{x}\right)$$ [Taking logarithm on both sides]

Differentiating both sides, w.r.t x, we get

$$\frac{1}{u}\frac{du}{dx} = 1.\log\left(x+\frac{1}{x}\right) + x.\frac{1}{x+\frac{1}{x}}.\left(1-\frac{1}{x^2}\right)$$

$$\frac{du}{dx} = u\left[\log\left(x+\frac{1}{x}\right) + x.\frac{x}{x^2+1}.\frac{x^2-1}{x^2}\right] = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$ ...(*ii*)

Now, $v = (x)^{x+\frac{1}{x}}$

$$\log v = \log(x)^{x+\frac{1}{x}} = \left(x+\frac{1}{x}\right)\log x$$ [Taking logarithm on both sides]

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \left(1-\frac{1}{x^2}\right)\log x + \left(x+\frac{1}{x}\right).\frac{1}{x} \quad \Rightarrow \quad \frac{dv}{dx} = x^{\left(x+\frac{1}{x}\right)}\left[\frac{x^2-1}{x^2}\log x + \frac{x^2+1}{x^2}\right]$$ ...(*iii*)

From (*i*), (*ii*) and (*iii*), we get

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right] + x^{\left(x+\frac{1}{x}\right)}\left[\frac{x^2-1}{x^2}\log x + \frac{x^2+1}{x^2}\right]$$

14. If $y = x^{\sin x} + (\sin x)^{\cos x}$, find $\frac{dy}{dx}$. ***[CBSE Delhi 2009, (F) 2013]***

OR

If $y = x^{\cos x} + (\cos x)^{\sin x}$, find $\frac{dy}{dx}$. ***[CBSE 2019 (65/4/1)]***

Sol. In $y = x^{\sin x} + (\sin x)^{\cos x}$, let $u = x^{\sin x}$, $v = (\sin x)^{\cos x}$

Now, $y = u + v$

and $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(*i*)

We have, $u = x^{\sin x}$

Taking log on both sides, we get

$$\log u = \sin x \log x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \sin x \times \frac{1}{x} + \log x . \cos x$$

$$\frac{du}{dx} = u\left(\frac{\sin x}{x} + \log x . \cos x\right) = x^{\sin x}\left[\frac{\sin x}{x} + \log x . \cos x\right] \qquad ...(ii)$$

Again, $\quad v = (\sin x)^{\cos x}$

Taking log on both sides with respect to x, we get

$$\log v = \cos x \log \sin x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x \times \frac{1}{\sin x} \times \cos x + \log \sin x (-\sin x)$$

$$= \cos x . \cot x + \log \sin x . (-\sin x) = \cos x . \cot x - \sin x . \log \sin x$$

$$\therefore \quad \frac{dv}{dx} = v[\cos x . \cot x - \sin x . \log \sin x]$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}[\cos x . \cot x - \sin x \log \sin x] \qquad ...(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x . \cos x\right] + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$$

OR

Solution is similar only values change.

Ans. $x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) + (\cos x)^{\sin x}[-\sin x \tan x + \cos x \log \cos x]$

15. Differentiate w.r.t x : $y = (x \cos x)^x + (x \sin x)^{1/x}$

Sol. We have $y = (x \cos x)^x + (x \sin x)^{1/x}$

$$y = e^{\log_e (x \cos x)^x} + e^{\log_e (x \sin x)^{\frac{1}{x}}} \quad \Rightarrow \quad y = e^{x \log (x \cos x)} + e^{\frac{1}{x}\log (x \sin x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x \log (x \cos x)} . \frac{d}{dx}[x \log (x \cos x)] + e^{\frac{1}{x}\log x \sin x} . \frac{d}{dx}\left(\frac{1}{x}\log (x \sin x)\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = (x \cos x)^x \left[1. \log (x \cos x) + x . \frac{1}{x \cos x}(1. \cos x - x \sin x)\right]$$

$$+ (x \sin x)^{1/x}\left[\frac{-1}{x^2}\log (x \sin x) + \frac{1}{x} . \frac{1}{x \sin x} . (1. \sin x + x \cos x)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = (x \cos x)^x [\log (x \cos x) + 1 - x \tan x] + (x \sin x)^{1/x}\left[\frac{x \cot x + 1 - \log (x \sin x)}{x^2}\right]$$

16. Find $\frac{dy}{dx}$ if $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$.

Sol. Given $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$

Differentiating both sides w.r.t. t, we have

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{\sin^3 t}{\sqrt{\cos 2t}}\right) = \frac{\sqrt{\cos 2t}\,(3 \sin^2 t \cos t) - \sin^3 t\left(\frac{-2 \sin 2t}{2\sqrt{\cos 2t}}\right)}{\cos 2t}$$

$$= \frac{3(\cos 2t)\sin^2 t\cos t + \sin 2t \sin^3 t}{(\cos 2t)^{3/2}}$$

$$= \frac{3(1-2\sin^2 t)\sin^2 t\cos t + 2\sin t\cos t.\sin^3 t}{(\cos 2t)^{3/2}} = \frac{3\sin^2 t\cos t - 4\sin^4 t\cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin t\cos t(3\sin t - 4\sin^3 t)}{(\cos 2t)^{3/2}} = \frac{\sin t\cos t(\sin 3t)}{(\cos 2t)^{3/2}} = \frac{\sin t\cos t\sin 3t}{(\cos 2t)^{3/2}}$$

Now, $\frac{dy}{dt} = \frac{d}{dt}\left(\frac{\cos^3 t}{\sqrt{\cos 2t}}\right) = \frac{\sqrt{\cos 2t}(-3\cos^2 t\sin t) - \cos^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{\cos 2t}$

$$= \frac{-3(\cos 2t)\cos^2 t\sin t + \cos^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{-3(2\cos^2 t - 1)\cos^2 t\sin t + \cos^3 t(2\sin t\cos t)}{(\cos 2t)^{3/2}}$$

$$= \frac{-6\cos^4 t\sin t + 3\cos^2 t\sin t + 2\cos^4 t\sin t}{(\cos 2t)^{3/2}} = \frac{3\cos^2 t\sin t - 4\cos^4 t\sin t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin t\cos t(3\cos t - 4\cos^3 t)}{(\cos 2t)^{3/2}} = \frac{\sin t\cos t(-\cos 3t)}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = -\frac{\sin t\cos t(\cos 3t)}{(\cos 2t)^{3/2}}\times\frac{(\cos 2t)^{3/2}}{\sin t\cos t\sin 3t} = -\cot 3t$$

17. If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$. **[CBSE (AI) 2012]**

Sol. $x = \sqrt{a^{\sin^{-1} t}}$ ...(*i*)

$y = \sqrt{a^{\cos^{-1} t}}$ ...(*ii*)

Multiplying, (*i*) and (*ii*) we get

$$x.y = \sqrt{a^{\sin^{-1} t}}\times\sqrt{a^{\cos^{-1} t}} \quad\Rightarrow\quad x.y = \sqrt{a^{\sin^{-1} t + \cos^{-1} t}}$$

$\Rightarrow \quad x.y = \sqrt{a^{\pi/2}}$ $\quad [\sin^{-1} t + \cos^{-1} t = \pi/2]$

On differentiating both sides, we get

$$x\frac{dy}{dx} + y\times 1 = 0 \quad\Rightarrow\quad \frac{dy}{dx} = \frac{-y}{x}$$

18. If $y = Ae^{mx} + Be^{nx}$, then show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$ **[CBSE (AI) 2007; (F) 2013]**

Sol. Given, $y = Ae^{mx} + Be^{nx}$

On differentiating with respect to x, we have

$$\frac{dy}{dx} = Ame^{mx} + Bne^{nx}$$

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$$

Now, LHS $= \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Amne^{mx} - Bmne^{nx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$

$= 0 =$ RHS

19. If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 y_2 + 2x(x^2+1) y_1 = 2$. ***[CBSE (AI) 2012; Delhi 2012]***

Sol. We have, $y = (\tan^{-1} x)^2$ ...(*i*)

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x . \frac{1}{1+x^2} \quad ...(ii)$$

or $(1 + x^2)\, y_1 = 2\tan^{-1} x$ $\left[\text{where } y_1 = \dfrac{dy}{dx}\right]$

Again differentiating with respect to x, we get

$$(1+x^2).\frac{dy_1}{dx} + y_1 \frac{d}{dx}(1+x^2) = 2.\frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2).y_2 + y_1.2x = \frac{2}{1+x^2}$$

or $(1 + x^2)^2\, y_2 + 2x\,(1 + x^2)\, y_1 = 2$ $\left[\text{where } y_2 = \dfrac{d^2y}{dx^2} \text{ and } y_1 = \dfrac{dy}{dx}\right]$

20. Verify mean value theorem if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Sol. Given, $f(x) = x^2 - 4x - 3, x \in [a, b]$ *i.e.*, $[1, 4]$

Since $f(x)$ is a polynomial so it is differentiable everywhere and so it is also continuous.

So, all conditions of mean value theorem are satisfied.

Now, $f(a) = f(1) = (1)^2 - 4\,(1) - 3 = -6$

$f(b) = f(4) = (4)^2 - 4 \times 4 - 3 = -3$

Now, for any $c \in (a, b)$, we have

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \Rightarrow \quad f'(c) = \frac{f(4)-f(1)}{4-1}$$

$$\Rightarrow \quad 2c - 4 = \frac{-3-(-6)}{3} \quad \Rightarrow \quad 2c - 4 = \frac{-3+6}{3} = 1$$

$$\Rightarrow \quad 2c = 5 \quad \Rightarrow \quad c = \frac{5}{2} \in (1,4)$$

21. If $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, $0 < x < \dfrac{\pi}{2}$ then find the value of $\dfrac{dy}{dx}$. ***[CBSE Delhi 2008; (F) 2013]***

Sol. Consider $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}, 0 < x < \dfrac{\pi}{2}$

$$= \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1+\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$= \frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{1+\sin x-(1-\sin x)} = \frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}$$

$$= \frac{2+2\cos x}{2\sin x} = \frac{1+\cos x}{\sin x} = \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \cot\left(\frac{x}{2}\right)$$

$$\therefore \quad y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \quad \therefore \quad y = \frac{x}{2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}$$

22. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is a continuous function. [*NCERT Exemplar*]

Sol. We note that domain of f is R. Let c be any real number then two cases arise.

Case I: If $c \neq 0$ then

$$\therefore \quad \lim_{x \to c} f(x) = \lim_{x \to c} x^2 \sin\frac{1}{x} = c^2 \sin\left(\frac{1}{c}\right) = f(c)$$

$\Rightarrow$ f is continuous for $c \in R$, where $c \neq 0$.

Case II : If $c = 0$, then $f(c) = f(0) = 0$

and $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \times$ (a number oscillating between -1 and 1)

$$= 0 = f(0)$$

$\Rightarrow$ f is continuous at $x = 0$.

Thus, f is a continuous function at every point of its domain.

23. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$, for all positive integer n. [*HOTS*]

Sol. Let $P(n)$ be statement such that

$$P(n): \frac{d}{dx}(x^n) = nx^{n-1}$$

For $P(1)$: Putting $n = 1$, we get

$$\frac{d}{dx}(x^1) = 1.x^{1-1} = 1 \quad \Rightarrow \quad P(1) \text{ is true.}$$

Let $P(m)$ be true $\quad \Rightarrow \quad \frac{d}{dx}(x^m) = mx^{m-1}$

Now for $P(m+1)$:

$$\frac{d}{dx}(x^{m+1}) = \frac{d}{dx}(x^m . x) = x^m . 1 + x.\frac{d}{dx}(x^m)$$

$$= x^m + x . m . x^{m-1} = x^m + m . x^m = (m+1) . x^m = (m+1) . x^{(m+1)-1}$$

$\Rightarrow$ $P(m+1)$ is true

Here, by PMI $P(n)$ is true.

i.e., $\quad \frac{d}{dx}(x^n) = n.x^{n-1}$

24. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, then prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$. [*HOTS*]

Sol. Given, $\quad y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$$y = (mc - nb)\, f(x) - (lc - na)\, g(x) + (lb - ma)\, h(x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (mc - nb)\, f'(x) - (lc - na)\, g'(x) + (lb - ma)\, h'(x)$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Multiple Choice Questions [1 mark]

Choose and write the correct option in the following questions.

1. **The function $f : R \rightarrow R$ given by $f(x) = -|x-1|$ is** [CBSE 2020 (65/2/1)]
   (*a*) continuous as well as differentiable at $x = 1$
   (*b*) not continuous but differentiable at $x = 1$
   (*c*) continuous but not differentiable at $x = 1$
   (*d*) neither continuous nor differentiable at $x = 1$

2. **The function $f(x) = e^{|x|}$ is** [NCERT Exemplar]
   (*a*) continuous everywhere but not differentiable at $x = 0$
   (*b*) continuous and differentiable everywhere
   (*c*) not continuous at $x = 0$
   (*d*) none of these

3. **The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at**
   (*a*) 4 (*b*) –2 (*c*) 1 (*d*) 1.5

4. **The number of points at which the function $f(x) = \frac{1}{x-[x]}$ is not continuous is** [NCERT Exemplar]
   (*a*) 1 (*b*) 2 (*c*) 3 (*d*) none of these

5. **The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is** [NCERT Exemplar]
   (*a*) 3 (*b*) 2 (*c*) 1 (*d*) 1.5

6. **The value of k which makes the function defined by $f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$, continuous at $x = 0$ is**
   (*a*) 8 (*b*) 1 (*c*) –1 (*d*) none of these

7. **The function $f(x) = \cot x$ is discontinuous on the set** [NCERT Exemplar]
   (*a*) $\{x = n\pi : n \in Z\}$ (*b*) $\{x = 2n\pi : n \in Z\}$
   (*c*) $\left\{x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$ (*d*) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

8. **Let $f(x) = |\sin x|$. Then** [NCERT Exemplar]
   (*a*) f is everywhere differentiable
   (*b*) f is everywhere continuos but not differentiable at $x = n\pi : n \in Z$
   (*c*) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in Z$
   (*d*) none of these

9. **The function $f(x) = \frac{x-1}{x(x^2-1)}$ is discontinuous at** [CBSE 2020 (65/2/2)]
   (*a*) exactly one point (*b*) exactly two points
   (*c*) exactly three points (*d*) no point

10. **If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is** [NCERT Exemplar]
    (*a*) 0 (*b*) –1 (*c*) 1 (*d*) None of these

11. **The function $f(x) = |x| + |x-1|$ is**

(a) continuous at $x = 0$ as well as at $x = 1$.
(b) continuous at $x = 1$ but not at $x = 0$.
(c) discontinuous at $x = 0$ as well as at $x = 1$.
(d) continuous at $x = 0$ but not at $x = 1$.

12. **The function $f(x) = \dfrac{4-x^2}{4x-x^3}$ is**

(a) discontinuous at only one point
(b) discontinuous at exactly two points
(c) discontinuous at exactly three points
(d) none of these

13. **The value of c in Rolle's Theorem for the function $f(x) = e^x \sin x$, in $[0, \pi]$ is** *[NCERT Exemplar]*

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{3\pi}{4}$

14. **The value of c in Mean Value Theorem for the function $f(x) = x(x-2)$, $x \in [1, 2]$ is**

(a) $\dfrac{3}{2}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{7}{4}$

15. **The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is**

(a) 1 (b) –1 (c) $\dfrac{3}{2}$ (d) $\dfrac{1}{3}$

16. **The set of points where the functions f given by $f(x) = |x-3| \cos x$ is differentiable is**

(a) R (b) $R - \{3\}$ (c) $(0, \infty)$ (d) none of these

17. **Differential coefficient of sec $(\tan^{-1}x)$ w.r.t. x is** *[NCERT Exemplar]*

(a) $\dfrac{x}{\sqrt{1+x^2}}$ (b) $\dfrac{x}{1+x^2}$ (c) $x\sqrt{1+x^2}$ (d) $\dfrac{1}{\sqrt{1+x^2}}$

18. **If $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then $\dfrac{du}{dv}$ is**

(a) $\dfrac{1}{2}$ (b) x (c) $\dfrac{1-x^2}{1+x^2}$ $\{4, -4\}, \phi$ (d) 1

19. **If $y = \log\sqrt{\tan x}$, then the value of $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$ is**

(a) 0 (b) 1 (c) $\dfrac{1}{2}$ (d) ∞

20. **If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to**

(a) $\dfrac{\cos x}{2y-1}$ (b) $\dfrac{\cos x}{1-2y}$ (c) $\dfrac{\sin x}{1-2y}$ (d) $\dfrac{\sin x}{2y-1}$

Answers

1. (c)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (d)
7. (a)	**8.** (b)	**9.** (c)	**10.** (a)	**11.** (a)	**12.** (c)
13. (d)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (a)	**18.** (d)
19. (b)	**20.** (a)				

Solutions of Selected Multiple Choice Questions

1. We have,

$$f(x) = -|x-1| = \begin{cases} x-1, & \text{if } x \le 1 \\ -(x-1), & \text{if } x > 1 \end{cases}$$

At $x = 1$

LHL $= \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h) - 1 = 0$

RHL = $\lim_{h \to 0} f(1+h) = \lim_{h \to 0} -(1+h-1) = 0$

$f(1) = 1 - 1 = 0$

$\therefore$ LHL = RHL = $f(0)$ $\Rightarrow$ $f(x)$ is continuous every where.

Now, at $x = 1$

$$\text{LHD} = \frac{d}{dx}(x-1) = 1\,; \quad \text{RHD} = \frac{d}{dx}\{-(x-1)\} = -1$$

LHD $\neq$ RHD

$\therefore$ $f(x)$ is not differentiable of $x = 1$.

6. Indeed $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist.

7. We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \dfrac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $\{n\pi, n \in Z\}$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

9. We have,

$$f(x) = \frac{x-1}{x(x^2-1)}$$

$\therefore$ $f(x)$ is discontinuous when $x(x^2-1) = 0$

$\Rightarrow$ $x = 0, \; x = \pm 1$

$\therefore$ $f(x)$ is discontinuous at $x = 0, -1, 1$

i.e., exactly at three points.

10. $\because f(x) = x^2 \sin\left(\frac{1}{x}\right)$, where $x \neq 0$; $\because \lim_{x \to 0} f(x) = 0$

Hence, value of the function f at $x = 0$, so that it is continuous at $x = 0$ is 0.

15. $\because$ $f'(c) = 0$ $[\because f'(x) = 3x^2 - 3]$

$\Rightarrow$ $3c^2 - 3 = 0$ $\Rightarrow c^2 = \dfrac{3}{3} = 1$

$\Rightarrow$ $c = \pm 1$, where $1 \in (0, \sqrt{3})$ $\Rightarrow$ $c = 1$

19. We have, $y = \log\sqrt{\tan x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{\tan x}} \times \frac{1}{2\sqrt{\tan x}} \times \sec^2 x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^2 x}{2\tan x} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{at } x = \frac{\pi}{4}} = \frac{(\sqrt{2})^2}{2 \times 1} = \frac{2}{2} = 1$$

20. $\because$ $y = (\sin x + y)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2} \cdot \frac{d}{dx}(\sin x + y) = \frac{1}{2} \cdot \frac{1}{(\sin x + y)^{1/2}} \cdot \left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \qquad [\because (\sin x + y)^{1/2} = y]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y} \cdot \frac{2y}{2y-1} = \frac{\cos x}{2y-1}$$

Fill in the Blanks [1 mark]

1. If $y = \tan^{-1} x + \cot^{-1} x, x \in R$, then $\frac{dy}{dx}$ is equal to ____________ . [CBSE 2020 (65/3/1)]

2. If $\cos(xy) = k$, where k is a constant and $xy \neq n\pi, n \in Z$, then $\frac{dy}{dx}$ is equal to ____________ . [CBSE 2020 (65/3/1)]

3. The number of points of discontinuity of f defined by $f(x) = |x| - |x+1|$ is ____________ . [CBSE 2020 (65/4/1)]

4. The function $f(x) = \frac{2-x^2}{9x - x^3}$ is discontinuous exactly at ____________ points.

Answers

1. 0 **2.** $\frac{-y}{x}$ **3.** 0 **4.** three

Solutions of Selected Fill in the Blanks

1. We have, $y = \tan^{-1} x + \cot^{-1} x, x \in R$

$$\Rightarrow \quad y = \frac{\pi}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} = 0$$

2. We have, $\cos(xy) = k$

Diff. w.r.t x, we get

$$-\sin(xy) \times \left\{x\frac{dy}{dx} + y\right\} = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0 \qquad \left(\begin{array}{l}\because xy \neq n\pi \\ \therefore \sin(xy) \neq 0\end{array}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

Very Short Answer Questions [1 mark]

1. If the function f defined as $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$ is continuous at $x = 3$, find the value of k. **[CBSE 2020 (65/5/1)]**

Sol. It is given that $f(x)$ is continuous at $x = 3$

$$\therefore \quad \lim_{x \to 3} f(x) = f(3)$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = k$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = k$$

$$\Rightarrow \quad \lim_{x \to 3} (x+3) = k \quad \Rightarrow \quad 3 + 3 = k$$

$$\Rightarrow \quad k = 6$$

2. For what value of 'k' is the function $f(x) = \begin{cases} \dfrac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ continuous at $x = 0$? *[CBSE (F) 2017]*

Sol. $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \left(\frac{\sin 5h}{3h} + \cos h \right)$$

$$= \lim_{h \to 0} \frac{\sin 5h}{5h} \times \frac{5}{3} + \lim_{h \to 0} \cos h = 1 \times \frac{5}{3} + 1 \qquad [\because h \to 0 \Rightarrow 5h \to 0]$$

$\Rightarrow \quad \lim_{x \to 0^+} f(x) = \frac{8}{3}$

Also, $\quad f(0) = k$

Since, $f(x)$ is continuous at $x = 0$.

$\Rightarrow \quad \lim_{x \to 0^+} f(x) = f(0) \quad \Rightarrow \quad \frac{8}{3} = k$

3. Determine value of the constant 'k' so that the function $f(x) = \begin{cases} \dfrac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$. *[CBSE Delhi 2017]*

Sol. $\because \; f(x)$ is continuous at $x = 0$

$\Rightarrow \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

Now, $\quad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{k(-h)}{|-h|} = \lim_{h \to 0} \frac{-kh}{h} = -k$$

Also, $f(0) = 3$

$\because \; \lim_{x \to 0^-} f(x) = f(0) \quad \Rightarrow -k = 3 \quad \Rightarrow \quad k = -3$

4. If $y = 2\sqrt{\sec(e^{2x})}$; then find $\dfrac{dy}{dx}$. *[CBSE 2019 (65/5/3)]*

Sol. Given, $y = 2\sqrt{\sec(e^{2x})}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(2\sqrt{\sec(e^{2x})}\right) = 2 \times \frac{1}{2\sqrt{\sec(e^{2x})}} \times \sec(e^{2x}) \tan(e^{2x}) \times 2e^{2x}$$

$$= 2\sqrt{\sec(e^{2x})} \tan(e^{2x}) . e^{2x} = 2e^{2x} \sqrt{\sec(e^{2x})} \tan(e^{2x})$$

5. If $y = \text{cosec}(\cot \sqrt{x})$, then find $\dfrac{dy}{dx}$. *[CBSE 2019 (65/4/2)]*

Sol. $y = \text{cosec}(\cot \sqrt{x})$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\text{cosec}(\cot \sqrt{x}))$$

$$= -\text{cosec}(\cot \sqrt{x}) \cot(\cot \sqrt{x}) \times \left(-\text{cosec}^2(\sqrt{x}) \times \frac{1}{2\sqrt{x}} \right)$$

$$= \frac{\text{cosec}(\cot \sqrt{x}) \cot(\cot \sqrt{x}) \times \text{cosec}^2(\sqrt{x})}{2\sqrt{x}}$$

6. Find the derivative of $\log_{10} x$ with respect to x. *[NCERT Exemplar]*

Sol. Let $y = \log_{10} x = \log_{10} e \,.\, \log_e x$

$$\therefore \quad \frac{dy}{dx} = \log_{10} e \times \frac{1}{x} = \frac{\log_{10} e}{x} \qquad \left[\because \frac{d}{dx}\log_e x = \frac{1}{x}\right]$$

7. If $y = 5e^{7x} + 6e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$. *[CBSE 2019 (65/5/1)]*

Sol. $\because \quad y = 5e^{7x} + 6e^{-7x} \quad \Rightarrow \quad \frac{dy}{dx} = 35e^{7x} - 42e^{-7x}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 245e^{7x} + 294e^{-7x} = 49(5e^{7x} + 6e^{-7x}) = 49y$$

8. If $y = \log(\cos e^x)$, then find $\frac{dy}{dx}$. *[CBSE 2019 (65/4/1)]*

Sol. $\because \quad y = \log(\cos e^x)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(\log(\cos e^x)) = \frac{1}{\cos(e^x)} \times (-\sin(e^x)) \times e^x = \frac{-e^x \sin(e^x)}{\cos(e^x)} = -e^x \tan(e^x)$$

$$\therefore \quad \frac{dy}{dx} = -e^x \tan(e^x)$$

Short Answer Questions-I [2 marks]

1. Find $\frac{dy}{dx}$ at $t = \frac{2\pi}{3}$ when $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$. *[CBSE (F) 2017]*

Sol. Given, $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$

$\because \quad x = 10(t - \sin t) \quad \Rightarrow \quad \frac{dx}{dt} = 10\,(1 - \cos t)$ (Differentiating w.r.t. t)

Again $y = 12(1 - \cos t) \quad \Rightarrow \quad \frac{dy}{dt} = 12(0 + \sin t) = 12 \sin t$ (Differentiating w.r.t. t)

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1 - \cos t)}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{t = \frac{2\pi}{3}} = \frac{12 \sin \frac{2\pi}{3}}{10\left(1 - \cos \frac{2\pi}{3}\right)} = \frac{6}{5} \times \frac{\sin\left(\pi - \frac{\pi}{3}\right)}{\left(1 - \cos\left(\pi - \frac{\pi}{3}\right)\right)}$$

$$= \frac{6}{5} \times \frac{\sin \frac{\pi}{3}}{1 + \cos \frac{\pi}{3}} = \frac{6}{5} \times \frac{\frac{\sqrt{3}}{2}}{1 + \frac{1}{2}} = \frac{6\sqrt{3}}{5 \times 3} = \frac{2\sqrt{3}}{5}$$

2. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$. *[CBSE Delhi 2017]*

Sol. $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x, we get

$$2 \sin y . \cos y \frac{dy}{dx} + (-\sin xy)(x.\frac{dy}{dx} + y) = 0 \quad \Rightarrow \quad \sin 2y.\frac{dy}{dx} - x \sin xy.\frac{dy}{dx} - y \sin xy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \sin xy}{(\sin 2y - x \sin xy)}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{x=1,\, y=\frac{\pi}{4}} = \frac{\frac{\pi}{4} \cdot \sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{\frac{\pi}{4} \cdot \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2}-1)}$$

3. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then prove that *[HOTS]*

(*i*) $C_1 + 2C_2 + ... + nC_n = n \,.\, 2^{n-1}$ (*ii*) $C_1 - 2C_2 + 3C_3 - .... + (-1)^n nC_n = 0$

Sol. We have, $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Differentiating both sides with respect to x, we have

$$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_nx^{n-1}$$

Putting $x = 1$ and $x = -1$ successively, we have

(*i*) $C_1 + 2C_2 + 3C_3 + ... + nC_n = n \,.\, 2^{n-1}$ and (*ii*) $C_1 - 2C_2 + 3C_3 - .... + (-1)^n nC_n = 0$

4. If $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$, then show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$. *[NCERT Exemplar]*

Sol. We have, $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$

$\Rightarrow \quad y = (\cos x)^y$

$\therefore \quad \log y = \log (\cos x)^y \quad \Rightarrow \quad \log y = y \log \cos x$

On differentiating w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{d}{dx} \log \cos x + \log \cos x \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{y}{\cos x} \cdot \frac{d}{dx} \cos x + \log \cos x \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y} - \log \cos x\right] = \frac{-y \sin x}{\cos x} = -y \tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2 \tan x}{(1 - y \log \cos x)} = \frac{y^2 \tan x}{y \log \cos x - 1}$$

5. If $x = a \cos \theta$; $y = b \sin \theta$, then find $\frac{d^2y}{dx^2}$. *[CBSE 2020 (65/5/1)]*

Sol. We have,

$$x = a \cos \theta \;\Rightarrow\; \frac{dx}{d\theta} = -a \sin \theta$$

and $$y = b \sin \theta \;\Rightarrow\; \frac{dy}{d\theta} = b \cos \theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{b \cos \theta}{-a \sin \theta} = -\frac{b}{a} \cot \theta$$

Again diff. w.r.t x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{b}{a}\frac{d}{dx}(\cot \theta) = -\frac{b}{a} \times (-\operatorname{cosec}^2 \theta)\frac{d\theta}{dx}$$

$$= -\frac{b}{a} \times (-\operatorname{cosec}^2 \theta) \times \left(-\frac{1}{a \sin \theta}\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{b}{a^2} \operatorname{cosec}^3 \theta$$

Short Answer Questions-II [3 marks]

CONTINUITY AND DIFFERENTIABILITY

1. Find the values of p and q, for which *[CBSE Delhi 2016]*

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$$ **is continuous at** $x=\dfrac{\pi}{2}$.

Sol. We have, $$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$$ is continuous at $x=\dfrac{\pi}{2}$.

Now, $\lim\limits_{x\to\frac{\pi^+}{2}} f(x)=\lim\limits_{h\to 0} f\left(\dfrac{\pi}{2}+h\right)$ $\quad\left[\text{Let } x=\dfrac{\pi}{2}+h,\ x\to\dfrac{\pi^+}{2}\Rightarrow h\to 0\right]$

$$=\lim_{h\to 0}\frac{q\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2}=\lim_{h\to 0}\frac{q\{1-\cos h\}}{\{\pi-\pi-2h\}^2}=\lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{q.2\sin^2\frac{h}{2}}{4h^2}=\lim_{h\to 0}\frac{q.\sin^2\frac{h}{2}}{2h^2}=q.\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\times\frac{1}{8}=\frac{q}{8}$$

Again $\lim\limits_{x\to\frac{\pi^-}{2}} f(x)=\lim\limits_{h\to 0} f\left(\dfrac{\pi}{2}-h\right)$ $\quad\left[\text{Let } x=\dfrac{\pi}{2}-h,\ x\to\dfrac{\pi^-}{2}\Rightarrow h\to 0\right]$

$$=\lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}=\lim_{h\to 0}\frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}.(1+1+1)}{3\sin^2 h}=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}.3}{3\sin^2 h}=\lim_{h\to 0}\frac{2.\sin^2\frac{h}{2}}{\sin^2 h}$$

Dividing N^r and D^r by h^2, we get

$$=\lim_{h\to 0}\frac{2.\frac{\sin^2\frac{h}{2}}{h^2}}{\frac{\sin^2 h}{h^2}}=\lim_{h\to 0}\frac{2.\frac{\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4}}{\frac{\sin^2 h}{h^2}}=\frac{1}{2}\frac{\left(\lim\limits_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2}{\left(\lim\limits_{h\to 0}\frac{\sin h}{h}\right)^2}=\frac{1}{2}$$

Also $f\left(\frac{\pi}{2}\right)=p$

$\because$ $f(x)$ is continuous at $x=\frac{\pi}{2}$ $\Rightarrow$ $\lim_{h\to\frac{\pi^+}{2}} f(x)=\lim_{h\to\frac{\pi^-}{2}} f(x)=f\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\frac{q}{8}=\frac{1}{2}=p$ $\Rightarrow$ $p=\frac{1}{2}$ and $q=4$

2. Show that the function $f(x)=2x-|x|$ is continuous but not differentiable at $x=0$. [*CBSE (F) 2013*]

Sol. Here $f(x)=2x-|x|$

For continuity at $x=0$

$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$=\lim_{h\to 0}\{2h-|h|\}=\lim_{h\to 0}(2h-h)$

$=\lim_{h\to 0} h$

$=0$ ...(*i*)

$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$=\lim_{h\to 0}\{2(-h)-|-h|\}=\lim_{h\to 0}\{-2h-h\}$

$=\lim_{h\to 0}(-3h)$

$=0$ ...(*ii*)

Also, $f(0)=2\times 0-|0|=0$ ...(*iii*)

(*i*), (*ii*) and (*iii*) $\Rightarrow$ $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)=f(0)$

Hence, $f(x)$ is continuous at $x=0$.

For differentiability at $x=0$

LHD $=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$

$=\lim_{h\to 0}\frac{(2(-h)-|-h|)-\{2\times 0-|0|\}}{-h}=\lim_{h\to 0}\frac{-2h-h-0}{-h}$

$=\lim_{h\to 0}\frac{-3h}{-h}=\lim_{h\to 0} 3$

LHD $=3$ ...(*iv*)

Again RHD $=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{2h-|h|-2\times 0-|0|}{h}=\lim_{h\to 0}\frac{2h-h}{h}=\lim_{h\to 0}\frac{h}{h}$

$=\lim_{h\to 0} 1$

RHD $=1$ ...(*v*)

From (*iv*) and (*v*), we get

LHD $\neq$ RHD *i.e.*, function $f(x)=2x-|x|$ is not differentiable at $x=0$.

Hence, $f(x)$ is continuous but not differentiable at $x=0$.

3. Find the value of 'a' for which the function f defined as

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1), & x\le 0\\ \frac{\tan x-\sin x}{x^3}, & x>0\end{cases}$$

is continuous at $x=0$. [*CBSE Delhi 2011; (South) 2016*]

Sol. $\because$ $f(x)$ is continuous at $x=0$.

$\Rightarrow$ (LHL of $f(x)$ at $x=0$) = (RHL of $f(x)$ at $x=0$) = $f(0)$

$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ ...(i)

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} a\sin\frac{\pi}{2}(x+1)$ $\left[\because f(x) = a\sin\frac{\pi}{2}(x+1), \text{ if } x \le 0\right]$

$$= \lim_{x\to 0^+} a\sin\left(\frac{\pi}{2}+\frac{\pi}{2}x\right) = \lim_{x\to 0^+} a\cos\frac{\pi}{2}x = a.\cos 0 = a \quad \text{...(ii)}$$

Again, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\tan x - \sin x}{x^3}$ $\left[\because f(x) = \frac{\tan x - \sin x}{x^3}, \text{ if } x > 0\right]$

$$= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x\to 0}\frac{\sin x - \sin x.\cos x}{\cos x.x^3} = \lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x.x^3}$$

$$= \lim_{x\to 0}\frac{1}{\cos x}.\lim_{x\to 0}\frac{\sin x}{x}.\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} \qquad \left[\because 1-\cos x = 2\sin^2\frac{x}{2}\right]$$

$$= \frac{1}{1}.1.\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = \frac{1}{2}\times 1 = \frac{1}{2} \quad \text{...(iii)}$$

Also, $f(0) = a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a$ ...(iv)

$\because$ f is continuous at $x = 0$.

$\therefore$ (i), (ii), (iii) and (iv) $\Rightarrow$ $a = \frac{1}{2}$

4. If $f(x) = \begin{cases} \frac{\sin(a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \frac{\sqrt{1+bx}-1}{x}, & x > 0 \end{cases}$ is continuous at $x = 0$, then find the values of a and b.

[*CBSE (North) 2016*]

Sol. We have, $f(x) = \begin{cases} \frac{\sin(a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \frac{\sqrt{1+bx}-1}{x}, & x > 0 \end{cases}$ is continuous at $x = 0$

Since, $f(x)$ is continuous at $x = 0$

$\Rightarrow \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$ ... (i)

Now, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$ [Let $x = 0 + h$, h is +ve small quantity $x \to 0^+ \Rightarrow h \to 0$]

$$= \lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{\sqrt{1+bh}-1}{h} = \lim_{h\to 0}\frac{\sqrt{1+bh}-1}{h}\times\frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$$

$$= \lim_{h\to 0}\frac{1+bh-1}{h(\sqrt{1+bh}+1)} = \lim_{h\to 0}\frac{bh}{h(\sqrt{1+bh}+1)} = \lim_{h\to 0}\frac{b}{\sqrt{1+bh}+1} = \frac{b}{2}$$

Again $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$ [Let $x = 0 - h$, h is +ve small quantity $x \to 0^- \Rightarrow h \to 0$]

$$= \lim_{h\to 0} f(-h) = \lim_{h\to 0}\frac{\sin(a+1)(-h) + 2\sin(-h)}{-h}$$

$$= \lim_{h\to 0}\left[\frac{-\sin(a+1)h - 2\sin h}{-h}\right] = \lim_{h\to 0}\left[\frac{\sin(a+1)h}{h} + \frac{2\sin h}{h}\right]$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h}{h} + 2\lim_{h \to 0} \frac{\sin h}{h} = \lim_{h \to 0} \frac{\sin(a+1)h}{(a+1)h} \times (a+1) + 2\lim_{h \to 0} \frac{\sin h}{h}$$

$$= 1 \times (a + 1) + 2 = a + 3$$

Also $\quad f(0) = 2$

Now from (i) $\frac{b}{2} = a + 3 = 2 \quad \Rightarrow \quad b = 4,\ a = -1$

5. Show that the function $f(x) = |x - 3|, x \in \mathbb{R}$, is continuous but not differentiable at $x = 3$.
[*CBSE Delhi 2013*]

Sol. Here, $f(x) = |x - 3| \quad \Rightarrow \quad f(x) = \begin{cases} -(x-3) & ,x < 3 \\ 0 & ,x = 3 \\ (x-3) & ,x > 3 \end{cases}$

For Continuity:

Now, $\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$ $\quad$ [Let $x = 3 + h$ and $x \to 3^+ \Rightarrow h \to 0$]

$$= \lim_{h \to 0} (3 + h - 3) = \lim_{h \to 0} h = 0$$

$\lim_{x \to 3^+} f(x) = 0$ ...(i)

$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3-h)$ $\quad$ [Let $x = 3 - h$ and $x \to 3^- \Rightarrow h \to 0$]

$$= \lim_{h \to 0} -(3 - h - 3) = \lim_{h \to 0} h = 0$$

$\lim_{x \to 3^+} f(x) = 0$ ...(ii)

Also, $\quad f(3) = 0$ ...(iii)

From equation (i), (ii) and (iii), we get

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

Hence, $f(x)$ is continuous at $x = 3$.

For Differentiability:

RHD $= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h-3) - 0}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$ ...(iv)

LHD $= \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h} = \lim_{h \to 0} \frac{-(3-h-3) - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1$ ...(v)

Equation (iv) and (v)

$\Rightarrow \quad$ RHD $\neq$ LHD at $x = 3$.

Hence, $f(x)$ is not differentiable at $x = 3$.

Therefore, $f(x) = |x - 3|, x \in \mathbb{R}$ is continuous but not differentiable at $x = 3$.

6. Discuss the continuity and differentiability of the function
$f(x) = |x| + |x - 1|$ in the interval $(-1, 2)$. **[*CBSE Ajmer 2015*]**

Sol. Given function is $f(x) = |x| + |x - 1|$

Function is also written as

$$f(x) = \begin{cases} -x-(x-1), & \text{if } -1 < x < 0 \\ 1, & \text{if } 0 \le x < 1 \\ x+(x-1), & \text{if } x \ge 1 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} -2x+1, & \text{if } x < 0 \\ 1, & \text{if } 0 \le x < 1 \\ 2x-1, & \text{if } x \ge 1 \end{cases}$$

Obviously, in given function we need to discuss the continuity and differentiability of the function $f(x)$ at $x = 0$ or 1 only.

For continuity at $x = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) \qquad [\text{Let } x = 0 + h \text{ and } x \to 0^+ \Rightarrow h \to 0]$$

$$= \lim_{h\to 0} f(h) = \lim_{h\to 0} 1 \qquad [\because h \text{ is very small positive quantity}]$$

$$= 1 \qquad \dots(i)$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) \qquad [\text{Let } x = 0 - h \text{ and } x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{h\to 0} f(-h) = \lim_{h\to 0}\{-2(-h)+1\} = \lim_{h\to 0}(2h+1)$$

$$\lim_{x\to 0^-} f(x) = 1 \qquad \dots(ii)$$

Also, $f(0) = 1$ $\dots(iii)$

(i), (ii) and (iii) $\Rightarrow$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

For differentiability at $x = 0$

$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \qquad [\because h \text{ is very small positive quantity}]$$

$$= \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 \qquad [\because |h| = h,\ |0| = 0]$$

$$\text{RHD} = 0 \qquad \dots(iv)$$

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-2(-h)+1-1}{-h} = \lim_{h\to 0} \frac{2h}{-h} = \lim_{h\to 0}(-2)$$

$$\text{LHD} = -2 \qquad \dots(v)$$

(iv) and (v) $\Rightarrow$ RHD $\neq$ LHD at $x = 0$.

Hence, $f(x)$ is not differentiable at $x = 0$ but continuous at $x = 0$.

Similarly, we can prove $f(x)$ is not differentiable at $x = 1$ but continuous at $x = 1$. (Do yourself)

7. Find 'a' and 'b', if the function given by $f(x) = \begin{cases} ax^2 + b, & \text{if } x < 1 \\ 2x + 1, & \text{if } x \geq 1 \end{cases}$ is differentiable at $x = 1$.

[*CBSE Sample Paper 2018*]

Sol. Since, f is differentiable at 1 $\Rightarrow$ f is also continuous at 1.

Now $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$ [Here h is +ve and very small quantity]

$$= \lim_{h\to 0} 2(1+h) + 1 = 2 + 1 = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\{a(1-h)^2 + b\} = a + b$$

Since $f(x)$ is continuous at $x = 1$

$\Rightarrow$ $a + b = 3$ $\dots(i)$

Again, since f is differentiable

$\Rightarrow$ LHD (at $x = 1$) = RHD (at $x = 1$) $\Rightarrow$ $\lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$\Rightarrow \lim_{h\to 0}\frac{a(1-h)^2+b-3}{-h}=\lim_{h\to 0}\frac{2(1+h)+1-3}{h} \quad \Rightarrow \lim_{h\to 0}\frac{a-2ah+ah^2+b-3}{-h}=\lim_{h\to 0}\frac{2+2h+1-3}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{-2ah+ah^2+(a+b)-3}{-h}=\lim_{h\to 0}\frac{2h}{h} \quad \Rightarrow \lim_{h\to 0}\frac{-2ah+ah^2+3-3}{-h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{ah(2-h)}{h}=2 \quad \Rightarrow \quad 2a=2 \quad \Rightarrow \quad a=1 \quad \Rightarrow \quad b=2 \quad \text{[From equation (}i\text{)]}$$

DERIVATIVES

1. If $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$, prove that $\frac{dy}{dx}=\frac{x+y}{x-y}$. ***[CBSE 2020 (65/2/1)]***

Sol. Given, $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log(x^2+y^2)$$

Differentiating w.r.t x, we have

$$\Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\times\left\{\frac{x\frac{dy}{dx}-y\times 1}{x^2}\right\}=\frac{1}{2}\times\frac{1}{x^2+y^2}\times\left(2x+2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{x^2}{x^2+y^2}\times\frac{\left(x\frac{dy}{dx}-y\right)}{x^2}=\frac{1}{x^2+y^2}\left(x+y\frac{dy}{dx}\right)$$

$$\Rightarrow x\frac{dy}{dx}-y=x+y\frac{dy}{dx}$$

$$\Rightarrow (x-y)\frac{dy}{dx}=x+y \quad \Rightarrow \quad \frac{dy}{dx}=\frac{x+y}{x-y}$$

2. Differentiate $\tan^{-1}\frac{3x-x^3}{1-3x^2}, |x|<\frac{1}{\sqrt{3}}$ w.r.t. $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$. ***[CBSE 2019 (65/5/1)]***

Sol. Let $y=\tan^{-1}\frac{3x-x^3}{1-3x^2}=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$ put $x=\tan\theta$

$$y=\tan^{-1}(\tan 3\theta)=3\theta=3\tan^{-1}x$$

$$\therefore \quad y=3\tan^{-1}x$$

and let $t=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$ put $x=\sin\theta$

$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)=\tan^{-1}(\tan\theta)=\theta$$

$$t=\sin^{-1}x$$

$$\therefore \quad \frac{dy}{dt}=\frac{d\,3\tan^{-1}x}{d\sin^{-1}x}=\frac{\frac{3d\tan^{-1}x}{dx}}{\frac{d\sin^{-1}x}{dx}}=\frac{\frac{3\times 1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}}=3\frac{\sqrt{1-x^2}}{1+x^2}$$

3. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$. **[CBSE (AI) 2013]**

Sol. Given, $y^x = e^{y-x}$

Taking logarithm both sides, we get $\log y^x = \log e^{y-x}$

$\Rightarrow \quad x.\log y = (y - x) . \log e \quad \Rightarrow \quad x . \log y = (y - x)$

$\Rightarrow \quad x(1 + \log y) = y \quad \Rightarrow \quad x = \frac{y}{1+\log y}$

$\left[\because \begin{array}{l}(i)\ \log_e mn = \log_e m + \log_e n \\ (ii)\ \log_e \frac{m}{n} = \log_e m - \log_e n \\ (iii)\ \log_e m^n = n\log_e m \\ (iv)\ \log e = 1\end{array}\right]$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = \frac{(1+\log y).1 - y.\left(0+\frac{1}{y}\right)}{(1+\log y)^2} = \frac{1+\log y - 1}{(1+\log y)^2} = \frac{\log y}{(1+\log y)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

4. If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$. **[CBSE Delhi 2012]**

Sol. Given, $(\cos x)^y = (\cos y)^x$

Taking logarithm both sides, we get $\log (\cos x)^y = \log (\cos y)^x$

$\Rightarrow \quad y . \log (\cos x) = x . \log (\cos y) \quad [\because \log m^n = n \log m]$

Differentiating both sides with respect to x, we get

$$y.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\frac{dy}{dx} = x.\frac{1}{\cos y}.(-\sin y).\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad -\frac{y\sin x}{\cos x} + \log(\cos x).\frac{dy}{dx} = -\frac{x\sin y}{\cos y}.\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad \log(\cos x).\frac{dy}{dx} + \frac{x\sin y}{\cos y}.\frac{dy}{dx} = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\log(\cos x) + \frac{x\sin y}{\cos y}\right] = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log(\cos y) + \frac{y\sin x}{\cos x}}{\log(\cos x) + \frac{x\sin y}{\cos y}} = \frac{\log(\cos y) + y\tan x}{\log(\cos x) + x\tan y}$$

5. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$ **[CBSE (AI) 2008, 2014]**

Sol. Given, $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$

Taking $x = ae^\theta(\sin\theta - \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = ae^\theta(\cos\theta + \sin\theta) + a(\sin\theta - \cos\theta).e^\theta = ae^\theta(\cos\theta + \sin\theta + \sin\theta - \cos\theta)$$

$$= 2ae^\theta\sin\theta \qquad \ldots (i)$$

Again, $y = ae^\theta(\sin\theta + \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dy}{d\theta} = ae^{\theta}(\cos\theta - \sin\theta) + a(\sin\theta + \cos\theta).e^{\theta} = ae^{\theta}(\cos\theta - \sin\theta + \sin\theta + \cos\theta)$$

$$= 2\,ae^{\theta}.\cos\theta \qquad \dots (ii)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta}.\cos\theta}{2ae^{\theta}.\sin\theta} \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\theta \quad \Rightarrow \quad \left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

6. Differentiate the following with respect to x: $(\sin x)^x + (\cos x)^{\sin x}$ **[*CBSE (F) 2013*]**

Sol. Let $u = (\sin x)^x$ and $v = (\cos x)^{\sin x}$

∴ Given differential equation becomes $y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ....(i)$$

Now, $u = (\sin x)^x$

Taking log on both sides, we get

$$\log u = x \log \sin x$$

Differentiating with respect to x, we get

$$\frac{1}{u}.\frac{du}{dx} = x\frac{1}{\sin x}.\cos x + \log\sin x \quad \Rightarrow \quad \frac{du}{dx} = u(x\cot x + \log\sin x)$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x\{x\cot x + \log\sin x\} \qquad ....(ii)$$

Again $v = (\cos x)^{\sin x}$

Taking log on both sides, we get

$$\log v = \sin x \,.\, \log\cos x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}.\frac{dv}{dx} = \sin x.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left\{-\frac{\sin^2 x}{\cos x} + \cos x.\log\cos x\right\} = (\cos x)^{\sin x}\left\{\cos x.\log(\cos x) - \frac{\sin^2 x}{\cos x}\right\}$$

$$= (\cos x)^{1+\sin x}\{\log(\cos x) - \tan^2 x\} \qquad ....(iii)$$

From (*i*), (*ii*) and (*iii*), we get

$$\frac{dy}{dx} = (\sin x)^x\{x\cot x + \log\sin x\} + (\cos x)^{1+\sin x}\{\log(\cos x) - \tan^2 x\}$$

7. If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. Hence show that

$$\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0.$$ **[*CBSE (F) 2014; (North) 2016*]**

Sol. Given, $\cos y = x\cos(a+y)$

$$\therefore \quad x = \frac{\cos y}{\cos(a+y)}$$

Differentiating with respect to y on both sides, we get

$$\frac{dx}{dy}=\frac{\cos(a+y)\times(-\sin y)-\cos y\times[-\sin(a+y)]}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{\cos y\sin(a+y)-\sin y\cos(a+y)}{\cos^2(a+y)} \quad \Rightarrow \quad \frac{dx}{dy}=\frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{\sin a}{\cos^2(a+y)} \qquad \therefore \quad \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{1}{\sin a}\left\{-2\cos(a+y).\sin(a+y).\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \sin a\frac{d^2y}{dx^2}=-\sin 2(a+y).\frac{dy}{dx} \quad \Rightarrow \quad \sin a\frac{d^2y}{dx^2}+\sin 2(a+y).\frac{dy}{dx}=0$$

8. If $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$, then show that $\left.\frac{dy}{dx}\right|_{\text{at } t=\frac{\pi}{4}}=\frac{b}{a}$. Also find the value of $\left(\frac{dy}{dx}\right)$ at $t=\frac{\pi}{3}$. *[CBSE Delhi 2016; (AI) 2014; Panchkula 2015; (Central) 2016]*

Sol. Given, $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$

$$\Rightarrow \quad \frac{dx}{dt}=a[\sin 2t\times(-2\sin 2t)+(1+\cos 2t)\times 2\cos 2t]=a[-2\sin^2 2t+2\cos 2t+2\cos^2 2t]$$

$$=a(2\cos 4t+2\cos 2t)=2a(\cos 4t+\cos 2t)$$

Again, $$\frac{dy}{dt}=b[\cos 2t\times 2\sin 2t+(1-\cos 2t)(-2\sin 2t)]$$

$$=b[\sin 4t-2\sin 2t+\sin 4t]=b[2\sin 4t-2\sin 2t]=2b(\sin 4t-\sin 2t)$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2b(\sin 4t-\sin 2t)}{2a(\cos 4t+\cos 2t)}=\frac{b}{a}\left[\frac{\sin 4t-\sin 2t}{\cos 4t+\cos 2t}\right]$$

So, $$\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}}=\frac{b}{a}\left(\frac{\sin\pi-\sin\frac{\pi}{2}}{\cos\pi+\cos\frac{\pi}{2}}\right)=\frac{b}{a}\times\left(\frac{-1}{-1}\right)=\frac{b}{a} \text{ and}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{3}}=\frac{b}{a}\left(\frac{\sin\frac{4\pi}{3}-\sin\frac{2\pi}{3}}{\cos\frac{4\pi}{3}+\cos\frac{2\pi}{3}}\right)=\frac{b}{a}\left(\frac{-\sin\frac{\pi}{3}-\sin\frac{\pi}{3}}{-\cos\frac{\pi}{3}-\cos\frac{\pi}{3}}\right)$$

$$=\frac{b}{a}\times\left(\frac{-2\sin\frac{\pi}{3}}{-2\cos\frac{\pi}{3}}\right)=\frac{b}{a}\tan\frac{\pi}{3}=\frac{\sqrt{3}\,b}{a}$$

9. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then show that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$. *[CBSE (F) 2009, 2019 (65/5/3)]*

Sol. Given, $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Putting $x=\sin\alpha \Rightarrow \alpha=\sin^{-1}x$ and $y=\sin\beta \Rightarrow \beta=\sin^{-1}y$, we get

$$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta) \quad \Rightarrow \quad \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow \quad 2\cos\frac{(\alpha+\beta)}{2}\cos\left(\frac{\alpha-\beta}{2}\right) = a.2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \cot\left(\frac{\alpha-\beta}{2}\right) = a \quad \Rightarrow \quad \frac{\alpha-\beta}{2} = \cot^{-1}a \quad \Rightarrow \quad \alpha-\beta = 2\cot^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

10. If $x = \cos t(3-2\cos^2 t)$ and $y = \sin t\,(3-2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [*CBSE (AI) 2014*]

Sol. Given, $x = \cos t(3-2\cos^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = \cos t\{0 + 4\cos t.\sin t\} + (3-2\cos^2 t).(-\sin t)$$

$$= 4\sin t.\cos^2 t - 3\sin t + 2\cos^2 t.\sin t$$

$$= 6\sin t\cos^2 t - 3\sin t = 3\sin t\,(2\cos^2 t - 1) = 3\sin t.\cos 2t$$

Again, $\because\ y = \sin t\,(3-2\sin^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dy}{dt} = \sin t.\{0 - 4\sin t\cos t\} + (3-2\sin^2 t).\cos t$$

$$= -4\sin^2 t.\cos t + 3\cos t - 2\sin^2 t.\cos t = 3\cos t - 6\sin^2 t.\cos t$$

$$= 3\cos t\,(1-2\sin^2 t) = 3\cos t.\cos 2t$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t.\cos 2t}{3\sin t.\cos 2t} \quad \Rightarrow \quad \frac{dy}{dx} = \cot t$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

11. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. [*CBSE Delhi 2014*]

Sol. Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}(2x\sqrt{1-x^2})$

We have to determine $\frac{du}{dv}$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

Now, $$u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) \quad \Rightarrow \quad u = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad u = \tan^{-1}(\cot\theta) \quad \Rightarrow \quad u = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$$

$$\Rightarrow \quad u = \frac{\pi}{2} - \theta \quad \Rightarrow \quad u = \frac{\pi}{2} - \sin^{-1}x$$

$\Rightarrow \quad \dfrac{du}{dx} = 0 - \dfrac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \dfrac{du}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

Again, $v = \cos^{-1}(2x\sqrt{1-x^2})$

$\because \quad x = \sin\theta$

$\therefore \quad v = \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) \quad \Rightarrow \quad v = \cos^{-1}(2\sin\theta \,.\, \cos\theta)$

$\Rightarrow \quad v = \cos^{-1}(\sin 2\theta) \quad \Rightarrow \quad v = \cos^{-1}\left(\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right)$

$\Rightarrow \quad v = \dfrac{\pi}{2} - 2\theta \quad \Rightarrow \quad v = \dfrac{\pi}{2} - 2\sin^{-1}x$

$\Rightarrow \quad \dfrac{dv}{dx} = 0 - \dfrac{2}{\sqrt{1-x^2}} \quad \Rightarrow \quad \dfrac{dv}{dx} = -\dfrac{2}{\sqrt{1-x^2}}$

$\therefore \quad \dfrac{du}{dv} = \dfrac{\frac{du}{dx}}{\frac{dv}{dx}} = \dfrac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}} = \dfrac{1}{2}$

$$\left[\begin{array}{l} \because \ -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}} \\ \Rightarrow \ \sin\left(-\dfrac{\pi}{4}\right) < \sin\theta < \sin\left(\dfrac{\pi}{4}\right) \\ \Rightarrow \ -\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4} \\ \Rightarrow \ -\dfrac{\pi}{2} < 2\theta < \dfrac{\pi}{2} \\ \Rightarrow \ \dfrac{\pi}{2} > -2\theta > -\dfrac{\pi}{2} \\ \Rightarrow \ \pi > \left(\dfrac{\pi}{2} - 2\theta\right) > 0 \\ \Rightarrow \ \left(\dfrac{\pi}{2} - 2\theta\right) \in (0,\pi) \subset [0,\pi] \end{array}\right]$$

[**Note:** Here the range of x is taken as $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$]

12. Differentiate the following function with respect to x:

$y = (\sin x)^x + \sin^{-1}\sqrt{x}$ **[*CBSE Delhi 2009, 2013, 2017*]**

Sol. Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$y = u + v$, where $u = (\sin x)^x$, $v = \sin^{-1}\sqrt{x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \quad \ldots(i)$

Now, $u = (\sin x)^x$

Taking log both sides, we get

$\log u = \log(\sin x)^x \quad \Rightarrow \quad \log u = x \,.\, \log(\sin x)$

Differentiating both sides with respect to x, we get

$\dfrac{1}{u} \cdot \dfrac{du}{dx} = x . \dfrac{1}{\sin x}\cos x + \log\sin x \quad \Rightarrow \quad \dfrac{du}{dx} = u\{x\cot x + \log\sin x\}$

$\Rightarrow \quad \dfrac{du}{dx} = (\sin x)^x\{x\cot x + \log\sin x\} \quad \ldots(ii)$

Also, $v = \sin^{-1}\sqrt{x}$

$\dfrac{dv}{dx} = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}} \times \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x(1-x)}} \quad \ldots(iii)$

From (*i*), (*ii*) and (*iii*), we get

$\therefore \quad \dfrac{dy}{dx} = (\sin x)^x\{x\cot x + \log\sin x\} + \dfrac{1}{2\sqrt{x(1-x)}}$

13. If $x^{13}y^7 = (x+y)^{20}$, then prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$. **[*CBSE (F) 2012*]**

OR

If $x^m y^n = (x+y)^{m+n}$, then prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$. **[*CBSE (F) 2014*]**

Sol. Given $x^{13}y^7 = (x+y)^{20}$

Taking logarithm on both sides, we get

$$\log (x^{13}y^7) = \log (x+y)^{20}$$

$$\Rightarrow \quad \log x^{13} + \log y^7 = 20 \log (x+y) \quad \Rightarrow \quad 13 \log x + 7 \log y = 20 \log (x+y)$$

Differentiating both sides with respect to x, we get

$$\frac{13}{x} + \frac{7}{y}.\frac{dy}{dx} = \frac{20}{x+y}.\left(1+\frac{dy}{dx}\right) \quad \Rightarrow \quad \frac{13}{x} - \frac{20}{x+y} = \left(\frac{20}{x+y} - \frac{7}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{13x+13y-20x}{x(x+y)} = \left(\frac{20y-7x-7y}{(x+y).y}\right)\frac{dy}{dx} \quad \Rightarrow \quad \frac{13y-7x}{x(x+y)} = \left(\frac{13y-7x}{y(x+y)}\right).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{13y-7x}{x(x+y)} \times \frac{y(x+y)}{13y-7x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

OR

Do yourself (similar question)

14. Differentiate with respect to x :

$$\sin^{-1}\left(\frac{2^{x+1}.3x}{1+(36)^x}\right)$$ **[CBSE (AI) 2013]**

Sol. Let $y = \sin^{-1}\left(\frac{2^{x+1}.3x}{1+(36)^x}\right) = \sin^{-1}\left(\frac{2.2^x.3^x}{1+(6^2)^x}\right) = \sin^{-1}\left(\frac{2.6^x}{1+(6^x)^2}\right)$

Let $6^x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}(6^x)$

$$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \quad \Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow \quad y = 2\theta \quad \Rightarrow \quad y = 2 . \tan^{-1}(6^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+(6^x)^2}.\log_e 6.6^x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2.6^x.\log_e 6}{1+36^x}$$

15. If $x \sin(a+y) + \sin a \cos(a+y) = 0$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$. **[CBSE (AI) 2013]**

Sol. Given $x \sin(a+y) + \sin a \cos(a+y) = 0$

$$\Rightarrow \quad x = -\frac{\sin a.\cos(a+y)}{\sin(a+y)} \quad \Rightarrow \quad x = -\sin a . \cot(a+y)$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = +\sin a.\text{cosec}^2(a+y) = \frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

16. If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + e^{y-x} = 0$. **[CBSE (F) 2014]**

Sol. Given, $e^x + e^y = e^{x+y}$

Differentiating both sides with respect to x, we get

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}\left\{1+\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad e^x + e^y . \frac{dy}{dx} = e^{x+y} + e^{x+y} . \frac{dy}{dx} \quad \Rightarrow \quad (e^{x+y} - e^y)\frac{dy}{dx} = e^x - e^{x+y}$$

$$\Rightarrow \quad (e^x + e^y - e^y)\frac{dy}{dx} = e^x - e^x - e^y \qquad [\because e^x + e^y = e^{x+y} \text{(given)}]$$

$$\Rightarrow \quad e^x . \frac{dy}{dx} = -e^y \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{e^y}{e^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x} \quad \Rightarrow \quad \frac{dy}{dx} + e^{y-x} = 0$$

17. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$. **[*CBSE (East) 2016*]**

Sol. We have $x = e^{\cos 2t}$

$$\frac{dx}{dt} = e^{\cos 2t}(-2\sin 2t) = -2x\sin 2t \qquad \text{[Differentiating w.r.t. } t\text{]}$$

Again $y = e^{\sin 2t}$

$$\frac{dy}{dt} = e^{\sin 2t} . 2\cos 2t = 2y\cos 2t \qquad \text{[Differentiating w.r.t. } t\text{]}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2y\cos 2t}{-2x\sin 2t} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y\cos 2t}{x\sin 2t}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y\log x}{x\log y} \qquad [\because x = e^{\cos 2t} \Rightarrow \log x = \cos 2t;\ y = e^{\sin 2t} \Rightarrow \log y = \sin 2t]$$

Hence proved.

18. If $x \in R - [-1, 1]$ then prove that the derivative of $\sec^{-1} x$ with respect to x is $\frac{1}{|x|\sqrt{x^2-1}}$. **[*HOTS*]**

Sol. Let $y = \sec^{-1} x$

Then, $\sec y = \sec(\sec^{-1} x) = x$

Differentiating both sides with respect to x, we have

$$\Rightarrow \quad \frac{d}{dx}\sec y = \frac{d}{dx}(x) \quad \Rightarrow \quad \frac{d}{dy}(\sec y)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \sec y \tan y \frac{dy}{dx} = 1 \qquad \text{[Using chain rule]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{|\sec y| . |\tan y|}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{|\sec y|\sqrt{\tan^2 y}} = \frac{1}{|\sec y|\sqrt{\sec^2 y - 1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

[If $x > 1$, then $y \in \left(0, \frac{\pi}{2}\right)$
$\therefore \quad \sec y > 0, \tan y > 0$
$\Rightarrow \quad |\sec y| . |\tan y| = \sec y \tan y$
If $x < -1$, then
$y \in \left(\frac{\pi}{2}, \pi\right) \therefore \quad \sec y < 0, \tan y < 0$
$\Rightarrow \quad |\sec y|\ |\tan y|$
$\Rightarrow \quad (-\sec y)(-\tan y) = \sec y \tan y$]

SECOND ORDER DERIVATIVES

1. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.
[*CBSE (F) 2014; Delhi 2015*]

Sol. Given, $x = a\cos\theta + b\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \qquad \ldots(i)$

Also, $y = a\sin\theta - b\cos\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta \qquad \ldots(ii)$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\cos\theta+b\sin\theta}{-a\sin\theta+b\cos\theta} \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{a\cos\theta+b\sin\theta}{b\cos\theta-a\sin\theta} \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{x}{y} \qquad ...(iii)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2}=-\frac{y-x.\frac{dy}{dx}}{y^2}$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2}=-y+x\frac{dy}{dx} \quad \Rightarrow \quad y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$$

2. If $y=Pe^{ax}+Qe^{bx}$, then show that $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0$ **[*CBSE (AI) 2014*]**

Sol. Given, $y=Pe^{ax}+Qe^{bx}$

On differentiating with respect to x, we have

$$\frac{dy}{dx}=Pae^{ax}+Qbe^{bx}$$

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=Pa^2e^{ax}+Qb^2e^{bx}$$

Now, LHS $=\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby$

$=Pa^2e^{ax}+Qb^2e^{bx}-(a+b)(Pae^{ax}+Qbe^{bx})+ab\,(Pe^{ax}+Qe^{bx})$

$=Pa^2e^{ax}+Qb^2e^{bx}-Pa^2e^{ax}-Pabe^{ax}-Qabe^{bx}-Qb^2e^{bx}+Pabe^{ax}+Qabe^{bx}$

$=0=$ RHS

3. If $y=e^{a\cos^{-1}x}, -1<x<1$, then show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$. **[*CBSE 2020 (65/2/1)*]**

Sol. Given, $y=e^{a\cos^{-1}x}, -1<x<1$

Differentiating w.r.t x, we have

$$\therefore \quad \frac{dy}{dx}=e^{a\cos^{-1}x}\times a\times -\frac{1}{\sqrt{1-x^2}}=-\frac{ay}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx}=-ay$$

Squaring both sides, we have

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

Again differentiating w.r.t. x, we have

$$(1-x^2)\times 2\frac{dy}{dx}\times\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\times(-2x)=a^2\times 2y\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Proved

4. If $y = \sin(\log x)$, then prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. **[*CBSE (F) 2013*]**

Sol. Given, $y = \sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

Again, $$\frac{d^2y}{dx^2} = \frac{x\left[-\sin(\log x) \times \frac{1}{x}\right] - \cos(\log x)}{x^2} = \frac{-\cos(\log x) - \sin(\log x)}{x^2}$$

Now, LHS $= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y$

$$= \frac{x^2\{-\cos(\log x) - \sin(\log x)\}}{x^2} + \frac{x\cos(\log x)}{x} + \sin(\log x)$$

$= -\cos(\log x) - \sin(\log x) + \cos(\log x) + \sin(\log x) = 0 = \text{RHS}$

5. If $y = \text{cosec}^{-1} x, x > 1$, then show that $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$ **[*CBSE (AI) 2010*]**

Sol. $\because \quad y = \text{cosec}^{-1} x$

Differentiating with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{x\sqrt{x^2-1}.0 + 1.\left\{x.\frac{2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}\right\}}{x^2(x^2-1)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x^2 + x^2 - 1}{x^2(x^2-1).\sqrt{x^2-1}} = \frac{2x^2-1}{\sqrt{x^2-1}.x^2(x^2-1)}$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} = \frac{2x^2-1}{x\sqrt{x^2-1}} = (2x^2-1)\left(-\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$$

6. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$ **[*CBSE Delhi 2009, 2012*]**

Sol. Given, $y = 3\cos(\log x) + 4\sin(\log x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} \quad \Rightarrow \quad y_1 = \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)]$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{x\left[\frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}\right] - [-3\sin(\log x) + 4\cos(\log x)]}{x^2}$$

$$= \frac{-3\cos(\log x) - 4\sin(\log x) + 3\sin(\log x) - 4\cos(\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2} \quad \Rightarrow \quad y_2 = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Now, LHS $= x^2y_2 + xy_1 + y$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x \times \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)] + 3\cos(\log x) + 4\sin(\log x)$$

$= -\sin(\log x) - 7\cos(\log x) - 3\sin(\log x) + 4\cos(\log x) + 3\cos(\log x) + 4\sin(\log x)$

$= 0 =$ RHS

7. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

[*CBSE (AI) 2012*]

Sol. Given, $x = a(\cos t + t\sin t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) \quad \Rightarrow \quad \frac{dx}{dt} = at\cos t \quad \text{...(i)}$$

Differentiating again with respect to t, we get

$$\frac{d^2x}{dt^2} = a(-t\sin t + \cos t) = a(\cos t - t\sin t)$$

Again, $y = a(\sin t - t\cos t)$

Differentiating with respect to t, we get

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) \quad \Rightarrow \quad \frac{dy}{dt} = at\sin t \quad \text{...(ii)}$$

Differentiating again with respect to t we get

$$\frac{d^2y}{dt^2} = a(t\cos t + \sin t)$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ [From (*i*) and (*ii*)]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{at\sin t}{at\cos t} \quad \Rightarrow \quad \frac{dy}{dx} = \tan t$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1}{dx/dt} = \frac{\sec^2 t}{at\cos t} = \frac{\sec^3 t}{at} \quad \text{[From (i)]}$$

Hence, $\frac{d^2x}{dt^2} = a(\cos t - t\sin t)$, $\frac{d^2y}{dt^2} = a(t\cos t + \sin t)$ and $\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$.

8. **If $y = \log\left[x+\sqrt{x^2+a^2}\right]$, show that $(x^2+a^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=0$.** **[CBSE Delhi 2013]**

Sol. Given $\quad y = \log\left[x+\sqrt{x^2+a^2}\right]$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}.\left[1+\frac{2x}{2\sqrt{x^2+a^2}}\right] = \frac{x+\sqrt{x^2+a^2}}{(x+\sqrt{x^2+a^2})(\sqrt{x^2+a^2})}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}} \quad \ldots(i)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+a^2)^{-\frac{3}{2}}.2x = \frac{-x}{(x^2+a^2)^{\frac{3}{2}}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-x}{(x^2+a^2).\sqrt{x^2+a^2}} \quad \Rightarrow \quad (x^2+a^2)\frac{d^2y}{dx^2} = -\frac{x}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad (x^2+a^2)\frac{d^2y}{dx^2}+x.\frac{dy}{dx}=0 \quad \text{[From (i)]}$$

9. **If $y = x^x$, then prove that $\dfrac{d^2y}{dx^2}-\dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2-\dfrac{y}{x}=0$.** **[CBSE Delhi 2014, 2016]**

Sol. Given, $\; y = x^x$

Taking logarithm on both sides, we get

$$\log y = x\,.\log x$$

Differentiating both sides, we get

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x}+\log x \quad \Rightarrow \quad \frac{dy}{dx} = y(1+\log x) \quad \ldots(i)$$

Again differentiating both sides, we get

$$\frac{d^2y}{dx^2} = y.\frac{1}{x}+(1+\log x).\frac{dy}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x}+\frac{1}{y}.\frac{dy}{dx}.\frac{dy}{dx} \quad \text{[From (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2 \quad \Rightarrow \quad \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$

10. **If $y = \left(x+\sqrt{1+x^2}\right)^n$, then show that $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=n^2y$.** **[CBSE (F) 2015]**

Sol. Given $\quad y = \left(x+\sqrt{1+x^2}\right)^n$

Differentiating with respect to x, we get

$$\Rightarrow \quad \frac{dy}{dx} = n\left(x+\sqrt{1+x^2}\right)^{n-1}.\left\{1+\frac{2x}{2\sqrt{1+x^2}}\right\} \quad \Rightarrow \quad \frac{dy}{dx} = n\left(x+\sqrt{1+x^2}\right)^{n-1}.\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{n\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{ny}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sqrt{1+x^2}.\frac{dy}{dx} = ny$$

Again differentiating with respect to x, we get

$$\sqrt{1+x^2}.\frac{d^2y}{dx^2}+\frac{2x}{2\sqrt{1+x^2}}.\frac{dy}{dx} = n\frac{dy}{dx} \quad \Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2}+x.\frac{dy}{dx} = n.\sqrt{1+x^2}.\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n.\sqrt{1+x^2}.\frac{ny}{\sqrt{1+x^2}} \quad \Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

11. If $y=\log\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$, then prove that $x(x+1)^2y_2+(x+1)^2y_1=2$. **[*CBSE Sample Paper 2018*]**

Sol. $\because \quad y=\log\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 \quad \Rightarrow \quad y=2\log\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)=2\log\left(\frac{x+1}{\sqrt{x}}\right)$

$$y=2\log(x+1))-2\log\sqrt{x} \quad \Rightarrow \quad y=2\log(x+1)-\log x$$

$$\Rightarrow \quad y_1=\frac{2}{x+1}-\frac{1}{x}=\frac{2x-x-1}{x(x+1)} \quad \Rightarrow \quad y_1=\frac{x-1}{x(x+1)}$$

$$\Rightarrow \quad y_2=\frac{x(x+1)-(x-1)(2x+1)}{x^2(x+1)^2} \quad \Rightarrow \quad y_2=\frac{x^2+x-2x^2-x+2x+1}{x^2(x+1)^2}$$

$$\Rightarrow \quad y_2=\frac{-x^2+2x+1}{x^2(x+1)^2}$$

Now, $\quad x(x+1)^2y_2+(x+1)^2y_1=x(x+1)^2.\frac{-x^2+2x+1}{x^2(x+1)^2}+(x+1)^2.\frac{(x-1)}{x(x+1)}$

$$=\frac{-x^2+2x+1}{x}+\frac{(x+1)(x-1)}{x}$$

$$=\frac{-x^2+2x+1+x^2-1}{x}=\frac{2x}{x}=2$$

Hence proved.

12. If $y=\sin(\sin x)$, prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2x=0$. **[*CBSE 2018*]**

Sol. $y=\sin(\sin x)$

$$\Rightarrow \quad \frac{dy}{dx}=\cos(\sin x)\frac{d}{dx}(\sin x) \quad \Rightarrow \quad \frac{dy}{dx}=\underset{\text{I}}{\cos(\sin x)}\;\underset{\text{II}}{\cos x}$$

Again differentiating w.r.t x on both sides, we get

$$\frac{d^2y}{dx^2}=\cos(\sin x)\frac{d}{dx}(\cos x)+(\cos x)\frac{d}{dx}\cos(\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\cos(\sin x)(-\sin x)-(\cos x)\{(\sin(\sin x)\}(\cos x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\sin x\cos(\sin x)-\cos^2x\sin(\sin x)$$

Putting these values in LHS, we get

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2x$$

$$=\{-\sin x\cos(\sin x)-\cos^2x\sin(\sin x)\}+\tan x\{\cos x\cos(\sin x)\}+y\cos^2x$$

$$=-\sin x\cos(\sin x)-\cos^2x\sin(\sin x)+\tan x\cos x\cos(\sin x)+y\cos^2x$$

$$=-\sin x\cos(\sin x)-\cos^2x\sin(\sin x)+\frac{\sin x}{\cos x}\cos x\cos(\sin x)+\sin(\sin x)\cos^2x=0$$

Hence proved.

13. If $x = \sin t$ and $y = \sin pt$, then prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$. *[NCERT Exemplar]*

Sol. We have, $x = \sin t$ and $y = \sin pt$

$\therefore \quad \frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = \cos pt.p$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p.\cos pt}{\cos t} \quad ...(i)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos t.\frac{d}{dt}(p.\cos pt)\frac{dt}{dx} - p\cos pt.\frac{d}{dt}\cos t.\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t.p.(-\sin pt).p - p\cos pt.(-\sin t)]\frac{dt}{dx}}{\cos^2 t} = \frac{[-p^2\sin pt.\cos t + p\sin t.\cos pt]\frac{1}{\cos t}}{\cos^2 t}$$

$$\frac{d^2y}{dx^2} = \frac{-p^2\sin pt.\cos t + p\cos pt.\sin t}{\cos^3 t} \quad ...(ii)$$

Since, we have to prove

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

$$\therefore \quad \text{LHS} = (1-\sin^2 t)\frac{[-p^2\sin pt.\cos t + p\cos pt.\sin t]}{\cos^3 t} - \sin t.\frac{p\cos pt}{\cos t} + p^2\sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}(1-\sin^2 t)(-p^2\sin pt.\cos t + p\cos pt.\sin t)\\ -p\cos pt.\sin t.\cos^2 t + p^2\sin pt.\cos^3 t\end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}-p^2\sin pt.\cos^3 t + p\cos pt.\sin t.\cos^2 t\\ -p\cos pt.\sin t.\cos^2 t + p^2\sin pt.\cos^3 t\end{bmatrix}[\because 1-\sin^2 t = \cos^2 t]$$

$$= \frac{1}{\cos^3 t}.0 = 0$$

Hence proved.

Rolle's and Mean Value Theorem

1. Verify Rolle's theorem for the function $f(x) = e^x \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *[CBSE 2020 (65/4/1)]*

Sol. Given function, $f(x) = e^x \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad f'(x) = e^x(\cos x - \sin x)$, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Clearly, $f(x)$ is differentiable in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, $f(x)$ is differentiable in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad f(x)$ must be continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Because every differentiable function must be continuous.

Now, $\quad f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}}\cos\left(-\frac{\pi}{2}\right) = 0$

and, $\quad f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}\cos\left(\frac{\pi}{2}\right) = 0$

$$\therefore \quad f\left(-\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}\right)$$

Therefore, Rolle's theorem is applicable.

So, there exists a real number c such that

$$f'(c) = 0 \quad \Rightarrow \quad e^c (\cos c - \sin c) = 0$$

$$\Rightarrow \quad \cos c - \sin c = 0 \qquad (\because e^c \neq 0)$$

$$\Rightarrow \quad \cos c = \sin c$$

$$\Rightarrow \quad \tan c = 1 = \tan\frac{\pi}{4} \quad \Rightarrow \quad c = \frac{\pi}{4}$$

$$\therefore \quad c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

2. Verify Mean Value theorem for the function $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$. [*CBSE (North) 2016*]

Sol. We have, $f(x) = 2\sin x + \sin 2x$

$f(x)$ is continuous in $[0, \pi]$ being trigonometric function and it is differentiable on $(0, \pi)$.

Hence, condition of Mean Value Theorem is satisfied.

Therefore, Mean Value Theorem is applicable.

So, there exist a real number c such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} \qquad \ldots(i)$$

Now $f(0) = 2\sin 0 + \sin 0 = 0$; $f(\pi) = 2\sin\pi + \sin 2\pi = 0$ and $f'(x) = 2\cos x + 2\cos 2x$

$\therefore \quad f'(c) = 2\cos c + 2\cos 2c$

From (i)

$$2\cos c + 2\cos 2c = \frac{0-0}{\pi}$$

$\Rightarrow \quad 2\cos c + 2\cos 2c = 0 \quad \Rightarrow \quad 2\cos c + 2(2\cos^2 c - 1) = 0$

$\Rightarrow \quad \cos c + 2\cos^2 c - 1 = 0 \quad \Rightarrow \quad 2\cos^2 c + \cos c - 1 = 0$

$\Rightarrow \quad 2\cos^2 c + 2\cos c - \cos c - 1 = 0 \quad \Rightarrow \quad 2\cos c(\cos c + 1) - 1(\cos c + 1) = 0$

$\Rightarrow \quad (\cos c + 1)(2\cos c - 1) = 0 \quad \Rightarrow \quad \cos c = -1 \text{ and } \cos c = \frac{1}{2}$

$\Rightarrow \quad c = \pi \text{ and } c = \frac{\pi}{3} \in (0, \pi) \quad \Rightarrow \quad c = \frac{\pi}{3} \in (0, \pi)$

Hence, Mean Value Theorem is verified.

3. Discuss the applicability of Rolle's theorem on the function given by [*NCERT Exemplar*]

$$f(x) = \begin{cases} x^2 + 1, \text{ if } 0 \le x \le 1 \\ 3 - x, \text{ if } 1 < x \le 2 \end{cases}$$

Sol. We have, $$f(x) = \begin{cases} x^2 + 1, \text{ if } 0 \le x \le 1 \\ 3 - x, \text{ if } 1 \le x \le 2 \end{cases}$$

We know that, polynomial function is everywhere continuous and differentiable.

So, $f(x)$ is continuous and differentiable at all points except possibly at $x = 1$.

Now, check the differentiability at $x = 1$,

At $x = 1$,

$$\text{LHD} = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x\to 1} \frac{(x^2+1) - (1+1)}{x-1} \qquad [\because f(x) = x^2 + 1, 0 \le \forall\, x \le 1]$$

$$= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = 2$$

and $$\text{RHD} = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{(3-x) - (1+1)}{(x-1)}$$

$$= \lim_{x\to 1} \frac{3 - x - 2}{x - 1} = \lim_{x\to 1} \frac{-(x-1)}{x-1} = -1$$

$\therefore$ LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 1$.

Hence, Rolle's theorem is not applicable on the interval [0, 2].

4. Find a point on the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1). **[*NCERT Exemplar*]**

Sol. We have, $y = (x - 3)^2$, which is continuous in $x_1 = 3$ and $x_2 = 4$ *i.e.*, [3, 4].

Also, $y' = 2(x - 3) \times 1 = 2(x - 3)$ which exists in (3, 4).

Hence, by mean value theorem there exists a point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

Thus, $$f'(c) = \frac{f(4) - f(3)}{4 - 3}$$

$$\Rightarrow \quad 2(c - 3) = \frac{(4-3)^2 - (3-3)^2}{4-3}$$

$$\Rightarrow \quad 2c - 6 = \frac{1 - 0}{1} \Rightarrow c = \frac{7}{2}$$

For $x = \frac{7}{2}$, $$y = \left(\frac{7}{2} - 3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

So, $\left(\frac{7}{2}, \frac{1}{4}\right)$ is the point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

PROFICIENCY EXERCISE

■ **Objective Type Questions:** **[1 mark each]**

1. Choose and write the correct option in each of the following questions.

(*i*) If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function?

(*a*) $f(x) + g(x)$ (*b*) $f(x) - g(x)$ (*c*) $f(x) \cdot g(x)$ (*d*) $\frac{g(x)}{f(x)}$

(*ii*) The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is [*NCERT Exemplar*]

(*a*) R (*b*) $R - \left\{\frac{1}{2}\right\}$ (*c*) $(0, \infty)$ (*d*) none of these

(iii) Let $f(x) = |\cos x|$. Then,

(a) f is everywhere differentiable.

(b) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

(c) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.

(d) none of these.

(iv) If $y = A e^{5x} + B e^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to [CBSE 2020 (65/5/1)]

(a) $25\,y$ (b) $5\,y$ (c) $-25\,y$ (d) $15\,y$

(v) For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is

(a) $\frac{1}{2}$ (b) 1 (c) -1 (d) 2

(vi) If $f'(1) = 2$ and $y = f(\log_e x)$, then $\frac{dy}{dx}$ at $x = e$ is

(a) 0 (b) 1 (c) e (d) $\frac{2}{e}$

2. Fill in the blanks.

(i) If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right) =$ ____________ .

(ii) If $f(x) = (x + 1)$, then $\frac{d}{dx} fof(x) =$ ____________ .

(iii) $\frac{d}{dx}\sec(\tan^{-1} x) =$ ____________ .

(iv) The number of points at which the function $f(x) = \frac{1}{\log|x|}$ is discontinuous is ________ .

■ Very Short Answer Questions: [1 mark each]

3. If $f(x) = \begin{cases} \frac{\sin^{-1} x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$, is continuous at $x = 0$, then write the value of k.

4. Determine the value of 'k' for which the following function is continuous at $x = 3$:

$$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ [CBSE (AI) 2017]

5. If $y = \left[\sin\frac{x}{2} + \cos\frac{x}{2}\right]^2$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$.

6. Find $\frac{dy}{dx}$, if $y = \tan^{-1}\left[\frac{1 + \tan x}{1 - \tan x}\right]$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

7. If $y = \log(e^x)$, then find $\frac{dy}{dx}$. [CBSE 2019 (65/4/3)]

■ Short Answer Questions–I: [2 marks each]

8. Examine the continuity at the indicated points.

$f(x) = |x| + |x - 1|$ at $x = 1$

9. Find k if $f(x)$ is continuous at $x = 0$.

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

10. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-3, 0]$. [CBSE (AI) 2017]

11. If $f(x) = |\cos x - \sin x|$, find $f'(\pi/6)$.

12. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

■ Short Answer Questions–II: [3 marks each]

13. Show that the function 'f' defined by $f(x) = \begin{cases} 3x-2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases}$ is continuous at $x = 2$, but not differentiable. [CBSE Delhi 2010]

14. Show that the function $f(x) = |x-1| + |x+1|$, for all $x \in R$, but is not differentiable at the points $x = -1$ and $x = 1$. [CBSE Allahabad 2015]

15. If $y = x^3 (\cos x)^x + \sin^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. [CBSE 2020 (65/3/1)]

16. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1} x^2$. [CBSE (South) 2016, 2019 (65/4/1)]

17. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x. [CBSE (AI) 2012]

18. Differentiate $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. [CBSE Delhi 2014]

19. If $y = \cos^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, then find $\frac{dy}{dx}$. [CBSE (F) 2010]

20. If $y = e^x (\sin x + \cos x)$, then show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ [CBSE (AI) 2009]

21. Verify Lagrange's Mean Value Theorem for the following function:

$f(x) = x^2 + 2x + 3$, for $[4, 6]$. [CBSE (AI) 2008]

22. If $y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right)$, then find $\frac{dy}{dx}$. [CBSE Delhi 2008]

23. Discuss the differentiability of the function $f(x) = \begin{cases} 2x - 1, & x < \frac{1}{2} \\ 3 - 6x, & x \ge \frac{1}{2} \end{cases}$ at $x = \frac{1}{2}$.

[CBSE Sample Paper 2017]

24. For what value of k is the following function continuous at $x = -\frac{\pi}{6}$? [CBSE Sample Paper 2017]

$$f(x) = \begin{cases} \dfrac{\sqrt{3}\sin x + \cos x}{x + \frac{\pi}{6}}, & x \ne -\frac{\pi}{6} \\ k, & x = -\frac{\pi}{6} \end{cases}$$

25. If $x\sqrt{1+y} + y\sqrt{1+x} = 0, -1 < x < 1, x \ne y$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$. [CBSE (F) 2012]

26. Differentiate the following function with respect to x : $(x)^{\cos x} + (\sin x)^{\tan x}$ [CBSE Delhi 2009]

27. If $y = \sin^{-1}(6x\sqrt{1-9x^2}), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$ then find $\frac{dy}{dx}$. [CBSE Delhi 2017]

28. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1,$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$ *[CBSE (F) 2012]*

29. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$. *[CBSE Delhi 2017]*

30. If $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, show that $\frac{dy}{dx} = \sec x$. Also find the value of $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{4}$. *[CBSE (F) 2010]*

31. If $x = \tan\left(\frac{1}{a}\log y\right)$, show that: *[CBSE (AI) 2011]*

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

32. If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$. *[CBSE (F) 2017]*

33. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2 \le 1$, then find $\frac{dy}{dx}$. *[CBSE Delhi 2015]*

34. If $(ax+b)e^{y/x} = x$, then show that $x^3\left(\frac{d^2y}{dx^2}\right) = \left(x\frac{dy}{dx} - y\right)^2$. *[CBSE Ajmer 2015]*

35. If $f(x) = \sqrt{x^2+1}; g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x-3$, then find $f'[h'\{g'(x)\}]$. *[CBSE Allahabad 2015]*

36. Let $f(x) = x - |x - x^2|, x \in [-1, 1]$. Find the point of discontinuity, (if any), of this function on $[-1, 1]$. *[CBSE Bhubaneshwar 2015]*

37. If $\frac{x}{x-y} = \log\frac{a}{x-y}$, then prove that $\frac{dy}{dx} = 2 - \frac{x}{y}$. *[CBSE Guwahati 2015]*

38. Let $y = (\log x)^x + x^{x\cos x}$, then find $\frac{dy}{dx}$. *[CBSE Sample Paper 2016]*

39. If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. *[CBSE (AI) 2017]*

40. If $x = a(\theta - \sin\theta), y = a(1+\cos\theta)$, then find $\frac{d^2y}{dx^2}$. *[CBSE Delhi 2011]*

41. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. *[CBSE (Central) 2016]*

42. Show that the function f given by *[CBSE (East) 2016]*

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

is discontinuous at $x = 0$.

43. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$. *[CBSE (North) 2016]*

44. Differentiate $(\sin 2x)^x + \sin^{-1}\sqrt{3x}$ with respect to x. *[CBSE (South) 2016]*

45. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, if $x \in (-1, 1)$. [CBSE (F) 2016]

46. If $x = a(\cos 2t + 2t \sin 2t)$ and $y = a(\sin 2t - 2t \cos 2t)$, then find $\frac{d^2y}{dx^2}$. [CBSE (Ajmer) 2015]

47. Find the values of a and b, if the function f is defined by [CBSE (F) 2016]

$$f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$$

is differentiable at $x = 1$.

Answers

1. (i) (d) (ii) (b) (iii) (c) (iv) (a) (v) (c) (vi) (d)

2. (i) $-\frac{1}{\sqrt{2}}$ (ii) 1 (iii) $\frac{x}{\sqrt{1+x^2}}$ (iv) three

3. $k = 1$ 4. $k = 12$ 5. $\frac{\sqrt{3}}{2}$ 6. 1 7. 1 8. Discontinuous

9. $k = 2$ 10. $c = -1$ 11. $-\frac{1}{2}(1+\sqrt{3})$ 15. $x^3(\cos x)^x\left[\frac{3}{x} - x\tan x + \log(\cos x)\right] + \frac{1}{2\sqrt{x - x^2}}$

16. $-\frac{1}{2}$ 17. $\frac{1}{2(1+x^2)}$ 18. $\frac{1}{2}$ 19. $\frac{-2^{x+1}.\log_e 2}{1+4^x}$ 22. $\frac{\sqrt{x^2+1}}{x}$

23. Not differentiable 24. $k = 2$

26. $x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) + (\sin x)^{\tan x}[1 + \sec^2 x \log \sin x]$ 27. $\frac{6}{\sqrt{1-9x^2}}$ 30. $\sqrt{2}$

33. $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^4}}$; **[Hint:** At first simplify as $y = \tan^{-1}\left(\frac{1+\sqrt{1-x^2}}{x^2}\right)$ by multiplying with $\sqrt{1+x^2} + \sqrt{1-x^2}$. Then let $x^2 = \sin\theta$ and then solve.]

34. **Hint:** $e^{y/x} = \frac{x}{ax+b} \Rightarrow \frac{y}{x} = \log\left(\frac{x}{ax+b}\right) \Rightarrow y = \log\left(\frac{x}{ax+b}\right)^x$

35. $\frac{2\sqrt{5}}{5}$; **Hint:** At first find $f'(x)$, $g'(x)$ and $h'(x)$ and then find $f'[h'\{g'(x)\}] = f'\left[h'\left\{\frac{-x^2-2x+1}{x^2+1}\right\}\right]$

36. No point of discontinuity.

Hint: $f(x) = \begin{cases} 2x - x^2, & -1 \le x < 0 \\ 0, & x = 0 \\ x^2, & 0 < x \le 1 \end{cases}$

37. **Hint:** $\frac{x}{x-y} = \log a - \log(x-y)$ then differentiate.

38. $(\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\} + x^{x\cos x}\{\cos x + \cos x(\log x) - x\sin x \log x\}$

40. $-\cot\frac{\theta}{2}$ 43. $\frac{2}{\sqrt{1-4x^2}}$ 44. $(\sin 2x)^x\{2x\cot 2x + \log(\sin 2x)\} + \frac{3}{2\sqrt{3x-9x^2}}$

45. $\frac{1}{4}$ 46. $\frac{\sec^3 2t}{2at}$ 47. $a = 3, b = 5$

SELF-ASSESSMENT TEST

Time allowed: 1 hour **Max. marks: 30**

1. Choose and write the correct option in the following questions. **(4 × 1 = 4)**

(*i*) If the function $f(x)$ defined by $f(x) = \begin{cases} \dfrac{\log(1+ax) - \log(1-bx)}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, then value of k is

(*a*) a (*b*) b (*c*) $a + b$ (*d*) 0

(*ii*) If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, then $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$ is

(*a*) $\dfrac{2}{\sqrt{3}}$ (*b*) $\dfrac{1}{\sqrt{3}}$ (*c*) $\sqrt{3}$ (*d*) $\dfrac{\sqrt{3}}{2}$

(*iii*) Derivatives of x^2 w.r.t. x^3 is

(*a*) $\dfrac{5}{3x}$ (*b*) $\dfrac{7x}{3}$ (*c*) $\dfrac{2}{3x}$ (*d*) 0

(*iv*) If $y = (\sin x + \cos x)^2$, then $\left.\dfrac{dy}{dx}\right|_{\text{at } x = \frac{\pi}{4}}$ equals to

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) 2

2. Fill in the blanks. **(2 × 1 = 2)**

(*i*) Derivative of $\sin x$ w.r.t $\cos x$ is ______________ .

(*ii*) If $f(x) = \begin{cases} ax+1, & \text{if } x \geq 1 \\ x+2, & \text{if } x < 1 \end{cases}$ is continuous, then a should be equal to ______________ .

■ **Solve the following questions.** **(2 × 1 = 2)**

3. Find the derivative of $\dfrac{e^x}{\sin x}$.

4. Find the value of the constant 'k' so that the function $f(x) = \begin{cases} \dfrac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$.

■ **Solve the following questions.** **(4 × 2 = 8)**

5. Write the derivative of $f(x) = |x^3|$ at $x = 0$.

6. Differentiate $\sin(\cos(xy))$ with respect to x.

7. Find k if $f(x)$ is continuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

8. If $f(x) = |\cos x - \sin x|$, find $f'(\pi/6)$.

■ **Solve the following questions.** **(3 × 3 = 9)**

9. For what value of k, the following function is continuous at $x = 0$?

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

10. Examine the continuity of the following function:

$$f(x) = \begin{cases} \dfrac{x}{2\,|\,x\,|}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases} \quad \text{at } x = 0$$

11. If the function f, as defined below is continuous at $x = 0$, find the values of a, b and c.

$$f(x) = \begin{cases} \dfrac{\sin(a+1)\,x + \sin x}{x}, & x < 0 \\ c, & x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}, & x > 0 \end{cases}$$

■ **Solve the following question.** **(1 × 5 = 5)**

12. Differentiate the following function with respect to x: $(\log x)^x + x^{\log x}$.

Answers

1. (*i*) (*c*) (*ii*) (*d*) (*iii*) (*c*) (*iv*) (*a*)

2. (*i*) $-\cot x$ (*ii*) $a = 2$

3. $\dfrac{e^x(\sin x - \cos x)}{\sin^2 x}$ 4. -3 5. 0 6. $\dfrac{-y\sin(xy)\cos(\cos xy)}{1 + x\sin(xy)\cos(\cos xy)}$

7. $k = 2$ 8. $-\dfrac{1}{2}(1 + \sqrt{3})$

9. 1 10. f is discontinuous 11. $a = -\dfrac{3}{2}$, $c = \dfrac{1}{2}$ and b is any real number

12. $(\log x)^x \left\{\dfrac{1}{\log x} + \log(\log x)\right\} + \dfrac{2\log x . x^{\log x}}{x}$