4 Surds, Indices and Logarithm

Measurement Scale: Richter, Decibel, etc.

- We're at the typical "logarithms in the real world" example: Richter scale and Decibel. The idea is to put events which can vary drastically (earthquakes) on a single 1–10 scale. Just like Page Rank, each 1-point increase is a 10x improvement in power.
- Decibels are similar, though it can be negative. Sounds can go from intensely quiet (pin drop) to extremely loud (airplane) and our brains can process it all. In reality, the sound of an airplane's engine is millions (billions, trillions) of times more powerful than a pin-drop, and it's inconvenient to have a scale that goes from 1 to a gazillion. Logs keep everything on a reasonable scale.

Powers: An expression that represents repeated multiplication of the same factor is called its power.

exponent
or index
or power
↓
base ⟶

Exponents are even used to describe the extent to which rain water is acidic and even determine whether it would be safe to go swimming in swimming pool (acidic water is not good for skin!). You would have probably heard of the term pH scale and Richter scale. A pH scale is a scale which is used to determine the acidity of basicity of water. The scale uses exponents to mark various acidity levels. Similarly, a Richter scale is used to measure the magnitude of earthquakes. One earthquake is different from another in terms of magnitude. This scale too uses exponents.

Exponential growth can also be seen in population. If one person has 4 children and then each of these children have 4 children, and so on, we get exponential increase in the population.

Generation	0	1	2	3	4
Children	1	4	16	64	216

We can correlate the above data in exponential form as follows:

Generation	$(2^0)^2$	$(2^1)^2$	$(2^2)^2$	$(2^3)^2$	$(2^4)^2$
Children	1	4	16	64	216

(Anything to the power 'zero' is actually 1, we will learn more about it in this concept)

Multiplication of Fractions:

The following rule for multiplying two fractions:

Product of two fractions

$$= \frac{\text{Product of numerators of the fractions}}{\text{Product of denominations of the fractions}}$$

Note

- The rules of multiplication given above are applicable to both, proper and improper fractions.
- To multiply mixed numbers, we first convent them into improper fractions and then multiply.

Division of Fractions: In our daily life we come across several situations/problems, wherein we need to perform division infractions. Some such situations are given in the following examples.

- A paper strip of length $7\frac{1}{2}$ cm is cut into two equal pieces. What is the length of each piece?
- A ribbon of length 44 cm is cut into equal pieces, each of length $5\frac{1}{2}$ cm. How many pieces are obtained?
- A $38\frac{1}{2}$ m long rope is cut into pieces of $5\frac{1}{2}$ m length each.

Indices and Surds: An **index** (plural: indices) is the power, or exponent, of a number. For example, has an index of 3 and a **surd** is an irrational number that can be expressed with roots. If a is not rational, $\sqrt[n]{a}$ is not a surd.

Laws of Indices

- $a^0 = 1,\ (a \neq 0)$

  $a^{-m} = \frac{1}{a^m},\ (a \neq 0)$
- $a^{m+n} = a^m . a^n$, where m and n are rational numbers.
- $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$
- $(a^m)^n = a^{mn}$
- $a^{p/q} = \sqrt[q]{a^p}$
- If $x = y$, then $a^x = a^y$, but the converse may not be true.

For example:

$(1)^6 = (1)^8$, but $6 \neq 8$

- If $a \neq \pm 1$, or 0, then $x = y$
- If $a = 1$, then x, y may be any real number.
- If $a = -1$, then x, y may be both even or both odd.
- If $a = 0$, then x, y may be any non-zero real number.
- $a^m . b^m = (ab)^m$ is not always true.

  In real domain, $\sqrt{a}\sqrt{b} = \sqrt{(ab)}$, only when $a \geq 0, b \geq 0$
- In complex domain, $\sqrt{a}.\sqrt{b} = \sqrt{(ab)}$, if at least one of a and b is positive.
- If $a^x = b^x$ then consider the following cases:
- If $a \neq \pm b$, then $x = 0$
- If $a = b \neq 0$, then x may have any real value.
- If $a = -b$, then x is even.

Note

If we have to solve the equation of the form $[f(x)]^{\phi(x)} = [g(x)]^{\phi(x)}$ *i.e.,* same index, different bases, then we have to solve (a) $f(x) = g(x)$, (b) $f(x) = -g(x)$, (c) $\phi(x) = 0$. Verification should be done in (b) and (c) cases.

Example 1. For $x \neq 0, \left(\frac{x^l}{x^m}\right)^{(l^2+lm+m^2)} \left(\frac{x^m}{x^n}\right)^{(m^2+nm+n^2)} \left(\frac{x^n}{x^l}\right)^{(n^2+nl+l^2)} = ?$

Solution: $\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2} \left(\frac{x^m}{x^n}\right)^{m^2+nm+n^2} \left(\frac{x^n}{x_l}\right)^{n^2+nl+l^2}$

$$= (x^{l-m})^{(l^2+lm+m^2)} (x^{m-n})^{m^2+nm+n^2} (x^{n-l})^{n^2+nl+l^2}$$

$$= x^{l^3-m^3} . x^{m^3-n^3} . x^{n^3-l^3} = x^{l^3-m^3+m^3-n^3+n^3-l^3} = x^0 = 1$$

Example 2. Simplify: $\left(\frac{81}{16}\right)^{-3/4} \times \left[\left(\frac{25}{9}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$

Solution: $\left(\frac{81}{16}\right)^{-3/4} \times \left[\left(\frac{25}{9}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$

$$= \left(\frac{3^4}{2^4}\right)^{-3/4} \times \left[\left(\frac{5^2}{3^2}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left[\left(\frac{3}{2}\right)^4\right]^{-3/4} \times \left\{\left[\left(\frac{5}{3}\right)^2\right]^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right\}$$

$$= \left(\frac{3}{2}\right)^{4\times\left(-\frac{3}{4}\right)} \times \left\{\left(\frac{5}{3}\right)^{2\times\left(-\frac{3}{2}\right)} \div \left(\frac{5}{2}\right)^{-3}\right\}$$

$$= \left(\frac{3}{2}\right)^{-3} \times \left\{\left(\frac{5}{3}\right)^{-3} \div \left(\frac{5}{2}\right)^{-3}\right\}$$

$$= \left(\frac{3}{2}\right)^{3} \times \left\{\left(\frac{3}{5}\right)^{3} \div \left(\frac{2}{5}\right)^{3}\right\} = \frac{2^3}{3^3} \times \left\{\frac{3^3}{5^3} \div \frac{2^3}{5^3}\right\}$$

$$= \frac{2^3}{3^3} \times \left\{\frac{3^3}{5^3} \times \frac{5^3}{2^3}\right\} = 1$$

Example 3. Show that $\left(\frac{x^a}{x^b}\right)^{a+b} . \left(\frac{x^b}{x^c}\right)^{b+c} . \left(\frac{x^c}{x^a}\right)^{c+a} = 1.$

Solution: $\left(\frac{x^a}{x^b}\right)^{a+b} . \left(\frac{x^b}{x^c}\right)^{b+c} . \left(\frac{x^c}{x^a}\right)^{c+a}$

$$= (x^{a-b})^{a+b} . (x^{b-c})^{b+c} . (x^{c-a})^{c+a}$$

$$= x^{a^2-b^2} . x^{b^2-c^2} . x^{c^2-a^2} = x^{a^2-b^2+b^2-c^2+c^2-a^2} = x^0$$

$$= 1 = \text{R.H.S.}$$

Example 4. If $2^x = 3^y = 6^{-z}$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$

Solution: Let $2^x = 3^y = 6^{-z} = k$

Then, $2 = k^{1/x}, 3 = k^{1/y}$ and $6 = k^{-1/z}$ . . .(*i*)

Now, $2 \times 3 = 6$

$\Rightarrow k^{1/x} \times k^{1/y} = k^{-1/z}$

[Using (*i*)]

$\Rightarrow k^{\frac{1}{x}+\frac{1}{y}} = k^{-\frac{1}{z}}$

$\Rightarrow \frac{1}{x} + \frac{1}{y} = -\frac{1}{z}$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

Example 5. Find the value of x if $\left(\frac{2}{3}\right)^x \left(\frac{3}{2}\right)^{2x} = \frac{81}{16}.$

Solution: $\left(\frac{2}{3}\right)^x \left(\frac{3}{2}\right)^{2x} = \frac{81}{16}$

$\Rightarrow \frac{2^x}{3^x} \times \frac{3^{2x}}{2^{2x}} = \frac{3^4}{2^4}$ $\qquad \Rightarrow \frac{3^{2x-x}}{2^{2x-x}} = \left(\frac{3}{2}\right)^4$

$$\Rightarrow \quad \frac{3^x}{2^x} = \left(\frac{3}{2}\right)^4 \quad \Rightarrow \left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^4 \Rightarrow x = 4$$

Types of Surds

- **Simple surd:** A surd consisting of a single term. For example $2\sqrt{3}, 6\sqrt{5}, \sqrt{5}$ etc.
- **Pure and mixed surds:** A surd consisting of wholly of an irrational number is called pure surd.

  Example: $\sqrt{5}, \sqrt[3]{7}.$ A surd consisting of the product of a rational number and an irrational number is called a mixed surd.

  Example: $5\sqrt{3}.$
- **Compound surds:** An expression consisting of the sum or difference of two or more surds.

  Example: $\sqrt{5} + \sqrt{2}, 2 - \sqrt{3} + 3\sqrt{5}$ etc.
- **Similar surds:** If the surds are different multiples of the same surd, they are called similar surds.

  Example: $\sqrt{45}, \sqrt{80}$ are similar surds because they are equal to $3\sqrt{5}$ and $4\sqrt{5}$ respectively.
- **Binomial surds:** A compound surd consisting of two surds is called a binomial surd.

  Example: $\sqrt{5} - \sqrt{2}, 3 + \sqrt[3]{2}$ etc.
- **Binomial quadratic surds:** Binomial surds consisting of pure (or simple) surds of order two i.e., the surds of the form $a\sqrt{b} \pm c\sqrt{d}$ or $a \pm b\sqrt{c}$ are called binomial quadratic surds.

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be conjugate or complementary to each other. The product of a binomial quadratic surd and its conjugate is always rational.

Example: The conjugate of the surd $2\sqrt{7} + 5\sqrt{3}$ is the surd $2\sqrt{7} - 5\sqrt{3}.$

Operation of Surds

Addition and Subtraction of Surds: Addition and subtraction of surds are possible only when order and radicand are same i.e. only for surds.

Example 6. Simplify

(i) $\sqrt{6} - \sqrt{216} + \sqrt{96}$ **(ii)** $5\sqrt[3]{250} + 7\sqrt[3]{16} - 14\sqrt[3]{54}$

(iii) $5\sqrt[3]{250} + 7\sqrt[3]{16} - 14\sqrt[3]{54}$ **(iv)** $4\sqrt{3} + 3\sqrt{48} - \frac{5}{2}\sqrt{\frac{1}{3}}$

Solution:(i) $\sqrt{6} - \sqrt{216} + \sqrt{96} = 15\sqrt{6} - \sqrt{6^2} \times 6 + \sqrt{16 \times 6}$

[Bring surd in simples form]

$$= 15\sqrt{6} = 6\sqrt{6} + 4\sqrt{6} = (15 - 6 + 4)\sqrt{6} = 13\sqrt{6}$$

(ii) $5\sqrt[3]{250} + 7\sqrt[3]{16} - 14\sqrt[3]{54} = 5\sqrt[3]{125 \times 2} + 7\sqrt[3]{8 \times 2} - 14\sqrt[3]{27 \times 2}$

$$= 5 \times 5\sqrt[3]{2} + 7 \times 2\sqrt[3]{2} - 14 \times 3 \times \sqrt[3]{2} = (25 + 14 - 42)\sqrt[3]{2}$$

$$= -3\sqrt[3]{2}$$

(iii) $5\sqrt[3]{250} + 7\sqrt[3]{16} - 14\sqrt[3]{54}$

$$= 5\sqrt[3]{125 \times 2} + 7\sqrt[3]{8 \times 2} - 14\sqrt[3]{27 \times 2}$$

$$= 5 \times 5\sqrt[3]{2} + 7 \times 2\sqrt[3]{2} - 14 \times 3 \times \sqrt[3]{2} = (25 + 14 - 42)\sqrt[3]{2}$$

$$= -3\sqrt[3]{2}$$

(iv) $4\sqrt{3} + 3\sqrt{48} - \frac{5}{2}\sqrt{\frac{1}{3}} = 4\sqrt{3} + 3\sqrt{16 \times 3} - \frac{5}{2}\sqrt{\frac{1 \times 3}{3 \times 3}}$

$$= 4\sqrt{3} + 3 \times 4\sqrt{3} - \frac{5}{2} \times \frac{1}{3}\sqrt{3} = 4\sqrt{3} + 12\sqrt{3} - \frac{5}{6}\sqrt{3}$$

$$= \left(4 + 12 - \frac{5}{6}\right)\sqrt{3} = \frac{91}{6}\sqrt{3}$$

Multiplication and Division of Surds

Example 7. Simplify

(i) $\sqrt[3]{4} \times \sqrt[3]{22} = \sqrt[3]{4 \times 22} = \sqrt[3]{2^3 \times 11} = 2\sqrt[3]{11}$

(ii) $\sqrt[3]{2} \times \sqrt[4]{3} = \sqrt[12]{2^4} \times \sqrt[12]{3^3} = \sqrt[12]{2^4 \times 3^3} = \sqrt[12]{16 \times 27} = \sqrt[12]{432}$

Example 8. Simplify $\sqrt{8a^5b^3} \times \sqrt[3]{4a^2b^2}$

Solution: $\sqrt[6]{8^3 a^{15} b^3} \times \sqrt[6]{4^2 a^4 b^4} = \sqrt[6]{2^{13} a^{19} b^7} = \sqrt[6]{2ab}.$

Example 9. Divide $\sqrt{24} \div \sqrt[3]{200}$

Solution: $\sqrt{24} \div \sqrt[3]{200} = \frac{\sqrt{24}}{\sqrt[3]{200}} = \frac{\sqrt[6]{(24)^3}}{\sqrt[6]{(200)^2}} = \sqrt[6]{\frac{216}{625}}$

Comparison of Surds: It is clear that if $x > y > 0$ and $n > 1$ is a positive integer then $\sqrt[n]{x} > \sqrt[n]{y}.$

Example 10. Which is greater is each of the following:

(i) $\sqrt[3]{16}$ and $\sqrt[5]{8}$ and **(ii)** $\sqrt{\frac{1}{2}}$ and $\sqrt[3]{\frac{1}{3}}$

Solution: (i) L.C.M. of 3 and 515.

$$\sqrt[3]{6} = \sqrt[3\times5]{6^5} = \sqrt[15]{7776}$$

$$\sqrt[5]{8} = \sqrt[3\times5]{8^5} = \sqrt[15]{512}$$

$\therefore \quad \sqrt[75]{7776} > \sqrt[15]{512}$

$\Rightarrow \quad \sqrt[3]{6} > \sqrt[5]{8}$

(ii) L.C.M. of 2 and 3 is 6.

$\sqrt[6]{\left(\frac{1}{2}\right)^3}$ and $\sqrt[3]{\left(\frac{1}{3}\right)^2}$

$\sqrt[6]{\frac{1}{8}}$ and $\sqrt[6]{\frac{1}{9}}$ $\quad \left[\text{As } 8 < 9 \therefore \frac{1}{8} > \frac{1}{9}\right]$

so, $\sqrt[6]{\frac{1}{8}} > \sqrt[6]{\frac{1}{9}} \Rightarrow \sqrt{\frac{1}{2}} > \sqrt[3]{\frac{1}{3}}$

Example 11. Arrange $\sqrt{2}, \sqrt[3]{3}$ and $\sqrt[4]{5}$ is ascending order.

Solution: L.C.M. of 2, 3, 4 is 12.

$\therefore \quad \sqrt{2} = \sqrt[2\times 6]{2^6} = \sqrt[12]{64}$

$\sqrt[3]{3} = \sqrt[3\times 4]{3^4} = \sqrt[12]{81}$

$\sqrt[4]{5} = \sqrt[4\times 3]{5^3} = \sqrt[12]{125}$

As, $64 < 81 < 125$.

$\therefore \quad \sqrt[12]{64} < \sqrt[12]{81} < \sqrt[12]{125}$

$\Rightarrow \quad \sqrt{2} < \sqrt[3]{3} < \sqrt[4]{5}$

Example 12. Which is greater $\sqrt{7}-\sqrt{3}$ or $\sqrt{5}-1$

Solution: $\sqrt{7}-\sqrt{3} = \frac{(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})}{(\sqrt{7}+\sqrt{3})}$

$= \frac{7-3}{\sqrt{7}+\sqrt{3}} = \frac{4}{\sqrt{7}+\sqrt{3}}$

And, $\sqrt{5}-1 = \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{(\sqrt{5}-1)} = \frac{5-1}{\sqrt{5}+1} = \frac{4}{\sqrt{5}+1}$

Now, we know that $\sqrt{7} > \sqrt{5}$ and $\sqrt{3} > 1$, add

So, $\sqrt{7}+\sqrt{3} > \sqrt{5}+1 \quad \Rightarrow \frac{1}{\sqrt{7}+\sqrt{3}} < \frac{1}{\sqrt{5}+1}$

$\Rightarrow \quad \frac{4}{\sqrt{7}+\sqrt{3}} < \frac{4}{\sqrt{5}+1} \quad \Rightarrow \sqrt{7}-\sqrt{3} < \sqrt{5}-1$

So, $\sqrt{5}-1 > \sqrt{7}-\sqrt{3}$

Properties of Quadratic Surds

- The square root of a rational number cannot be expressed as the sum or difference of a rational number and a quadratic surd.
- If two quadratic surds cannot be reduced to others, which have not the same irrational part, their product is irrational.
- One quadratic surd cannot be equal to the sum or difference of two others, not having the same irrational part.
- If $a+\sqrt{b} = c+\sqrt{d}$, where a and c are rational, and $\sqrt{b}, \sqrt{d}$ are irrational, then $a = c$ and $b = d$.

Example 13. The greatest number among $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$ is:

Solution: $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$

$\because \quad$ L.C.M of 3, 4, 6 is 12

$\therefore \quad \sqrt[3]{9} = 9^{1/3} = (9^4)^{1/12} = (6561)^{1/12}$

$\sqrt[4]{11} = (11)^{1/4}(11^3)^{1/12} = (1331)^{1/12}$

$\sqrt[6]{17} = (17)^{1/6} = (17^2)^{1/2} = (289)^{1/12}$

Hence $\sqrt[3]{9}$ is the greatest number.

Rationalisation Factors

Example 14. Find the R.F. (rationalising factor) of the following:

(i) $\sqrt{10}$ **(ii)** $\sqrt{12}$ **(iii)** $\sqrt{162}$ **(iv)** $\sqrt[3]{4}$;

(v) $\sqrt[3]{16}$ **(vi)** $\sqrt[4]{162}$ **(vii)** $2+\sqrt{3}$ **(viii)** $7-4\sqrt{3}$

(ix) $3\sqrt{3}+2\sqrt{2}$ **(x)** $\sqrt[3]{3}+\sqrt[3]{2}$ **(xi)** $1+\sqrt{2}+\sqrt{3}$

Solution:(i) $\sqrt{10}$ $\left[\therefore \sqrt{10}\times\sqrt{10} = \sqrt{10\times 10} = 10\right]$ as 10 is rational number.

$\therefore \quad$ R.F. of $\sqrt{10}$ is $\sqrt{10}$

(ii) $\sqrt{12}$

First write its simplest from i.e. $2\sqrt{3}$.

Now find R.F. (i.e. R.F. of $\sqrt{3}$ is $\sqrt{3}$)

$\therefore \quad$ R.F. of $\sqrt{12}$ is .$\sqrt{3}$

(iii) $\sqrt{162}$

Simplest form of $\sqrt{162}$ is $9\sqrt{2}$.

R.F. of $\sqrt{2}$ is $\sqrt{2}$.

$\therefore \quad$ R.F. of $\sqrt{162}$ is $\sqrt{2}$

(iv) $\sqrt[3]{4}$ $\sqrt[3]{4}\times\sqrt[3]{4^2} = \sqrt[3]{4^3} = 4$

$\therefore \quad$ **R.F.** of $\sqrt[3]{4}$ is $\sqrt[3]{4^2}$

(v) $\sqrt[3]{16}$

Simplest form of $\sqrt[3]{16}$ is $2\sqrt[3]{2}$

Now R.F. of $\sqrt[3]{2}$ is $\sqrt[3]{2^2}$

$\therefore \quad$ R.F. of $\sqrt[3]{16}$ is $\sqrt[3]{2^2}$

(vi) $\sqrt[4]{162}$

Simplest form of $\sqrt[4]{162}$ is $3\sqrt[4]{2}$

Now R.F. of $\sqrt[4]{2}$ is $\sqrt[4]{2^3}$

$\therefore \quad$ R.F. of $\left(\sqrt[4]{162}\right)$ is $\sqrt[4]{2^3}$

(vii) $2+\sqrt{3}$

As, $(2+\sqrt{3})(2-\sqrt{3})=(2)^2-(\sqrt{3})^2=4-3=1,$

which is rational.

$\therefore$ R.F. of $(2+\sqrt{3})$ is $(2-\sqrt{3})$

(viii) $7-4\sqrt{3}$

As , $(7-4\sqrt{3})(7+4\sqrt{3})=(7)^2-(4-\sqrt{3})^2=49-48=1,$ which is rational

$\therefore$ R.F. of $(7-4\sqrt{3})$ is $(7+4\sqrt{3})$

(ix) $3\sqrt{3}+2\sqrt{2}$

As $(3\sqrt{3}+2\sqrt{2})(3\sqrt{3}-2\sqrt{2})=(3\sqrt{3})^2-(2\sqrt{2})^2=27-8=19,$

which is rational.

$\therefore$ R.F. of $(3\sqrt{3}+2\sqrt{2})$ is $(3\sqrt{3}-2\sqrt{2})$

(x) $\sqrt[3]{3}+\sqrt[3]{2}$

As $(\sqrt[3]{3}+\sqrt[3]{2})(\sqrt[3]{3^2}-\sqrt[3]{3}\times\sqrt[3]{2}+\sqrt[3]{2^2})=(\sqrt[3]{3^3}+\sqrt[3]{2^3})=3+2=5,$

which is rational.

$\therefore$ R.F. of $(\sqrt[3]{3}+\sqrt[3]{2})$ is $(\sqrt[3]{3^2}-\sqrt[3]{3}\times\sqrt[3]{2}+\sqrt[3]{2^2})$

(xi) $1\times\sqrt{2}+\sqrt{3}$

$(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})=(1+\sqrt{2})^2-(\sqrt{3})^2$

$=(1)^2+(\sqrt{2})^2+2(1)(\sqrt{2})-3=1+2+2\sqrt{2}-3$

$=3+2\sqrt{2}-3=2\sqrt{2}$

$2\sqrt{2}\times\sqrt{2}=2\times2=4$

$\therefore$ R.F. of $1+\sqrt{2}+\sqrt{3}$ is

$(1+\sqrt{2}-\sqrt{3})$ and $\sqrt{2}$.

Note

R.F. of $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$ type surds are also called conjugate surds and vice versa.

- $2-\sqrt{3}$ is conjugate of $2+\sqrt{3}$
- $\sqrt{5}+1$ is conjugate of $\sqrt{5}-1$

Sometimes conjugate surds and reciprocals are same

- $2+\sqrt{3}$, it's conjugate is $2-\sqrt{3}$, its reciprocal is $2-\sqrt{3}$ and vice versa.
- $5-2\sqrt{6}$, it's conjugate is $5+2\sqrt{6}$, its reciprocal is $5-2\sqrt{6}$ and vice versa.

Example 15. Express the following surd with a rational denominator.

Solution: $\dfrac{8}{\sqrt{15}+1-\sqrt{5}-\sqrt{3}}$

$$=\left[\frac{8}{\left(\sqrt{15}+1\right)-\left(\sqrt{5}+\sqrt{3}\right)}\right]\times\left[\frac{\left(\sqrt{15}+1\right)+\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{15}+1\right)+\left(\sqrt{5}+\sqrt{3}\right)}\right]$$

$$=\frac{8\left(\sqrt{15}+1+\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{15}+1\right)^2-\left(\sqrt{5}+\sqrt{3}\right)^2}$$

$$=\frac{8\left(\sqrt{15}+1+\sqrt{5}+\sqrt{3}\right)}{15+1+2\sqrt{15}-\left(5+3+2\sqrt{15}\right)}$$

$$=\frac{8\left(\sqrt{15}+1+\sqrt{5}+\sqrt{3}\right)}{8}$$

$$=\left(\sqrt{15}+1+\sqrt{5}+\sqrt{3}\right)$$

Example 16. Evaluate each of the following:

(i) $5^2\times5^4$; **(ii)** $5^8\div5^3$;

(iii) $(3^2)^2$; **(iv)** $\left(\dfrac{11}{12}\right)^3$;

(v) $\left(\dfrac{3}{4}\right)^{-3}$

Solution: Using the laws of indices, we have

(i) $5^2.5^4=5^{2+4}=5^6=15625$

$\therefore$ $a^m\times a^n=a^{m-n}$

(ii) $5^8\div5^3=\dfrac{5^8}{5^3}=5^{8-3}=5^5=3125$

$\therefore$ $a^m+a^n=a^{m-n}$

(iii) $\left(3^2\right)^3=3^{2\times3}=3^6=729$

$\therefore$ $\left(a^m\right)^n=a^{m\times n}$

(iv) $\left(\dfrac{11}{12}\right)^3=\dfrac{11^3}{12^3}=\dfrac{1331}{1728}$

$\therefore$ $\left(\dfrac{a}{b}\right)^m=\dfrac{a^m}{b^m}$

(v) $\left(\dfrac{3}{4}\right)^{-3}=\dfrac{1}{\left(\dfrac{3}{4}\right)^3}=\dfrac{1}{\dfrac{3^3}{4^3}}=\dfrac{1}{\dfrac{27}{64}}=\dfrac{64}{27}$

$\therefore$ $a^{-n}=\dfrac{1}{a^n}$

Example 17. Evaluate: $\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{5}\right)^{-3} \times \left(\frac{3}{5}\right)^2$

Solution: We have, $\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{5}\right)^{-3} \times \left(\frac{3}{5}\right)^{-2} = \frac{2^3}{3^3} \times \frac{1}{(2/5)} \times \frac{3^2}{5^2}$

$$= \frac{2^3}{3^3} \times \frac{1}{2^3/5^3} \times \frac{3^2}{5^2} = \frac{2^3 \times 5^3 \times 3^2}{3^3 \times 2^3 \times 5^2} = \frac{5}{3}$$

Example 18. Simplify $\left(\frac{81}{16}\right)^{-3/4} \times \left[\left(\frac{25}{9}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$

Solution: We have

$$\left(\frac{81}{16}\right)^{-3/4} \times \left[\left(\frac{25}{9}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right] = \left(\frac{3^4}{2^4}\right)^{-3/4} \times \left[\left(\frac{5^2}{3^2}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left[\left(\frac{3}{2}\right)^4\right]^{-3/4} \times \left[\left(\frac{5}{3}\right)^2\right]^{-3/2} \div \left[\left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{4x-3/4} \times \left[\left(\frac{5}{3}\right)^{2x-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{-3} \times \left[\left(\frac{5}{3}\right)^{-3} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{2}{3}\right)^{3} \times \left[\left(\frac{5}{3}\right)^{-3} \times \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \frac{2^3}{3^3} \times \left[\frac{3^3}{5^3} \div \frac{2^3}{5^3}\right] = \frac{2^3}{3^3} \times \left[\frac{3^3}{5^3} \times \frac{5^3}{2^3}\right] = 1$$

Square root of *a* + √*b* + √*c* + √*d* where√*b*, √*c* and√*d* are surds:

Let $\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y}$, where $x, y > 0$ are rational numbers.

Then squaring both sides we have, $a + \sqrt{b} = x + y + 2\sqrt{x}\sqrt{y}$

$\Rightarrow \ a = x + y,\ \sqrt{b} = 2\sqrt{xy}$

$\Rightarrow b = 4xy$

So, $(x-y)^2 = (x+y)^2 - 4xy = a^2 - b$

After solving we can find *x* and *y*.

Similarly square root of $a - \sqrt{b}$ can be found by taking $\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y},\ x > y$

To find square root of *a* + √*b* + √*c* + √*d*:

Let $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})} = \sqrt{x} + \sqrt{y} + \sqrt{z}, (x, y, z > 0)$ and take $\sqrt{(a+\sqrt{b}-\sqrt{c}-\sqrt{d})} = \sqrt{x} + \sqrt{y} - \sqrt{z}$. Then by squaring and equating, we get equations in *x, y, z*. On solving these equations, we can find the required square roots.

Note

- If $a^2 - b$ is not a perfect square, the square root of $a + \sqrt{b}$ is complicated *i.e.*, we can't find the value of $\sqrt{(a+\sqrt{b})}$ in the form of a compound surd.
- If $\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y}, x > y$ then $\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y}$
- $\sqrt{a+\sqrt{(b)}} = \sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)} + \sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$
- $\sqrt{a-\sqrt{(b)}} = \sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)} - \sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$
- If *a* is a rational number, $\sqrt{b}, \sqrt{c}, \sqrt{d}$, are surds then
- $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}} = \sqrt{\frac{bd}{4c}} + \sqrt{\frac{bc}{4d}} + \sqrt{\frac{cd}{4b}}$
- $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}} = \sqrt{\frac{bd}{4c}} + \sqrt{\frac{cd}{4b}} + \sqrt{\frac{bc}{4d}}$
- $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}} = \sqrt{\frac{bc}{4d}} - \sqrt{\frac{bd}{4c}} - \sqrt{\frac{cd}{4b}}$

Cube Root of a Binomial Quadratic: If $(a+\sqrt{b})^{1/3} = x + \sqrt{y}$ then $(a-\sqrt{b})^{2/3} = x - \sqrt{y}$, where *a* is a rational number and *b* is a surd. Procedure of finding $(a+\sqrt{b})^{1/3}$ is illustrated with the help of an example:

Taking $(37 - 30\sqrt{3})^{1/3} = x + \sqrt{y}$ we get on cubing both sides,

$37 - 30\sqrt{30} = x^3 + 3xy - \left(3x^2 + y\right)\sqrt{y}$

$\therefore \ x^3 + 3xy = 37$

$(3x^2 + y)\sqrt{y}30\sqrt{3} = 15\sqrt{12}$

As $\sqrt{3}$ cannot be reduced, let us assume $y = 3$ we get $3x^2 + y = 3x^2 + 3 = 30$

$\therefore \ x = 3$

Which doesn't satisfy $x^3 + 3xy = 37$

Again taking $y = 12$, we get $3x^2 + 12 = 15$,

$\therefore \quad x=1 \; x=1, y=12$ satisfy $x^3+3xy=37$

$\therefore \quad \sqrt[3]{37-30\sqrt{3}}=1-\sqrt{12}=1-2\sqrt{3}$

Logarithm: The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number.If $a>0$ and $\neq 1$, then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N *i.e.,* $\log_a N = x$ *iff* $a^x = N \Rightarrow a^{\log_a N} = N, a>0, a \neq 1$ and $N>0$

It is also known as fundamental logarithmic identity.Its domain is $(0,\infty)$ and range is R.a is called the base of the logarithmic function.When base is 'e' then the logarithmic function is called natural or Napierian logarithmic function and when base is 10, then it is called common logarithmic function.

Characteristic and mantissa

(1) The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underset{\text{Characterstics}}{\underset{\downarrow}{\text{integer}}} + \underset{\text{Mantissa}}{\underset{\downarrow}{\text{fraction}}}\ (+ve)$$

(2) The mantissa part of log of a number is always kept positive.

(3) If the characteristics of $\log_{10} N$ be n, then the number of digits in N is (n+1).

(4) If the characteristics of $\log_{10} N$ be ($-n$) then there exists ($n-1$) number of zeros after decimal part of N.

Properties of logarithms: Let m and n be arbitrary positive numbers such that $a>0,\ a\neq 1,\ b>0,\ b\neq 1$ then

(1) $\log_a a = 1,\ \log_a 1 = 0$

(2) $\log_a b . \log_b a = 1 \Rightarrow \log_a b = \dfrac{1}{\log_b a}$

(3) $\log_c a = \log_b a.\ \log_c b$ or $\log_c a = \dfrac{\log_b a}{\log_b c}$

(4) $\log_a (mn) = \log_a m + \log_a n$

(5) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n \log_a m$

(7) $a^{\log_a m} = m$

(8) $\log_a \left(\dfrac{1}{n}\right) = -\log_a n$

(9) $\log_{a^\beta} n = \dfrac{1}{\beta}\log_a n$

(10) $\log_{a^\beta} n^\alpha = \dfrac{\alpha}{\beta}\log_a n$, $(\beta \neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}$, $(a,b,c>0$ and $c\neq 1)$

Logarithmic inequalities

(1) If $a>1, p>1 \Rightarrow \log_a p > 0$

(2) If $0<a<1, p>1 \Rightarrow \log_a p < 0$

(3) If $a>1,\ 0<p<1 \Rightarrow \log_a p < 0$

(4) If $p>a>1 \Rightarrow \log_a p > 1$

(5) If $a>p>1 \Rightarrow 0<\log_a p < 1$

(6) If $0<a<p<1 \Rightarrow 0<\log_a p<1$

(7) If $0<p<a<1 \Rightarrow \log_a p > 1$

(8) If $\log_m a > b \Rightarrow \begin{cases} a>m^b, \text{ if } m>1 \\ a<m^b, \text{ if } 0<m<1 \end{cases}$

(9) $\log_m a < b \Rightarrow \begin{cases} a<m^b, \text{ if } m>1 \\ a>m^b, \text{ if } 0<m<1 \end{cases}$

(10) $\log_p a > \log_p b \Rightarrow a \geq b$ if base p is positive and >1 or $a \leq b$ if base p is positive and < 1 *i.e.,* $0<p<1$.

In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

Example 19. Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is

Solution: Let x be the required logarithm, then by definition

$(2\sqrt{2})^x = 32\sqrt[5]{4}$

$(2.2^{1/2})^x = 2^5.2^{2/5}$;

$\therefore \quad 2^{\frac{3x}{2}} = 2^{5+\frac{2}{5}}$

Here, by equating the indices, $\dfrac{3}{2}x = \dfrac{27}{5}$,

$\therefore \quad x = \dfrac{18}{5} = 3.6$

Example 20. If $\log_7 2 = m$, then $\log_7 2 = m$, is?

Solution: $\log_{49} 28 = \dfrac{\log 28}{\log 49} = \dfrac{\log 7 + \log 4}{2\log 7}$

$= \dfrac{\log 7}{2\log 7} + \dfrac{\log 4}{2\log 7} = \dfrac{1}{2} + \dfrac{1}{2}\log_7 4$

$= \dfrac{1}{2} + \dfrac{1}{2}.2\log_7 2 = \dfrac{1}{2} + \log_7 2 = \dfrac{1}{2} + m = \dfrac{1+2m}{2}$

Multiple Choice Questions

1. If $2^x = 4^y = 8^z$ and $xyz = 288$, then $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=$

a. 11/48 **b.** 11/24 **c.** 11/8 **d.** 11/96

2. $\frac{2.3^{n+1}+7.3^{n-1}}{3^{n+2}-2(1/3)^{l-n}}=$

a. 1 **b.** 3 **c.** -1 **d.** 0

3. If $\left(\frac{2}{3}\right)^{x+2}=\left(\frac{3}{2}\right)^{2-2x}$, then x=

a. 1 **b.** 3 **c.** 4 **d.** 0

4. The equation $4^{(x^2+2)}-9.2^{(x^2+2)}+8=0$ has the solution

a. $x=1$ **b.** $x=-1$ **c.** $x=\sqrt{2}$ **d.** $x=-\sqrt{2}$

5. The value of $\frac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$ is:

a. $\sqrt{5}(5+\sqrt{2})$ **b.** $\sqrt{5}(2+\sqrt{2})$

c. $\sqrt{5}(1+\sqrt{2})$ **d.** $\sqrt{5}(3+\sqrt{2})$

6. If $x=\sqrt[3]{(\sqrt{2}+1)}-\sqrt[3]{(\sqrt{2}-1)}$; then $x^3+3x=$

a. 2 **b.** 6

c. $6x$ **d.** None of these

7. $\sqrt{(3+\sqrt{5})}$ is equal to:

a. $\sqrt{5}+1$ **b.** $\sqrt{3}+\sqrt{2}$

c. $\left(\sqrt{5}+1\right)/\sqrt{2}$ **d.** $\frac{1}{2}\left(\sqrt{5}+1\right)$

8. $\sqrt{[10-\sqrt{(24)}-\sqrt{(40)}+\sqrt{(60)}]}=$

a. $\sqrt{5}+\sqrt{3}+\sqrt{2}$ **b.** $\sqrt{5}+\sqrt{3}-\sqrt{2}$

c. $\sqrt{5}-\sqrt{3}+\sqrt{2}$ **d.** $\sqrt{2}+\sqrt{3}-\sqrt{5}$

9. $\sqrt[4]{(17+12\sqrt{2})}=$

a. $\sqrt{2}+1$ **b.** $2^{1/4}(\sqrt{2}+1)$

c. $2\sqrt{2}+1$ **d.** None of these

10. $\left(\frac{2}{11}\right)^4\times\left(\frac{11}{3}\right)^2\times\left(\frac{3}{2}\right)^3$

a. 1 **b.** 0 **c.** 2 **d.** $\frac{6}{121}$

11. $\left(\frac{1}{2}\right)^5\times\left(\frac{-2}{3}\right)^4\times\left(\frac{3}{5}\right)^{-1}$

a. 0 **b.** 20 **c.** $\frac{5}{486}$ **d.** 1

12. $2^{55}\times 2^{60}-2^{97}\times 2^{18}$

a. 0 **b.** 1

c. 2 **d.** -1

13. $\frac{(25)^{3/2}\times(243)^{3/5}}{(16)^{5/4}\times(8)^{4/3}}$

a. $\frac{3375}{512}$ **b.** 0

c. 1 **d.** None of these

14. $\frac{16\times 2^{n+1}-4\times 2^n}{16\times 2^{n+2}-2\times 2^{n+2}}$

a. $\frac{1}{2}$ **b.** 1 **c.** 0 **d.** 2

15. For $y=\log_a x$ to be defined 'a' must be

a. Any positive real number

b. Any number

c. $\geq e$

d. Any positive real number $\neq 1$

16. Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is:

a. 3.6 **b.** 5

c. 5.6 **d.** None of these

17. The number $\log_2 7$ is:

a. An integer **b.** A rational number

c. An irrational number **d.** A prime number

18. If $\log_7 2 = m$, then $\log_{49} 28$ is equal to:

a. $2(1+2m)$ **b.** $\frac{1+2m}{2}$ **c.** $\frac{2}{1+2m}$ **d.** $1+m$

19. If $\log_e\left(\frac{a+b}{2}\right)=\frac{1}{2}(\log_e a+\log_e b)$, then relation between a and b will be:

a. $a=b$ **b.** $a=\frac{b}{2}$ **c.** $2a=b$ **d.** $a=\frac{b}{3}$

20. Which is the correct order for a given number α in increasing order?

a. $\log_2\alpha, \log_3\alpha, \log_e\alpha, \log_{10}\alpha$

b. $\log_{10}\alpha, \log_3\alpha, \log_e\alpha, \log_2\alpha$

c. $\log_{10}\alpha, \log_e\alpha, \log_2\alpha, \log_3\alpha$

d. $\log_3\alpha, \log_e\alpha, \log_2\alpha, \log_{10}\alpha$

21. $\log ab-\log|b|=$

a. $\log a$ **b.** $\log|a|$

c. $-\log a$ **d.** None of these

22. The value of $\sqrt{(\log_{0.5}^2 4)}$ is:

a. -2 **b.** $\sqrt{(-4)}$

c. 2 **d.** None of these

23. The value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$ is:

a. 1 **b.** 2 **c.** 3 **d.** 4

24. $\log_7 \log_7 \sqrt{7(\sqrt{7\sqrt{7}})} =$

a. $3\log_2 7$ **b.** $1-3\log_3 7$

c. $1-3\log_7 2$ **d.** None of these

25. The value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$ is equal to

a. 49 **b.** 625 **c.** 216 **d.** 890

26. $7\log\left(\frac{16}{15}\right) + 5\log\left(\frac{25}{24}\right) + 3\log\left(\frac{81}{80}\right)$ is equal to

a. 0 **b.** 1 **c.** $\log 2$ **d.** $\log 3$

27. If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to

a. $\frac{1}{2a+1}$ **b.** $\frac{1}{2b+1}$ **c.** $2ab+1$ **d.** $\frac{1}{2ab-1}$

28. If $\log_k x.\log_5 k = \log_x 5, k \neq 1, k > 0$, then x is equal to

a. k **b.** $\frac{1}{5}$

c. 5 **d.** None of these

29. If $\log_5 a.\log_a x = 2$, then x is equal to

a. 125 **b.** a^2

c. 25 **d.** None of these

30. If $a^2 + 4b^2 = 12ab$, then $\log(a+2b)$ is

a. $\frac{1}{2}[\log a + \log b - \log 2]$ **b.** $\log\frac{a}{2} + \log\frac{b}{2} + \log 2$

c. $\frac{1}{2}[\log a + \log b + 4\log 2]$ **d.** $\frac{1}{2}[\log a - \log b + 4\log 2]$

ANSWERS

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
d	a	c	a, b	c	a	c	b	a	d
11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**
c	a	a	a	d	a	c	b	a	b
21.	**22.**	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**	**29.**	**30.**
b	c	b	c	d	c	d	b,c	c	c

SOLUTIONS

1. **(d)** $2^x = 2^{2y} = 2^{3z}$ *i.e.*, $x = 2y = 3z = k$ (say)

Then $xyz = \frac{k^3}{6} = 288,$

So, $k = 12$

$\therefore$ $x = 12, y = 6, z = 4.$

Therefore, $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{11}{96}$

2. **(a)** $\frac{2.3^{n+1} + 7.3^{n-1}}{3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n}} = \frac{2.3^{n-1}.3^2 + 7.3^{n-1}}{3^{n-1}.3^3 - 2.3^{n-1}}$

$= \frac{3^{n-1}[18+7]}{3^{n-1}[27-2]} = 1$

3. **(c)** $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x}$

$\Rightarrow \left(\frac{2}{3}\right)^{x+2} = \left(\frac{2}{3}\right)^{2x-2}.$

Clearly $x+2 = 2x-2 \Rightarrow x = 4$

4. **(a, b)** $4^{(x^2+2)} - 9.2^{(x^2+2)} + 8 = 0$

$\Rightarrow (2^{(x^2+2)})^2 - 9.2^{(x^2+2)} + 8 = 0$

Put $2^{(x^2+2)^2} = y.$

Then $y^2 - 9y + 8 = 0$ which gives $y = 8, y = 1$

When $y = 8 \Rightarrow 2^{x^2+2} = 8 \Rightarrow 2^{x^2+2} = 2^3$

$\Rightarrow x^2 + 2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$

When $y = 1 \Rightarrow 2^{x^2+2}$

$= 1 \Rightarrow 2^{x^2+2} = 2^\circ \Rightarrow x^2 + 2$

$= 0 \Rightarrow x^2 = -2$, which is not possible.

5. **(c)** Given fraction $= \frac{15}{\sqrt{10} + \sqrt{20} + \sqrt{40} - \sqrt{5} - \sqrt{80}}$

$= \frac{15}{\sqrt{10} + 2\sqrt{5} + 2\sqrt{10} - \sqrt{5} - 4\sqrt{5}}$

$= \frac{15}{3\sqrt{10} - 3\sqrt{5}} = \frac{5}{\sqrt{10} - \sqrt{5}} . \frac{\sqrt{10} + \sqrt{5}}{\sqrt{10} + \sqrt{5}}$

$= \sqrt{10} + \sqrt{5} = \sqrt{5}(\sqrt{2} + 1)$

6. **(a)** $x = (\sqrt{2}+1)^{1/3} - (\sqrt{2}-1)^{1/3}$

$\Rightarrow x^3 = (\sqrt{2}+1) - (\sqrt{2}-1) - 3(\sqrt{2}+1)^{1/3}$

$(\sqrt{2}-1)^{1/3}\left[\sqrt[3]{(\sqrt{2}+1)} - \sqrt[3]{\sqrt{2}-1}\right]$

$\Rightarrow x^3 = 2 - 3(2-1)^{1/3} x$

$\Rightarrow x^3 + 3x = 2.$

7. **(c)** Let $\sqrt{3+\sqrt{5}}=\sqrt{x}+\sqrt{y}$

$3+\sqrt{5}=x+y+2\sqrt{xy}$.

Obviously $x+y=3$ and $4xy=5$.

So, $(x-y)^2=9-5=4$ or $(x-y)=2$

After solving $x=\frac{5}{2}, y=\frac{1}{2}$.

Hence, $\sqrt{3+\sqrt{5}}=\sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}}=\frac{\sqrt{5}+1}{\sqrt{2}}$

8. **(b)** Let $10-\sqrt{24}-\sqrt{40}+\sqrt{60}$

$=\left(\sqrt{a}-\sqrt{b}+\sqrt{c}\right)^2$ $10-\sqrt{24}-\sqrt{40}+\sqrt{60}$

$=a+b+c-2\sqrt{ab}-2\sqrt{bc}+2\sqrt{ac}, a,b,c>0$.

Then $a+b+c=10, ab=6, bc=10, ca=15$

$a^2b^2c^2=900$

$\Rightarrow$ $abc=30(\neq \pm 30)$. So, $a=3, b=2, c=5$

Therefore, $\sqrt{(10-\sqrt{24}-\sqrt{40}+\sqrt{60})}$

$=\pm(\sqrt{3}+\sqrt{5}-\sqrt{2})$

9. **(a)** $\sqrt{\left(17+12\sqrt{2}\right)}=\sqrt{\left[3^2+\left(2\sqrt{2}\right)^2+2.3.2\sqrt{2}\right]}$

$=3+2\sqrt{2}$

$\therefore$ $\sqrt[4]{\left(17+12\sqrt{2}\right)}=\sqrt{\left(3+2\sqrt{2}\right)}=\sqrt{2}+1$.

10. **(d)** We have

$\left(\frac{2}{11}\right)^4\times\left(\frac{11}{3}\right)^2\times\left(\frac{3}{2}\right)^3=\frac{2^4}{11^4}\times\frac{11^2}{3^2}\times\frac{3^3}{2^3}$

$=\frac{2\times 3}{11^2}=\frac{6}{121}$

11. **(c)** We have, $\left(\frac{1}{2}\right)^5\times\left(\frac{-2}{3}\right)^4\times\left(\frac{3}{5}\right)^{-1}$

$=\left(\frac{1}{2}\right)^5\times\left(\frac{-2}{3}\right)^4\times\left(\frac{1}{\frac{3}{5}}\right)=\frac{1^5}{2^5}\times\frac{(-2)^2}{3^4}\times\frac{5}{3}$

$=\frac{1\times 16\times 5}{32\times 81\times 3}=\frac{5}{2\times 81\times 3}=\frac{5}{486}$

12. **(a)** We have, $2^{55}\times 2^{60}-2^{97}\times 2^{18}={}^{55+60}-2^{97+18}$

$=2^{15}-2^{115}=0$

13. **(a)** $\frac{(25)^{3/2}\times(243)^{3/5}}{(16)^{5/4}\times(8)^{4/3}}=\frac{\left(5^2\right)^{3/2}\times\left(3^5\right)^{3/5}}{\left(2^4\right)^{5/4}\times\left(2^3\right)^{4/3}}$

$=\frac{5^{2\times 3/2}\times 3^{5\times 3/5}}{2^{4\times 5/4}\times 2^{3\times 4/3}}=\frac{5^3\times 3^3}{2^5\times 2^4}=\frac{125\times 27}{32\times 16}=\frac{3375}{512}$

14. **(a)** $\frac{16\times 2^{n+1}-4\times 2^n}{16\times 2^{n+2}-2\times 2^{n+2}}=\frac{2^4\times 2^{n+1}-2^2\times 2^n}{2^4\times 2^{n+2}-2\times 2^{n+2}}$

$=\frac{2^{n+5}-2^{n+2}}{2^{n+6}-2^{n+3}}=\frac{2^{n+5}-2^{n+2}}{2.2^{n+5}-2.2^{n+2}}=\frac{2^{n+5}-2^{n+2}}{2\left(2^{n+5}-2^{n+2}\right)}=\frac{1}{2}$

15. **(d)** It is obvious.

16. **(a)** Let x be the required logarithm ,

then by definition $(2\sqrt{2})^x=32\sqrt[5]{4}$

$\Rightarrow$ $(2.2^{1/2})^x=2^5.2^{2/5}$;

$\therefore$ $2^{\frac{3x}{2}}=2^{5+\frac{2}{5}}$

Here, by equating the indices, $\frac{3}{2}x=\frac{27}{5}$

$\therefore$ $x=\frac{18}{5}=3.6$.

17. **(c)** Suppose, if possible, $\log_2 7$ is rational, say p/q where p and q are integers, prime to each other. Then, $\frac{p}{q}=\log_2 7\Rightarrow 7=2^{p/q}\Rightarrow 2^p=7^q$,

which is false since L.H.S is even and R.H.S is odd. Obviously $\log_2 7$ is not an integer and hence not a prime number.

18. **(b)** $\log_{49} 28=\frac{\log 28}{\log 49}=\frac{\log 7+\log 4}{2\log 7}$

$=\frac{\log 7}{2\log 7}+\frac{\log 4}{2\log 7}=\frac{1}{2}+\frac{1}{2}\log_7 4$

$=\frac{1}{2}+\frac{1}{2}.2\log_7 2$

$=\frac{1}{2}+\log_7 2=\frac{1}{2}+m=\frac{1+2m}{2}$

19. **(a)** $\log_e\left(\frac{a+b}{2}\right)=\frac{1}{2}(\log_e a+\log_e b)$

$=\frac{1}{2}\log_e(ab)=\log_e\sqrt{ab}$

$\Rightarrow$ $\frac{a+b}{2}=\sqrt{ab}\Rightarrow a+b=2\sqrt{ab}$

$\Rightarrow$ $\left(\sqrt{a}-\sqrt{b}\right)^2=0\Rightarrow\sqrt{a}-\sqrt{b}=0\Rightarrow a=b$.

20. **(b)** Since 10, 3, e, 2 are in decreasing order. Obviously, $\log_{10}\alpha, \log_3\alpha, \log_e\alpha, \log_2\alpha$ are in increasing order.

21. **(b)** $\log ab-\log|b|$

$=\log\left(\frac{ab}{|b|}\right)=\log|a|$.

22. **(c)** $\sqrt{\log_{0.5}^2 4}$

$=\sqrt{\{\log_{0.5}(0.5)^{-2}\}^2}$

$=\sqrt{(-2)^2}=2$.

23. **(b)** $\log_3 4.\log_4 5.\log_5 6.\log_6 7.\log_7 8.\log_8 9$

$= \frac{\log 4}{\log 3}.\frac{\log 5}{\log 4}.\frac{\log 6}{\log 5}.\frac{\log 7}{\log 6}.\frac{\log 8}{\log 7}.\frac{\log 9}{\log 8}$

$= \frac{\log 9}{\log 3} = \log_3 9 = \log_3 3^2 = 2$.

24. **(c)** $\log_7 \log_7 \sqrt{7\sqrt{7\sqrt{7}}}$

$= \log_7 \log_7 7^{7/8}$

$= \log_7 (7/8)$

$= \log_7 7 - \log_7 8$

$= 1 - \log_7 2^3$

$= 1 - 3\log_7 2$.

25. **(d)** $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$

$= 3^{4\log_3 5} + 3^{3.\frac{1}{2}\log_3 36} + 3^{4\log_9 7}$

$= 3^{\log_3 5^4} + 3^{\log_3 36^{3/2}} + 3^{\log_3 7^{4/2}}$

$= 5^4 + 36^{3/2} + 7^2 = 890$.

26. **(c)** Given expression

$= \log\left(\frac{16^7}{15^7}.\frac{25^5}{24^5}.\frac{81^3}{80^3}\right) = \log 2$.

27. **(d)** $ab = \log_4 5.\log_5 6 = \log_4 6 = \frac{1}{2}\log_2 6$

$ab = \frac{1}{2}(1 + \log_2 3) \Rightarrow 2ab - 1 = \log_2 3$

$\therefore \quad \log_3 2 = \frac{1}{2ab - 1}$.

28. **(b, c)** $\log_k x.\log_5 k = \log_x 5$

$\Rightarrow \quad \log_5 x = \log_x 5$

$\Rightarrow \quad \log_x 5 = \frac{1}{\log_x 5}$

$\Rightarrow \quad (\log_x 5)^2 = 1$

$\Rightarrow \quad \log_x 5 = \pm 1$

$\Rightarrow \quad x^{\pm 1} = 5$

$\Rightarrow \quad x = 5, \frac{1}{5}$.

29. **(c)** $\log_5 a.\log_a x = 2$

$\Rightarrow \quad \log_5 x = 2$

$\Rightarrow \quad x = 5^2 = 25$.

30. **(c)** $a^2 + 4b^2 = 12ab$

$\Rightarrow \quad a^2 + 4b^2 + 4ab = 16ab$

$\Rightarrow \quad (a + 2b)^2 = 16ab$

$\Rightarrow \quad 2\log(a + 2b) = \log 16 + \log a + \log b$

$\therefore \quad \log(a + 2b) = \frac{1}{2}[\log a + \log b + 4\log 2]$

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