

## DAY SIX

# Determinants

### Learning & Revision for the Day

- Determinants
- Properties of Determinants
- Cyclic Determinants
- Area of Triangle by using Determinants
- Minors and Cofactors
- Adjoint of a Matrix
- Inverse of a Matrix
- Solution of System of Linear Equations in Two and Three Variables

## Determinants

Every square matrix  $A$  can be associated with a number or an expression which is called its determinant and it is denoted by  $\det(A)$  or  $|A|$  or  $\Delta$ .

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \text{ then } \det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

- If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- If  $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ , then  $|A| = \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$   
 $= a \begin{vmatrix} q & r \\ v & w \end{vmatrix} - b \begin{vmatrix} p & r \\ u & w \end{vmatrix} + c \begin{vmatrix} p & q \\ u & v \end{vmatrix}$  [expanding along  $R_1$ ]  
 $= a(qw - vr) - b(pw - ur) + c(pv - uq)$

There are six ways of expanding a determinant of order 3 corresponding to each of three rows ( $R_1, R_2, R_3$ ) and three columns ( $C_1, C_2, C_3$ ).

- NOTE**
- Rule to put + or - sign in the expansion of determinant of order 3.  $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$
  - A square matrix  $A$  is said to be singular, if  $|A| = 0$  and non-singular, if  $|A| \neq 0$ .

## Properties of Determinants

- (i) If each element of a row (column) is zero, then  $\Delta = 0$ .
- (ii) If two rows (columns) are proportional, then  $\Delta = 0$ .
- (iii)  $|A^T| = |A|$ , where  $A^T$  is a transpose of a matrix.
- (iv) If any two rows (columns) are interchanged, then  $\Delta$  becomes  $-\Delta$ .
- (v) If each element of a row (column) of a determinant is multiplied by a constant  $k$ , then the value of the new determinant is  $k$  times the value of the original determinant
- (vi)  $\det(kA) = k^n \det(A)$ , if  $A$  is of order  $n \times n$ .
- (vii) If each element of a row (column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants i.e.

$$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

- (viii) If a scalar multiple of any row (column) is added to another row (column), then  $\Delta$  is unchanged

$$\text{i.e. } \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p + ka & q + kb & r + kc \\ u & v & w \end{vmatrix}, \text{ which is}$$

obtained by the operation  $R_2 \rightarrow R_2 + kR_1$

## Product of Determinants

$$\text{If } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } |B| = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}, \text{ then}$$

$$|A| \times |B| = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix} = |AB|$$

[multiplying row by row]

We can multiply rows by columns or columns by rows or columns by columns

**NOTE** •  $|AB| = |A| |B| = |BA| = |A^T B| = |AB^T| = |A^T B^T|$   
 •  $|A^n| = |A|^n, n \in \mathbb{Z}^+$

## Cyclic Determinants

In a cyclic determinant, the elements of row (or column) are arranged in a systematic order and the value of a determinant is also in systematic order.

$$(i) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$$

$$(ii) \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x - y)(y - z)(z - x)(x + y + z)$$

$$(iii) \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x - y)(y - z)(z - x)(xy + yz + zx)$$

$$(iv) \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ = -(a^3 + b^3 + c^3 - 3abc)$$

$$(v) \begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a - b)(b - c)(c - a)$$

## Area of Triangle by using Determinants

If  $A(x_1, y_1), B(x_2, y_2)$  and  $C(x_3, y_3)$  are vertices of  $\Delta ABC$ , then

$$\text{Area of } \Delta ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ = \frac{1}{2} |[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

If these three points are collinear, then  $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$   
 and *vice-versa*.

## Minors and Cofactors

The **minor**  $M_{ij}$  of the element  $a_{ij}$  is the determinant obtained by deleting the  $i$ th row and  $j$ th column of  $\Delta$ .

$$\text{If } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

then  $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$  etc.

The **cofactor**  $C_{ij}$  of the element  $a_{ij}$  is  $(-1)^{i+j} M_{ij}$ .

$$\text{If } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then } C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ etc.}$$

The sum of product of the elements of any row (or column) with their corresponding cofactors is equal to the value of determinant.

$$\text{i.e. } \Delta = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} \\ = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$$

But if elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

## Adjoint of a Matrix

If  $A = [a_{ij}]_{n \times n}$ , then adjoint of  $A$ , denoted by  $\text{adj}(A)$ , is defined as  $[C_{ij}]_{n \times n}^T$ , where  $C_{ij}$  is the cofactor of  $a_{ij}$ .

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } \text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

**NOTE** • If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then  $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

## Properties of Adjoint of a Matrix

Let  $A$  be a square matrix of order  $n$ , then

- (i)  $(\text{adj } A)A = A(\text{adj } A) = |A| \cdot I_n$
- (ii)  $|\text{adj } A| = |A|^{n-1}$ , if  $|A| \neq 0$
- (iii)  $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- (iv)  $\text{adj}(A^T) = (\text{adj } A)^T$
- (v)  $\text{adj}(\text{adj } A) = |A|^{n-2} A$ , if  $|A| \neq 0$
- (vi)  $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$ , if  $|A| \neq 0$

## Inverse of a Matrix

Let  $A$  be any non-singular (i.e.  $|A| \neq 0$ ) square matrix, then inverse of  $A$  can be obtained by following two ways.

### 1. Using determinants

In this,  $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

### 2. Using Elementary operations

In this, first write  $A = IA$  (for applying row operations) or  $A = AI$  (for applying column operations) and then reduce  $A$  of LHS to  $I$ , by applying elementary operations simultaneously on  $A$  of LHS and  $I$  of RHS. If it reduces to  $I = PA$  or  $I = AP$ , then  $P = A^{-1}$ .

## Properties of Inverse of a Matrix

- (i) A square matrix is invertible if and only if it is non-singular.
- (ii) If  $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ , then  $A^{-1} = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1})$  provided  $\lambda_i \neq 0 \forall i = 1, 2, \dots, n$ .

## Solution of System of Linear Equations in Two and Three Variables

Let system of linear equations in three variables be

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2$$

and  $a_3x + b_3y + c_3z = d_3$ .

Now, we have two methods to solve these equations.

### 1. Matrix Method

In this method we first write the above system of equations in matrix form as shown below.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ or } AX = B$$

where,  $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

#### Case I When system of equations is non-homogeneous (i.e. when $B \neq 0$ ).

- If  $|A| \neq 0$ , then the system of equations is consistent and has a unique solution given by  $X = A^{-1}B$ .
- If  $|A| = 0$  and  $(\text{adj } A) \cdot B \neq 0$ , then the system of equations is inconsistent and has no solution.
- If  $|A| = 0$  and  $(\text{adj } A) \cdot B = 0$ , then the system of equations may be either consistent or inconsistent according as the system have infinitely many solutions or no solution.

#### Case II When system of equations is homogeneous (i.e. when $B = 0$ ).

- If  $|A| \neq 0$ , then system of equations has only trivial solution, namely  $x = 0, y = 0$  and  $z = 0$ .
- If  $|A| = 0$ , then system of equations has non-trivial solution, which will be infinite in numbers.

### 2. Cramer's Rule Method

In this method we first determine

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

#### Case I When system of equations is non-homogeneous

- If  $D \neq 0$ , then it is consistent with unique solution given by  $x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$ .
- If  $D = 0$  and atleast one of  $D_1, D_2$  and  $D_3$  is non-zero, then it is inconsistent (no solution).
- If  $D = D_1 = D_2 = D_3 = 0$ , then it may be consistent or inconsistent according as the system have infinitely many solutions or no solution.

#### Case II When system of equations is homogeneous

- If  $D \neq 0$ , then  $x = y = z = 0$  is the only solution, i.e. the trivial solution.
- If  $D = 0$ , then it has infinitely many solutions.

Above methods can be used, in a similar way, for the solution of system of linear equations in two variables.



- 16** If the system of linear equations  
 $x + ky + 3z = 0$ ,  $3x + ky - 2z = 0$   
and  $2x + 4y - 3z = 0$   
has a non-zero solution  $(x, y, z)$ , then  $\frac{xz}{y^2}$  is equal to  
→ JEE Mains 2018
- (a) -10 (b) 10  
(c) -30 (d) 30

- 17** If  $A = [a_{ij}]_{n \times n}$  and  $a_{ij} = (i^2 + j^2 - ij)(j - i)$ ,  $n$  is odd, then which of the following is not the value of  $\text{tr}(A)$ .  
(a) 0 (b)  $|A|$   
(c)  $2|A|$  (d) None of these

- 18** If the coordinates of the vertices of an equilateral triangle with sides of length  $a$  are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ , then  $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$  is equal to  
→ NCERT Exemplar
- (a)  $\frac{a^4}{4}$  (b)  $\frac{3a^2}{4}$  (c)  $\frac{5a^4}{4}$  (d)  $\frac{3a^4}{4}$

- 19** The points  $A(a, b + c)$ ,  $B(b, c + a)$  and  $C(c, a + b)$  are  
(a) vertices of an isosceles triangle  
(b) vertices of an equilateral triangle  
(c) collinear  
(d) None of the above

- 20** If  $A_1, B_1, C_1, \dots$  are respectively the co-factors of the elements  $x_1, y_1, z_1, \dots$  of the determinant

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ then } \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} \text{ equals to}$$

- (a)  $x_1 \Delta$  (b)  $x_1 x_3 \Delta$   
(c)  $(x_1 + y_1) \Delta$  (d) None of these
- 21** If  $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$ , then  $\text{adj}(3A^2 + 12A)$  is equal to  
→ JEE Mains 2017
- (a)  $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$  (b)  $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$   
(c)  $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$  (d)  $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

- 22** Which of the following is/are incorrect?  
(i) Adjoint of a symmetric matrix is symmetric  
(ii) Adjoint of a unit matrix is a unit matrix  
(iii)  $A(\text{adj } A) = (\text{adj } A)A = |A|I$   
(iv) Adjoint of a diagonal matrix is a diagonal matrix.  
(a) (i) (b) (ii)  
(c) (iii) and (iv) (d) None of these

- 23** If  $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$  is the adjoint of a  $3 \times 3$  matrix  $A$  and  $|A| = 4$ , then  $\alpha$  is equal to  
→ JEE Mains 2013
- (a) 4 (b) 11 (c) 5 (d) 0

- 24** If  $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$  and  $A \text{ adj } A = AA^T$ , then  $5a + b$  is equal to  
→ JEE Mains 2016
- (a) -1 (b) 5 (c) 4 (d) 13

- 25** If  $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ ,  $B = \begin{bmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{bmatrix}$  and if  $A$  is invertible, then which of the following is not true?  
(a)  $|A| = -|B|$  and  $|\text{adj } A| \neq |\text{adj } B|$   
(b)  $|A| = -|B|$  and  $|\text{adj } A| = |\text{adj } B|$   
(c)  $A$  is invertible iff  $B$  is invertible  
(b) None of the above

- 26** If  $A$  an  $3 \times 3$  non-singular matrix such that  $AA' = A'A$  and  $B = A^{-1}A'$ , then  $BB'$  is equal to  
→ JEE Mains 2014
- (a)  $I + B$  (b)  $I$  (c)  $B^{-1}$  (d)  $(B^{-1})'$

- 27** If  $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , then  
(a)  $a = 1, b = 1$   
(b)  $a = \sin 2\theta, b = \cos 2\theta$   
(c)  $a = \cos 2\theta, b = \sin 2\theta$   
(d) None of the above

- 28** If  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$  and  $u_1, u_2$  are column matrices such

$$\text{that } Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ then } u_1 + u_2 \text{ is equal to}$$

- (a)  $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$  (b)  $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$  (c)  $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$  (d)  $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

- 29** The number of values of  $k$ , for which the system of equations  $(k + 1)x + 8y = 4k$  and  $kx + (k + 3)y = 3k - 1$  has no solution, is  
→ JEE Mains 2013
- (a) infinite (b) 1 (c) 2 (d) 3

- 30** The system of equations  
 $\alpha x + y + z = \alpha - 1$ ,  $x + \alpha y + z = \alpha - 1$   
and  $x + y + \alpha z = \alpha - 1$   
has no solution, if  $\alpha$  is  
(a) 1 (b) not -2  
(c) either -2 or 1 (d) -2

- 31** If the system of linear equations  
 $x_1 + 2x_2 + 3x_3 = 6$ ,  $x_1 + 3x_2 + 5x_3 = 9$   
 $2x_1 + 5x_2 + ax_3 = b$   
is consistent and has infinite number of solutions, then  
(a)  $a = 8, b$  can be any real number  
(b)  $b = 15, a$  can be any real number  
(c)  $a \in R - \{8\}$  and  $b \in R - \{15\}$   
(d)  $a = 8, b = 15$

- 32** If the trivial solution is the only solution of the system of equations

$$x - ky + z = 0, \quad kx + 3y - kz = 0$$

and  $3x + y - z = 0$

Then, the set of all values of  $k$  is

- (a)  $\{2, -3\}$  (b)  $R - \{2, -3\}$  (c)  $R - \{2\}$  (d)  $R - \{-3\}$

- 33** Let  $A$ , other than  $I$  or  $-I$ , be a  $2 \times 2$  real matrix such that  $A^2 = I$ ,  $I$  being the unit matrix. Let  $\text{tr}(A)$  be the sum of diagonal elements of  $A$ . → JEE Mains 2013

**Statement I**  $\text{tr}(A) = 0$

**Statement II**  $\det(A) = -1$

- (a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I  
 (b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I  
 (c) Statement I is true; Statement II is false  
 (d) Statement I is false; Statement II is true

- 34** The set of all values of  $\lambda$  for which the system of linear equations  $2x_1 - 2x_2 + x_3 = \lambda x_1$ ,  $2x_1 - 3x_2 + 2x_3 = \lambda x_2$  and  $-x_1 + 2x_2 = \lambda x_3$  has a non-trivial solution.

- (a) is an empty set → JEE Mains 2015  
 (b) is a singleton set  
 (c) contains two elements  
 (d) contains more than two elements

- 35 Statement I** Determinant of a skew-symmetric matrix of order 3 is zero.

**Statement II** For any matrix  $A$ ,  $\det(A^T) = \det(A)$  and  $\det(-A) = -\det(A)$ .

Where,  $\det(A)$  denotes the determinant of matrix  $A$ . Then, → JEE Mains 2013

- (a) Statement I is true and Statement II is false  
 (b) Both statements are true  
 (c) Both statements are false  
 (d) Statement I is false and Statement II is true

## DAY PRACTICE SESSION 2

# PROGRESSIVE QUESTIONS EXERCISE

- 1** If  $a^2 + b^2 + c^2 = -2$  and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

then  $f(x)$  is a polynomial of degree.

- (a) 2 (b) 3 (c) 0 (d) 1

- 2** If  $A$  is a square matrix of order 3 such that  $|A| = 2$ , then  $|(\text{adj } A^{-1})^{-1}|$  is

- (a) 1 (b) 2 (c) 4 (d) 8

- 3** The equations  $(k-1)x + (3k+1)y + 2kz = 0$ ,

$$(k-1)x + (4k-2)y + (k+3)z = 0$$

and  $2x + (3k+1)y + 3(k-1)z = 0$

gives non-trivial solution for some values of  $k$ , then the ratio  $x : y : z$  when  $k$  has the smallest of these values.

- (a) 3:2:1 (b) 3:3:2 (c) 1:3:1 (d) 1:1:1

- 4** If  $x = 1 + 2 + 4 + \dots$  upto  $k$  terms,  $y = 1 + 3 + 9 + \dots$  upto  $k$  terms and  $c = 1 + 5 + 25 + \dots$  upto  $k$  terms. Then,

$$\Delta = \begin{vmatrix} x & 2y & 4z \\ 3 & 3 & 3 \\ 2^k & 3^k & 5^k \end{vmatrix} \text{ equals to}$$

- (a)  $(20)^k$  (b)  $5^k$  (c) 0 (d)  $2^k + 3^k + 5^k$

- 5** Product of roots of equation  $\begin{vmatrix} 1+2x & 1 & 1-x \\ 2-x & 2+x & 3+x \\ x & 1+x & 1-x^2 \end{vmatrix} = 0$

- (a)  $\frac{1}{2}$  (b)  $\frac{3}{4}$  (c)  $\frac{4}{3}$  (d)  $\frac{1}{4}$

- 6** If the equations  $a(y+z) = x$ ,  $b(z+x) = y$ ,  $c(x+y) = z$  have non-trivial solution, then  $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$  is equal to

- (a) 1 (b) 2 (c) -1 (d) -2

- 7** Let  $a, b$  and  $c$  be positive real numbers. The following system of equations in  $x, y$  and  $z$ .

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has

- (a) no solution (b) unique solution  
 (c) infinitely many solutions (d) finitely many solutions

- 8** If  $S$  is the set of distinct values of  $b$  for which the following system of linear equations

$$x + y + z = 1, \quad x + ay + z = 1 \text{ and } ax + by + z = 0$$

has no solution, then  $S$  is → JEE Mains 2017

- (a) an infinite set  
 (b) a finite set containing two or more elements  
 (c) singleton set  
 (d) an empty set

- 9** If  $M$  is a  $3 \times 3$  matrix, where  $M^T M = I$  and  $\det(M) = 1$ , then the value of  $\det(M - I)$  is

- (a) -1 (b) 1  
 (c) 0 (d) None of these

- 10** If  $a_1, a_2, \dots, a_n, \dots$  are in GP, then the determinant

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix} \text{ is equal to}$$

- (a) 2 (b) 4 (c) 0 (d) 1

**11** Let  $A$  be a square matrix of order 2 with  $|A| \neq 0$  such that  $|A + |A|\text{adj}(A)| = 0$ , then the value of  $|A - |A|\text{adj}(A)|$  is

- (a) 1 (b) 2  
(c) 3 (d) 4

**12** Let  $P$  and  $Q$  be  $3 \times 3$  matrices  $P \neq Q$ . If  $P^3 = Q^3$  and  $P^2Q = Q^2P$ , then the determinant of  $(P^2 + Q^2)$  is

- (a) -2 (b) 1  
(c) 0 (d) -1

**13** If  $\alpha, \beta \neq 0$ ,  $f(n) = \alpha^n + \beta^n$  and  $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$= K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$ , then  $K$  is equal to

- (a)  $\alpha\beta$  (b)  $\frac{1}{\alpha\beta}$  (c) 1 (d) -1

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**14** Area of triangle whose vertices are  $(a, a^2), (b, b^2), (c, c^2)$  is  $\frac{1}{2}$ , and the area of triangle whose vertices are

$(p, p^2), (q, q^2)$  and  $(r, r^2)$  is 4, then the value of

$$\begin{vmatrix} (1+ap)^2 & (1+bp)^2 & (1+cp)^2 \\ (1+aq)^2 & (1+bq)^2 & (1+cq)^2 \\ (1+ar)^2 & (1+br)^2 & (1+cr)^2 \end{vmatrix} \text{ is}$$

- (a) 2 (b) 4 (c) 8 (d) 16

**15** Let  $\det A = \begin{vmatrix} l & m & n \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$  and if  $(l-m)^2 + (p-q)^2 = 9$ ,

$(m-n)^2 + (q-r)^2 = 16$ ,  $(n-l)^2 + (r-p)^2 = 25$ , then the value of  $(\det A)^2$  equals to

- (a) 36 (b) 100 (c) 144 (d) 169

## ANSWERS

### SESSION 1

- |        |        |        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1 (b)  | 2 (b)  | 3 (a)  | 4 (a)  | 5 (a)  | 6 (a)  | 7 (a)  | 8 (a)  | 9 (c)  | 10 (c) |
| 11 (c) | 12 (c) | 13 (c) | 14 (a) | 15 (b) | 16 (b) | 17 (d) | 18 (d) | 19 (c) | 20 (a) |
| 21 (b) | 22 (d) | 23 (b) | 24 (b) | 25 (a) | 26 (b) | 27 (c) | 28 (d) | 29 (d) | 30 (d) |
| 31 (d) | 32 (b) | 33 (b) | 34 (c) | 35 (a) |        |        |        |        |        |

### SESSION 2

- |        |        |        |        |        |       |       |       |       |        |
|--------|--------|--------|--------|--------|-------|-------|-------|-------|--------|
| 1 (a)  | 2 (c)  | 3 (d)  | 4 (c)  | 5 (a)  | 6 (b) | 7 (d) | 8 (c) | 9 (c) | 10 (c) |
| 11 (d) | 12 (c) | 13 (c) | 14 (d) | 15 (c) |       |       |       |       |        |

# Hints and Explanations

## SESSION 1

- 1** Given equation can be rewritten as

$$x - cy - bz = 0$$

$$cx - y + az = 0$$

$$bx + ay - z = 0$$

Since,  $x, y$  and  $z$  are not all zero

$\therefore$  The above system have non-trivial solution.

$$\therefore \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$$

$$\Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 = 1 - 2abc$$

- 2** If we interchange any two rows of a determinant in the set  $B$ , its value becomes  $-1$  and hence it is in  $C$ . Likewise, for every determinant in  $C$ , there is corresponding determinant in  $B$ . So,  $B$  and  $C$  have the same number of elements.

**3** Let  $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} \frac{\log x}{\log y} & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & \frac{\log z}{\log z} \end{vmatrix}$$

By taking common factors from the rows, we get

$$\Delta = \frac{1}{\log x \cdot \log y \cdot \log z}$$

$$\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

Now, by taking common factor from the

columns, we get  $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

**4** Let  $\Delta = \begin{vmatrix} e^a & e^{2a} & e^{3a} \\ e^b & e^{2b} & e^{3b} \\ e^c & e^{2c} & e^{3c} \end{vmatrix} - \begin{vmatrix} e^a & e^{2a} & 1 \\ e^b & e^{2b} & 1 \\ e^c & e^{2c} & 1 \end{vmatrix}$

$$= e^a \cdot e^b \cdot e^c$$

$$\begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} + \begin{vmatrix} e^a & 1 & e^{2a} \\ e^b & 1 & e^{2b} \\ e^c & 1 & e^{2c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} - \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix}$$

$$= 0 \quad [\because a + b + c = 0 \Rightarrow e^{a+b+c} = 1]$$

- 5** Put  $x = 0$  in the given equation, we get

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 12 = 0$$

- 6** On taking common factors  $x$  from  $C_2$ ,  $(x + 1)$  from  $C_3$  and  $(x - 1)$  from  $R_3$ , we get

$$f(x) = x(x^2 - 1) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x - 1 & x \\ 3x & x - 2 & x \end{vmatrix}$$

$$= x(x^2 - 1) \begin{vmatrix} 1 & 0 & 0 \\ 2x & -(x + 1) & 1 \\ 3x & -2(x + 1) & 2 \end{vmatrix}$$

$$= 0 \quad \begin{matrix} [C_3 \rightarrow C_3 - C_2] \\ [C_2 \rightarrow C_2 - C_1] \end{matrix}$$

$$\therefore f(50) = 0$$

- 7** Clearly,  $\alpha + \beta + \gamma = 0$

On applying  $C_1 \rightarrow C_1 + C_2 + C_3$  in the given determinant, we get

$$\therefore \Delta = \begin{vmatrix} \alpha + \beta + \gamma & \beta & \gamma \\ \alpha + \beta + \gamma & \gamma & \alpha \\ \alpha + \beta + \gamma & \alpha & \beta \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

- 8** On applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$\begin{vmatrix} x & \omega & \omega^2 \\ x & x + \omega^2 & 1 \\ x & 1 & x + \omega \end{vmatrix} = 0$$

$$[\because 1 + \omega + \omega^2 = 0]$$

$$\Rightarrow \begin{vmatrix} 1 & \omega & \omega^2 \\ x & x + \omega^2 & 1 \\ 1 & 1 & x + \omega \end{vmatrix} = 0$$

$$\therefore x = 0$$

**9**  $\Delta = \begin{vmatrix} a(1 + \omega) & b\omega^2 & a\omega \\ b(\omega + \omega^2) & c & b\omega^2 \\ c(\omega^2 + 1) & a\omega & c \end{vmatrix}$

$$= \begin{vmatrix} -a\omega^2 & b\omega^2 & a\omega \\ -b & c & b\omega^2 \\ -c\omega & a\omega & c \end{vmatrix} \quad [\because C_1 \rightarrow C_1 + C_3]$$

$$= \omega^2 \cdot \omega \begin{vmatrix} -a & b & a\omega^2 \\ -b & c & b\omega^2 \\ -c & a & c\omega^2 \end{vmatrix}$$

$$= -\omega^5 \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} = 0$$

**10** Given,  $\begin{vmatrix} x - 4 & 2x & 2x \\ 2x & x - 4 & 2x \\ 2x & 2x & x - 4 \end{vmatrix}$

$$= (A + Bx)(x - A)^2$$

On applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$\begin{vmatrix} 5x - 4 & 2x & 2x \\ 5x - 4 & x - 4 & 2x \\ 5x - 4 & 2x & x - 4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

On taking common  $(5x - 4)$  from  $C_1$ , we get

$$(5x - 4) \begin{vmatrix} 1 & 2x & 2x \\ 1 & x - 4 & 2x \\ 1 & 2x & x - 4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

On applying  $R_2 \rightarrow R_2 - R_1$

and  $R_3 \rightarrow R_3 - R_1$ , we get

$$\therefore (5x - 4) \begin{vmatrix} 1 & 2x & 2x \\ 0 & -x - 4 & 0 \\ 0 & 0 & -x - 4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

On expanding along  $C_1$ , we get

$$(5x - 4)(x + 4)^2 = (A + Bx)(x - A)^2$$

On comparing, we get

$$A = -4 \text{ and } B = 5$$

**11** Let  $\Delta = \begin{vmatrix} 1 + x & 1 & 1 \\ 1 + y & 1 + 2y & 1 \\ 1 + z & 1 + z & 1 + 3z \end{vmatrix} = 0$

On applying  $C_1 \rightarrow C_1 - C_3$

and  $C_2 \rightarrow C_2 - C_3$ , we get

$$\Delta = \begin{vmatrix} x & 0 & 1 \\ y & 2y & 1 \\ -2z & -2z & 1 + 3z \end{vmatrix}$$

On expanding along  $R_1$ , we get

$$\Delta = x[2y(1 + 3z) + 2z]$$

$$+ 1[-2yz + 4yz] = 0$$

$$\Rightarrow 2[xy + 3xyz + xz] + 2yz = 0$$

$$\Rightarrow xy + yz + zx + 3xyz = 0$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -3$$

**12** Let  $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$$

[using  $C_1 \rightarrow C_1 + C_2 + C_3$ ]

$$= 2 \begin{vmatrix} a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \\ a+b+c & b+c & c+a \end{vmatrix}$$

[taking common 2 from  $C_1$ ]

$$= 2 \begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -c & -a \\ a+b+c & -a & -b \end{vmatrix}$$

[using  $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$ ]

$$= 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

[using  $C_1 \rightarrow C_1 + C_2 + C_3$ ]

On comparing,  $k = 2$

**13**  $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$$

[ $\therefore |A'| = |A|$ ]

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$$

$$= [1 + (-1)^{n+2}] \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

This is equal to zero only, if  $n + 2$  is an odd i.e.  $n$  is an odd integer.

**14**  $\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$

On applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we get

$$\begin{vmatrix} x^2-4 & x^2-4 & x^2-4 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x^2 - 4) \begin{vmatrix} 1 & 1 & 1 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix} = 0$$

Now, on applying  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we get

$$(x^2 - 4) \begin{vmatrix} 1 & 0 & 0 \\ 2 & x^2 - 15 & 0 \\ x^2 - 13 & 16 - x^2 & 20 - x^2 \end{vmatrix} = 0$$

On expanding along  $R_1$ , we get  $(x^2 - 4)(x^2 - 15)(x^2 - 20) = 0$

Thus, other five roots are  $-2, \pm\sqrt{15}, \pm 2\sqrt{5}$

Hence, sum of other five roots is  $-2$ .

**15** Let  $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -(a^3 + b^3 + c^3 - 3abc)$$

$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= -\frac{1}{2}(a+b+c) \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

$< 0$ , where  $a \neq b \neq c$

**16** We have,

$$x + ky + 3z = 0; 3x + ky - 2z = 0; 2x + 4y - 3z = 0$$

System of equation has non-zero solution, if

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$$

$$\Rightarrow (-3k + 8) - k(-9 + 4) + 3(12 - 2k) = 0$$

$$\Rightarrow -3k + 8 + 9k - 4k + 36 - 6k = 0$$

$$\Rightarrow -4k + 44 = 0 \Rightarrow k = 11$$

Let  $z = \lambda$ , then we get

$$x + 11y + 3\lambda = 0 \quad \dots(i)$$

$$3x + 11y - 2\lambda = 0 \quad \dots(ii)$$

$$\text{and } 2x + 4y - 3\lambda = 0 \quad \dots(iii)$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{5\lambda}{2}, y = \frac{-\lambda}{2}, z = \lambda$$

$$\Rightarrow \frac{xz}{y^2} = \frac{5\lambda^2}{2 \times \left(\frac{-\lambda}{2}\right)^2} = 10$$

**17** We have,

$$a_{ij} = (i^2 + j^2 - ij)(j - i)$$

$$\therefore a_{ji} = (i^2 + j^2 - ij)(i - j) = -(i^2 + j^2 - ij)(j - i) = -a_{ij}$$

$\Rightarrow A$  is a skew-symmetric matrix of odd order.

$$\therefore \text{tr}(A) = 0 \text{ and } |A| = 0$$

**18** If  $(x_1, y_1), (x_2, y_2)$  and  $(x_3, y_3)$  are the vertices of a triangle, then Area

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \dots(i)$$

Also, we know that, if  $a$  is the length of an equilateral triangle, then

$$\text{Area} = \frac{\sqrt{3}}{4} a^2 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{\sqrt{3}}{4} a^2 = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \frac{\sqrt{3}}{2} a^2 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

On squaring both sides, we get

$$\frac{3}{4} a^4 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

**19** Clearly, area of

$$(\Delta ABC) = \frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2} (a+b+c) \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0$$

[ $\therefore C_2 \rightarrow C_2 + C_1$ ]

**20** Clearly,  $B_2 = \begin{vmatrix} x_1 & z_1 \\ x_3 & z_3 \end{vmatrix} = x_1 z_3 - x_3 z_1$

$$C_2 = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -(x_1 y_3 - x_3 y_1)$$

$$B_3 = -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} = -(x_1 z_2 - x_2 z_1)$$

$$\text{and } C_3 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - y_1 x_2$$

$$\therefore \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 z_3 - x_3 z_1 & -(x_1 y_3 - y_1 x_3) \\ -(x_1 z_2 - x_2 z_1) & x_1 y_2 - x_2 y_1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 z_3 & -x_1 y_3 \\ -x_1 z_2 + x_2 z_1 & x_1 y_2 - x_2 y_1 \end{vmatrix}$$

$$+ \begin{vmatrix} -x_3 z_1 & y_1 x_3 \\ -x_1 z_2 + x_2 z_1 & x_1 y_2 - x_2 y_1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 z_3 & -x_1 y_3 \\ -x_1 z_2 & x_1 y_2 \end{vmatrix} + \begin{vmatrix} x_1 z_3 & -x_1 y_3 \\ x_2 z_1 & -x_2 y_1 \end{vmatrix}$$

$$+ \begin{vmatrix} -x_3 z_1 & y_1 x_3 \\ -x_1 z_2 & x_1 y_2 \end{vmatrix} + \begin{vmatrix} -x_3 z_1 & y_1 x_3 \\ x_2 z_1 & -x_2 y_1 \end{vmatrix}$$

$$= x_1^2 (z_3 y_2 - z_2 y_3) - x_1 x_2 (z_3 y_1 - z_1 y_3)$$

$$- x_1 x_3 (z_1 y_2 - z_2 y_1) + x_2 x_3 (z_1 y_1 - z_1 y_1)$$

$$= x_1 [x_1 (z_3 y_2 - z_2 y_3) - x_2 (z_3 y_1 - z_1 y_3)$$

$$+ x_3 (z_2 y_1 - z_1 y_2)]$$

$$= x_1 \Delta$$

**21** We have,  $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$   
 $\therefore A^2 = A \cdot A$   
 $= \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$   
 $= \begin{bmatrix} 4 + 12 & -6 - 3 \\ -8 - 4 & 12 + 1 \end{bmatrix} = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$   
Now,  $3A^2 + 12A = 3 \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$

$$+ 12 \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

$$\therefore \text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$$

**22** All the given statements are true.

**23** Given,  $\text{adj } A = P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$

Find the determinant of  $P$  and apply the formula

$$|\text{adj } A| = |A|^{n-1}$$

**24** Given,  $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$

and  $A \text{ adj } A = AA^T$

Clearly,  $A(\text{adj } A) = |A| I_2$

[ $\because$  if  $A$  is square matrix of order  $n$ , then  $A(\text{adj } A) = (\text{adj } A) \cdot A = |A| I_n$ ]

$$= \begin{vmatrix} 5a & -b \\ 3 & 2 \end{vmatrix} I_2$$

$$= (10a + 3b) I_2$$

$$= (10a + 3b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} \quad \dots(i)$$

and  $AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} \quad \dots(ii)$$

$$\therefore A(\text{adj } A) = AA^T$$

$$\therefore \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix}$$

$$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$$

[using Eqs. (i) and (ii)]

$$\Rightarrow 15a - 2b = 0$$

$$\Rightarrow a = \frac{2b}{15} \quad \dots(iii)$$

$$\text{and } 10a + 3b = 13 \quad \dots(iv)$$

On substituting the value of 'a' from Eq. (iii) in Eq. (iv), we get

$$10 \left( \frac{2b}{15} \right) + 3b = 13$$

$$\Rightarrow \frac{20b + 45b}{15} = 13$$

$$\Rightarrow \frac{65b}{15} = 13 \Rightarrow b = 3$$

Now, substituting the value of  $b$  in Eq. (iii), we get  $5a = 2$

$$\text{Hence, } 5a + b = 2 + 3 = 5$$

**25** Clearly,  $|B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$

$$= - \begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix}$$

(taken  $(-1)$  common from  $R_2$ )

$$= (+) \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix}$$

(taken  $(-1)$  common from  $C_2$ )

$$= - \begin{vmatrix} p & a & x \\ q & b & y \\ r & c & z \end{vmatrix} \quad (\because R_1 \leftrightarrow R_2)$$

$$= - \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$$

( $\because C_1 \leftrightarrow C_2$  and then  $C_2 \leftrightarrow C_3$ )

$$= -|A^T| = -|A|$$

Thus,  $|A| = -|B|$

Hence,  $|A| \neq 0$  iff  $|B| \neq 0$

$\therefore A$  is invertible iff  $B$  is invertible

Also,  $|\text{adj } A| = |A|^2 = |B|^2 = |\text{adj } B|$

**26** Given,  $A$  is non-singular matrix

$\Rightarrow |A| \neq 0$ .

Also we have,  $AA^T = A^T A$  and

$$B = A^{-1} A^T$$

$$\text{Now, } BB^T = (A^{-1} A^T) (A^{-1} A^T)^T$$

$$= A^{-1} A^T A (A^{-1})^T \quad [\because (A^T)^T = A]$$

$$= A^{-1} AA^T (A^{-1})^T \quad [\because AA^T = A^T A]$$

$$= I A^T (A^{-1})^T = A^T (A^{-1})^T$$

$$= (A^{-1} A)^T \quad [\because (AB)^T = B^T A^T]$$

$$= I^T = I$$

**27** We have,

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \cdot \frac{1}{1 + \tan^2\theta}$$

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \frac{1}{1 + \tan^2\theta} \begin{bmatrix} 1 - \tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1 - \tan^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1 - \tan^2\theta}{1 + \tan^2\theta} & \frac{-2\tan\theta}{1 + \tan^2\theta} \\ \frac{2\tan\theta}{1 + \tan^2\theta} & \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow a = \cos 2\theta \text{ and } b = \sin 2\theta$$

**28** On adding  $Au_1$  and  $Au_2$ , we get

$$Au_1 + Au_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 0 \\ 0 + 1 \\ 0 + 0 \end{bmatrix}$$

$$\Rightarrow A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since,  $A$  is non-singular matrix, i.e.  $|A| \neq 0$ , on multiplying both sides by  $A^{-1}$ , we get

$$A^{-1} A(u_1 + u_2) = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \dots(i)$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1(1 - 0) - 0 + 0 = 1$$

$\therefore$  adj of  $A$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad [\because |A| = 1]$$

From Eq. (i),

$$u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 + u_2 = \begin{bmatrix} 1 + 0 + 0 \\ -2 + 1 + 0 \\ 1 - 2 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

**29** Given equations can be written in matrix form as  $AX = B$

$$\text{where, } A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$$

For no solution,  $|A| = 0$

and  $(\text{adj } A) B \neq 0$

$$\text{Now, } |A| = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow (k+1)(k+3) - 8k &= 0 \\ \Rightarrow k^2 + 4k + 3 - 8k &= 0 \\ \Rightarrow k^2 - 4k + 3 &= 0 \\ \Rightarrow (k-1)(k-3) &= 0 \\ \therefore k &= 1, k = 3 \end{aligned}$$

$$\text{Also, } \text{adj } A = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}$$

$$\begin{aligned} \therefore (\text{adj } A)B &= \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix} \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix} \\ &= \begin{bmatrix} (k+3)(4k) - 8(3k-1) \\ -4k^2 + (k+1)(3k-1) \end{bmatrix} \\ &= \begin{bmatrix} 4k^2 - 12k + 8 \\ -k^2 + 2k - 1 \end{bmatrix} \end{aligned}$$

Put  $k = 1$ , then

$$(\text{adj } A)B = \begin{bmatrix} 4-12+8 \\ -1+2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ not true.}$$

Put  $k = 3$ , then  $(\text{adj } A)$

$$B = \begin{bmatrix} 36-36+8 \\ -9+6-1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \neq 0, \text{ true.}$$

Hence, the required value of  $k$  is 3.

**Alternate Method** Condition for the system of equations has no solution is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow \frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$$

$$\text{Take } \frac{k+1}{k} = \frac{8}{k+3}$$

$$\begin{aligned} \Rightarrow k^2 + 4k + 3 &= 8k \\ \Rightarrow k^2 - 4k + 3 &= 0 \\ \Rightarrow (k-1)(k-3) &= 0 \end{aligned}$$

$$\therefore k = 1, 3$$

$$\text{If } k = 1, \text{ then } \frac{8}{1+3} \neq \frac{4 \cdot 1}{2}, \quad [\text{false}]$$

$$\text{and } k = 3, \text{ then } \frac{8}{6} \neq \frac{4 \cdot 3}{9-1}, \quad [\text{true}]$$

$$\therefore k = 3$$

Hence, only one value of  $k$  exists.

**30** The system of given equations has no solution, if

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

On applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$\begin{vmatrix} \alpha+2 & 1 & 1 \\ \alpha+2 & \alpha & 1 \\ \alpha+2 & 1 & \alpha \end{vmatrix} = 0$$

On applying  $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$ ,

$$(\alpha+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-1 \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+2)(\alpha-1)^2 = 0$$

$$\therefore \alpha = 1, -2$$

But  $\alpha = 1$  makes given three equations same. So, the system of equations has infinite solution. Hence, answer is  $\alpha = -2$  for which the system of equations has no solution.

**31** For consistency  $|A| = 0$  and  $(\text{adj } A)B = O$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{vmatrix} = 0$$

$$\Rightarrow 1(3a-25) - 2(a-10) + 3(5-6) = 0$$

$$\Rightarrow a = 8$$

On solving,  $(\text{adj } A)B = O$ , we get  $b = 15$

**32** Since,  $x - ky + z = 0$ ,

$$kx + 3y - kz = 0 \text{ and}$$

$$3x + y - z = 0 \text{ has trivial solution.}$$

$$\therefore \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} \neq 0$$

$$\Rightarrow 1(-3+k) + k(-k+3k) + 1(k-9) \neq 0$$

$$\Rightarrow k - 3 + 2k^2 + k - 9 \neq 0$$

$$\Rightarrow 2k^2 + 2k - 12 \neq 0$$

$$\Rightarrow k^2 + k - 6 \neq 0$$

$$\Rightarrow (k+3)(k-2) \neq 0$$

$$\therefore k \neq 2, -3$$

$$k \in R - \{2, -3\}$$

**33** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $A \neq I, -I$

$$\text{and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow b(a+d) = 0, c(a+d) = 0$$

$$\text{and } a^2+bc = 1, bc+d^2 = 1$$

$$\Rightarrow a = 1, d = -1, b = c = 0 \text{ or}$$

$$a = -1, d = 1, b = c = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ then}$$

$$\det(A) = -1 \text{ and } \text{tr}(A) = 1 - 1 = 0$$

**34** Given system of linear equations are

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$

$$\Rightarrow (2-\lambda)x_1 - 2x_2 + x_3 = 0 \quad \dots(\text{i})$$

$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$

$$\Rightarrow 2x_1 - (3+\lambda)x_2 + 2x_3 = 0 \quad \dots(\text{ii})$$

$$\text{and } -x_1 + 2x_2 = \lambda x_3$$

$$\Rightarrow -x_1 + 2x_2 - \lambda x_3 = 0 \quad \dots(\text{iii})$$

Since, the system has non-trivial solution.

$$\therefore \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3\lambda + \lambda^2 - 4) + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 + 3\lambda - 4) + 4(1-\lambda) + (1-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(\lambda+4)(\lambda-1) + 5(1-\lambda) = 0$$

$$\Rightarrow (\lambda-1)[(2-\lambda)(\lambda+4) - 5] = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 + 2\lambda - 3) = 0$$

$$\Rightarrow (\lambda-1)[(\lambda-1)(\lambda+3)] = 0$$

$$\Rightarrow (\lambda-1)^2(\lambda+3) = 0$$

$$\Rightarrow \lambda = 1, 1, -3$$

**35** Determinant of skew-symmetric matrix of odd order is zero and of even order is perfect square.

$$\text{Also, } \det(A^T) = \det(A)$$

$$\text{and } \det(-A) = (-1)^n \det(A)$$

Hence, Statement II is false.

## SESSION 2

**1** Given that,

$$f(x) = \begin{vmatrix} 1 + a^2x & (1 + b^2)x & (1 + c^2)x \\ (1 + a^2)x & 1 + b^2x & (1 + c^2)x \\ (1 + a^2)x & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

On applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$f(x) = \begin{vmatrix} 1 + a^2x + x + b^2x + x + c^2x & (1 + b^2)x & (1 + c^2)x \\ x + a^2x + 1 + b^2x + x + c^2x & 1 + b^2x & (1 + c^2)x \\ x + a^2x + x + b^2x + 1 + c^2x & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

$$(1 + b^2)x \quad (1 + c^2)x$$

$$1 + b^2x \quad (1 + c^2)x$$

$$(1 + b^2)x \quad 1 + c^2x$$

$$= \begin{vmatrix} 1 + (a^2 + b^2 + c^2 + 2)x & (1 + b^2)x \\ 1 + (a^2 + b^2 + c^2 + 2)x & (1 + b^2)x \\ 1 + (a^2 + b^2 + c^2 + 2)x & (1 + b^2)x \end{vmatrix}$$

$$(1 + c^2)x$$

$$(1 + c^2)x$$

$$1 + c^2x$$

$$= \begin{vmatrix} 1 & (1 + b^2)x & (1 + c^2)x \\ 1 & 1 + b^2x & (1 + c^2)x \\ 1 & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

$$[\because a^2 + b^2 + c^2 = -2, \text{ given}]$$

On applying

$R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$ , we get

$$= \begin{vmatrix} 0 & 0 & x-1 \\ 0 & 1-x & x-1 \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x-1 \\ 1-x & x-1 \end{vmatrix} = (x-1)^2$$

Hence,  $f(x)$  is of degree 2.

**2** Clearly,  $|\text{adj}A^{-1}| = |A^{-1}|^2 = \frac{1}{|A|^2}$

and  $|(\text{adj}A^{-1})^{-1}| = \frac{1}{|(\text{adj}A^{-1})|} = |A|^2 = 2^2 = 4$

**3** For non-trivial solution,

$$\begin{vmatrix} k-1 & 3k+1 & 2k \\ k-1 & 4k-2 & k+3 \\ 2 & 3k+1 & 3(k-1) \end{vmatrix} = 0$$

$\Rightarrow k = 0$  or  $3$

Now, when  $k = 0$ , then the equation becomes

$-x + y = 0$  ... (i)

$-x - 2y + 3z = 0$  ... (ii)

and  $2x + y - 3z = 0$  (iii)

From (i), we get  $x = y = \lambda$  (say), Then, from Eq. (ii), we get

$-\lambda - 2\lambda + 3z = 0$

$\Rightarrow 3z = 3\lambda$

$\Rightarrow z = \lambda$

$\therefore x : y : z = 1 : 1 : 1$

**4** Clearly,  $x = 2^k - 1, y = \frac{3^k - 1}{2}$

and  $z = \frac{5^k - 1}{4}$

$\therefore$  sum to  $n$  terms of  $a$  GP is

given by  $\frac{a(r^n - 1)}{r - 1}$

Now,  $\Delta = \begin{vmatrix} 2^k - 1 & 3^k - 1 & 5^k - 1 \\ 3 & 3 & 3 \\ 2^k & 3^k & 5^k \end{vmatrix}$

$= \begin{vmatrix} 2^k & 3^k & 5^k \\ 3 & 3 & 3 \\ 2^k & 3^k & 5^k \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2^k & 3^k & 5^k \end{vmatrix}$

$= 0 - 3 \times 0 = 0$

**5** Let  $p(x) = \begin{vmatrix} 1+2x & 1 & 1-x \\ 2-x & 2+x & 3+x \\ x & 1+x & 1-x^2 \end{vmatrix} = 0$

Clearly, product of roots =  $\frac{\text{Constant term}}{\text{Coefficient of } x^4}$

Here, constant term is given by

$P(0) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -1$

and coefficient of  $x^4$  is  $-2$

$\therefore$  Product of roots is  $\frac{1}{2}$

**6** Here,  $\begin{vmatrix} -1 & a & a \\ b & -1 & b \\ c & c & -1 \end{vmatrix} = 0$

On applying,  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we get

$\begin{vmatrix} -1 & a+1 & a+1 \\ b & -(b+1) & 0 \\ c & 0 & -(1+c) \end{vmatrix} = 0$

On applying  $R_1 \rightarrow \frac{R_1}{a+1}, R_2 \rightarrow \frac{R_2}{b+1}$

and  $R_3 \rightarrow \frac{R_3}{c+1}$ , we get

$\begin{vmatrix} -\frac{1}{a+1} & 1 & 1 \\ \frac{b}{b+1} & -1 & 0 \\ \frac{c}{c+1} & 0 & -1 \end{vmatrix} = 0$

$\Rightarrow -\frac{1}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 0$

$\therefore -\frac{1}{a+1} + 1 - \frac{1}{b+1} + 1 - \frac{c}{c+1} = 0$

$\Rightarrow \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$

**7** Let  $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y, \frac{z^2}{c^2} = Z$

Then,  $X + Y - Z = 1, X - Y + Z = 1, -X + Y + Z = 1$

Now, coefficient matrix is

$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Here,  $|A| \neq 0$

$\therefore$  It has unique solution, viz.,  $X = 1, Y = 1$  and  $Z = 1$

Hence,  $x = \pm a; y = \pm b$  and  $z = \pm c$ .

**8** Here,  $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix}$

$= 1(a-b) - 1(1-a) + 1(b-a^2)$

$= -(a-1)^2$

$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{vmatrix}$

$= 1(a-b) - 1(1) + 1(b) = (a-1)$

$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 1 \end{vmatrix}$

$= 1(1) - 1(1-a) + 1(0-a) = 0$

and  $\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 0 \end{vmatrix}$

$= 1(-b) - 1(-a) + 1(b-a^2)$

$= -a(a-1)$

For  $a = 1$

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

For  $b = 1$  only

$x + y + z = 1, x + y + z = 1$

and  $x + y + z = 0$

i.e. no solution ( $\because$  RHS is not equal)

Hence, for no solution,  $b = 1$  only.

**9** Clearly,

$\det(M-I) = \det(M-I) \cdot \det(M)$   
 $[\because \det(M) = 1]$

$= \det(M-I) \det(M^T)$

$[\because \det(A^T) = \det(A)]$

$= \det(MM^T - M^T)$

$= \det(I - M^T) [\because MM^T = I]$

$= -\det(M^T - I)$

$= -\det(M^T - I)^T$

$= -\det(M - I)$

$\Rightarrow 2\det(M - I) = 0$

$\Rightarrow \det(M - I) = 0$

**10** Since,  $a_1, a_2, \dots, a_n, \dots$  are in GP, then,

$\log a_n, \log a_{n+1}, \log a_{n+2}, \dots,$

$\log a_{n+8}, \dots$  are in AP.

Given that,

$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$

$\therefore \Delta = \begin{vmatrix} a & a+d & a+2d \\ a+3d & a+4d & a+5d \\ a+6d & a+7d & a+8d \end{vmatrix}$

where  $a$  and  $d$  are the first term and common difference of AP.

On applying  $R_2 \rightarrow 2R_2$ , we get

$\Delta = \frac{1}{2} \begin{vmatrix} a & a+d & a+2d \\ 2a+6d & 2a+8d & 2a+10d \\ a+6d & a+7d & a+8d \end{vmatrix}$

On applying  $R_2 \rightarrow R_2 - R_3$ , we get

$\Delta = \frac{1}{2} \begin{vmatrix} a & a+d & a+2d \\ a & a+d & a+2d \\ a+6d & a+7d & a+8d \end{vmatrix} = 0$

**11** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then,

$|A| = ad - bc = k$  (say)

and  $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now,  $|A + |A| \text{adj}(A)| = 0$

$\Rightarrow \begin{vmatrix} a+kd & (1-k)b \\ (1-k)c & d+ka \end{vmatrix} = 0$

$\Rightarrow (a+kd)(d+ka) - (1-k)^2 bc$

$\Rightarrow ad + a^2k + kd^2 + k^2ad - (1+k^2-2k)bc$

$\Rightarrow (ad - bc) + (ad - bc)k^2 + k(ad^2 + d^2a - 2kbc) + 2kbc = 0$

$$\begin{aligned} \Rightarrow (ad - bc) + (ad - bc)k^2 + k(a^2 + d^2) + 2(ad - k) &= 0 \\ [\because bc = ad - k] \\ \Rightarrow (ad - bc) + (ad - bc)k^2 + k(a + d)^2 - 2k^2 &= 0 \\ \Rightarrow (k + k^3 - 2k^2) + k(a + d)^2 &= 0 \\ \Rightarrow k[(k - 1)^2 + (a + d)^2] &= 0 \\ \Rightarrow k = 1 \text{ and } a + d = 0 \end{aligned}$$

Now,  $|A - |A|\text{adj}A|$

$$\begin{aligned} &= \begin{vmatrix} a - kd & (1 + k)b \\ (1 + k)c & d - ak \end{vmatrix} = \begin{vmatrix} a - d & 2b \\ 2c & d - a \end{vmatrix} \\ &= -(a - d)^2 - 4bc \\ &= -((a + d)^2 - 4ad) - 4bc \\ &= 4ad - 4bc = 4k = 4 \end{aligned}$$

**12** On subtracting the given equation, we get

$$\begin{aligned} P^3 - P^2Q &= Q^3 - Q^2P \\ \Rightarrow P^2(P - Q) &= Q^2(Q - P) \\ \Rightarrow (P - Q)(P^2 + Q^2) &= 0 \quad \dots(i) \end{aligned}$$

Now, if  $|P^2 + Q^2| \neq 0$   
 $(P^2 + Q^2)$  is invertible.

On post multiply both sides by  $(P^2 + Q^2)^{-1}$ , we get

$P - Q = 0$ , which is a contradiction.  
Hence,  $|P^2 + Q^2| = 0$

**13** Let  $\Delta = \begin{vmatrix} 3 & 1 + f(1) & 1 + f(2) \\ 1 + f(1) & 1 + f(2) & 1 + f(3) \\ 1 + f(2) & 1 + f(3) & 1 + f(4) \end{vmatrix}$

$$\begin{aligned} \Rightarrow \Delta &= \begin{vmatrix} 3 & 1 + \alpha + \beta \\ 1 + \alpha + \beta & 1 + \alpha^2 + \beta^2 \\ 1 + \alpha^2 + \beta^2 & 1 + \alpha^3 + \beta^3 \end{vmatrix} \\ &= \begin{vmatrix} 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot \alpha + 1 \cdot \beta \\ 1 \cdot 1 + \alpha \cdot 1 + \beta \cdot 1 & 1 \cdot 1 + \alpha \cdot \alpha + \alpha \cdot \beta \\ 1 \cdot 1 + 1 \cdot \alpha^2 + 1 \cdot \beta^2 & 1 \cdot 1 + \alpha^2 \cdot \alpha + \beta^2 \cdot \beta \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2 \end{aligned}$$

On expanding, we get

$$\Delta = (1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2$$

$$\begin{aligned} \text{Hence, } K &= (1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2 \\ &= (1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2 \end{aligned}$$

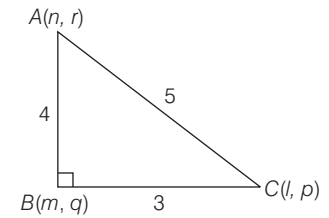
$$\therefore K = 1$$

**14** Let  $\Delta = \begin{vmatrix} (1 + ap)^2 & (1 + bp)^2 & (1 + cp)^2 \\ (1 + aq)^2 & (1 + bq)^2 & (1 + cq)^2 \\ (1 + ar)^2 & (1 + br)^2 & (1 + cr)^2 \end{vmatrix}$

$$\begin{aligned} &= \begin{vmatrix} 1 + 2ap + a^2p^2 & 1 + 2bp + b^2p^2 & 1 + 2cp + c^2p^2 \\ 1 + 2aq + a^2q^2 & 1 + 2bq + b^2q^2 & 1 + 2cq + c^2q^2 \\ 1 + 2ar + a^2r^2 & 1 + 2br + b^2r^2 & 1 + 2cr + c^2r^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 + 2cp + c^2p^2 & 1 + 2cq + c^2q^2 & 1 + 2cr + c^2r^2 \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix} \times \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \\ &= 2 \left( \begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \right) \\ &= 2(2A_1 \times 2A_2) = 2(8 \times 1) = 16 \end{aligned}$$

**15** According to given conditions we get a right angled triangle whose vertices are  $(n, r), (m, q)$  and  $(l, p)$ .



Also, we have,  $|A| = \begin{vmatrix} l & m & n \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

$$\begin{aligned} \Rightarrow |A|^2 &= \begin{vmatrix} l & p & 1 \\ m & q & 1 \\ n & r & 1 \end{vmatrix}^2 = [2ar(\Delta ABC)]^2 \\ &= \left[ 2 \times \frac{1}{2} \times 3 \times 4 \right]^2 = 144 \end{aligned}$$