VECTORS

1. Physical quantities are broadly divided in two categories viz (a) Vector Quantities & (b) Scalar quantities.

(a) Vector quantities :

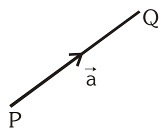
Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

2. REPRESENTATION :

Vectors are represented by directed straight line segment



magnitude of $\vec{a} = |\vec{a}|$ = length PQ

direction of $\vec{a}$ = P to Q.

3. (a) ZERO VECTOR OR NULL VECTOR :

A vector of zero magnitude i.e. which has the same initial & terminal point is called a ZERO VECTOR . It is denoted by $\vec{O}$.

(b) UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

(c) COLLINEAR VECTORS

Two vectors are said to be collinear if their supports are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

(d) COPLANAR VECTORS

A given number of vectors are called coplanar if their supports are all parallel to the same plane.

Note that "TWO VECTORS ARE ALWAYS COPLANAR".

(e) EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(i) the same length,

(ii) the same or parallel supports and

(iii) the same sense.

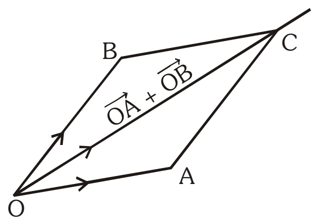
(f) Free vectors : If a vector can be translated anywhere in space without changing its magnitude & direction, then such a vector is called free vector. In other words, the initial point of free vector can be taken anywhere in space keeping its magnitude & direction same.

(g) Localized vectors : For a vector of given magnitude and direction, if its initial point is fixed in space, then such a vector is called localised vector. Unless & until stated, vectors are treated as free vectors.

4. ADDITION OF VECTORS :

(a) It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

(i) If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.



(ii) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative)

(iii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associativity)

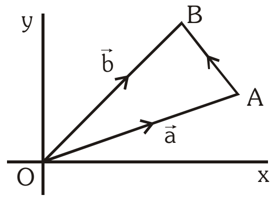
(b) Multiplication of vector by scalars :

(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$ **(ii)** $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$ **(iv)** $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

5. POSITION VECTOR :

Let O be a fixed origin, then the position vector of a point P is the vector $\vec{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then,



$\vec{AB} = \vec{b} - \vec{a} =$ pv of B – pv of A.

6. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$.

7. VECTOR EQUATION OF A LINE :

Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ where t is a parameter. If the line pass through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\vec{r} = \vec{a} + t\vec{b}$, where t is a parameter.

8. TEST OF COLLINEARITY OF THREE POINTS :

(a) Three points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$

(b) Three points A, B, C are collinear, if any two vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ are parallel.

9. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT):

(a) $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \;\; (0 \le \theta \le \pi)$, θ is angle between $\vec{a}$ & $\vec{b}$.

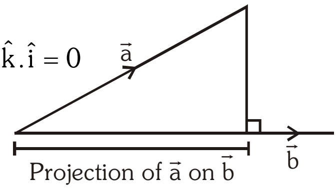
Note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

(b) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

(c) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$; $(\vec{a}, \vec{b} \ne 0)$

(d) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(e) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$.

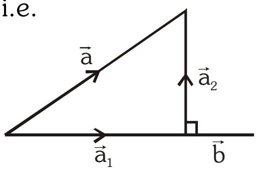


Note :

(i) The vector component of $\vec{a}$ along $\vec{b}$ i.e.

$\vec{a}_1 = \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular

to $\vec{b}$ i.e. $\vec{a}_2 = \vec{a} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ $\qquad (\vec{a} = \vec{a}_1 + \vec{a}_2)$



(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by

$$\cos\phi = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \quad 0 \le \phi \le \pi$$

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}, \; |\vec{b}| = \sqrt{{b_1}^2 + {b_2}^2 + {b_3}^2}$$

(iv) $-|\vec{a}|\,|\vec{b}| \le \vec{a}.\vec{b} \le |\vec{a}|\,|\vec{b}|$

(v) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

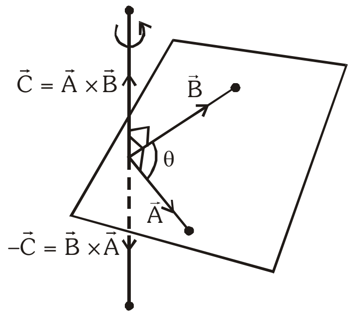
(vi) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}-\hat{b})$, $\lambda \in R^+$

(vii) $|\vec{a}\pm\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2\pm 2\vec{a}.\vec{b}$

(viii) $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

10. VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT) :

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system .



(b) Lagranges Identity : For any two vectors $\vec{a}$ & $\vec{b}$;

$$(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}|=\sqrt{\vec{a}^2\vec{b}^2-(\vec{a}.\vec{b})^2}$ **(ii)** $\vec{c}.\vec{a}=0;\ \vec{c}.\vec{b}=0$ and

(iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

(d) $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0,\ \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar

(i) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

(ii) $(m\vec{a}) \times \vec{b} = \vec{a}x(m\vec{b}) = m(\vec{a} \times \vec{b})$ where m is a scalar.

(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive)

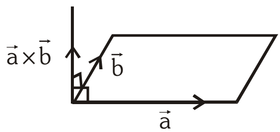
(vi) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

(e) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(f) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



(g) **(i)** Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is

$$\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

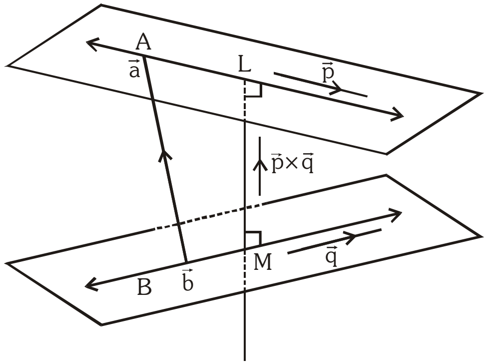
(h) Vector area :

(i) If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC $= \dfrac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$.

The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

(ii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$. Area of $\Delta = \dfrac{1}{2}|\vec{a} \times \vec{b}|$

DISTANCE BETWEEN TWO LINES :



Lines which do not intersect & are also not parallel are called skew lines. In other words the lines which are not coplanar are skew lines. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\vec{AB}$ along the direction of the line of shortest distance, $\vec{LM}$ is parallel to $\vec{p}\times\vec{q}$

i.e. $\vec{LM}$ = | Projection of $\vec{AB}$ on $\vec{LM}$ |

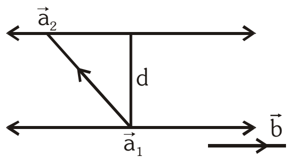
= | Projection of $\vec{AB}$ on $\vec{p}\times\vec{q}$ |

$$= \left|\frac{\vec{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(a) The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0

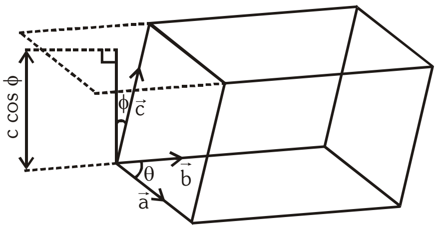
i.e. $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$ $\Rightarrow \left[\left(\vec{b}-\vec{a}\right)\ \vec{p}\ \ \vec{q}\right] = 0$

(b) If two lines are given by $\vec{r}_1 = \vec{a}_1 + K_1\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K_2\vec{b}$ i.e. they are parallel then, $d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$



12. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

(a) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as: $(\vec{a} \times \vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a} \times \vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.



(b) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(c) $\vec{a}.(\vec{b} \times \vec{c}) = -\vec{a}.(\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(d) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(e) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(f) $[i\,j\,k] = 1; [K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]; [(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(g) **(i)** The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(ii) Volume of parallelopiped whose co-terminus edges are $\vec{a}, \vec{b}$ & $\vec{c}$ is $[\vec{a}\ \vec{b}\ \vec{c}]$.

(h) **Remember that :**

(i) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

(ii) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

13. VECTOR TRIPLE PRODUCT :

Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ be any three vectors, then that expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product.

(a) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(b) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} . \vec{c})\vec{b} - (\vec{b} . \vec{c})\vec{a}$

(c) $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

14. LINEAR COMBINATIONS / LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS :

Linear combination of vectors :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots\ldots\ldots$ then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots\ldots\ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots\ldots$ for any x, y, z .... $\in R$. We have the following results :

(a) If $\vec{x}_1, \vec{x}_2, \ldots.\vec{x}_n$ are n non zero vectors, & $k_1, k_2, \ldots.k_n$ are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots..k_n\vec{x}_n = 0$ $\Rightarrow k_1 = 0, k_2 = 0 \ldots\ldots k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2, \ldots.\vec{x}_n$ are **linearly independent vectors**

(b) If $\vec{x}_1, \vec{x}_2, \ldots\ldots.\vec{x}_n$ are **not linearly independent** then they are said to be linear dependent vectors. i.e. if $k_1\vec{x}_1, k_2\vec{x}_2 + \ldots\ldots.k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, \ldots\vec{x}_n$ are said to be **linearly dependent.**

(c) **Fundamental theorem in plane** : let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$

(d) **Fundamental theorem in space** : let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique $x, y, z \in R$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

15. COPLANARITY OF FOUR POINTS :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$

where, $x + y + z + w = 0$

16. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}\,.\,\vec{a}' = \vec{b}\,.\,\vec{b}' = \vec{c}\,.\,\vec{c}' = 1$ then the two systems are called Reciprocal System of vectors.

Note : $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

17. TETRAHEDRON :

(i) Lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and this point of concurrecy is called the centre of the tetrahedron.

(ii) In a tetrahedron, straight lines joining the mid points of each pair of opposite edges are also concurrent at the centre of the tetrahedron.

(iii) The angle between any two plane faces of regular tetrahedron is

$$\cos^{-1}\frac{1}{3}$$