Question Set 4

PAIR OF STRAIGHT LINES

(Marks with option : 06)

Theory Questions	3 or 4 marks each

Q. 1. Prove that every homogeneous equation of second degree in x and y, i.e. $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin, if $h^2 - ab \geqslant 0$.

Proof : Consider a homogeneous equation of the second degree in x and y,

$ax^2 + 2hxy + by^2 = 0$ ... (1)

Case I : If $b = 0$ (i.e. $a \neq 0$, $h \neq 0$), then the equation (1) reduces to $ax^2 + 2hxy = 0$, i.e. $x(ax + 2hy) = 0$.

This represents the two lines $x = 0$ and $ax + 2hy = 0$, both passing through the origin.

Case II : If $b \neq 0$, then the equation (1), on dividing it by b, becomes

$$\frac{a}{b}x^2 + \frac{2hxy}{b} + y^2 = 0 \qquad \therefore\ y^2 + \frac{2h}{b}xy = -\frac{a}{b}x^2$$

On completing the square and adjusting, we get

$$y^2 + \frac{2h}{b}xy + \frac{h^2x^2}{b^2} = \frac{h^2x^2}{b^2} - \frac{a}{b}x^2$$

$$\therefore \left(y + \frac{h}{b}x\right)^2 = \left(\frac{h^2 - ab}{b^2}\right)x^2$$

$$\therefore\ y + \frac{h}{b}x = \pm\frac{\sqrt{h^2 - ab}}{b}x$$

$$\therefore\ y = \frac{-h}{b}x \pm \frac{\sqrt{h^2 - ab}}{b}x \qquad \therefore\ y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

$\therefore$ the equation represents the two lines

$$y = \left(\frac{-h + \sqrt{h^2 - ab}}{b}\right)x \text{ and } y = \left(\frac{-h - \sqrt{h^2 - ab}}{b}\right)x$$

Since none of these equations contains a constant term, both these lines pass through the origin.

Thus, the homogeneous equation (1) represents a pair of lines through the origin, if $h^2 - ab \geqslant 0$.

Q. 2. Show that the acute angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$, if $a + b \neq 0$.

Find the condition, if *(Sept. '21)*

(i) the lines are perpendicular to each other.

(ii) the lines are parallel (coincident).

Proof : Let m_1 and m_2 be the slopes of the lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0. \qquad \text{... (1)}$$

Then their separate equations are $y = m_1x$ and $y = m_2x$.

$\therefore$ their combined equation is $(m_1x - y)(m_2x - y) = 0$

i.e. $m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$ ... (2)

Since (1) and (2) represent the same two lines, comparing the coefficients, we have

$$\frac{m_1m_2}{a} = \frac{-(m_1+m_2)}{2h} = \frac{1}{b} \qquad \therefore\ m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

$$\therefore\ (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 = \frac{4h^2}{b^2} - \frac{4a}{b} = \frac{4(h^2 - ab)}{b^2}$$

$$\therefore\ |m_1 - m_2| = \left|\frac{2\sqrt{h^2 - ab}}{b}\right|$$

If θ is the acute angle between the lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|, \quad \text{if } m_1m_2 \neq -1$$

$$= \left|\frac{2\sqrt{h^2 - ab}/b}{1 + (a/b)}\right|, \quad \text{if } \frac{a}{b} \neq -1$$

$$= \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|, \quad \text{if } a + b \neq 0.$$

(i) If the lines are perpendicular to each other, then

$$m_1m_2 = -1 \qquad \therefore\ \frac{a}{b} = -1 \qquad \therefore\ a = -b \qquad \therefore\ a + b = 0$$

i.e. (coeff. of x^2) + (coeff. of y^2) = 0.

This is the condition for the lines to be perpendicular to each other.

(ii) If the lines are parallel (coincident), then the angle θ between them is 0.

$$\therefore\ \tan\theta = 0 \qquad \therefore\ \frac{2\sqrt{h^2 - ab}}{a+b} = 0 \qquad \therefore\ h^2 - ab = 0$$

This is the condition for the lines to be parallel (coincident).

Remarks :

(i) The slopes of the lines whose joint equation is $ax^2+2hxy+by^2=0$ are

$$m_1=\frac{-h+\sqrt{h^2-ab}}{b} \text{ and } m_2=\frac{-h-\sqrt{h^2-ab}}{b}.$$

$$\therefore\ m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b} \quad \text{... (1)}$$

Now, m_1 and m_2 are the roots of the quadratic equation

$(m-m_1)(m-m_2)=0,$

i.e. $m^2-(m_1+m_2)\,m+m_1m_2=0$

i.e. $m^2-\left(-\frac{2h}{b}\right)m+\frac{a}{b}=0$ ... [By (1)]

i.e. $bm^2+2hm+a=0$

This is called the *auxiliary equation* of $ax^2+2hxy+by^2=0$.

It is obtained by putting $y=mx$, i.e. $m=y/x$ in the above equation.

(ii) The separate equations of the lines represented by $ax^2+2hxy+by^2=0$ are

$$y=\left(\frac{-h+\sqrt{h^2-ab}}{b}\right)x \text{ and}$$

$$y=\left(\frac{-h-\sqrt{h^2-ab}}{b}\right)x, \text{ provided } h^2-ab\geqslant 0.$$

The nature of the lines depends on the value of h^2-ab which is called the *discriminant*. Lines represented by $ax^2+2hxy+by^2=0$ are

(i) real, if and only if, $h^2-ab\geqslant 0$.

(ii) real and distinct, if and only if, $h^2-ab>0$.

(iii) real and coincident, if and only if, $h^2-ab=0$.

Note : If $h^2-ab<0$, then $ax^2+2hxy+by^2=0$ does not represent a pair of lines, i.e. the lines do not exist.

Solved Examples	1 mark or 2 marks each

Ex. 1. Find the separate equations of the lines represented by the following equations :

(1) $3x^2-10xy-8y^2=0$

(2) $5x^2-9y^2=0$ *(March '22)*

(3) $x^2-y^2+x-y=0$. *(Sept. '21)*

Solution :

(1) $3x^2 - 10xy - 8y^2 = 0$

$\therefore\ 3x^2 - 12xy + 2xy - 8y^2 = 0$

$\therefore\ 3x(x - 4y) + 2y(x - 4y) = 0$

$\therefore (x - 4y)(3x + 2y) = 0$

$\therefore$ the separate equations of the lines are $x - 4y = 0$ and $3x + 2y = 0$.

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(2) $5x^2 - 9y^2 = 0$

$\therefore\ (\sqrt{5}x)^2 - (3y)^2 = 0$

$\therefore\ (\sqrt{5}x - 3y)(\sqrt{5}x + 3y) = 0$

$\therefore$ the separate equations of the lines are

$\sqrt{5}x - 3y = 0$ and $\sqrt{5}x + 3y = 0$

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(3) $x^2 - y^2 + x - y = 0$

$\therefore\ (x - y)(x + y) + (x - y) = 0$

$\therefore\ (x - y)(x + y + 1) = 0$

$\therefore$ the separate equations of the lines are

$x - y = 0$ and $x + y + 1 = 0$.

Ex. 2. Find the combined equation of the lines bisecting the angles between coordinate axes.

Solution :

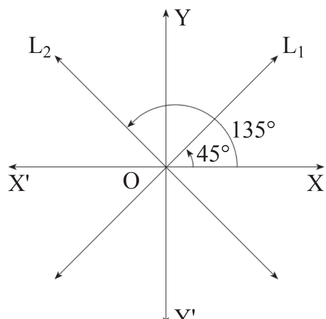
Let L_1 and L_2 be the lines bisecting the angles between the coordinate axes. Then these lines make angles of 45° and 135° with positive direction of X-axis.

$\therefore$ slope of line $L_1 = \tan 45° = 1$ and

slope of line $L_2 = \tan 135°$

$= \tan(180° - 45°)$

$= -\tan 45° = -1$



Since these lines pass through the origin,

their equations are $y = x$ and $y = -x$

i.e. $x - y = 0$ and $x + y = 0$

$\therefore$ the combined equation of the lines is $(x - y)(x + y) = 0$

$\therefore\ x^2 - y^2 = 0$.

Ex. 3. Find the joint equation of the lines passing through the origin and perpendicular to the lines $x+2y=19$ and $3x+y=18$.

Solution : Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines $x+2y=19$ and $3x+y=18$ respectively.

Slopes of the lines $x+2y=19$ and $3x+y=18$ are $-\frac{1}{2}$ and $-\frac{3}{1}=-3$ respectively.

$\therefore$ slopes of the lines L_1 and L_2 are 2 and $\frac{1}{3}$ respectively.

Since the lines L_1 and L_2 pass through the origin, their equations are

$y=2x$ and $y=\frac{1}{3}x$

i.e. $2x-y=0$ and $x-3y=0$

$\therefore$ their combined equation is

$(2x-y)(x-3y)=0$

$\therefore 2x^2-6xy-xy+3y^2=0$

$\therefore 2x^2-7xy+3y^2=0.$

Ex. 4. Find the value of k if $2x+y=0$ is one of the lines represented by $3x^2+kxy+2y^2=0$. ***(March '22)***

Solution : The auxiliary equation of the lines represented by $3x^2+kxy+2y^2=0$ is $2m^2+km+3=0$

Since one of the lines is $2x+y=0$ whose slope is $m=-2$,

$m=-2$ is the root of the auxiliary equation $2m^2+km+3=0$.

$\therefore 2(-2)^2+k(-2)+3=0$

$\therefore 8-2k+3=0$

$\therefore 2k=11 \qquad \therefore k=\frac{11}{2}.$

Ex. 5. Find k, if the slopes of the lines represented by $3x^2+kxy-y^2=0$ differ by 4.

Solution : Comparing the equation $3x^2+kxy-y^2=0$ with $ax^2+2hxy+by^2=0$, we get, $a=3$, $2h=k$, $b=-1$.

Let m_1 and m_2 be the slopes of the lines represented by $3x^2+kxy-y^2=0$.

$\therefore m_1+m_2=-\frac{2h}{b}=-\frac{k}{-1}=k$

and $m_1 m_2 = \frac{a}{b} = \frac{3}{-1} = -3$

$\therefore\ (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$= k^2 - 4\,(-3)$

$= k^2 + 12$ ... (1)

But $|m_1 - m_2| = 4$

$\therefore\ (m_1 - m_2)^2 = 16$ ... (2)

$\therefore$ from (1) and (2)

$k^2 + 12 = 16$

$\therefore\ k^2 = 4 \qquad \therefore\ k = \pm 2$.

Ex. 6. If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other, prove that $16h^2 = 25ab$.

Solution : Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$\therefore\ m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

We are given that $m_2 = 4m_1$

$\therefore\ m_1 + 4m_1 = -\frac{2h}{b} \qquad \therefore\ 5m_1 = -\frac{2h}{b}$

$\therefore\ m_1 = -\frac{2h}{5b}$ ... (1)

Also, $m_1(4m_1) = \frac{a}{b} \qquad \therefore\ 4{m_1}^2 = \frac{a}{b}$

$\therefore\ 4\left(-\frac{2h}{5b}\right)^2 = \frac{a}{b}$ ... [By (1)]

$\therefore\ \frac{16h^2}{25b^2} = \frac{a}{b}$

$\therefore\ 16h^2 = 25ab$, as $b \neq 0$.

Ex. 7. Find the measure of the acute angle between the lines represented by $(a^2 - 3b^2)\,x^2 + 8abxy + (b^2 - 3a^2)\,y^2 = 0$.

Solution : Comparing the equation

$(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$, with

$Ax^2 + 2Hxy + By^2 = 0$, we have

$A = a^2 - 3b^2,\ H = 4ab,\ B = b^2 - 3a^2.$

$$\therefore\ H^2 - AB = 16a^2b^2 - (a^2 - 3b^2)(b^2 - 3a^2)$$
$$= 16a^2b^2 + (a^2 - 3b^2)(3a^2 - b^2)$$
$$= 16a^2b^2 + 3a^4 - 10a^2b^2 + 3b^4$$
$$= 3a^4 + 6a^2b^2 + 3b^4 = 3(a^4 + 2a^2b^2 + b^4)$$
$$= 3(a^2 + b^2)^2$$

$$\therefore\ \sqrt{H^2 - AB} = \sqrt{3}(a^2 + b^2)$$

Also, $A + B = (a^2 - 3b^2) + (b^2 - 3a^2) = -2(a^2 + b^2)$

If θ is the acute angle between the lines, then

$$\tan\theta = \left|\frac{2\sqrt{H^2 - AB}}{A + B}\right| = \left|\frac{2\sqrt{3}(a^2 + b^2)}{-2(a^2 + b^2)}\right|$$
$$= \sqrt{3} = \tan 60^\circ \qquad \therefore\ \theta = 60^\circ.$$

Ex. 8. If the lines given by $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, show that $(3a + b)(a + 3b) = 4h^2$.

Solution : Since the lines $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, the angle between the lines

$ax^2 + 2hxy + by^2 = 0$ is 60°.

$$\therefore\ \tan 60^\circ = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$\therefore\ \sqrt{3} = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$\therefore\ 3(a + b)^2 = 4(h^2 - ab)$$

$$\therefore\ 3(a^2 + 2ab + b^2) = 4h^2 - 4ab$$

$$\therefore\ 3a^2 + 6ab + 3b^2 + 4ab = 4h^2$$

$$\therefore\ 3a^2 + 10ab + 3b^2 = 4h^2$$

$$\therefore\ 3a^2 + 9ab + ab + 3b^2 = 4h^2$$

$$\therefore\ 3a(a + 3b) + b(a + 3b) = 4h^2$$

$$\therefore\ (3a + b)(a + 3b) = 4h^2$$

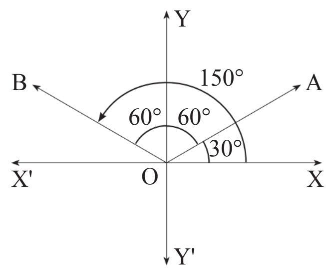
This is the required condition.

Ex. 9. Find the combined equation of the lines passing through the origin and each of which making an angle of 60° with the Y-axis.

Solution : Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis.

Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

$\therefore$ slope of OA $= \tan 30° = \dfrac{1}{\sqrt{3}}$



$\therefore$ equation of the line OA is

$y = \dfrac{1}{\sqrt{3}}x$, i.e. $x - \sqrt{3}y = 0$

Slope of OB $= \tan 150° = \tan(180° - 30°)$

$= -\tan 30° = -\dfrac{1}{\sqrt{3}}$

$\therefore$ equation of the line OB is

$y = -\dfrac{1}{\sqrt{3}}x$, i.e. $x + \sqrt{3}y = 0$

$\therefore$ required combined equation is

$(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$

i.e. $x^2 - 3y^2 = 0$.

Examples for Practice	**2 marks each**

1. Find the separate equations of the lines represented by :

(1) $3x^2 - 7xy + 4y^2 = 0$ (2) $3y^2 + 7xy = 0$

(3) $2x^2 + 2xy - y^2 = 0$.

2. Find the combined equation of the lines :

(1) passing through the origin having the inclinations 30° and 120°.

(2) passing through (3, 2) and parallel to the lines $x = 2$ and $y = 3$.

3. Find the combined equation of the lines passing through the origin such that

(1) one is parallel to $x + 2y = 5$ and other is perpendicular to $2x - y + 3 = 0$.

(2) which are perpendicular to the lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$.

4. (1) Find the value of k, if the sum of slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product. ***(Sept. '21)***

(2) Find k, if one of the lines given by $3x^2 - kxy + 5y^2 = 0$ is perpendicular to the line $5x + 3y = 0$.

5. Find the condition that the line $3x - 2y = 0$ coincides with one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

6. Find the combined equation of the lines passing through the origin and each of which making an angle of 60° with the X-axis.

7. Find the measure of the acute angle between the lines represented by :

   (1) $3x^2 - 4\sqrt{3}xy + 3y^2 = 0$ (2) $2x^2 - 6xy + y^2 = 0$.

8. If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisect an angle between the coordinate axes, then show that $(a + b)^2 = 4h^2$.

ANSWERS

1. (1) $x - y = 0$, $3x - 4y = 0$ (2) $y = 0$, $7x + 3y = 0$

   (3) $(1 + \sqrt{3})x - y = 0$, $(1 - \sqrt{3})x - y = 0$.

2. (1) $\sqrt{3}x^2 - 2xy - \sqrt{3}y^2 = 0$ (2) $xy - 2x - 3y + 6 = 0$.

3. (1) $x^2 + 4xy + 4y^2 = 0$ (2) $6x^2 - 7xy - 3y^2 = 0$.

4. (1) $k = -2$ (2) $k = 8$

5. $4a + 12h + 9b = 0$

6. $3x^2 - y^2 = 0$

7. (1) 30° (2) $\tan^{-1}\left(\frac{2\sqrt{7}}{3}\right)$.

Solved Examples	3 or 4 marks each

Ex. 10. Prove that the combined equation of the pair of lines passing through the origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Solution : Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$$\left.\begin{aligned} \therefore\ m_1 + m_2 &= -\frac{2h}{b} \\ \text{and } m_1 m_2 &= \frac{a}{b} \end{aligned}\right\} \quad \dots (1)$$

Now, required lines are perpendicular to these lines.

$\therefore$ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$$y = -\frac{1}{m_1}x \quad \text{and} \quad y = -\frac{1}{m_2}x$$

i.e. $m_1 y = -x$ and $m_2 y = -x$

i.e. $x + m_1 y = 0$ and $x + m_2 y = 0$

$\therefore$ their combined equation is

$(x + m_1 y)(x + m_2 y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$

$\therefore x^2 - \frac{2h}{b}xy + \frac{a}{b}y^2 = 0$ ... [By (1)]

$\therefore bx^2 - 2hxy + ay^2 = 0.$

Ex. 11. Show that the lines $x^2 - 4xy + y^2 = 0$ and $x + y = 10$ contain the sides of an equilateral triangle.

Solution : We find the joint equation of the pair of lines OA and OB through origin, each making an angle of 60° with $x + y = 10$ whose slope is -1.

Let OA (or OB) has slope m.

$\therefore$ its equation is $y = mx$ ... (1)

Also, $\tan 60° = \left|\frac{m - (-1)}{1 + m(-1)}\right|$

$\therefore \sqrt{3} = \left|\frac{m + 1}{1 - m}\right|$

Squaring both sides, we get

$$3 = \frac{(m + 1)^2}{(1 - m)^2}$$

$\therefore 3(1 - 2m + m^2) = m^2 + 2m + 1$

$\therefore 3 - 6m + 3m^2 = m^2 + 2m + 1$

$\therefore 2m^2 - 8m + 2 = 0 \qquad \therefore m^2 - 4m + 1 = 0$

$\therefore \left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) + 1 = 0$ ... [By (1)]

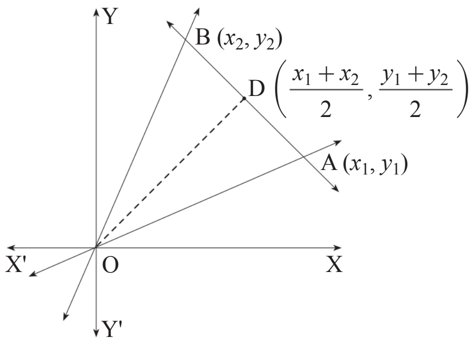
$\therefore y^2 - 4xy + x^2 = 0$

$\therefore x^2 - 4xy + y^2 = 0$ is the joint equation of the two lines through the origin each making an angle of 60° with $x + y = 10$

$\therefore x^2 - 4xy + y^2 = 0$ and $x + y = 10$ form a triangle OAB which is equilateral.

Ex. 12. **$\triangle$OAB is formed by the lines $x^2 - 4xy + y^2 = 0$ and the line AB. The equation of line AB is $2x + 3y - 1 = 0$. Find the equation of the median of the triangle drawn from O.**

Solution :



Let D be the midpoint of seg AB where A is (x_1, y_1) and B is (x_2, y_2).

Then D has coordinates

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The joint (combined) equation of the lines OA and OB is

$x^2 - 4xy + y^2 = 0$ and the equation of the line AB is $2x + 3y - 1 = 0$.

$\therefore$ points A and B satisfy the equations $2x + 3y - 1 = 0$ and $x^2 - 4xy + y^2 = 0$ simultaneously.

We eliminate x from the above equations, i.e. put $x = \frac{1-3y}{2}$ in the equation $x^2 - 4xy + y^2 = 0$, we get

$$\therefore \left(\frac{1-3y}{2}\right)^2 - 4\left(\frac{1-3y}{2}\right)y + y^2 = 0$$

$$\therefore (1-3y)^2 - 8(1-3y)y + 4y^2 = 0$$

$$\therefore 1 - 6y + 9y^2 - 8y + 24y^2 + 4y^2 = 0$$

$$\therefore 37y^2 - 14y + 1 = 0$$

The roots y_1 and y_2 of the above quadratic equation are the y-coordinates of the points A and B.

$$\therefore y_1 + y_2 = \frac{-b}{a} = \frac{14}{37}$$

$$\therefore y\text{-coordinate of D} = \frac{y_1+y_2}{2} = \frac{7}{37}.$$

Since D lies on the line AB, we can find the x-coordinate of D as

$$2x+3\left(\frac{7}{37}\right)-1=0$$

$$\therefore\ 2x=1-\frac{21}{37}=\frac{16}{37}$$

$$\therefore\ x=\frac{8}{37}$$

$\therefore$ D is (8/37, 7/37)

$\therefore$ equation of the median OD is $\dfrac{x}{8/37}=\dfrac{y}{7/37}$, i.e. $7x-8y=0$.

Ex. 13. If the angle between the lines represented by $ax^2+2hxy+by^2=0$ is equal to the angle between the lines $2x^2-5xy+3y^2=0$, then show that $100\,(h^2-ab)=(a+b)^2$. ***(March '22)***

Solution : The acute angle θ between the lines $ax^2+2hxy+by^2=0$ is given by

$$\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right| \qquad \text{... (1)}$$

Comparing the equation $2x^2-5xy+3y^2=0$ with $ax^2+2hxy+by^2=0$, we get

$a=2$, $2h=-5$, i.e. $h=-\dfrac{5}{2}$ and $b=3$.

Let α be the acute angle between the lines $2x^2-5xy+3y^2=0$.

$$\therefore\ \tan\alpha=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|=\left|\frac{2\sqrt{\left(-\frac{5}{2}\right)^2-2(3)}}{2+3}\right|=\left|\frac{2\sqrt{\frac{25}{4}-6}}{5}\right|=\left|\frac{2\times\frac{1}{2}}{5}\right|$$

$$\therefore\ \tan\alpha=\frac{1}{5} \qquad \text{... (2)}$$

If $\theta=\alpha$, then $\tan\theta=\tan\alpha$

$$\therefore\ \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|=\frac{1}{5} \qquad \text{... [By (1) and (2)]}$$

$$\therefore\ \frac{4(h^2-ab)}{(a+b)^2}=\frac{1}{25} \qquad \therefore\ 100\,(h^2-ab)=(a+b)^2$$

This is the required condition.

Examples for Practice	3 or 4 marks each

1. Find the combined equation of the lines through the origin :

(1) each making an angle of 45° with the line $3x+y=2$.

(2) each making an angle of $\pi/6$ with the line $3x+y-6=0$.

(3) which form an equilateral triangle with the line $3x+4y=8$.

2. Prove that the lines represented by $3x^2-8xy-3y^2=0$ and $x+2y=3$ form the sides of an isosceles right angled triangle.

3. Find the combined equation of the pair of lines through the origin and perpendicular to the lines represented by :

(1) $5x^2-8xy+3y^2=0$ (2) $3x^2+2xy-y^2=0$.

4. If the slope of one of the lines given by $ax^2+2hxy+by^2=0$ is square of the other line, show that $a^2b+ab^2+8h^3=6abh$.

5. Show that the combined equation of the pair of lines passing through the origin and each making an angle α with the line $x+y=0$ is $x^2+2(\sec 2\alpha)xy+y^2=0$.

ANSWERS

1. (1) $2x^2+3xy-2y^2=0$ (2) $13x^2+12xy-3y^2=0$

(3) $39x^2-96xy+11y^2=0$.

3. (1) $3x^2+8xy+5y^2=0$ (2) $x^2+2xy-3y^2=0$.

GENERAL SECOND DEGREE EQUATION IN *x* AND *y*

1. The combined equation of two lines is a general equation of second degree but not conversely.

2. The general equation of second degree in x and y viz.

$ax^2+2hxy+by^2+2gx+2fy+c=0$ will represent a pair of lines, if

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

i.e. $abc+2fgh-af^2-bg^2-ch^2=0$ and $h^2-ab \geqslant 0$.

3. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ ... (1)

represents a pair of lines, then these lines are parallel to the pair of lines $ax^2+2hxy+by^2=0$ through the origin and hence

(i) angle between the lines represented by (1) is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|, \text{ if } a + b \neq 0$$

(ii) The lines represented by (1) are :

(a) parallel or coincident, if $h^2 - ab = 0$

(b) perpendicular, if $a + b = 0$

(iii) The point of intersection of lines represented by (1) is obtained by solving the equations $ax + hy + g = 0$ and $hx + by + f = 0$ simultaneously.

Solved Examples	3 or 4 marks each

Ex. 14. Show that the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of lines. Also, find the acute angle between them.

Solution : Comparing the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = 2,\ h = \frac{1}{2},\ b = -1,\ g = \frac{1}{2},\ f = 2,\ c = -3.$$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & -3 \end{vmatrix}$$

Taking $\frac{1}{2}$ common from each row, we get

$$D = \frac{1}{8}\begin{vmatrix} 4 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -6 \end{vmatrix}$$

$$= \frac{1}{8}[4(12 - 16) - 1(-6 - 4) + 1(4 + 2)]$$

$$= \frac{1}{8}[4(-4) - 1(-10) + 1(6)]$$

$$= \frac{1}{8}(-16 + 10 + 6) = 0$$

Also, $h^2 - ab = \left(\frac{1}{2}\right)^2 - 2(-1) = \frac{1}{4} + 2 = \frac{9}{4} > 0$

Hence, the given equation represents a pair of lines.

Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2-2(-1)}}{2-1}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4}+2}}{1}\right| = 2 \times \frac{3}{2} = 3$$

$$\therefore \theta = \tan^{-1}(3).$$

Ex. 15. Find the value of k, if the equation $x^2+3xy+2y^2+x-y+k=0$ represents a pair of lines.

Solution : Comparing the given equation with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get

$$a=1,\ h=\frac{3}{2},\ b=2,\ g=\frac{1}{2},\ f=-\frac{1}{2},\ c=k.$$

Now, given equation represents a pair of lines.

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

i.e. $$\begin{vmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & k \end{vmatrix} = 0$$

Taking out $\frac{1}{2}$ common from each row, we get

$$\frac{1}{8}\begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$$

$\therefore\ 2(8k-1)-3(6k+1)+1(-3-4)=0$

$\therefore\ 16k-2-18k-3-7=0$

$\therefore\ -2k-12=0$

$\therefore\ -2k=12 \qquad \therefore\ k=-6.$

Ex. 16. Find p and q, if the equation $px^2-8xy+3y^2+14x+2y+q=0$ represents a pair of perpendicular lines.

Solution : The given equation represents a pair of lines perpendicular to each other

$\therefore$ (coefficient of x^2) + (coefficient of y^2) = 0

$\therefore\ p+3=0 \qquad \therefore\ p=-3$

With this value of p, the given equation is

$-3x^2-8xy+3y^2+14x+2y+q=0.$

Comparing this equation with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we have

$a=-3$, $h=-4$, $b=3$, $g=7$, $f=1$ and $c=q$.

$$\therefore\ D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\begin{vmatrix} -3 & -4 & 7 \\ -4 & 3 & 1 \\ 7 & 1 & q \end{vmatrix}$$

$$=-3(3q-1)+4(-4q-7)+7(-4-21)$$

$$=-9q+3-16q-28-175$$

$$=-25q-200=-25(q+8)$$

Since the given equation represents a pair of lines, $D=0$

$\therefore\ -25(q+8)=0 \qquad \therefore\ q=-8.$

Hence, $p=-3$ and $q=-8$.

Examples for Practice	**3 or 4 marks each**

1. Show that the equation $x^2-6xy+5y^2+10x-14y+9=0$ represents a pair of lines. Find the acute angle between them. Also, find the point of their point of intersection.
2. Find k, if the following equations represents a pair of lines :
   (1) $2x^2+4xy-2y^2+4x+8y+k=0$ (2) $kxy+10x+6y+4=0$.
3. Find p and q, if the following equations represent a pair of perpendicular lines :
   (1) $2x^2+4xy-py^2+4x+qy+1=0$
   (2) $px^2-8xy+3y^2+14x+2y+q=5$.

4. Find p and q, if the equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ represents a pair of parallel lines.

5. Find the condition that the equation $ay^2 + bxy + ex + dy = 0$ may represent a pair of lines.

ANSWERS

1. $\tan^{-1}\left(\frac{2}{3}\right)$, (1, 2)

2. (1) $k = 1$ (2) $k = 15$

3. (1) $p = 2, q = 0$ or 8 (2) $p = -3, q = -3$

4. $p = 8, q = 1$

5. $e = 0$ or $bd = ae$.

MULTIPLE CHOICE QUESTIONS	2 marks each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The combined equation of the lines, trisecting the angle in second and fourth quadrants, is

   (a) $\sqrt{3}(x^2 + y^2) - 4xy = 0$ (b) $\sqrt{3}(x^2 - y^2) + 4xy = 0$

   (c) $\sqrt{3}(x^2 + y^2) + 4xy = 0$ (d) $\sqrt{3}(x^2 - y^2) - 4xy = 0$

2. The joint equation of the pair of lines passing through (2, 3) and parallel to the coordinate axes is

   (a) $xy - 3x - 2y + 6 = 0$ (b) $xy + 3x + 2y + 6 = 0$

   (c) $xy = 0$ (d) $xy - 3x - 2y - 6 = 0$

3. The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is

   (a) 2 (b) 1 (c) 3 (d) 4

4. If $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, then the value of k is

   (a) $\frac{1}{2}$ (b) $\frac{11}{2}$ (c) $\frac{5}{2}$ (d) $-\frac{11}{2}$

5. If $h^2 = ab$, then slopes of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio

   (a) 1 : 2 (b) 2 : 1 (c) 2 : 3 (d) 1 : 1

6. If the sum of slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product, then the value of 'k' is

(a) 2 (b) 1 (c) -1 (d) -2

7. If the sum of squares of the slopes of the lines $ax^2 - 3xy + y^2 = 0$ is 5, then the value of a is

(a) 2 (b) -2 (c) ± 2 (d) 1

8. If the lines represented by $x^2 - 4xy + y^2 = 0$ make angles α and β with the X-axis, then $\tan^2\alpha + \tan^2\beta$ is equal to

(a) 14 (b) -14 (c) 16 (d) -16

9. If the equation $hxy + gx + fy + c = 0$ represents a pair of straight lines, then

(a) $fg = ch$ (b) $gh = fc$ (c) $fh = gc$ (d) $fh = -gc$

10. If the equation $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$ represents a pair of perpendicular lines, then the values of p and q are respectively

(a) -3 and -7 (b) -7 and -3

(c) 3 and 7 (d) -7 and 3

ANSWERS

1. (c) $\sqrt{3}(x^2 + y^2) + 4xy = 0$ **2.** (a) $xy - 3x - 2y + 6 = 0$

3. (a) 2 **4.** (b) $\frac{11}{2}$ **5.** (d) 1 : 1

6. (d) -2 **7.** (a) 2 **8.** (a) 14

9. (a) $fg = ch$ **10.** (b) -7 and -3.