CHAPTER 5

TWO DIMENSIONAL ANALYTICAL GEOMETRY

THEOREM 1

The circle passing through the points of intersection of the line $lx + my + n = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the circle of the form $x^2 + y^2 + 2gx + 2fy + c + \lambda (lx + my + n) = 0, \lambda \in \mathbb{R}^1$

THEOREM 2

The equation of a circle with (x_1, y_1) and (x_2, y_2) as extremities of one of the diameters of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

THEOREM 3

The position of a point $P(x_1, y_1)$ with respect to a given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in the plane containing the circle is outside or on or inside the circle according as

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \text{ is } \begin{cases} > 0, & or \\ = 0, & or \\ < 0, & \end{cases}$$

THEOREM 4 From any point outside the circle $x^2 + y^2 = a^2$ two tangent can be drawn.

THEOREM 5 The sum of the focal distances of any points on the ellipse is equal to length of the major axis.

THEOREM 6 Three normal can be drawn to a parabola $y^2 = 4ax$ from a given point, one of which is always real.

TANGENT AND NORMAL

CURVE	EQUATION	EQUATION OF TANGENT	EQUATION OF NORMAL
CIRCLE	$x^2 + y^2 = a^2$	(i) Cartesian form $xx_1 + yy_1 = a^2$ (ii) parametric form $x\cos\theta + y\sin\theta = a$	(i) Cartesian form $xy_1 - yx_1 = 0$ (ii) parametric from $x\sin\theta - y\cos\theta = 0$
PARABOLA	$y^2 = 4ax$	(i) $yy_1 = 2a(x + x_1)$ (ii) $yt = x + at^2$	(i) $xy_1 + 2y = 2ay_1 + x_1y_1$ (ii) $y + xt = at^3 + 2at$
ELLIPSE	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	(i) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ (ii) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$	(i) $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 - b^2$ (ii) $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$
HYPERBOLA	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	(i) $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ (ii) $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$	(i) $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$ (ii) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

CONDITION FOR THE SINE y = mx + c TO BE A TANGENT TO THE CONICS

CONIC	EQUATION	CONDITION TO BE TANGENT	POINT OF CONTACT	EQUATION OF TANGENT
CIRCLE	$X^2 + y^2 = a^2$	$c^2 = a^2(l + m^2)$	$\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$	$y = mx \pm \sqrt{1+ m^2}$
PARABOLA	$Y^2 = 4ax$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
ELLIPSE	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 + b^2$	$\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 + b^2}$
HYPERBOLA	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 - b^2$	$\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 - b^2}$

PARAMETRIC FORMS

CONIC	PARAMETRIC EQUATIONS	PARAMETER	RANGE OF PARAMETER	ANY POINT ON THE CONIC
CIRCLE	$x = a \cos \theta$ $y = a \sin \theta$	θ	$0 \le \theta \le 2\pi$	' θ ' or ($a \cos \theta$, $b \sin \theta$)
PARABOLA	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	' t ' or (at^2 , $2at$)
ELLIPSE	$x = a \cos \theta$ $y = b \sin \theta$	θ	$0 \le \theta \le 2\pi$	' θ ' or ($a \cos \theta$, $b \sin \theta$)
HYPERBOLA	$x = a \sec \theta$ $y = b \tan \theta$	θ	$-\pi \le 0 \le \pi$ Except $\theta = \pm\frac{\pi}{2}$	' θ ' or ($a \sec \theta$, $b \tan \theta$)

PARABOLA

EQUATION	VERTICES	FOCUS	AXIS OF SYMMETRY	EQUATION OF DIRECTRIX	LENGTH OF LATUS RECTUM
$(y – k)^2 = 4a(x – h)$	(h, k)	(h + a, 0 + k)	y = k	x = h - a	4a
$(y – k)^2 = -4a(x – h)$	(h, k)	(h –a, 0 + k)	y = k	x = h + a	4a
$(x– h)^2 = 4a(y – k)$	(h, k)	(0 + h, a + k)	x = h	y = k -a	4a
$(x –h)^2 = - 4a(y – k)$	(h, k)	(0 +h , -a +k)	x = h	y = k + a	4a

PARAMETRIC FORMS

Identifying the conic from the general equation of conic

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The graph of the second degree equation is one of a circle, parabola, an ellipse, a hyperbola, a point, an empty set, a single line or a pair of lines. When,

1) A = C = 1 , B =0 , D = -2h, E = -2k , $F = h^2 + k^2 - r^2$ the general equation reduces to $(x - h)^2 + (y - h)^2 = r^2$, which is a circle .

2) B = 0 and either A or C = 0, the general equation yields a parabola under study, at this level

3) A ≠ C and A and C are of the same sign the general equation yields an ellipse.

4) A ≠ C and A and C are of the opposite signs the general equation yields a hyperbola

ELLIPSE

EQUATION	CENTRE	MAJOR AXIS	VERTICES	FOCI
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $a^2 > b^2$ a) Major axis parallel to the x – axis foci are c units right and c units left of centre , where $c^2 = a^2 - b^2$	(h , k)	Parallel to the x - axis	(h – a , k) (h + a , k)	(h – c , k) (h + c , k)
$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $a^2 > b^2$ a) Major axis parallel to the y – axis foci are c units right and c units left of centre , where $c^2 = a^2 - b^2$	(h , k)	Parallel to the y - axis	(h, k – a) (h, k + a)	(h , k – c) (h , k + c)

HYPERBOLA

a) Transverse axis parallel to the x – axis	a) Transverse axis parallel to the x- axis The equation of a hyperbola with centre C (h, k) and transverse axis parallel to the x- axis is given by $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$ The coordinates of the vertices are A(h+a, k) and A'(h – a , k) . the coordinates of the foci are S(h + c , k) and S'(h – c , k) where $c^2 = a^2 + b^2$ The equations of directrices are x = $\pm\frac{a}{e}$
b) Transverse axis parallel to the y – axis	b) Transverse axis parallel to the y- axis The equation of a hyperbola with centre C (h , k) and transverse axis parallel to the y- axis is given by $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$ The coordinates of the vertices are A(h, k+a) and A'(h , k - a) . the coordinates of the foci are S(h, k+c) and S'(h, k-c) where $c^2 = a^2 + b^2$ The equations of directrices are y = $\pm\frac{a}{e}$