20. INVERSE TRIGONOMETRIC FUNCTIONS

1. INTRODUCTION TO INVERSE TRIGONOMETRY

The inverse trigonometric functions are the inverse functions of the trigonometric functions. They are sometimes referred to as cyclometric functions.

2. IMPORTANT DEFINITIONS

Given two non-empty sets X and Y, let f:X $\to$ Y be a function, such that y = f(x). The set X is called as the domain of f while the set Y is called as the co-domain of f. The set {f(x): x $\in$ X} is called as range of f.A map f: A$\to$B is said to be one-one or injective, if and only if, distinct elements of A have distinct images in B, i.e. if, and only if, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for all $x_1, x_2 \in A$

Onto map or Surjective map: A map f: A$\to$B is said to be an onto map or Surjective map if, and only if, each element of B is the image of some element of A, i.e. if, and only if, Range of f = co-domain of f.

Objective map: A map f : A$\to$B is an objective map if, and only if, it is both one – one and onto.

3. INVERSE FUNCTIONS

If f : X$\to$Y is one-to-one and onto (i.e. f is objective), then, we can define a unique function g : Y$\to$ X, such that g(y) = x, where x $\in$X is such that y = f (x). Thus, the domain of g = range of f and range of g = domain of f. The function is called the inverse of f and is denoted by f^{-1}.

(a) Trigonometric functions are many-one functions but these become one-one, onto, if we restrict the domain of trigonometric functions. Similarly, co-domain is equated to range to make it an onto function. We can say that the inverse of trigonometric functions are defined within restricted domains of corresponding trigonometric functions.

(b) Inverse of sin (sine functions) is denoted by $\sin^{-1}$ (arc sine function). We also write it as $\sin^{-1} x$. Similarly, other inverse trigonometric functions are given by $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\cot^{-1}x$ and $\mathrm{cosec}^{-1} x$.

(c) Note that $\sin^{-1} x \neq \frac{1}{\sin x}$ and $(\sin^{-1} x)^2 \neq \sin^{-2} x$, Also $\sin^{-1} x \neq (\sin x)^{-1}$

(d) Domain and Range of Inverse Trigonometric Functions:

	Function	Domain	Range (Principal value branch)
(i)	$y=\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

	Function	Domain	Range (Principal value branch)
(ii)	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} \le y < \frac{\pi}{2}$
(iv)	$y = \operatorname{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
(v)	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

(e) The principal value of an inverse trigonometric function is the value of that inverse trigonometric function which lies in the range of principal branch.

PLANCESS CONCEPTS

If no branch of an inverse trigonometric function is mentioned, then it can be implied that the principal value branch of that function.

You can remember range as set of angles that have the smallest absolute values satisfying for all the values of domain.

Vaibhav Gupta (JEE 2009 AIR 54)

4. TRANSFORMATION OF TRIGONOMETRIC FUNCTIONS TO INVERSE TRIGONOMETRIC FUNCTIONS

4.1 sin x to $\sin^{-1}$ x

The graph of an inverse trigonometric function can be obtained from the graph of the original by interchanging x and y axes.

Note: It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as the mirror image in the line y = x.

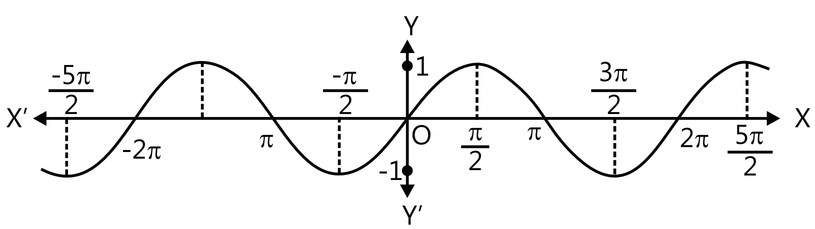
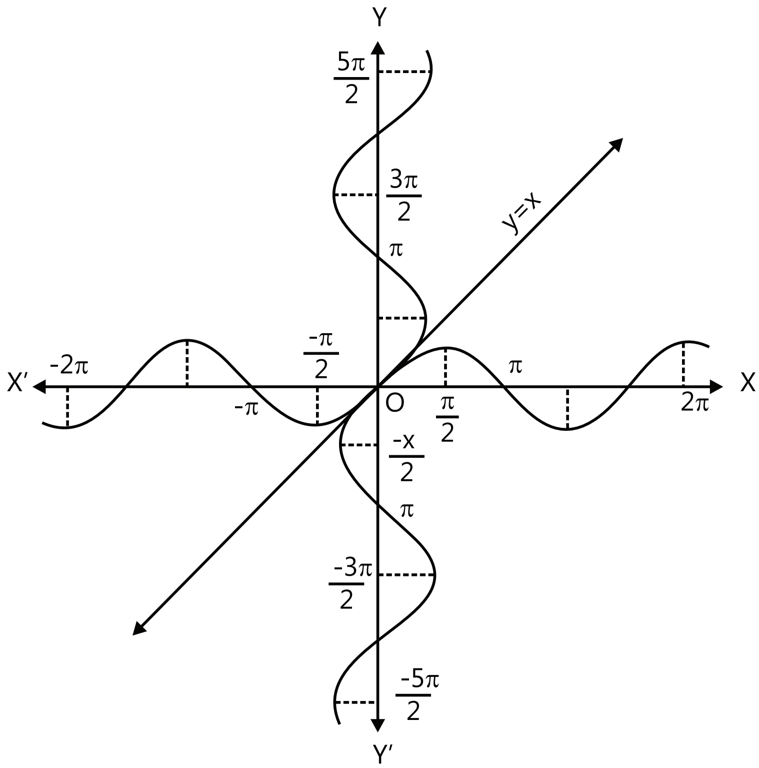


Figure 20.1

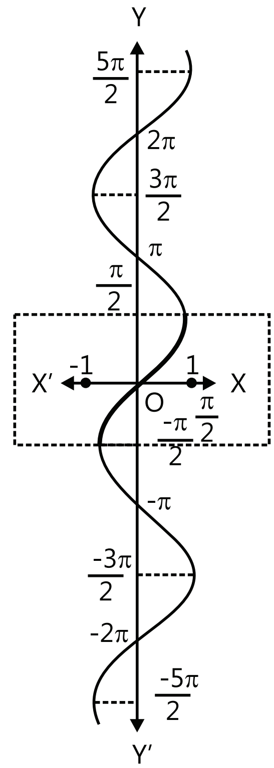
(a) y = sin x, x ∈ R and $|y| \le 1$; $y = \sin^{-1} x, |x| \le 1,\ y \in [-\pi/2, \pi/2]$

4.2 cos x to cos⁻¹ x

(b) $y = \cos x, x \in R$ and $|y| \le 1$ $y = \cos^{-1} x, x \in [-1, 1]$ and $y \in [0, \pi]$

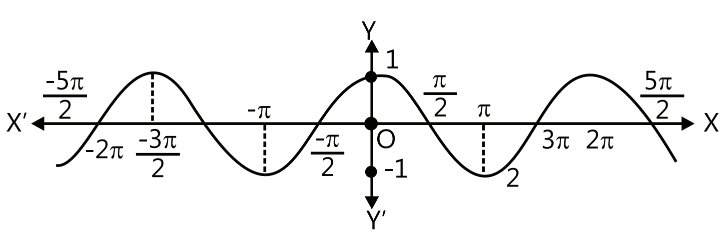


y-sin x and y-sin^{-1}(x)



y-sin^{-1}(x)

Figure 20.2



y=cos x

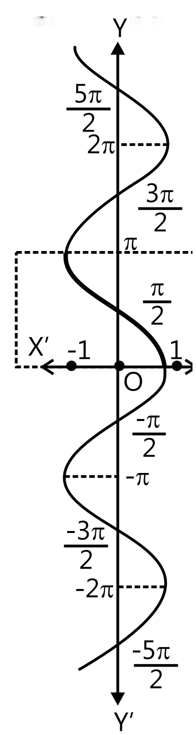


Figure 20.3

4.3 tan x to $\tan^{-1}$ x

(c) $y = \tan x, x \in R - \left\{x : x = (2\pi + 1)\frac{\pi}{2}, n \in Z\right\}$ and $y \in R$ $y = \tan^{-1} x, x \in R$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

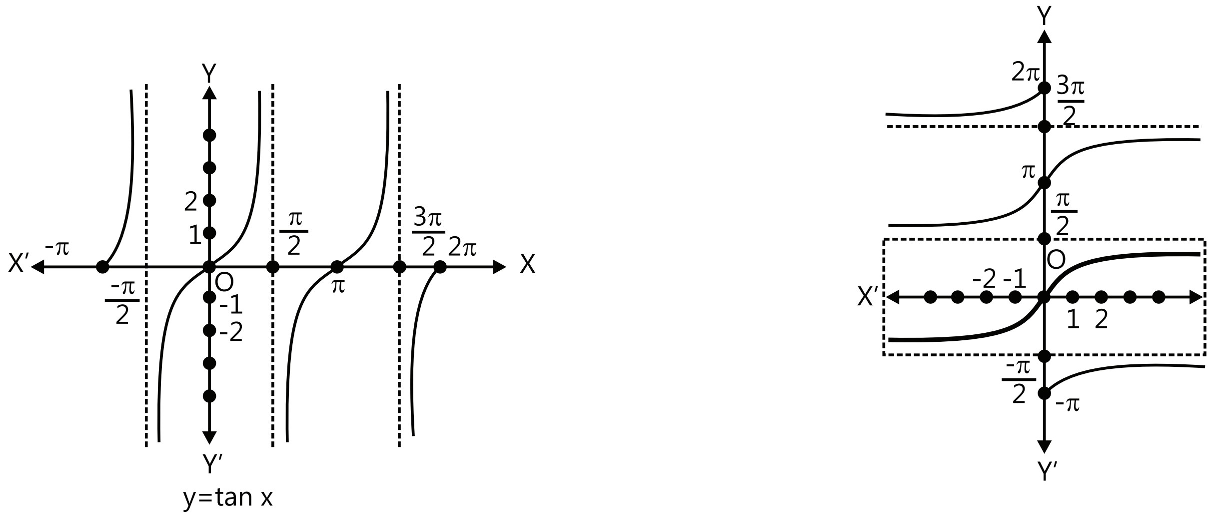


Figure 20.4

4.4 cot x to $\cot^{-1}$ x

(d) $y = \cot x, x \in R - \{x : x = n\pi, n \in Z\}$ and $y \in R$ $y = \cot^{-1} x, x \in R$ and $y \in (0, \pi)$

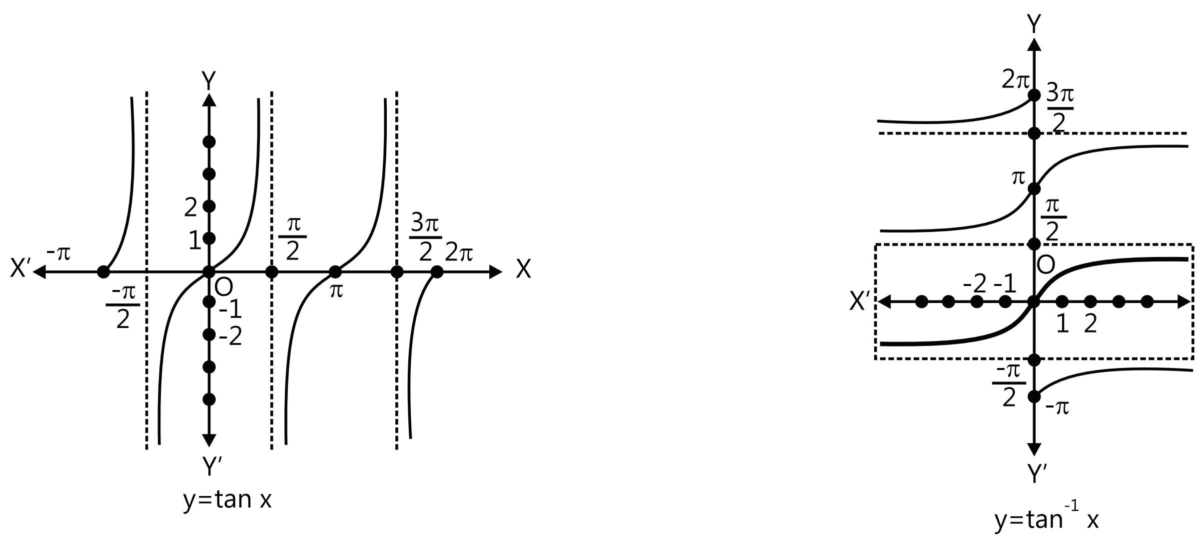


Figure 20.5

4.5 sec x to $\sec^{-1}$ x

(e) $y = \sec x, x \in R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in z\right\}$ and $y \in R - (-1, 1)$ $y = \sec^{-1} x, x \in R - (-1, 1)$ and $y \in [0, \pi]\left\{\frac{\pi}{2}\right\}$

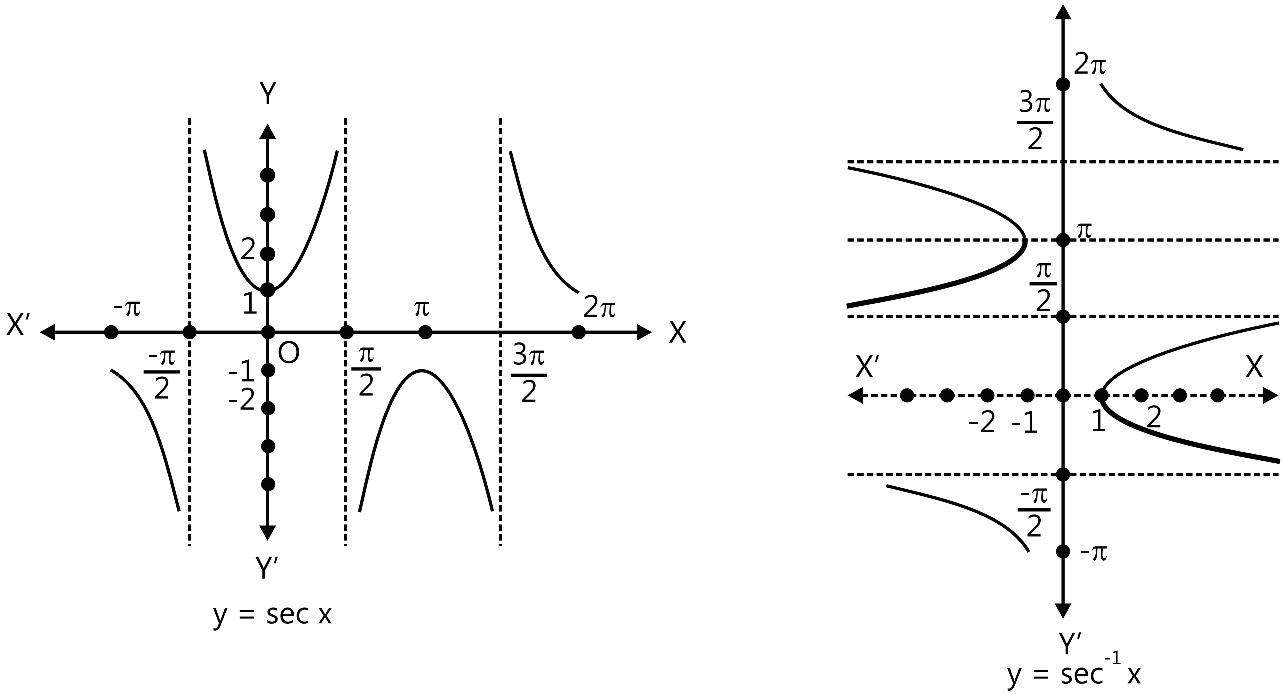


Figure 20.6

4.6 cosec x to cosec⁻¹ x

(f) $y = \text{cosec}\,x, x \in \{x : x = n\pi, n \in Z\}$ and $y \in R - (-1,1)$ $y = \text{cosec}^{-1} x, x \in R - (-1,1)$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

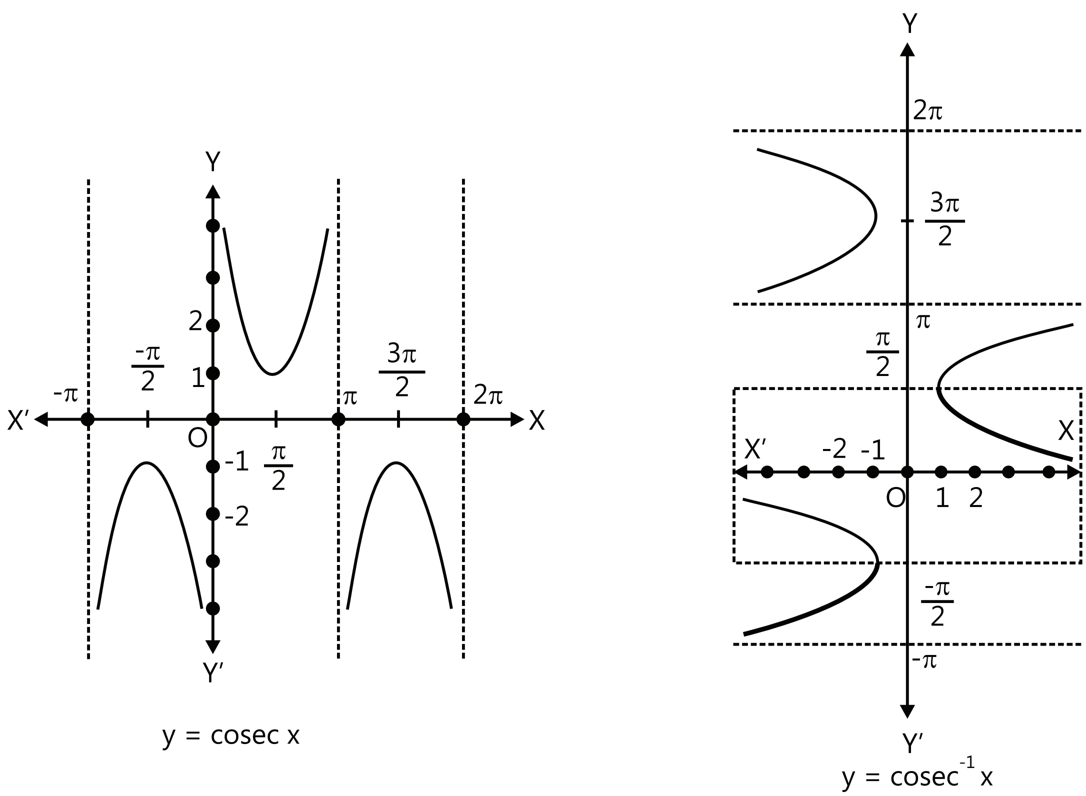


Figure 20.7

Illustration 1: Find the domain of definition of the function f(x) = $\sqrt{3\cos^{-1}(4x)-\pi}$. **(JEE MAIN)**

Sol: Use the condition that the expression inside the square root is ≥ zero.

For domain of f(x) = $\sqrt{3\cos^{-1}(4x)-\pi}$, we must have

$4x \ge \cos\left(\frac{\pi}{3}\right) \Rightarrow 4x \ge \frac{1}{2} \quad \Rightarrow x \ge \frac{1}{8}$ .....(i)

Also $-1 \le 4x \le 1 \Rightarrow \frac{-1}{4} \le x \le \frac{1}{4}$ .....(ii)

∴ From (i) and (ii), we get $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

PLANCESS CONCEPTS

In case of confusion, try solving problems by replacing inverse functions with angles and applying trigonometric identities.

Shrikant Nagori (JEE 2009 AIR 30)

Illustration 2: If $0 < \cos^{-1}x < 1$ and $1+\sin(\cos^{-1}x)+\sin^2(\cos^{-1}x)+\sin^3(\cos^{-1}x)+......\infty = 2$, then find the value of x. **(JEE MAIN)**

Sol: Use summation of infinite GP series.

We have $1+\sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + ...\infty = 2$

$\Rightarrow \frac{1}{1-\sin(\cos^{-1}x)} = 2 \Rightarrow \frac{1}{2} = 1-\sin(\cos^{-1}x) \Rightarrow \sin(\cos^{-1}x) = \frac{1}{2} \Rightarrow \cos^{-1}x = \frac{\pi}{6} \qquad \Rightarrow x = \frac{\sqrt{3}}{2}$

Illustration 3: Let $f(x) = \frac{2}{\pi}(\sin^{-1}[x]+\tan^{-1}[x]+\cot^{-1}[x])$, where [x] denotes the greatest integer less than or equal to x. If A and B denote the domain and range of f(x) respectively, find the number of integers in A∪B. **(JEE ADVANCED)**

Sol: Use $\tan^{-1}[x]+\cot^{-1}[x] = \frac{\pi}{2}$ and proceed.

For domain of f(x), we must have $-1 \le [x] \le 1 \Rightarrow -1 \le x < 2$, so set A = [-1,2)

$f(x) = \frac{2}{\pi}\left(\sin^{-1}[x]+\frac{\pi}{2}\right) \qquad \left(\text{As} \quad \tan^{-1}[x]+\cot^{-1}[x] = \frac{\pi}{2}, \forall x \in A\right)$

So, set B = {0, 1, 2} = Range of f(x). Now, $A \cup B = [-1,2) \cup \{0,1,2\} = [-1,2]$

Hence, number of integers in $(A \cup B) = 4$

5.PROPERTIES/IDENTITIES OF INVERSE TRIGONOMETRIC FUNCTIONS

5.1 Complementary Angles

(a) $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1,1]$

(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in R$

(c) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$

5.2 Negative Arguments

(a) $\sin^{-1}(-x) = -\sin^{-1}x,\ \forall x \in [-1,1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ \forall x \in [-1,1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, \forall x \in R$

(d) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall x \in R$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(f) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x \forall x \in (-\infty, -1) \cup [1, \infty)$

5.3 Reciprocal Arguments

(a) $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}; |x| \geq 1$ (Both the functions are identical)

and $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}; |x| \leq 1, x \neq 0$ (Both the functions are not identical)

(b) $\sec^{-1}x = \cos^{-1}\frac{1}{x}; |x| \geq 1$ (Both the functions are identical)

and $\cos^{-1}x = \sec^{-1}\frac{1}{x}; |x| \leq 1$ (Both the functions are not identical)

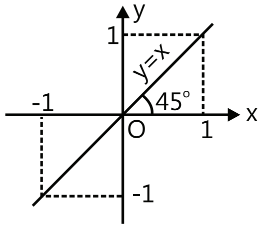
(c) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right), \quad x \in (0, \infty) = -\pi + \cot^{-1}\left(\frac{1}{x}\right), \quad x \in (-\infty, 0)$,

and $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), \quad x \in (0, \infty) = \pi + \tan^{-1}\left(\frac{1}{x}\right), \quad x \in (-\infty, 0)$

5.4 Forward Inverse Identities

(a) $y = \sin(\sin^{-1}x) = x, x \in [-1,1], y \in [-1,1]$, y is aperiodic

(b) $y = \cos(\cos^{-1}x) = x, x \in [-1,1], y \in [-1,1]$, y is aperiodic



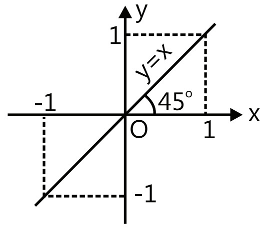
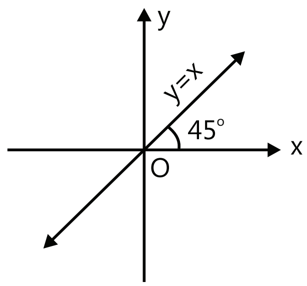


Figure 20.8

(c) $y = \tan(\tan^{-1} x) = x,\ x \in R,\ y \in R$, y is aperiodic



(d) $y = \cot(\cot^{-1} x) = x,\ x \in R,\ y \in R$, y is aperiodic

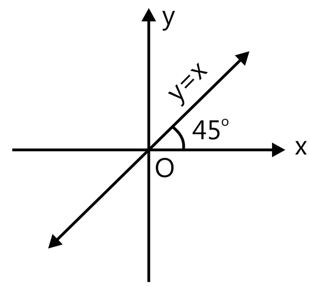
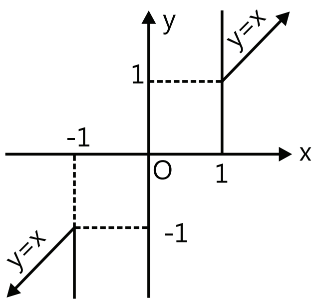


Figure 20.9

(e) $y = \text{cosec}(\text{cosec}^{-1} x) = x,\ |x| \ge 1, |y| \ge 1$, y is aperiodic



(f) $y = \sec(\sec^{-1} x) = x,\ |x| \ge 1, |y| \ge 1$, y is aperiodic

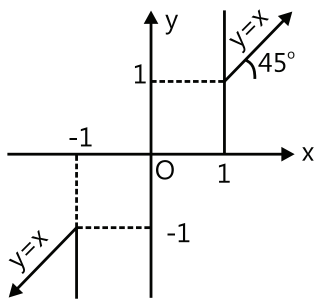


Figure 20.10

Also,

$\cos(\sin^{-1} x) = \sqrt{1-x^2}$

$\sin(\cos^{-1} x) = \sqrt{1-x^2}$

$\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}$

$\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$

$\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{1+x^2}}$

$\tan\left(\sin^{-1} x\right) = \dfrac{x}{\sqrt{1-x^2}}$

5.5 Inverse Forward Identities

(a) $y = \sin^{-1}(\sin x) = x, \lceil x \in R \rceil, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Periodic with period 2 π

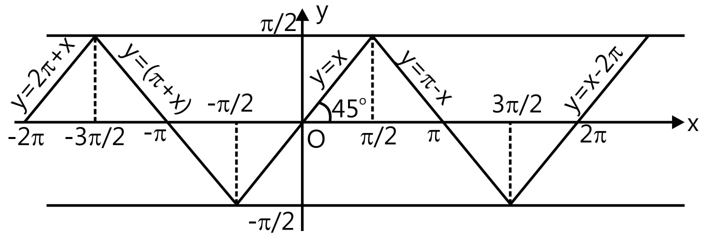


Figure 20.11

(b) $y = \cos^{-1}(\cos x) = x, x \in R, y \in [0, \pi]$, Periodic with period 2 π

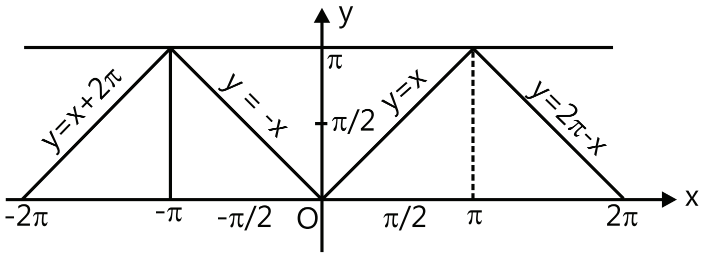


Figure 20.12

(c) $y = \tan^{-1}(\tan x) = x, x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, Periodic with period π

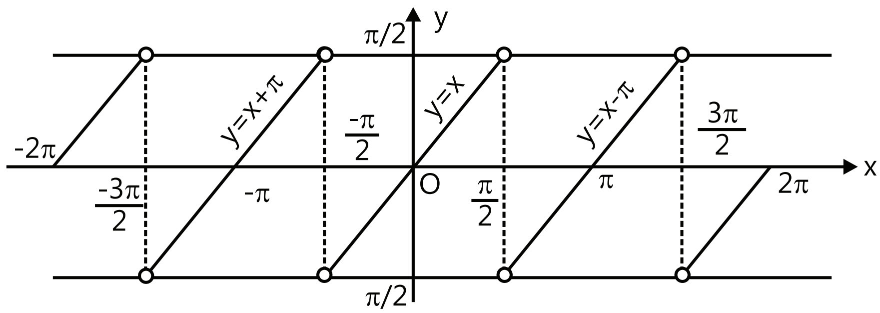


Figure 20.13

(d) $y = \cot^{-1}(\cot x) = x, x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with π

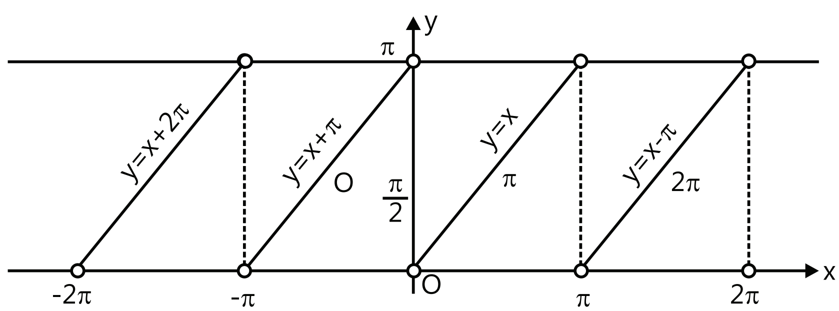


Figure 20.14

(e) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x) = x \in R - \{n\pi, n \in I\}$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ y is periodic with period 2π

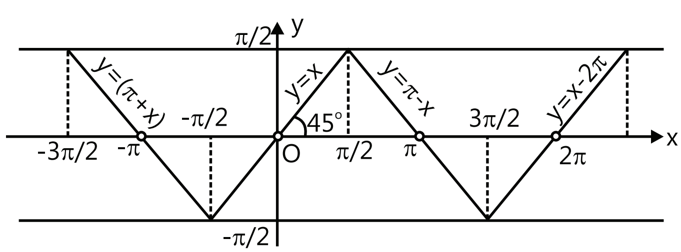


Figure 20.15

(f) $y = \sec^{-1}(\sec x) = x$, y is periodic,

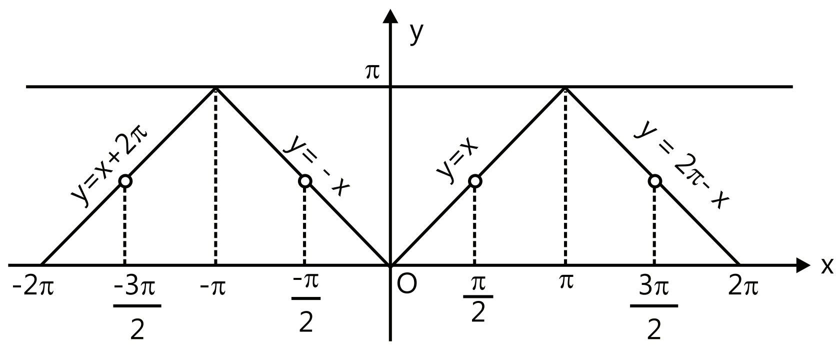


Figure 20.16

$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ with period 2π

(i) $\tan^{-1}(\cot x) = \frac{1}{2}\pi - x$ for $x \in [0, \pi]$

(ii) $\sin^{-1}(\operatorname{cosec} x) = \frac{1}{2}\pi - x$ for $x \in [0, \pi]$

(iii) $\sec^{-1}(\cos x) = \frac{1}{2}\pi - x$ for $x \in \left[0, \frac{1}{2}\pi\right]$.

5.6 Sum of Angles

(a) $\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) & \text{if } x \ge 0; y \ge 0 \text{ and } x^2 + y^2 \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) & \text{if } x \ge 0; y \ge 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$ $x > 0; y > 0$

(c) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$ if x, y >0 and $x^2 + y^2 \le 1$

(d) $\cos^{-1} x \pm \cos^{-1} y = \pi - \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$ if x, y > 0 and $x^2+y^2 \le 1$

(e) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy} & x > 0\, y > 0 \text{ and } xy < 1 \Rightarrow 0 < \tan^{-1} x + \tan^{-1} y < \frac{\pi}{2} \\ \pi - \tan^{-1}\frac{x+y}{1-xy} & x > 0\, y > 0 \text{ and } xy > 1 \Rightarrow \frac{\pi}{2} < \tan^{-1} x + \tan^{-1} y < \pi \end{cases}$

(f) x > 0 & y > 0 then $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$ (with no other restriction)

(g) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$; $\tan^{-1} - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

PLANCESS CONCEPTS

The above results can be generalized as follows:

$$\tan^{-1}x_1+\tan^{-1}x_2+...+\tan^{-1}x_n=\tan^{-1}\left[\frac{S_1-S_3+S_5-.............}{1-S_2+S_4-S_6+.......}\right]$$

where S_k denotes the sum of products of $x_1, x_2, \ldots, x_n$ taken k at a time

Rohit Kumar (JEE 2012 AIR 78)

Illustration 4: Evaluate: $\sin\left(\tan^{-1}\frac{15}{8}\right)$ **(JEE MAIN)**

Sol: Convert $\tan^{-1}\frac{15}{8}$ to $\sin^{-1}$.

We know that $\sin(\sin^{-1}x)=x$, for all $x \in$ [-1, 1], So, will convert each expression in the form sin $(\sin^{-1}x)$ by using

$\cos^{-1}\frac{b}{h}=\sin^{-1}\frac{p}{h}$, $\tan^{-1}\frac{p}{b}=\sin^{-1}\frac{p}{h}$, $\cot^{-1}\frac{p}{b}=\sin^{-1}\frac{b}{h}$ etc.

Where b, p and h denote the base, perpendicular and hypotenuse of a right triangle.

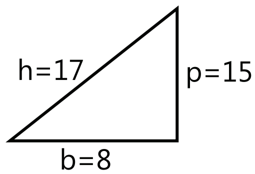


Figure 20.17

$$\sin\left(\tan^{-1}\frac{15}{8}\right)=\sin\left(\sin^{-1}\frac{15}{17}\right)=\frac{15}{17}$$

Illustration 5: Evaluate: $\cos\left(\operatorname{cosec}^{-1}\frac{13}{12}\right)$ **(JEE MAIN)**

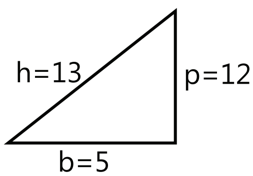


Figure 20.18

Sol: Write $\operatorname{cosec}^{-1}$ in terms of $\cos^{-1}$.

$$\cos\left(\operatorname{cosec}^{-1}\frac{13}{12}\right)=\cos\left(\cos^{-1}\frac{5}{13}\right)=\frac{5}{13}\left[\because \operatorname{cosec}^{-1}\frac{13}{12}=\cos^{-1}\frac{5}{13}\right]$$

Illustration 6: Find the principal value of $\cot^{-1}(-\sqrt{3})$ **(JEE MAIN)**

Sol: The principal value of $\cot^{-1}x$ lies in between 0 to π.

Let $\cot^{-1}(-\sqrt{3})=\theta$

Then $\cot\theta=-\sqrt{3}=-\cot\frac{\pi}{6}$

Since principal value branch of $\cot^{-1}x$ is $0<\theta<\pi$. Therefore, we want to find the value of θ such that $0<\theta<\pi$.

Now, $\cot\theta=-\cot\frac{\pi}{6}=\cot\left(\pi-\frac{\pi}{6}\right)=\cot\frac{5\pi}{6}$

Therefore, principal value of $\cot^{-1}(-\sqrt{3})=\frac{5\pi}{6}$.

Illustration 7: $\sin^{-1}\left(\sin\frac{10\pi}{7}\right)=\frac{10\pi}{7}$ **(JEE MAIN)**

Sol: Write $\frac{10\pi}{7}$ as $\pi+\frac{3\pi}{7}$ and expand.

$$=\sin^{-1}\left(\sin\frac{10\pi}{7}\right)=\sin^{-1}\left(-\sin\left(\frac{3\pi}{7}\right)\right)=\sin^{-1}\left(\sin\left(-\frac{3\pi}{7}\right)\right)=-\frac{3\pi}{7}$$

Illustration 8: $\cos^{-1}\left(\sin\left(-\frac{\pi}{9}\right)\right)$ **(JEE MAIN)**

Sol: $=\cos^{-1}\left(\cos\left(\frac{\pi}{2}+\frac{\pi}{9}\right)\right)=\cos^{-1}\left(\cos\left(\frac{11\pi}{18}\right)\right)=\frac{11\pi}{18}$

Illustration 9: $\sin^{-1}\left(\cos\frac{13\pi}{10}\right)$ **(JEE MAIN)**

Sol: Similar to previous example.

$$=\sin^{-1}\cos\frac{13\pi}{10}=\sin^{-1}\left(-\cos\frac{3\pi}{10}\right)=\sin^{-1}\left(-\sin\left(\frac{5\pi}{10}-\frac{2\pi}{10}\right)\right)=\sin^{-1}\left(-\sin\frac{\pi}{5}\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right)=-\frac{\pi}{5}$$

Illustration 10: Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$. **(JEE MAIN)**

Sol: Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=y$. Then $\sin\, y=\frac{1}{\sqrt{2}}\Rightarrow\quad y=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

We know that, the range of the principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$.

Therefore, principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

Illustration 11: Find the integral solution of the inequality $3x^2+8x<2\sin^{-1}(\sin 4)-\cos^{-1}(\cos 4)$.
(JEE ADVANCED)

Sol: Use inverse forward identities to simplify the equation.

$3x^2+8x<-4 \quad\Rightarrow\quad 3x^2+8x+4<0$

$\Rightarrow\quad 3x^2+6x+2x+4<0\Rightarrow\quad 3x(x+2)+2(x+2)<0$

$(x+2)(3x+2)<0 \qquad x=-1$

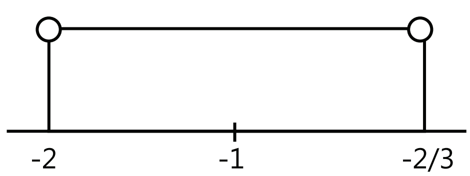


Figure 20.19

Illustration 12: Find the largest integral value of k, for which $(k-2)x^2+8x+k+4>\sin^{-1}(\sin 12)+\cos^{-1}(\cos 12)$, for all $x\in R$. **(JEE ADVANCED)**

Sol: Use inverse forward identities.

$\sin^{-1}(\sin 12)=\sin^{-1}(\sin(12-4\pi))=12-4\pi$

$\cos^{-1}(\cos 12)=\cos^{-1}(\cos(4\pi-12))=4\pi-12$

$\therefore\ (k-2)x^2+8x+k+4>0,\quad \forall x\in R$

If k = 2, then 8x+4>0, (not possible)

and if $k\neq 2$, then k-2>0 $\Rightarrow$ k>2

and $64-4(k-2)(k+4)<0 \Rightarrow 16<k^2+2k-8$

$\Rightarrow k^2+2k-24>0 \Rightarrow (k+6)(k-4)>0$

K = 5

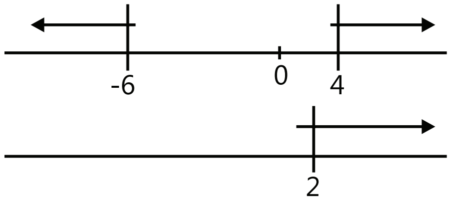


Figure 20.20

Illustration 13: Find domain of $f(x)=\dfrac{1}{\sqrt{\ln\left(\cot^{-1}x\right)}}$. **(JEE MAIN)**

Sol: Find the range of x for which $\ln(\cot^{-1}x)>0 \Rightarrow \cot^{-1}x>1 \Rightarrow x<\cot 1 \Rightarrow x\in(-\infty,\cot 1)$

Illustration 14: Evaluate the following: **(JEE MAIN)**

(i) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right)$ (ii) $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right)$ (iii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ (iv) $\cos\left\{\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)+\dfrac{\pi}{6}\right\}$

Sol: Recall that, $\sin^{-1}(\sin\theta)=\theta$, if $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$, $\cos^{-1}(\cos\theta)=\theta$, if $0\le\theta\le\pi$ and

$\tan^{-1}(\tan\theta)=\theta$, if $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$. Therefore,

(i) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right)=\dfrac{\pi}{3}$ (ii) $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right)=\dfrac{\pi}{4}$

(iii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)\neq\dfrac{7\pi}{6}$, because $\dfrac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi-\dfrac{5\pi}{6}\right)\right)$ $\left[\because \dfrac{7\pi}{6}=2\pi-\dfrac{5\pi}{6}\right]$ $=\cos^{-1}\left(\cos\dfrac{5\pi}{6}\right)$ $[\because\cos(2\pi-\theta)=\cos\theta]$ $=\dfrac{5\pi}{6}$

(iv) $\cos\left\{\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)+\dfrac{\pi}{6}\right\}=\cos\left(\dfrac{5\pi}{6}+\dfrac{\pi}{6}\right)$ $\left[\because\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{5\pi}{6}\right]$

Illustration 15: Evaluate the following:

(i) $\sin\left(\cos^{-1}\dfrac{3}{5}\right)$ (ii) $\sin\left(\dfrac{\pi}{2}-\sin^{-1}\left(-\dfrac{1}{2}\right)\right)$ (iii) $\sin(\cot^{-1}x)$ **(JEE MAIN)**

Sol: (i) Let $\cos^{-1}\dfrac{3}{5}=\theta$. Then, $\cos\theta=\dfrac{3}{5} \Rightarrow \sin\theta=\dfrac{4}{5}$

$\therefore\ \sin\left(\cos^{-1}\dfrac{3}{5}\right)=\sin\theta=\dfrac{4}{5}$

(ii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right)=\sin\left(\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

(iii) Let $\cot^{-1}x=\theta$, Then, $x=\cot\theta$

Now, $\cot\theta=x \Rightarrow \sin\theta=\frac{1}{\sqrt{1+x^2}}$ $\quad\therefore \sin(\cot^{-1}x)=\sin\theta=\frac{1}{\sqrt{1+x^2}}$

Illustration 16: Evaluate the following:

(i) $\sin^{-1}(\sin 5)$ (ii) $\cos^{-1}(\cos 10)$ **(JEE MAIN)**

Sol: Notice that the angle is in radians.

(i) Here, $\theta=5$ radians. Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But

$2\pi-5$ and $5-2\pi$ both lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that

$\sin(5-2\pi)=\sin(-(2\pi-5))=-\sin(2\pi-5)=-(-\sin 5)=\sin 5$

$\Rightarrow \sin^{-1}(\sin 5)=\sin^{-1}(\sin(5-2\pi))=5-2\pi.$

(ii) We know that $\cos^{-1}(\cos\theta)=\theta$, if $0\le\theta\le\pi$. Here, $\theta=10$ radians. Clearly, it does not lie between 0 and π such that, $(4\pi-10)=\cos 10 \Rightarrow \cos^{-1}(\cos 10)=\cos^{-1}(\cos(4\pi-10))=4\pi-10$

Illustration 17: Evaluate the following:

(i) $\sin^{-1}(2\sin^{-1}0.8)$ (ii) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$ **(JEE MAIN)**

Sol: Write the term inside the brackets in (i) and (ii) as $\sin^{-1}$ and $\tan^{-1}$ respectively.

(i) We know that: $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

$\therefore\ 2\sin^{-1}0.8=\sin^{-1}(2\times 0.8\times\sqrt{1-0.64})$

$\Rightarrow \sin^{-1}(2\sin^{-1}0.8)=\sin\{\sin^{-1}(0.96)\}=0.96$

(ii) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$

$=\tan\left(\tan^{-1}\frac{5}{12}-\frac{\pi}{4}\right)$ $\quad\left[\text{From (ii) we have, } 2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{5}{12}\right]$

$=\tan\left(\tan^{-1}\frac{5}{12}-\tan^{-1}1\right)$ $\left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right) \text{ if } xy>-1\right]$ $=\tan\left[\tan^{-1}\left(\frac{-7}{17}\right)\right]=-\frac{7}{17}$

Illustration 18: Write the following in their simplest forms:

(i) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ (ii) $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$ **(JEE ADVANCED)**

Sol: (i) Use the formula $1-\cos x=2\sin^2 x/2$ and $1+\cos x=2\cos^2 x/2$

(ii) Write the term inside the square bracket in terms of $\sin^{-1}$.

(i) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\dfrac{2\sin^2 x/2}{2\cos^2 x/2}} = \tan^{-1}\left|\tan\dfrac{x}{2}\right| = \dfrac{|x|}{2}$

(ii) $\sin[\cot^{-1}\{\cos(\tan^{-1} x)\}]$

$$= \sin\left[\cot^{-1}\left\{\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}\right] = \sin\left(\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \sin\left\{\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} \qquad \left[\because \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$= \sqrt{\frac{1+x^2}{2+x^2}}$$

Illustration 19: Express $\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right), -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ in the simplest form. **(JEE ADVANCED)**

Sol: Convert the term inside the bracket in terms of $\tan\dfrac{x}{2}$ and proceed.

We write, $\Rightarrow \tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\dfrac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$

$$= \left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right] = \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Alternatively,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right] = \tan^{-1}\left[\frac{\sin\left(\frac{\pi-2x}{2}\right)}{1-\cos\left(\frac{\pi-2x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi-2x}{4}\right)\cos\left(\frac{\pi-2x}{4}\right)}{2\sin^2\left(\frac{\pi-2x}{4}\right)}\right] = \tan^{-1}\left[\cot\left(\frac{\pi-2x}{4}\right)\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\pi-2x}{4}\right)\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Illustration 20: If $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1} x\right) = 1$, find the value of x. **(JEE MAIN)**

Sol: From the question, we have $\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1} x\right) = \dfrac{\pi}{2}$ and proceed.

We have $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1} x\right) = 1$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1 \quad \Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5} \quad \Rightarrow \quad \cos^{-1}x=\cos^{-1}\frac{1}{5} \Rightarrow x=\frac{1}{5}$$

Illustration 21: Find the value of cos $(\sec^{-1}x+\operatorname{cosec}^{-1}x), |x| \geq 1$. **(JEE MAIN)**

Sol: Use $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

We have $\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right) = \cos\left(\frac{\pi}{2}\right) = 0$

Illustration 22: Find maximum & minimum values of $\left(\sec^{-1}x\right)^2 + \left(\operatorname{cosec}^{-1}x\right)^2$. **(JEE ADVANCED)**

Sol: Apply the identity $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ and then use suitable substitution to form a quadratic.

$y=(\sec^{-1}x)^2+(\operatorname{cosec}^{-1}x)^2$

$=(\sec^{-1}x+\operatorname{cosec}^{-1}x)^2-2\sec^{-1}x\ \operatorname{cosec}^{-1}x$

put $t=\sec^{-1}x$; $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$

$$y=\frac{\pi^2}{4}-2t\left(\frac{\pi}{2}-t\right)=2t^2-\pi t+\frac{\pi^2}{4}$$

$$y=2\left[t^2-\frac{\pi}{2}t+\frac{\pi^2}{8}\right]=2\left[\left(t-\frac{\pi}{4}\right)^2+\frac{\pi^2}{16}\right]=\frac{\pi^2}{8}+2\left(t-\frac{\pi}{4}\right)^2 \therefore y_{min}=\frac{\pi^2}{8};\quad y_{max}=\frac{5\pi^2}{4} \text{ at } t=\pi$$

Illustration 23: Find the range of $f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$. **(JEE MAIN)**

Sol: Find the domain of the given function and then find the range.

$f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$

Here domain is only x = 1or –1;

So range will contain only 2 elements $\{3\pi/4,\ \pi/4\}$

Illustration 24: Find the number of solutions of the equation $\tan^{-1}x^3+\cot^{-1}(e^x)=\frac{\pi}{2}$. **(JEE ADVANCED)**

Sol: Use $\tan^{-1}A+\cot^{-1}A=\frac{\pi}{2}$ to simplify the given equation and then take the help of graph to find the number of solution.

$$\cot^{-1}(e^x)=\frac{\pi}{2}-\tan^{-1}(x^3)=\cot^{-1}(x^3) \Rightarrow e^x=x^3 \Rightarrow x^3e^{-x}=1$$

Plotting the graph of y = I and $y=x^3e^{-x}$ we can see that the line intersects the curve at two points. Hence there are 2 solutions for the above equation.

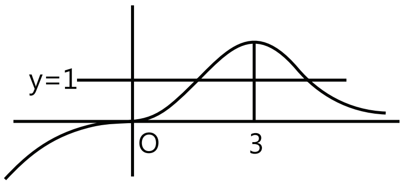


Figure 20.21

Illustration 25: Find the number of values of x satisfying the equation

$$\tan^{-1}\left(x-\frac{x^3}{4}+\frac{x^5}{16}-......\right)+\cot^{-1}\left(x+\frac{x^2}{2}+\frac{x^3}{4}-......\right)=\frac{\pi}{2} \text{ for } 0<|x|<2.$$ **(JEE ADVANCED)**

Sol: Use $\tan^{-1}A+\cot^{-1}A=\frac{\pi}{2}$.

We must have $x-\frac{x^3}{4}+\frac{x^5}{16}-......=x+\frac{x^2}{2}+\frac{x^3}{4}+......$

$$\Rightarrow \frac{x}{1+\frac{x^2}{4}}=\frac{x}{1-\frac{x}{2}} \Rightarrow \frac{4x}{4+x^2}=\frac{2x}{2-x} \Rightarrow 2x^2(x+2)=0$$

$\therefore\ x=0,\ -2 \quad (\text{As } 0<|x|<2)$

Clearly no value of x satisfies given equation.

Illustration 26: Prove that $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$ **(JEE MAIN)**

Sol: Use the formula $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

We have, $=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}x\frac{7}{24}}\right\}$ $\left[\because \tan^{-1}x+\tan^{-1}y =\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ If } xy<1\right]$

$$=\tan^{-1}\left\{\frac{48+77}{264-14}\right\}=\tan^{-1}\left(\frac{125}{250}\right)=\tan^{-1}\left(\frac{1}{2}\right)$$

Illustration 27: If $\tan^{-1}4+\tan^{-1}5=\cot^{-1}(\lambda)$ then find λ. **(JEE MAIN)**

Sol: Write the L.H.S. in terms of $\cot^{-1}$ and compare.

We have $\tan^{-1}4+\tan^{-1}5=\tan^{-1}\frac{4+5}{1-20}=\pi-\tan^{-1}\frac{9}{19}=\pi-\cot^{-1}\frac{19}{9}$

$$=\cot^{-1}\left(-\frac{19}{9}\right) \qquad \Rightarrow \lambda=-\frac{19}{9}$$

Illustration 28: Prove that: $\tan^{-1}\frac{1-x}{1+x}-\tan^{-1}\frac{1-y}{1+y}=\sin^{-1}\left(\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}\right)$ **(JEE MAIN)**

Sol: Use the formula $\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}x+\tan^{-1}y$.

We have, LHS $=\tan^{-1}\frac{1-x}{1+x}-\tan^{-1}\frac{1-y}{1+y}=(\tan^{-1}1-\tan^{-1}x)-(\tan^{-1}1-\tan^{-1}y)=\tan^{-1}y-\tan^{-1}x$

$$= \tan^{-1}\left(\frac{y-x}{1+yx}\right) = \sin^{-1}\left(\frac{y-x}{\sqrt{(1+yx)^2+(y-x)^2}}\right) = \sin^{-1}\left\{\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right\} = \text{RHS}$$

Illustration 29: Prove that: (i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$ (ii) $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$

(iii) $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$ **(JEE ADVANCED)**

Sol: Same as above.

(i) LHS $=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}$

$$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\text{x}\frac{1}{13}}\right\} \quad \left[\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right] = \tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\frac{2}{9}=\text{R.H.S.}$$

(ii) L.H.S. $= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}$

$$=\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right)-\tan^{-1}\frac{8}{19} = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\text{x}\frac{3}{5}}\right)-\tan^{-1}\frac{8}{19}$$

$$=\tan^{-1}\left(\frac{27}{11}\right)-\tan^{-1}\frac{8}{19} = \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\text{x}\frac{8}{19}}\right) = \tan^{-1}\frac{425}{425} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

(iii) L.H.S.$= \tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$

$$=\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right) \quad = \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\text{x}\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\text{x}\frac{1}{8}}\right)$$

$$=\tan^{-1}\frac{6}{17}+\tan^{-1}\frac{11}{23} \quad =\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\text{x}\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1} = \frac{\pi}{4}=\text{R.H.S.}$$

Illustration 30: Show that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{3}{4}$. **(JEE MAIN)**

Sol: We have, L.H.S. $= \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\text{x}\frac{2}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4}=\text{R.H.S.}$

Illustration 31: Simplify $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$ if $\frac{a}{b}\tan x>-1$. **(JEE MAIN)**

Sol: Divide the numerator and denominator inside the bracket by bcos x and expand.

We have, $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left[\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x-a\sin x}{b\cos x}}\right]=\tan^{-1}\left[\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right]=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\ \frac{a}{b}-x$

Illustration 32: Solve the following equations:

(i) $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ (ii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ **(JEE ADVANCED)**

Sol: Write $\frac{\pi}{4}$ as $\tan^{-1}1$ and simplify.

(i) $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\tan^{-1}1 \Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}1-\tan^{-1}\ \frac{x+1}{x+2}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right) \Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\ \frac{x+2-x-1}{x+2+x+1}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{1}{2x+3} \Rightarrow \frac{x-1}{x-2}=\frac{1}{2x+3} \Rightarrow (2x+3)(x-1)=x-2$

$\Rightarrow 2x^2+x-3=x-2 \Rightarrow 2x^2-1=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

(ii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \frac{2\cos x}{\sin^2 x}=2\operatorname{cosec} x \Rightarrow \cos x=\sin x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}$

Illustration 33: Prove that: $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}$ **(JEE MAIN)**

Sol: Covert the L.H.S. in terms of $\cos^{-1}$.

We have $\sin^{-1}\ \frac{3}{5}-\sin^{-1}\ \frac{8}{17}\ =\cos^{-1}\ \frac{4}{5}-\cos^{-1}\ \frac{15}{17}$ $\left[\because \sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5}\ \&\ \sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}\right]$

$=\cos^{-1}\left\{\frac{4}{5}x\frac{15}{17}+\sqrt{1-\left(\frac{4}{5}\right)^2}\ x\sqrt{1-\left(\frac{15}{17}\right)^2}\right\}=\cos^{-1}\left\{\frac{4}{5}x\frac{15}{17}+\frac{3}{5}x\frac{8}{17}\right\}=\cos^{-1}\left\{\frac{60}{85}+\frac{24}{85}\right\}=\cos^{-1}\frac{84}{85}$

Illustration 34: Prove that: $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$ **(JEE MAIN)**

Sol: We have $\sin^{-1}\frac{4}{5}-\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\left\{\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}\right\}+\sin^{-1}\frac{16}{65}$

$$=\sin^{-1}\left\{\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right\}+\sin^{-1}\frac{16}{25}$$

$$=\sin^{-1}\left\{\frac{4}{5}\text{x}\frac{12}{13}+\frac{5}{13}\text{x}\frac{3}{5}\right\}+\sin^{-1}\frac{16}{25}=\sin^{-1}\frac{63}{65}+\sin^{-1}\frac{16}{25}$$

$$=\cos^{-1}\frac{16}{65}+\sin^{-1}\frac{16}{25}\left[\because \sin^{-1}\frac{63}{65}=\cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2}=\cos^{-1}\frac{16}{65}\right]$$

$$=\frac{\pi}{2}\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$$

6. SIMPLIFICATION OF INVERSE FUNCTIONS BY ELEMENTARY SUBSTITUTION

(a) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ if $-1\le x\le 1$

(b) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$ if $-1\le x\le 1$

(c) $$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right) & -1\le x\le 1\\ \sin^{-1}\left(\frac{2x}{1+x^2}\right) & 0\le x\le 1\\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & 0\le x<\infty\end{cases}$$

(d) $$\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}2\tan^{-1}x & -1\le x\le 1\\ \pi-2\tan^{-1}x & x\ge 1\\ -\pi-2\tan^{-1}x & x\le -1\end{cases}$$

(e) $$\cos^{-1}\frac{1-x^2}{1+x^2}=\left[\begin{array}{ll}2\tan^{-1}x & x\ge 0\\ -2\tan^{-1}x & x<0\end{array}\right.$$

(f) $$\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}\pi+2\tan^{-1}x & x<-1\\ 2\tan^{-1}x & -1<x<1\\ 2\tan^{-1}x-\pi & x>1\end{cases}$$

(g) $$\sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\text{cosec}^{-1}\left(\frac{1}{x}\right)$$

(h) $\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(i) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

(j) $f(x)=\sin^{-1}\frac{2x}{1+x^2}+2\tan^{-1}x = \pi$, if $x \ge 1$

(k) $f(x) = \sin^{-1}\frac{2x}{1+x^2}+2\tan^{-1}x = -\pi$ if $x \le -1$

(l) $\sin^{-1}(3x-4x^3) = \begin{cases} -(\pi+3\sin^{-1}x) & -1 \le x \le 1/2 \\ 3\sin^{-1}x & -1/2 \le x \le 1/2 \\ \pi - 3\sin^{-1}x & 1/2 \le x \le 1 \end{cases}$

(m) $\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}x - 2\pi & -1 \le x \le -1/2 \\ 2\pi - 3\cos^{-1}x & -1/2 \le x \le 1/2 \\ 3\cos^{-1}x & 1/2 \le x \le 1 \end{cases}$

(n) $\tan^{-1}\frac{3x-x^3}{1-3x^2} = \begin{cases} 3\tan^{-1}x & -\frac{1}{\sqrt{3}} < x\frac{1}{\sqrt{3}} \\ -\pi+3\tan^{-1}x & x > \frac{1}{\sqrt{3}} \\ \pi+3\tan^{-1}x & x < -\frac{1}{\sqrt{3}} \end{cases}$

PLANCESS CONCEPTS

While writing inverse trigonometric functions in their simplest forms, we use the following substitutions.

- For $\sqrt{a^2-x^2}$, we substitute $x = a\sin\theta$ or $x = a\cos\theta$
- For $\sqrt{a^2+x^2}$, we substitute $x = a\tan\theta$ or $x = a\cot\theta$
- For $\sqrt{x^2-a^2}$, we substitute $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
- For $\sqrt{a+x}$ and $\sqrt{a-x}$ occurring together or separately, we substitute $x = a\cos\theta$

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Illustration 35: Solve for x: $\sin\left(2\cos^{-1}\left(\cot(2\tan^{-1}x)\right)\right) = 0$ **(JEE ADVANCED)**

Sol: The R.H.S. is equal to zero implies $\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = \frac{n\pi}{2}$ and proceed accordingly to find the value of x.

$$\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n=0 \\ \frac{\pi}{2} & \text{if } n=1 \\ \pi & \text{if } n=2 \end{cases} \Rightarrow \cot\left(2\tan^{-1}x\right) = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

$$\Rightarrow 2\tan^{-1}x = \begin{cases} n\pi + \frac{\pi}{4} \\ n\pi + \frac{\pi}{2} \\ n\pi - \frac{\pi}{4} \end{cases} \Rightarrow \tan^{-1}x = \begin{cases} \frac{n\pi}{2} + \frac{\pi}{8} \\ \frac{n\pi}{2} + \frac{\pi}{4} \\ \frac{n\pi}{2} - \frac{\pi}{8} \end{cases} \Rightarrow \tan^{-1}x = \begin{cases} \frac{\pi}{8}, -\frac{3\pi}{8} \\ \frac{\pi}{4}, -\frac{\pi}{4} \\ -\frac{\pi}{8}, \frac{3\pi}{8} \end{cases}$$

$$\Rightarrow x = \pm 1, \pm(\sqrt{2}-1), \pm(\sqrt{2}+1)$$

Illustration 36: Solve the system of inequalities involving inverse circular functions arc $\tan^2 x - 3$ arc $\tan x + 2 > 0$ and $[\sin^{-1}x] > [\cos^{-1}x]$ where [ ] denotes the greatest integer function. **(JEE ADVANCED)**

Sol: Substitute $\tan^{-1}x$ equal to t.

$\Rightarrow (t-2)(t-1) > 0$

$\Rightarrow t > 2 \;\text{ or }\; t > 1$

$\Rightarrow \tan^{-1}x > 2 \;\text{ or }\; \tan^{-1}x > 1$

$x \in (-\infty, \tan 1)\;\; x > \tan 1$

Again $[\sin^{-1}x] > [\cos^{-1}x]$

$[\sin^{-1}x]$ can take the values $\{-2,-1,0,1\}$

And $[\cos^{-1}x]$ can take the values $\{0,1,2,3\}$

Hence $[\sin^{-1}x]$ can be greater than $[\cos^{-1}x]$ only

If $[\sin^{-1}x] = 1$ and $[\cos^{-1}x] = 1$

Now, $[\sin^{-1}x] = 1 \Rightarrow 1 \le \sin^{-1}x \le \pi/2 \qquad (1 \le \sin^{-1}x < 2)$

$\sin 1 \le x \le 1$

And $[\cos^{-1}x] = 0 \Rightarrow 0 \le \cos^{-1}x < 1$

$\cos 1 < x \le 1$

Now, x must satisfy

From this $x \in [\sin 1, 1]$

PROBLEM-SOLVING TACTICS

- Making a habit of writing angle values in radians rather in degrees makes the calculation of inverse trigonometric functions easier.
- Try to remember graphs of inverse trigonometric functions. Sometimes it is easier to approximate answers using graphical methods.
- Always verify whether the results are in the range or domain of the respective function.
- In some cases, constructing a right angled triangle for the given inverse function and then solving using properties of triangle is much helpful.
- In case of identities in inverse circular functions, principal values should be taken. As such signs of x, y, etc., will determine the quadrant in which the angles will fall. In order to bring the angles of both sides in the same quadrant, one should make an adjustment by π.

FORMULAE SHEET

1.	If y = sin x, then x = $\sin^{-1}$ y, similarly for other inverse T-functions.

2. Domain and Range of Inverse T-functions:		
Function	**Domain(D)**	**Range (R)**
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\cot^{-1} x$	$-\infty < x < \infty$	$0 < \theta < \pi$
$\sec^{-1} x$	$x \le -1,\ x \ge 1$	$0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$
$\operatorname{cosec}^{-1} x$	$x \le -1,\ x \ge 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$

3.	Properties of Inverse T-functions:
	(i) $\sin^{-1}(\sin\theta) = \theta$ provided $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\cos^{-1}(\cos\theta) = \theta$ provided $0 \le \theta \le \pi$ $\tan^{-1}(\tan\theta) = \theta$ provided $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\cot^{-1}(\cot\theta) = \theta$ provided $0 < \theta < \pi$

	$\sec^{-1}(\sec\theta) = \theta$ provided $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$ $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ provided $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$
	(ii) $\sin(\sin^{-1}x) = x$ provided $-1 \le x \le 1$ $\cos(\cos^{-1}x) = x$ provided $-1 \le x \le 1$ $\tan(\tan^{-1}x) = x$ provided $-\infty < x < \infty$ $\cot(\cot^{-1}x) = x$ provided $-\infty < x < \infty$ $\sec(\sec^{-1}x) = x$ provided $-\infty < x \le -1$ or $1 \le x < \infty$ $\text{cosec}(\text{cosec}^{-1}x) = x$ provided $-\infty < x \le -1$ or $1 \le x < \infty$
	(iii) $\sin^{-1}(-x) = -\sin^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$ $\tan^{-1}(-x) = -\tan^{-1}x$ $\cot^{-1}(-x) = \pi - \cot^{-1}x$ $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$ $\sec^{-1}(-x) = \pi - \sec^{-1}x$ **(iv)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in [-1, 1]$ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in R$ $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in (-\infty,-1] \cup [1,\infty)$

4.	Value of one inverse function in terms of another inverse function:
	(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$ $= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1}\frac{1}{x}, \; 0 \le x \le 1$ **(ii)** $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$ $= \sec^{-1}\frac{1}{x} = \text{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}, \; 0 \le x \le 1$ **(iii)** $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}, \; x \ge 0$ **(iv)** $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x, \; \forall\, x \in (-\infty,1] \cup [1,\infty)$ **(v)** $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \; \forall\, x \in (-\infty,1] \cup [1,\infty)$ **(vi)** $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & \text{for } x > 0 \\ -\pi + \cot^{-1}x & \text{for } x < 0 \end{cases}$

5.	Formulae for sum and difference of inverse trigonometric function:
	(i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$; if $x>0,\ y>0,\ xy<1$
	(ii) $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$; if $x>0,\ y>0,\ xy>1$
	(iii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1+xy}\right)$; if $xy>-1$
	(iv) $\tan^{-1}x-\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right)$; if $x>0,\ y<0,\ xy<-1$
	(v) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$
	(vi) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$; If $x,y,\geq 0\ \&\ x^2+y^2\leq 1$
	(vii) $\sin^{-1}x\pm\sin^{-1}y=\pi-\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$; If $x,y,\geq 0\ \&\ x^2+y^2>1$
	(viii) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left[xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right]$; If $x,y,>0\ \&\ x^2+y^2\leq 1$
	(ix) $\cos^{-1}x\pm\cos^{-1}y=\pi-\cos^{-1}\left[xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right]$; If $x,y,>0\ \&\ x^2+y^2>1$

6.	Inverse trigonometric ratios of multiple angles
	(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, if $-1\leq x\leq 1$
	(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$, if $-1\leq x\leq 1$
	(iii) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
	(iv) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$
	(v) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$
	(vi) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Solved Examples

JEE Main/Boards

Example 1: Evaluate the following

(a) $\tan^{-1}(-1)$ (b) $\cot^{-1}(-1)$ (c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Sol: Do it yourself.

(a) $\tan^{-1}(-1) = \frac{-\pi}{4}$ as $\tan\left(-\frac{\pi}{4}\right) = -1$

(b) $\cot^{-1}(-1) = \frac{3\pi}{4}$ as $\cot\left(\frac{3\pi}{4}\right) = -1$

(c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ as $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Example 2: Find the angle $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

Sol: Write the angle $\frac{2\pi}{3}$ as $\pi - \frac{\pi}{3}$ and proceed.

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) \neq \frac{2\pi}{3}\left(\text{as } \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}.$$

Example 3: Find the value of

$\cos\ [2\sin^{-1}x + \cos^{-1}x]$ at $x = \frac{1}{5}$.

Sol: Use $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$\cos[2\sin^{-1}x + \cos^{-1}x]$

$= \cos[\cos^{-1}x + \sin^{-1}x + \sin^{-1}x]$

$= \cos\left[\frac{\pi}{2} + \sin^{-1}x\right] = -\sin(\sin^{-1}(x))$

$= -x = -\frac{1}{5}$

Example 4: Prove that $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$

Sol: Use substitution.

Let $2\sin^{-1}x = \theta$, where $\theta \in [-\pi, \pi]$;

then $x = \sin\frac{\theta}{2}$

$\therefore \sin(2\sin^{-1}x) = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\sin\frac{\theta}{2}\sqrt{1-\sin^2\frac{\theta}{2}} \quad = 2x\sqrt{1-x^2}$

Example 5: Find the angle

(a) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$; (b) $\sin^{-1}(\sin 5)$

Sol: (a) Write $\frac{3\pi}{4}$ as $\pi - \frac{\pi}{4}$

(b) Write 5 as 5 - 2π

(a) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$

(b) We know $\sin^{-1}(\sin\theta) = \theta$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = [-1.57, 1.57]$

$5 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, while $5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin 5 = \sin(5 - 2\pi + 2\pi) = \sin(5 - 2\pi)$

$\therefore \sin^{-1}\sin 5 = 5 - 2\pi$

Example 6: If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Sol: Take one of the term to the R.H.S. and take cosine on both sides.

$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

Taking cosine on both sides we get

$$xy-\sqrt{1-x^2}\,\sqrt{1-y^2}=-z$$

$$\Rightarrow \quad xy+z=\sqrt{1-x^2}\,\sqrt{1-y^2}$$

Squaring we get

$$x^2y^2+z^2+2xyz=1-x^2-y^2+x^2y^2$$

$$\Rightarrow \quad x^2+y^2+z^2+2xyz=1.$$

Example 7: Prove that $f(x)=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ is a constant for all $x\ge 1$. Find this constant.

Sol: Convert the $\sin^{-1}$ function on the R.H.S. to $\tan^{-1}$ and proceed.

If $0\le x\le 1$ then $\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x$.

Hence If $x\ge 1(0<1/x\le 1)$ then $\sin^{-1}\frac{2x}{1+x^2}$

$$=\sin^{-1}\left(\frac{\frac{2}{x}}{1+\frac{1}{x^2}}\right)=2\tan^{-1}\frac{1}{x}\text{ For } x\ge 1$$

$$f(x)=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x+2\tan^{-1}\frac{1}{x}$$

$$=2[\tan^{-1}x+\cot^{-1}x]=2.\pi/2=\pi=\text{constant}.$$

Example 8: Solve the equation: $2\tan^{-1}(\cos x)$ $=\tan^{-1}(2\operatorname{cosec}x)$.

Sol: Substitute a variable in place of $\tan^{-1}(\cos x)$ and take tan on both sides.

If $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x):\sin x\ne 0$

$$\tan[2\tan^{-1}(\cos x)]=2\operatorname{cosec}x \qquad \text{....(i)}$$

Assume $\tan^{-1}(\cos x)=\theta:\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

$\therefore$ L.H.S. $=\tan[2\tan^{-1}(\cos x)]$

$$=\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\cos x}{1-\cos^2 x}=\frac{2\cos x}{\sin^2 x}$$

Substituting this value (i) we get

$$\frac{2\cos x}{\sin^2 x}=2\operatorname{cosec}x;\ \cos x=\sin x;\ \tan x=1;$$

$$\therefore \quad x=n\pi+\frac{\pi}{4};\ n\in I$$

Example 9: Solve $\cos^{-1}x+\cos^{-1}y=\pi/2$ and $\tan^{-1}x-\tan^{-1}y=0$

Sol: From the second of the given equations, we have

$x=y.\Rightarrow \quad \tan^{-1}-\tan^{-1}y=0\Rightarrow x=y$

Substituting x = y in the first, we have

$2\cos^{-1}x=\pi/2$ or $\cos^{-1}x=\pi/4$

or $x=\cos\pi/4=1/\sqrt{2}=y$

It is clearly evident that these values satisfy the given equations. Hence the solution set of the given equations is $(x=1/\sqrt{2},\ y=1/\sqrt{2})$

Example 10: If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$ prove that

$$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta.$$

Sol: Do it yourself.

Let $\cos^{-1}\frac{x}{2}=\alpha$ and $\cos^{-1}\frac{y}{3}=\beta$

$\therefore \cos\alpha=\frac{x}{2}$ and $\cos\beta=\frac{y}{3}$

Given $\alpha+\beta=\theta \quad \therefore \cos(\alpha+\beta)=\cos\theta$

or $\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos\theta$

or $\frac{x}{2}.\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}.\sqrt{1-\frac{y^2}{9}}=\cos\theta$

or, $\frac{xy}{6}-\frac{\sqrt{4-x^2}\,\sqrt{9-y^2}}{6}=\cos\theta$

or $(xy-6\cos\theta)^2=(4-x^2)(9-y^2)$

or, $x^2y^2+36\cos^2\theta-12xy\cos\theta=36-9x^2-4y^2+x^2y^2$

or, $9x^2-12xy\cos\theta+4y^2=36(1-\cos\theta^2)$

JEE Advanced/Boards

Example 1: If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$

prove that $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$

Sol: In the given equation take cosine on both sides and proceed.

$\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$

$\cos\left(\cos^{-1}\frac{x}{a}\right)\cos\left(\cos^{-1}\frac{y}{b}\right)-\sin\left(\cos^{-1}\frac{x}{a}\right)\sin\left(\cos^{-1}\frac{y}{b}\right)$

$=\cos\alpha$

$\Rightarrow \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$

$\Rightarrow \frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}$

Squaring both sides

$\left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$

or $\frac{x^2y^2}{a^2b^2}+\cos^2\alpha-\frac{2xy}{ab}\cos\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=1-\cos^2\alpha$

or $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

Example 2: Prove

$\tan^{-1}x=2\tan^{-1}(\text{cosec}\tan^{-1}x-\tan\cot^{-1}x)$.

Sol: Use substitution.

R.H.S. $=2\tan^{-1}(\text{cosec}\tan^{-1}x-\tan\cot^{-1}x)$

$=2\tan^{-1}\left[\begin{matrix}\text{cosec}\,\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)\\ -\tan\tan^{-1}\left(\frac{1}{x}\right)\end{matrix}\right]$ or

$2\tan^{-1}\left[\text{cosec}\,\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)-\tan\left\{\pi+\tan^{-1}\frac{1}{x}\right\}\right]$

depending on x > 0 or x < 0

$=2\tan^{-1}\left[\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right]=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Let $\tan^{-1}x=\theta:\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$; then $x=\tan\theta$

R.H.S. $=2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]=2\tan^{-1}\tan\frac{\theta}{2}$

$=2.\frac{\theta}{2}=\tan^{-1}x=$ L.H.S.

Example 3: Prove that

$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=\begin{cases}0 & \text{if } \frac{\pi}{4}<A<\frac{\pi}{2}\\ \pi & \text{if } 0<A<\frac{\pi}{4}\end{cases}$

Sol: Divide the solution in two cases when $\frac{\pi}{4}<A<\frac{\pi}{2}$ and $0<A<\frac{\pi}{4}$ and use the definition accordingly.

Case I: $\frac{\pi}{4}<A<\frac{\pi}{2}$

$0<\cot A<1$ and $0<\cot^3 A<1$

$\therefore\ \tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=\tan^{-1}\left[\frac{\cot A+\cot^3 A}{1-\cot^4 A}\right]=\tan^{-1}\left[-\frac{\sin 2A}{2\cos 2A}\right]$

$=\tan^{-1}\left[-\frac{\sin 2A}{2\cos 2A}\right]=-\tan^{-1}\left[\frac{1}{2}\tan 2A\right]$

$\Rightarrow \tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0$

Case II: $0<A<\frac{\pi}{4}$

$\cot A>1$ and $\cot^3 A>1\Rightarrow \cot A.\cot^3 A>1$

Hence, $\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=$

$\pi+\tan^{-1}\left(\frac{\cot A+\cot^3 A}{1-\cot^4 A}\right)$

[As $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ If $x>0$. $y>0$ and $xy>1$]

$=\pi-\tan^{-1}\left(\frac{1}{2}\tan 2A\right)$ [From case 1]

$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=\pi$

Example 4: Find the sum

$\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+$ .... to Infinity.

Sol: Write the general term of the series and express it as a difference of two terms (telescopic series).

Let T_n denote the nth term of the series

$$\therefore \quad T_r=\cot^{-1}(2r^2)=\cot^{-1}\left(\frac{4r^2}{2}\right)$$

$$=\cot^{-1}\left(\frac{1+4r^2-1}{2}\right)=\cot^{-1}\left(\frac{1+(2r+1)(2r-1)}{(2r+1)-(2r-1)}\right)$$

$$=\tan^{-1}\left[\frac{(2r+1)-(2r-1)}{1+(2r+1)\,(2r-1)}\right]$$

$$=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$$

$\therefore \quad \cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\ldots\ldots\ldots+$

$\cot^{-1}2n^2=\tan^{-1}(2n+1)-\tan^{-1}(1)$ As

$n\to\infty \quad \tan^{-1}(2n+1)\to\pi/2$

Hence, required sum $=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$

Example 5: If X_1, X_2, X_3, X_4 are the roots of the equation

$X^4-X^3\sin2\beta+X^2\cos2\beta-X\cos\beta-\sin\beta=0$

where $\sin\beta\neq\frac{1}{2}$ prove that $\tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4=n\pi+\frac{\pi}{2}-\beta$ for some $n\in I$.

Sol: Use theory of equations.

X_1, X_2, X_3, X_4 are the roots of the given equation

$\therefore \sum X_1=\sin2\beta, \sum X_1X_2=\cos2\beta$

$\Sigma X_1X_2X_3=\cos\beta,\ X_1X_2X_3X_4=-\sin\beta$

$\tan\left[\tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4\right]$

$$=\frac{\Sigma X_1-\Sigma X_1X_2X_3}{1-\Sigma X_1X_2+X_1X_2X_3X_4}=\frac{\sin2\beta-\cos\beta}{1-\cos2\beta-\sin\beta}$$

$$\frac{\cos\beta(2\sin\beta-1)}{\sin\beta(2\sin\beta-1)}=\cos\beta$$

$\tan\left[\tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4\right]$

$$=\cos\beta=\tan\left(\frac{\pi}{2}-\beta\right)$$

$\therefore \tan^{-1}X_1+\tan^{-1}X_2+\tan^{-1}X_3+\tan^{-1}X_4=$

$n\pi+\frac{\pi}{2}-\beta$ for some $n\in I$.

Example 6: Find the value of

$$\sin^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{\left(x^2+k^2-kx\right)}}\right\}$$

$$-\cos^{-1}\left\{\left(\cos\frac{\pi}{6}\right)\frac{x}{\sqrt{\left(x^2+k^2-kx\right)}}\right\}$$

Where $\left(\frac{k}{2}<x<2k, k>0\right)$

Sol: We have

$$\text{Sin}^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{\left(x^2+k^2-kx\right)}}\right\}$$

$$-\cos^{-1}\left\{\left(\cos\frac{\pi}{6}\right)\frac{x}{\sqrt{\left(x^2+k^2-kx\right)}}\right\}$$

$$=\sin^{-1}\left\{\frac{\sqrt{3}x}{2\sqrt{x^2+k^2-kx}}\right\}-\cos^{-1}\left\{\frac{\sqrt{3}x}{2\sqrt{x^2+k^2-kx}}\right\}$$

$$=\frac{\pi}{2}\text{-}2\cos^{-1}\left\{\frac{\sqrt{3}x}{\sqrt{\left(4x^2-4kx+4k^2\right)}}\right\}$$

$$=\frac{\pi}{2}-\cos^{-1}\left\{\frac{6x^2}{4x^2-4kx+4k^2}-1\right\}$$

$$=\sin^{-1}\left(\frac{2X^2+4kx-4k^2}{4X^2-4kx+4k^2}\right)=\sin^{-1}\left(\frac{X^2+2xk-2k^2}{2X^2-2xk+2k^2}\right)$$

Example 7: Find the number of real solutions of the equation $\sqrt{1+\cos2x}=\sqrt{2}\sin^{-1}(\sin x)-\pi\le x\le\pi$

Sol: Divide the solution into three cases when $-\frac{\pi}{2}\le X\le\frac{\pi}{2}$, $\frac{\pi}{2}<X\le\pi$ and $-\pi\le X<-\frac{\pi}{2}$ and proceed.

Here $|\cos x|=\sin^{-1}(\sin x)$.

If $-\frac{\pi}{2} \le X \le \frac{\pi}{2}$, then $\cos^{-1}\cos x = x$

In the case there is one solution obtained graphically.

If $\frac{\pi}{2} < X \le \pi$ then $-\cos x = \sin^{-1}\{\sin(\pi - x)\} = \pi - x$

$\therefore \cos x = x - \pi$

In the case there is one solution obtained graphically.

If $-\pi \le X < -\frac{\pi}{2}$ then

$-\cos x = \sin^{-1}\{\sin(-\pi - x)\} = -x - \pi$

i.e. $\cos x = x + \pi$

This gives no solution as can be seen from their graphs.

Example 8: Find the integral values of p at which the system of equations $\cos^{-1} x + (\sin^{-1} y)^2 = p\pi^2/4$; and $(\cos^{-1} x)(\sin^{-1} y)^2 = \pi^2/16$ possess solutions. Also find these solutions.

Sol: Start with the range of $\cos^{-1} x$ and $\sin^{-1} y$ and use it in the two given equations.

The given system of the equation is

$\cos^{-1} x + (\sin^{-1} y)^2 = p\pi^2/4$ ....(i)

$(\cos^{-1} x)(\sin^{-1} y)^2 = \pi^4/16$ ....(ii)

It is clear that

$0 < \cos^{-1} x \le \pi$; $-\pi/2 \le \sin^{-1} y \le \pi/2$.

So $0 < (\sin^{-1} y)^2 \le \pi^2/4 \sin^{-1} y \ne 0$ [From ii]

$\therefore 0 < \cos^{-1} x + (\sin^{-1} y)^2 \le \pi + \pi^2/4$

i.e. $0 < \frac{p\pi}{4} \le \pi + \frac{\pi^2}{4}$ ....(iii)

From (i) and (ii) we get $p \not\le 0$

$\cos^{-1} x + \frac{\pi^4}{16\cos^{-1} x} = \frac{p\pi^2}{4}$

Or 16 $(\cos^{-1} x)^2 - 4p\pi^2 \cos^{-1} x + \pi^4 = 0$ ....(iv)

As $\cos^{-1} x$ is real $16p^2\pi^2 \ge 0$

Or $p^2 \ge 4$ i.e. $p \le -2$ ....(v)

From (iii) and (v)

$p^2 \le (\pi/4) + 1, p \ge 20$

p is integer so p = 2 for p = 2 (4) gives

$16(\cos^{-1} x)^2 - 8\pi^2 \cos^{-1} x + \pi^4 = 0$ or

$\left[4\cos^{-1} x - \pi^2\right]^2 = 0$ or

$\cos^{-1} x - \pi^2/4$ i.e. $x = \cos(\pi^2/4)$ ....(vii)

Then (ii) gives $(\sin^{-1} y)^2 = \pi^2/4$

Or $\sin^{-1} y = \pm\pi^2/2$ i.e. ± 1 ....(viii)

Hence p=2 and (x,y)= $\left[\cos(\pi^2/4), \pm 1\right]$

JEE Main/Boards

Exercise 1

Q.1 Evaluate: $\sin^{-1}(\sin \pi/4)$

Q.2 Evaluate: $\tan^{-1}(\tan(-6))$

Q.3 Evaluate: $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\right)\right)$

Q.4 Prove that: $\tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4}$

Q.5 Evaluate: $\cos\left\{\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right\}$

Q.6 Evaluate: $\sin(\cos^{-1} 3/5)$

Q.7 Prove that: $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$

Q.8 Prove that: $4\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$

Q.9 Solve for x: $\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\frac{5\pi^2}{8}$

Q.10 Solve for x: $\tan^{-1}(x+1)+\tan^{-1}x+\tan^{-1}(x-1)=\tan^{-1}3$

Q.11 Find the value of

$\tan^{-1}\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

Q.12 Prove that: $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Q.13 Differentiate $\tan^{-1}\left[\frac{1-\cos x}{\sin x}\right]$ w.r.t.x.

Q.14 Express $\tan^{-1}\left(\frac{1-\sin x}{\cos x}\right)\frac{\pi}{2}<x<\frac{\pi}{2}$ in the simplest form.

Q.15 Find the principle value $\cos^{-1}\left(-\frac{1}{2}\right)$

Q.16 Write the following functions in the simplest form: $\cot^{-1}\left(\sqrt{1+x^2}-x\right)$.

Q.17 Find the principle value of $\cot^{-1}\left(-\sqrt{3}\right)$.

Q.18 Prove that $3\cos^{-1}x=\cos^{-1}(4\times 3-3x)\left(-\sqrt{3}\right)$

$x\in\left[\frac{1}{2},1\right]$

Q.19 Write the following function in the simplest form:

$\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right], x<\pi$

Q.20 If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ then find the value of x.

Q.21 Write the following function in the simplest form:

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$.

Q.22 Prove that: $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Q.23 Solve $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

Q24 Prove that: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Q25 If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\theta$ prove that

$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta=\frac{y^2}{b^2}=\sin^2\theta$

Q.26 Find the value of the following:

$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Q.27 Prove that: $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(2\frac{\sqrt{2}}{3}\right)$

Q.28 The value of

$\cos^{-1}\left(\sqrt{\frac{1}{3}}\right)-\cos^{-1}\left(\sqrt{\frac{1}{6}}\right)+\cos^{-1}\left(\frac{\sqrt{10}-1}{3\sqrt{2}}\right)$ Is__________.

Q.29 The number of roots of the equation $\sqrt{\sin x}=\cos^{-1}(\cos x)$ is

Exercise 2

Single Correct Choice Type

Q.1 $\tan\cos^{-1}x$ is equal to

(A) $\frac{\sqrt{1-x^2}}{x}$ (B) $\frac{x}{\sqrt{1+x^2}}$

(C) $\frac{\sqrt{1+x^2}}{x}$ (D) $x\sqrt{1+x^2}$

Q.2 $\left|\sin^{-1}x\right|^2+\left|\sin^{-1}y\right|^2+2\left|\sin^{-1}x\right|\left|\sin^{-1}y\right|=\pi^2$

then x^2+y^2 is equal to

(A) 1 (B) 3/2 (C) 2 (D) ½

Q.3 Number of solution(s) of the equation

$\cot^{-1}\sqrt{x^2+3x+2}+\cos^{-1}\sqrt{x^2-3x-3}=\frac{\pi}{2}$ is

(A) 2 (B) 1

(C) 4 (D) Infinite

Q.4 If $\cos\left(\tan^{-1}x\right)=x$, then x^2 is equal to

(A) $\frac{\sqrt{5}+1}{4}$ (B) $\frac{\sqrt{5}-1}{2}$

(C) $\frac{\sqrt{5}+1}{2}$ (D) $\frac{\sqrt{5}-1}{4}$

Q.5 The value of a for which $x^2+ax+\sin^{-1}\left(x^2-4x+5\right)+\cos^{-1}\left(x^2-4x+5\right)=0$, is

(A) $\frac{\pi}{4}+2$ (B) $\frac{\pi}{4}+1$

(C) $-\left|\frac{\pi}{4}+1\right|$ (D) $-(\frac{\pi}{4}+2)$

Q.6 Domain of the function

$f(x)=\sqrt{\sin^{-1}(\sin x)}+\sqrt{\cos^{-1}(\cos x)}$ is

(A) $\left|2n\pi,2n\pi+\frac{\pi}{2}\right|,n\in I$

(B) $\left[(2n+1)\pi,(2n+2)\pi\right],n\in I$

(C) $\left[2n\pi,(2n+1)\pi\right],n\in I$

(D) $\left|2n\pi+\frac{\pi}{2},2n\pi+\frac{3\pi}{2}\right|,n\in I$

Q.7 If $\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x, x\in[0,1]$

Then the interval in which θ lies is given by

(A) $\left[0,\frac{\pi}{3}\right]$ (B) $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

(C) $\left[0,\frac{\pi}{4}\right]$ (D) $\left[\frac{\pi}{2},\frac{3\pi}{4}\right]$

Q.8 $\tan^{-1}2+\tan^{-1}3=\operatorname{cosec}^{-1}x$ then x is equal to

(A) 4 (B) $\sqrt{2}$ (C) $-\sqrt{2}$ (D) None of these

Q. 9 The function

$f(x)=\cot^{-1}\sqrt{(x+3)x}+\cos^{-1}\sqrt{x^2+3x+1}$

is defined on the set S where S=

(A) $\{0,3\}$ (B) $(0,3)$ (C) $\{0,-3\}$ (D) $(-3,0)$

Q.10 $\alpha=\sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)$ and

$\beta=\cos^{-1}\left(\sin\left(\cos^{-1}x\right)\right)$ then

(A) $\tan\alpha=\cot\beta$ (B) $\tan\alpha=-\cot\beta$

(C) $\tan\alpha=\tan\beta$ (D) $\tan\alpha=-\tan\beta$

Q.11 If $x=2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\sqrt{3}\right)$

and $y=\cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right)$ then which of the following statements hold good?

(A) $y=\cos\frac{3\pi}{16}$ (B) $y=\cos\frac{5\pi}{16}$

(C) $x=4\cos^{-1}y$ (D) None of these

Q.12 The set values of x satisfying the equation $\tan^2\left(\sin^{-1}x\right)>1$ is

(A) $[-1,1]$ (B) $\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]$

(C) $(-1,1)-\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]$ (D $[-1,1]-\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

Q.13 The equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for

(A) All real values of a (B) a<-1

(C) a > 1 (D) $\frac{-1}{\sqrt{2}}\le a\le\frac{1}{\sqrt{2}}$

Q.14 If $\left(\sin^{-1}x+\sin^{-1}w\right)\left(\sin^{-1}y+\sin^{-1}z\right)=\pi^2$

then $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}$; $(N_1,N_2,N_3,N_4 \text{ E } N)-$

(A) Has a maximum value 2.

(B) Has a minimum value 2.

(C) Is independent of N_1,N_2,N_3,N_4

(D) None of these

Q.15 If $\theta = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$ then cot θ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Q.16 Which of the following function(s) is/are periodic?

(A) f(x)=x-[x], [x] denotes integral part of x

(B) g(x)= sin(1/x) $x \neq 0$ and g(0) =0

(C) h(x)=x cosx

(D) sin $(\sin^{-1}x)$

Q.17 $\cos\left(2\tan^{-1}\left(\frac{1}{7}\right)\right)$ equals

(A) $\sin(4\cot^{-1}3)$ (B) $\sin(3\cot^{-1}4)$

(C) $\cos(3\cot^{-1}4)$ (D) $\cos(4\cot^{-1}4)$

Q.18 $\sin^{-1}\left(2\times\sqrt{1-x^2}\right) = 2\sin^{-1} x$ is true if: $x \in$

(A) [0,1] (B) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

(C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

Q.19 If the sum $\sum_{k=1}^{k=n} \tan^{-1}\frac{2k}{2+k^2+k^4} = \tan^{-1}\frac{6}{7}$

then the value of n is equal to :

(A) 2 (B) 3 (C) 4 (D) 5

Q.20 The domain of definition of the function

$f(x) = \text{arc}\cos\left[\frac{3x^2-7x+8}{1+x^2}\right]$ where [x]

denotes the greatest integer function is:

(A) (1, 6) (B) [1, 6]

(C) [0, 1] (D) (-2, 5)

Q.21 Consider two geometric progressions $a_1, a_2, a_3 \ldots\ldots a_n$ & $b_1, b_2, b_3 \ldots\ldots b_n$ with $a_r = \frac{1}{b_r} = 2^{r-1}$

and another sequence $t_1, t_2, t_3 \ldots\ldots t_n$ such that

$t_r = \cot^{-1}\left(2a_r + b_r\right)$. Then $\lim_{n\to\infty}\sum_{r=1}^{n} t_r$ is :

(A) 0 (B) $\pi/4$ (C) $\tan^{-1}2$ (D) $\pi/2$

Q.22 Number of point(s) where f(x) $=\sin^{-1}(3x-4x^3)$ is not differentiable is

(A) 1 (B) 2 (C) 3 (D) 4

Q.23 Solution of the equation

$\sec^{-1} x = \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(\frac{5}{3\sqrt{3}}\right)$ is

(A) $\frac{18}{3-\sqrt{6}}$ (B) $\frac{18}{\sqrt{6}-3}$

(C) $\frac{\sqrt{6}+3}{8}$ (D) None of these

Q.24 The value of

$\left[\tan\left\{\frac{\pi}{4} + \sin^{-1}\left(\frac{a}{b}\right)\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}\right]^{-1}$

Where (0<a<b) is

(A) $\frac{b}{2a}$ (B) $\frac{a}{2b}$

(C) $\frac{\sqrt{b^2-a^2}}{2b}$ (D) $\frac{\sqrt{b^2-a^2}}{2a}$

Q.25 If $x = \tan^{-1}1 - \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\frac{1}{2}$;

$y = \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ then:

(A) $x = \pi y$ (B) $y = \pi x$

(C) $\tan x = -(4/3)y$ (D) $\tan x = (4/3)y$

Q.26 Which of the following satisfy the equation?

$2\cos^{-1} x = \cot^{-1}\left(\frac{2x^2-1}{\sqrt{4x^2-4x^2}}\right)$

(A) (-1, 0) (B) (0, 1)

(C) $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ (D) [-1, 1]

Q.27 Find values of x if $\sin^{-1} x = \cos^{-1} x + \sin^{-1}(3x-2)$?

(A) $\left\{\frac{1}{2}, 1\right\}$ (B) $\left[\frac{1}{2}, 1\right]$

(C) $\left[\frac{1}{3}, 1\right]$ (D) $\left\{\frac{1}{3}, 1\right\}$

Q.28 $f(x)=\sin^{-1}\left|\frac{1-x^2}{1+x^2}\right|$ and

$g(x)=\cot^{-1}x-\tan^{-1}x$ are identical for:

(A) $x\in[0,1]$ (B) $x\in(-\infty,0]$

(C) $x\in[-1,1]$ (D) $x\in(-\infty,-1]\cup[1,\infty)$

Q.29 $\tan\left[\cos^{-1}\left\{\sin\left(2\tan^{-1}2\right)\right\}\right]$ is equal to

(A) $\frac{4}{3}$ (B) $\frac{4}{5}$ (C) $\frac{3}{5}$ (D) $\frac{3}{4}$

Q.30 $\sum_{n=1}^{\infty}\left|\frac{\sin^{-1}x+\cos^{-1}x}{\pi r}^{n}\right|$ is finite.

$x\in[-1,1]$ and r>0.then the possible values of 'r' is.

(A) $\left|\frac{1}{2},\infty\right|$ (B) $(2,\infty)$

(C) $(1,\infty)$ (D) $(0,\infty)$

Q.31 $y=\sin^{-1}(\sin x)$, x is the element of [0 π] divides the region bounded by coordinate axes

$x=\pi$ and $y=\frac{\pi}{2}$ into 3 region whose areas are

A_1,A_2,A_3 with $A_1\le A_2\le A_3$ then

(A) $A_1+A_2+2A_3=\pi^2$

(B) $A_1+A_3-A_2=\frac{\pi^2}{2}$

(C) $A_1+A_2-A_3=0$

(D) $2(A_1+A_2)-A_3=0$

Q.32 The sum $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4-2n^2+2}$ is equal to:

(A) $\tan^{-1}2+\tan^{-1}3$ (B) $4\tan^{-1}1$

(C) $\frac{\pi}{2}$ (D) $\sec^{-1}\left(1-\sqrt{2}\right)$

Previous Years' Questions

Q.1 The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is ***(1983)***

(A) $\frac{6}{17}$ (B) $\frac{17}{6}$

(C) $\frac{16}{7}$ (D) None of above

Q.2 The principle value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is ***(1986)***

(A) $-\frac{2\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{3}$

Q.3 The number of real solutions of

$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is ***(1999)***

(A) Zero (B) One

(C) Two (D) Infinite

Q.4 If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\right)$

$=\frac{\pi}{2}$, for $0<|x|<\sqrt{2}$, then x equals ***(2001)***

(A) 1/2 (B) 1 (C) -1/2 (D) -1

Q.5 The value of x for which

$\sin\left[\cos^{-1}(1+x)\right]=\cos\left(\tan^{-1}x\right)$ is ***(2004)***

(A) $\frac{1}{2}$ (B) 1 (C) 0 (D) $-\frac{1}{2}$

Q.6 If 0<x<1, then

$\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2}$ is equal to ***(2008)***

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x

(C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

Q.7 Let a, b, c be positive real number

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c+)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c+)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c+)}{ab}}.$$

Then $\tan\theta$ equal .... ***(1981)***

Q.8 The numerical value of

$\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]$ is equal to.... ***(1984)***

Q.9 The greater of the two angle A = $2\tan^{-1}\left(2\sqrt{2}-1\right)$

B= $3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)$ is .... ***(1989)***

Q.10 AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point c on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD = 7m. From D the angle of elevation of the point A is 45°. Then the height of the pole is ***(2008)***

(A) $\frac{7\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}-1}$m (B) $\frac{7\sqrt{3}}{2}\cdot(\sqrt{3}+1)$m

(C) $\frac{7\sqrt{3}}{2}\cdot(\sqrt{3}-1)$m (D) $\frac{7\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}-1}$

Q.11 The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$ is ***(2008)***

(A) $\frac{6}{17}$ (B) $\frac{3}{17}$ (C) $\frac{4}{17}$ (D) $\frac{5}{17}$

Q.12 Let $\cos(\alpha+\beta)=\frac{4}{5}$ and let $\sin(\alpha-\beta)=\frac{5}{13}$, where $0\le\alpha,\beta\le\frac{\pi}{4}$, then $\tan 2\alpha=$ ***(2010)***

(A) $\frac{56}{33}$ (B) $\frac{19}{12}$ (C) $\frac{20}{7}$ (D) $\frac{25}{16}$

Q.13 For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is ***(2010)***

(A) There is a regular polygon with $\frac{r}{R}=\frac{1}{\sqrt{2}}$

(B) There is a regular polygon with $\frac{r}{R}=\frac{2}{3}$

(C) There is a regular polygon with $\frac{r}{R}=\frac{\sqrt{3}}{2}$

(D) There is a regular polygon with $\frac{r}{R}=\frac{1}{2}$

Q.14 A line AB in three-dimensional space makes angle 45° and 120° with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals ***(2010)***

(A) 45° (B) 60° (C) 75° (D) 30°

Q.15 Let $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x|<\frac{1}{\sqrt{3}}$. Then the value of y is ***(2015)***

(A) $\frac{3x-x^3}{1-3x^2}$ (B) $\frac{3x+x^3}{1-3x^2}$

(C) $\frac{3x-x^3}{1+3x^2}$ (D) $\frac{3x+x^3}{1+3x^2}$

Q.16 If $0\le x<2\pi$, then the number of real values of x, which satisfy the equation cosx + cos2x + cos3x + cos4x = 0, is ***(2016)***

(A) 5 (B) 7 (C) 9 (D) 3

Q.17 Consider $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right)$.

A normal to $y=f(x)=\frac{\pi}{6}$ also passes through the point: ***(2016)***

(A) $\left(0,\frac{2\pi}{3}\right)$ (B) $\left(\frac{\pi}{6},0\right)$

(C) $\left(\frac{\pi}{4},0\right)$ (D) (0, 0)

JEE Advanced/Boards

Exercise 1

Q.1 If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

For 0<x<1 then prove that $\alpha + \beta = \pi$ what is the value of $\alpha + \beta$ will be if x>1?

Q.2 If $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ prove that

x^2= sin 2 y.

Q.3 Find the sum of following series upto n terms where x > 0.

(i) $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+.....+\tan^{-1}\frac{2^{n-1}}{1+2^{n-1+}}...0$

(ii) $\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}$

$+\tan^{-1}\frac{1}{x^2+5x+7}+\tan^{-1}\frac{1}{x^2+7x+13}$

Q.4 If $x \in \left[-1, -\frac{1}{2}\right]$ then express the function

$f(x) = \sin^{-1}(3x-4x^3)+\cos^{-1}(4x^3-3x)$ in the form of a

$\cos^{-1}x + b\pi$ where a and b are rational numbers.

Q.5 Solve the following equations:

(i) $\sin^{-1}x+\sin^{-1}2x = \frac{\pi}{3}$

(ii) $\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1) = \tan^{-1}(3x)$

(iii) $\tan^{-1}\frac{1-x}{1+x}+\tan^{-1}\frac{2x-x}{2x+x} = \tan^{-1}\frac{23}{36}$

(iv) $\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$

Q.6 Find all the positive integral solution of

$\tan^{-1}x+\cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

Q.7 If α and β are the roots of the equation

$x^2-4x+1=0\,(\alpha > \beta)$ then find the value of

$f(\alpha,\beta) = \frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^2\frac{\beta}{\alpha}\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^2\frac{\alpha}{\beta}\right)$

Q.8 Consider the functions $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$g(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $h(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(i) If x ∈ (-1, 1) then the find the solution of the Equation $f(x)+g(x)+h(x) = \pi/2$

(ii) Find the value of f(2) +g(2)=h(2).

Q.9 Solve the following inequalities

(i) arc $\cot^2 x - 5\,\text{arc}\cot x + 6 > 0$

(ii) arc sin x>arc cos x

(iii) $\tan^2$ (arc sin x)>1

Q.10 Show that roots r, s and t of the cubic x(x-2)(3x-7)=2 are real and positive.

Also compute the value of $\tan^{-1}(r)+\tan^{-1}(s)\tan^{-1}(t)$.

Q.11 Let $f(x) = \frac{\pi}{4}+\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)-\tan^{-1}x$

and $a_i\,(a_i < a_{i+1}\,\forall i = 1,2,3,....n)$ be the

positive integral values of x for which

$\text{sgn}(f(x)) = 1$, where sgn (y) denotes signum

function of y. Find $\sum_{i=1}^{n} a_i^2$.

Q.12 Solve for x: $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$.

Q.13 Let $f(x) = \tan^{-1}(\cot x - 2\cot 2x)$ and

$\sum_{r=1}^{5} f(r) = a - b\pi$ where a, b E N. Find the value of (a+b).

Q.14 Let $f(x)=(2a+b)\cos^{-1}x+(a+2b)\sin^{-1}x$

Where a, b $\in$ R and a>b.

If domain of and range of f are the same set then find the value of $\pi(a-b)$.

Q.15 Identify the pairs(s) of functions which are identical. Also plot the graphs in each case.

(i) $y=\tan(\cos^{-1}x)$; $y=\dfrac{\sqrt{1-x^2}}{x}$

(ii) $y=\tan(\cot^{-1}x)$; $y=\dfrac{1}{x}$

(iii) y = sin (arc tan x); $y=\dfrac{x}{\sqrt{1-x^2}}$

(iv) y = cos (arc tan x); y= sin (arc cot x)

Q.16 Find the domain and the following functions.

(i) $f(x)=\cot^{-1}\left(2x-x^2\right)$

(ii) $f(x)=\sec^{-1}\left(\log_3\tan x+\log_{\tan x}3\right)$

(iii) $f(x)=\cos^{-1}\left(\dfrac{\left[2x^2+1\right]}{x^2+1}\right)$

(iv) $f(x)=\tan^{-1}\left(\log_4\left(5x^2-8x+4\right)\right)$

Q.17 Let $y=\sin^{-1}(\sin 8)^5-\tan^{-1}(\tan 1)+\cos^{-1}$ (cos 12) – $\sec(\sec 9)+\cot^{-1}(\cot 6)-\operatorname{cosec}^{-1}$ – (cosec 7). If y simplifies to $a\pi+b$ then find (a-b).

Q.18 Let $\alpha=\sin^{-1}\left(\dfrac{36}{85}\right)$ $\beta=\cos^{-1}\left(\dfrac{4}{5}\right)$ and $\gamma=\tan^{-1}\left(\dfrac{8}{15}\right)$ find $(\alpha+\beta+\gamma)$ and hence

Prove that (i) $\Sigma\cot\alpha=\prod\cot\alpha$ (ii) $\Sigma\tan\alpha.\tan\beta=1$

Q.19 Show that:

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)=\frac{13\pi}{7}$$

Q.20 Prove that:

(i) $\cos^{-1}\left(\dfrac{5}{13}\right)+\cos^{-1}\left(-\dfrac{7}{25}\right)+\sin^{-1}\dfrac{36}{325}=\pi$

(ii) arc $\cos\sqrt{\dfrac{2}{3}}-$ arc $\cos\dfrac{\sqrt{6}+1}{2\sqrt{3}}=\dfrac{\pi}{6}$

Q.21 If a>b>c>0 then find the value of:

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

Q.22 If α and β are the roots of the equation $x^2+5x-49=0$ then find the value of $\cot\left(\cot^{-1}\alpha+\cot^{-1}\beta\right)$.

Q.23 Find all value of k for which there is a triangle Whose angles have measure $\tan^{-1}\left(\dfrac{1}{2}\right)$ $\tan^{-1}\left(\dfrac{1}{2}+k\right)$ and $\tan^{-1}\left(\dfrac{1}{2}+2k\right)$

Q.24 In a ΔABC if $\angle A=\angle B$ $=\dfrac{1}{2}\left(\sin^{-1}\left(\dfrac{\sqrt{6}+1}{2\sqrt{3}}\right)+\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\right)$ and $C=6.3^{\frac{1}{4}}$ then find the area of ΔABC.

Q.25 Find the simplest value of

(i) f (x) =arc cos x +arc $\cos\left(\dfrac{x}{2}+\dfrac{1}{2}\sqrt{3-3x^2}\right), x\in\left(\dfrac{1}{2},1\right)$

(ii) $f(x)=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right), x\in R-\{0\}$

Q.26 Let $f(x)=\cot^{-1}\left(x^2+4x+\alpha^2-\alpha\right)$ be a function defined $R\to(0,\pi/2]$ then find the complete set of real values of α for which $f(x)$ is onto.

Q.27 Prove that: $\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9}=\dfrac{1}{2}\cos^{-1}\dfrac{3}{5}=\dfrac{1}{2}\sin^{-1}\dfrac{4}{5}$.

Q.28 Prove that

$$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right\}=\frac{x}{2}<x<\frac{\pi}{2}$$

Q.29 Find the domain of definition the following functions.

(Real the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x)=\text{arc}\cos\frac{2x}{1+x}$

(ii) $f(x)=\sqrt{\cos(\sin x)}+\sin^{-1}\frac{1+x^2}{2x}$

(iii) $f(x)=\sin^{-1}\left(\frac{x-3}{2}\right)-\log_{10}(4-x)$

(iv) $f(x)=\sin^{-1}(2x+x^2)$

(v) $f(x)=\frac{\sqrt{1-\sin^{-1}x}}{\log_5(1-4x^2)}+\cos^{-1}(1-\{x\})$

where $\{x\}$ is the fractional part of x.

(vi) $f(x)=\sqrt{3-x}+\cos^{-1}\left(\frac{3-2x}{5}\right)$

$+\log_6(2|x|-3)+\sin^{-1}(\log_2 x)$

(vii) $f(x)=\log_{10}\left(1-\log_7(x^2-5x+13)\right)$

$+\cos^{-1}\left(\frac{3}{2+\sin\frac{9\pi x}{2}}\right)$

(viii) $f(x)=e^{\sin^{-1}\left(\frac{x}{2}\right)}+\tan^{-1}\left(\frac{x}{2}-1\right)+\ln\left(\sqrt{x-[x]}\right)$

Exercise 2

Single Correct Choice Type

Q.1 Solution set of the inequality x^2-4x+5 $>\sin^{-1}(\sin 3)+\cos^{-1}(\cos 2)-\pi$ is.

(A) R (B) R-{1}

(C) R-{2} (D) R-{-2}

Q.2 If x_1, x_2, x_3, x_4 are roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x\cos\beta - \sin\beta = 0$ then $\sum_{i=1}^{4}$ tan-1 x_i is equal to

(A) $x-\beta$ (B) $\pi-2\beta$

(C) $\left(\frac{\pi}{2}\right)-\beta$ (D) $\left(\frac{\pi}{2}\right)-2\beta$

Q.3 Range of the function,

$f(x)=\cot^{-1}\left(\log_{4/5}\left(5x^2-8x+4\right)\right)$ is.

(A) $(0,\pi)$ (B) $\left[\frac{\pi}{4},\pi\right)$

(C) $\left(0,\frac{\pi}{4}\right]$ (D) $\left(0,\frac{\pi}{2}\right)$

Q.4 Domain of the explicit form of the function y represented implicitly by the equation.

$(1+x)\cos y - x^2 = 0$

(A) (-1, 1] (B) $\left(-1,\frac{1-\sqrt{5}}{2}\right]$

(C) $\left[\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right]$ (D) $\left[0,\frac{1+\sqrt{5}}{2}\right]$

Q.5 Number of integral value(s) of x satisfying

$4(\tan^{-1}x)^2-(\tan^{-1}x)-3\le 0$ is –

(A) 1 (B) 2 (C) 3 (D) 4

Q.6 The area of the region bounded by the curves $y=x^2$ and $\sec^{-1}[-\sin^{-2}x]$ (where [.] denotes greatest integer function) is

(A) $\pi\sqrt{\pi}$ (B) $\frac{4}{3}\pi\sqrt{\pi}$

(C) $\frac{2}{3}\pi\sqrt{\pi}$ (D) $\frac{1}{3}\pi\sqrt{\pi}$

Q.7 If $\tan^{-1}\frac{a}{x}+\tan^{-1}\frac{b}{x}+\tan^{-1}\frac{c}{x}+\tan^{-1}\frac{d}{x}=\frac{\pi}{2}$

Then $x^4 - x^2\sum ab + abcd$ is equal to

(A) – 1 (B) 0 (C) 1 (D) 2

Q.8 The solution set of the equation

$$\sin^{-1}\sqrt{1-x^2}+\cos^{-1}x=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)-\sin^{-1}x$$

(A) [-1, 1] –{0} (B) (0, 1] U {-1}

(C) [-1, 0) U {1} (D) [-1, 1]

Q.9 The domain and range of the function

$f(x)=\operatorname{cosec}^{-1}\sqrt{\log\frac{3-4\sec x^2}{1-2\sec x}}$ are respectively

(A) $R;\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(B) $R^{+};\left(0,\frac{\pi}{2}\right)$

(C) $\left(2\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}\right)-\{2n\pi\};\left(0,\frac{\pi}{2}\right)$

(D) $\left(2\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}\right)-\{2n\pi\};\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

Q.10 Solution set of equation $[\sin^{-1}x]=[\cos^{-1}x]$ where [*] represents integral part function is

(A) (cos 1sin1) (B) [cos 1sin1]

(C) (sin 1 cos 1) (D) [sin 1 cos 1]

Multiple Correct Choice Type

Q.11 Which of the following statement (s) is/ are meaningless?

(A) $\cos^{-1}\left(\ln\left(\frac{2e+4}{3}\right)\right)$ (B) $\operatorname{cosec}^{-1}\left(\frac{\pi}{4}\right)$

(C) $\cot^{-1}\left(\frac{\pi}{2}\right)$ (D) $\sec^{-1}(\pi)$

Q.12 If the numerical value of tan

$\left(\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{2}\right)\right)$ is $\frac{a}{b}$ then

(A) a + b = 23 (B) a – b = 11

(C) 3b = a + 1 (D) 2a = 3b

Q.13 Which of the following equation represents a circle

(A) $y^2=\sin(\cos^{-1}x)$ (B) $y=\sin(\cos^{-1}(1-x))$

(C) $y^2=\sin^2(\cos^{-1}x)$ (D) $y=\sin^{-1}(\cos^2 x)]$

Q.14 If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ then

(A) $x^2+y^2+z^2+2xyz=1$

(B) $2(\sin^{-1}x+\sin^{-1}y+\sin^{-1}z)$

$=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z$

(C) $xy+yz+zx=x+y+z-1$

(D) $\left(x+\frac{1}{x}\right)+\left(x+\frac{1}{y}\right)+\left(x+\frac{1}{z}\right)\ge 6$

Match the Columns

Q.15 Column I contains functions and column II contains their range. Match the entries of column I with the entries of column II.

	Column I		Column II
(A)	$f(x)=\sin^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(p)	$(0,\pi)$
(B)	$g(x)=\cos^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(q)	$\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$
(C)	$h(x)=\tan^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(r)	$\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$
(D)	$k(x)=\cot^{-1}\left(\frac{x}{1+\lvert x\rvert}\right)$	(s)	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Previous Years' Questions

Q.1 Match the conditions/expressions in column I with statement in column II.

Let (x, y) be such that $\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$

	Column I		Column II
(A)	If a=1 and b=0, then (x, y)	(P)	lies on the circle $x^2+y^2=1$
(B)	If a=1 and b=1, then (x, y)	(q)	lies on $(x^2-1)(y^2-1)=0$
(C)	If a=1 and b=2, then (x, y)	(r)	lies on y=x
(D)	If a=2 and b=2, then (x, y)	(s)	lies on $(4x^2-1)(y^2-1)=0$

Q.2 Solve the following equation for x

$\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ ***(1978, 3M)***

Q.3 Find the value of $\cos\left(2\cos^{-1}x+\sin^{-1}x\right)$ at $x=\frac{1}{5}$ where $0\le\cos^{-1}x\le\pi$ and $-\pi/2\le\sin^{-1}x\le\pi/2$. ***(1981)***

Q.4 Prove that $\cos\tan^{-1}[\sin(\cot^{-1}x)]=\sqrt{\frac{x^2+1}{x^2+2}}$ ***(2002)***

Q.5 If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the length of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is ***(2010)***

Q.6 If $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) ***(2015)***

(A) $\cos\beta>0$ (B) $\sin\beta>0$

(C) $\cos(\alpha+\beta)>0$ (D) $\cos\alpha<0$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.5 Q.10 Q.13

Q.16 Q.22 Q.29

Exercise 2

Q.3 Q.10 Q.18

Q.21 Q.24 Q.29

Q.30 Q. 32

Previous Years' Questions

Q.3 Q.4 Q.5

Q.7

JEE Advanced/Boards

Exercise 1

Q.3 Q.11 Q.15

Q.17 Q.21 Q.24

Q.29 Q.30

Exercise 2

Q.3 Q.4 Q.6

Q.9 Q.10 Q.12

Previous Years' Questions

Q.1 Q.3

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\pi / 4$	**Q.2** $2\pi - 6$	**Q.3** $\frac{\sqrt{3}}{2}$
Q.5 -1	**Q.6** $\frac{4}{5}$	**Q.9** -1
Q.10 $-1;\ 5 \pm \sqrt{19}$	**Q.11** $z = \frac{x+y}{1-xy}$	**Q.13** 1/2
Q.14 $y = \frac{\pi}{4} - \frac{x}{2}$	**Q.15** $\frac{2\pi}{3}$	**Q.16** $y = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}x$
Q.17 $\frac{5\pi}{6}$	**Q.19** $\frac{\pi}{4} - x$	**Q.20** $\pm\frac{1}{\sqrt{2}}$
Q.21 $3\tan^{-1}\frac{x}{a}$	**Q.23** $x = 1/6$	**Q.26** $\frac{3\pi}{4}$
Q.28 $= \cos^{-1}\left(\frac{1}{3}\right)$	**Q.29** ∞	

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** D	**Q.5** D	**Q.6** C
Q.7 B	**Q.8** D	**Q.9** C	**Q.10** A	**Q.11** A	**Q.12** C
Q.13 D	**Q.14** C	**Q.15** C	**Q.16** A	**Q.17** A	**Q.18** B
Q.19 B	**Q.20** A	**Q.21** B	**Q.22** B	**Q.23** D	**Q.24** C
Q.25 C	**Q.26** B	**Q.27** A	**Q.28** A	**Q.29** D	**Q.30** A
Q.31 C	**Q.32** A				

Previous Years' Questions

Q.1 B	**Q.2** C	**Q.3** C	**Q.4** B	**Q.5** D	**Q.6** C
Q.7 0	**Q.8** $-\frac{7}{17}$	**Q.9** A	**Q.10** B	**Q.11** A	**Q.12** A
Q.13 B	**Q.14** B	**Q.15** A	**Q.16** B	**Q.17** A	

JEE Advanced/Boards

Exercise 1

Q.1 0

Q.2 $x^2 = \frac{2\tan y}{1+\tan^2 y} = \sin 2y$

Q.3 (i) $\frac{\pi}{4}$ (ii) $\text{arc tan }(x+n) - \text{arc tan} x$

Q.4 $6\cos^{-1} x - \frac{9\pi}{2}$ so $a = 6$ $b = -\frac{9}{2}$

Q.5 (i) $x = \frac{1}{2}\sqrt{\frac{3}{7}}$; (ii) $x = 0, \frac{1}{2}, -\frac{1}{2}$; (iii) $x = \frac{4}{3}$; (iv) $x = 2 - \sqrt{3}$ or $\sqrt{3}$

Q.6 x=1; y=2 & x=2 ; y=7

Q.7 56

Q.8 (i) $2-\sqrt{3}$; (ii) $\cot^{-1}\left(\frac{-3}{4}\right)$

Q.9 (i) $(\cot 2, \infty) \cup (-\infty, \cot 3)$ (ii) $\left|\frac{\sqrt{2}}{2}, 1\right|$ (iii) $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, \frac{\sqrt{2}}{2}\right)$

Q.10 $\frac{3\pi}{4}$

Q.11 5

Q.12 $x \in (-1, 1)$

Q.13 20

Q.14 –2

Q.15 (i), (ii), (iii) and (iv) all are identical

Q.16 (i) D:$x \in R$ R $:[\pi/4, \pi)$ (ii) D:$\in \left(n\pi . n\pi + \frac{\pi}{2}\right) - \left\{x \middle| x + \frac{\pi}{2}\right\} n \in I$: R:$\left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left\{\frac{\pi}{2}\right\}$

(iii) D:x $\in R$ R $:\left[0, \frac{\pi}{2}\right)$ (iv) D:x $\in R$ R $:\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$

Q.17 53

Q.21 0

Q.22 10

Q.23 $k = \frac{11}{4}$

Q.24 27

Q.25 (i) $\frac{\pi}{3}$; (ii) $\frac{\tan^{-1} x}{2}$

Q.26 $\frac{1 \pm \sqrt{17}}{2}$

Q.27 $\frac{1}{2}\sin^{-1}\frac{4}{5}$ = RHS

Q.28 $\frac{x}{2}$

Q.29 (i) $-1/3 \le x \le 1$; (ii) $\{1, -1\}$; (iii) $1 \le x < 4$; (iv) $[-(1+\sqrt{2}), (\sqrt{2}, -1)]$; (v) $x \in (-1/2, 1/2), x \ne 0$; (vi) $(3/2, 2]$; (vii) $\{7/3, 25/9\}$; (viii) $(-2, 2) - \{-1, 0, 1\}$

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** C **Q.3** B **Q.4** C **Q.5** B **Q.6** B

Q.7 B **Q.8** C **Q.9** C **Q.10** A

Multiple Correct Choice Type

Q.11 A, B **Q.12** A, B, C **Q.13** B, C **Q.14** A, B

Match the Columns

Q.15 A → s; B → p; C → r; D → q

Previous Years' Questions

Q.1 A → p; B → q; C → p; D → s **Q.2** $x = \frac{1}{6}$ **Q.3** $-\frac{2\sqrt{6}}{5}$ **Q.5** $\sqrt{3}$ **Q.6** B, C, D

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Sol 2: $\tan^{-1}(\tan(-6)) = \tan^{-1}\tan(2\pi - 6) = 2\pi - 6$

Sol 3: $\sin\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{2}\right)$

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

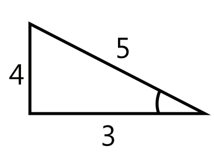
Sol 4: $\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\frac{2+3}{1-6}$ xy> 1

$= \pi + \tan^{-1}\frac{5}{-5} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Sol 5: $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right\}$

$\cos\{\pi\} = -1$

Sol 6:



$= \sin\cos^{-1}\left(\frac{3}{5}\right) = \frac{4}{5}$

Sol 7: $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{13}}{1 - \frac{1}{13\times 7}}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

Sol 8: $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{70}\right) + \tan^{-1}\left(\frac{1}{99}\right)$

$= 2\tan^{-1}\left(\frac{5}{12}\right) - \left\{\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right\}$

$= \tan^{-1}\frac{2\times 5/2}{1-\left(\frac{5}{2}\right)^2} - \tan^{-1}\frac{\left(\frac{1}{70} - \frac{1}{99}\right)}{1 + \frac{1}{70}\times\frac{1}{99}}$

$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{29}{6931}\right)$

$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$

$= \tan^{-1}\left\{\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\times\frac{1}{239}}\right\} \approx \tan^{-1}(1)$

Sol 9: $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$(\tan^{-1}x)^2 + \frac{\pi^2}{4} + (\tan^{-1}x)^2 - 2\frac{\pi}{2}(\tan^{-1}x) = \frac{5\pi^2}{8}$

$2(\tan^{-1}x)^2 - \pi(\tan^{-1}x) - \frac{3\pi^2}{8} = 0$

$\tan^{-1}x = \frac{\pi \pm \sqrt{\pi^2 + 3\pi^2}}{4} = \frac{\pi \pm 2\pi}{4}$

$\tan^{-1}x = \frac{3\pi}{4}$ or $= \frac{-\pi}{4}$

$x = -1$

Sol 10: $\tan^{-1}(x + 1) + \tan^{-1}x + \tan^{-1}(x - 1)$

$= \tan^{-1}(3)$

$\tan^{-1}\left[\frac{(x+1)+x+(x-1)-x(x^2-1)}{1-x(x+1)-(x^2-1)-x(x-1)}\right] = \tan^{-1}(3)$

$\frac{3x - x^3 + x}{1 - x^2 - x - x^2 + 1 - x^2 + x} = 3$

$\Rightarrow \frac{4x - x^3}{2 - 3x^2} = 3$

$\Rightarrow 4x - x^3 = 6 - 9x^2$

$\Rightarrow x^3 - 9x^2 - 4x + 6 = 0$

$\Rightarrow (x + 1)(x^2 - 10x + 6) = 0$

$\Rightarrow x = -1, 5 \pm \sqrt{19}$

Sol 11: $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$= \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \sin^{-1}\frac{2y}{1+y^2}\right]$

$= \frac{\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}\right) + \tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2y}{1+y^2}\right)\right)}{1 - \tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}\right)\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2y}{1+y^2}\right)\right)}$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$

$= \frac{x+y}{1-xy}$

Sol 12: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{14}{264}}\right)$

$= \tan^{-1}\left(\frac{48 + 77}{250}\right) = \tan^{-1}\frac{1}{2}$

Sol 13: $f(x) = \tan^{-1}\left(\frac{1 - \cos x}{\sin x}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right)$

$f'(x) = \frac{1}{2}$

Sol 14: $\tan^{-1}\left(\frac{1 - \sin x}{\cos x}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}.\tan\frac{x}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$

$= \frac{\pi}{4} - \frac{x}{2}$

Sol 15: $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

Sol 16: $\cot^{-1}(\sqrt{1+x^2} - x)$

Put $x = \tan y$

$= \cot^{-1}(\sec y - \tan y)$

$= \cot^{-1}\left(\frac{1 - \sin y}{\cos y}\right) = \cot^{-1}\frac{\cos\frac{y}{2} - \sin\frac{y}{2}}{\cos\frac{y}{2} + \sin\frac{y}{2}}$

$= \cot^{-1}\tan\left(\frac{\pi}{4} - \frac{y}{2}\right) = \frac{\pi}{4} + \frac{y}{2} = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}x$

Sol 17: $\cot^{-1}(-\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Sol 18: $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

put $x = \cos x$

L. H. S. $= 3x$

R. H. S. $= \cos^{-1}(4\cos^3x - 3\cos x) = \cos^{-1}\cos 3x = 3x$

Sol 19: $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right) = \tan^{-1}\left(\dfrac{1-\tan x}{1+\tan x}\right)$

$= \tan^{-1}\tan\left(\dfrac{\pi}{4}-x\right) = \dfrac{\pi}{4}-x$

Sol 20: $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{x^2-1}{x^2-4}}\right) = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{-2+x-2-x+2x^2}{-4+1}\right) = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\dfrac{2x^2-4}{-3} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{2x^2-4}{-3} = 1 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm\dfrac{1}{\sqrt{2}}$

Sol 21: $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^2-3ax^2}\right) = \tan^{-1}\left[\dfrac{\dfrac{3x}{a}-\left(\dfrac{x}{a}\right)^3}{1-3\left(\dfrac{x}{a}\right)^2}\right]$

$\dfrac{x}{a} = \tan y$

$\Rightarrow \tan^{-1}\left(\dfrac{3\tan y-\tan^3 y}{1-3\tan^2 y}\right) = \tan^{-1}\tan 3y = 3\tan^{-1}\dfrac{x}{a}$

Sol 22: $\sin^{-1}\dfrac{12}{13} + \cos^{-1}\dfrac{4}{5} + \tan^{-1}\dfrac{63}{16}$

$= \tan^{-1}\dfrac{12}{5} + \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{63}{16}$

$= \pi + \tan^{-1}\left(\dfrac{\dfrac{12}{5}+\dfrac{3}{4}}{1-\dfrac{36}{20}}\right) + \tan^{-1}\dfrac{63}{16}$

$= \pi + \tan^{-1}\dfrac{63}{-16} + \tan^{-1}\dfrac{63}{16} = \pi$

Sol 23: $\tan^{-1}2x + \tan^{-1}3x = \dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{2x+3x}{1-6x^2}\right) = \dfrac{\pi}{4}$

$\dfrac{5x}{1-6x^2} = 1 \Rightarrow 6x^2 - 1 + 5x = 0$

$(6x-1)(x+1) = 0 \Rightarrow x = 1/6, -1$

(-1) does not satisfy,

so answer is $x = \dfrac{1}{6}$

Sol 24: $2\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{7}\right)$

$= \tan^{-1}\dfrac{\dfrac{1}{2}+\dfrac{1}{2}}{1-\dfrac{1}{4}} + \tan^{-1}\left(\dfrac{1}{7}\right)$

$= \tan^{-1}\dfrac{\dfrac{4}{3}+\dfrac{1}{7}}{1-\dfrac{4}{21}} = \tan^{-1}\dfrac{31}{17}$

Sol 25: $\dfrac{xy}{ab} - \sqrt{1-\dfrac{x^2}{a^2}}\sqrt{1-\dfrac{y^2}{b^2}} = \cos\theta$

$\dfrac{xy-\sqrt{a^2-x^2}\sqrt{b^2-y^2}}{ab} = \cos\theta$

$1 - \sin^2\theta =$

$\dfrac{x^2y^2+(a^2-x^2)(b^2-y^2)-2xy\sqrt{a^2-x^2}\sqrt{b^2-y^2}}{a^2b^2}$

$1 - \sin^2\theta$

$= \dfrac{2x^2y^2-a^2y^2-b^2x^2-2xy(xy-ab\cos\theta)}{a^2b^2} + 1$

$\Rightarrow \sin^2\theta = \dfrac{a^2y^2+b^2x^2-2xyab\cos\theta}{a^2b^2}$

$\Rightarrow \sin^2\theta = \dfrac{y^2}{b^2}+\dfrac{x^2}{a^2}-\dfrac{2xy}{ab}\cos\theta$

Sol 26: $\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right)$

$= \dfrac{\pi}{4}+\dfrac{2\pi}{3}-\dfrac{\pi}{6} = \dfrac{\pi}{4}+\dfrac{\pi}{2} = \dfrac{3\pi}{4}$

Sol 27: L. H. S. $\Rightarrow \dfrac{9\pi}{8}-\dfrac{9}{4}\sin^{-1}\left(\dfrac{1}{3}\right)$

$= \dfrac{9}{4}\left[\sin^{-1}(1)-\sin^{-1}\left(\dfrac{1}{3}\right)\right]$

So L. H. S. $= \dfrac{9}{4}\left[\dfrac{\pi}{2}-\sin^{-1}\dfrac{1}{3}\right]$

$= \frac{9}{4}.\cos^{-1}\frac{1}{3} = \frac{9}{4}.\sin^{-1}\frac{2\sqrt{2}}{3}$

Sol 28: $\cos^{-1}\left(\sqrt{\frac{1}{3}}\right) - \cos^{-1}\left(\sqrt{\frac{1}{6}}\right) + \cos^{-1}\frac{\sqrt{10}-1}{3\sqrt{2}}$

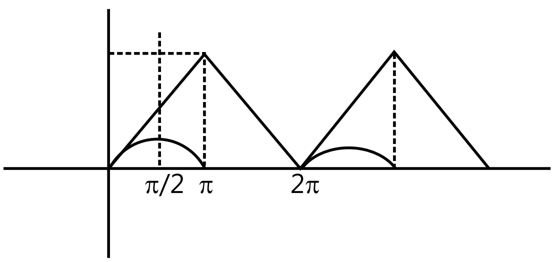
$= \cos^{-1}\left(\frac{1}{\sqrt{3}\sqrt{6}} + \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{5}}{\sqrt{6}}\right) + \cos^{-1}\frac{\sqrt{10}-1}{\sqrt{18}}$

$= \cos^{-1}\frac{1+\sqrt{10}}{\sqrt{18}} + \cos^{-1}\frac{\sqrt{10}-1}{\sqrt{18}}$

$= \cos^{-1}\left[\frac{(\sqrt{10}-1)(\sqrt{10}+1)}{18} - \frac{\sqrt{(7-\sqrt{40})(7+\sqrt{40})}}{18}\right]$

$= \cos^{-1}\left(\frac{1}{3}\right)$

Sol 29: $\sqrt{\sin x} = \cos^{-1}\cos x \Rightarrow 0 \le \sqrt{\sin x} \le 1$



x = 2nπ always satisfy

so infinite roots.

Exercise 2

Single Correct Choice Type

Sol 1: (A) $= \tan\cos^{-1}x = \frac{\sqrt{1-x^2}}{x}$

Sol 2: (C) $(|\sin^{-1}x| + |\sin^{-1}y|^2) = \pi^2$

$\Rightarrow (|\sin^{-1}x| + |\sin^{-1}y|) = \pi$

$\Rightarrow |\sin^{-1}x| = \frac{\pi}{2} = |\sin^{-1}y|$

$\Rightarrow x = \pm y = \pm 1$

$\Rightarrow x^2 + y^2 = 2$

Sol 3: (A)

$\cot^{-1}\sqrt{(x-1)(x-2)} + \cos^{-1}\sqrt{\left(x-\frac{3}{4}\right)^2 + \frac{3}{4}} = \frac{\pi}{2}$

Domain for $\cot^{-1}\sqrt{(x-1)(x-2)}$ is

$x \in (-\infty, 1] \cup [2, \infty)$

while $\cos^{-1}\sqrt{\left(x-\frac{3}{2}\right)^2 + \frac{3}{4}}$ is defined for x = [1,2]

At x = 1

$\Rightarrow \cot^{-1}(0) + \cos^{-1}(1) = \frac{\pi}{2}$

At x = 2

$\Rightarrow \cot^{-1}(0) + \cot^{-1}(1) = \frac{\pi}{2}$

Hence two solutions.

Sol 4: (D) $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}} = x$

$\frac{1}{1+x^2} = x^2 \quad \Rightarrow x^4 + x^2 - 1 = 0$

$x^2 = t \quad \Rightarrow t^2 + t - 1 = 0$

$\Rightarrow \left(t+\frac{1}{2}\right)^2 = \frac{5}{4}$

$x^2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ = positive

$x^2 = \frac{\sqrt{5}-1}{2}$

Sol 5: (D) $x^2 - 4x + 5 = (x-2)^2 + 1$

x = 2, to define $\sin^{-1}(x^2-4x+5)$

So $4 + 2a + \frac{\pi}{2} + 0 = 0 \Rightarrow a = -\frac{\pi}{4} - 2$

Sol 6: (C) $f(x) = \sqrt{\sin^{-1}\sin x} + \sqrt{\cos^{-1}\cos x}$

sin x must not be negative to define f(x). So the domain is

$x \in [2n\pi, (2n+1)\pi]$, $n \in I$

Sol 7: (B) $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

$x \in [0, 1] \Rightarrow \frac{\pi}{4} \Rightarrow \theta \le \frac{\pi}{2}$

Sol 8: (D) $\tan^{-1}(2) + \tan^{-1}(3)$

$= \pi - \tan^{-1}\left(\frac{3+2}{1-6}\right) = \frac{3\pi}{4} = \text{cosec}^{-1}(x)$

$\frac{-\pi}{2} \le \text{cosec}^{-1}(x) \le \frac{\pi}{2}$

So none of these.

Sol 9: (C) $f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2+3x+1}$

$= \cot^{-1}\sqrt{x(x+3)} + \cos^{-1}\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{5}{4}}$

$x(x + 3) \ge 0 \Rightarrow x \in (-\infty, -3] \cup [0, \infty)$

$\left(x+\frac{3}{2}\right)^2 - \frac{5}{4} \Rightarrow 0$

$\Rightarrow x \in \left(-\infty, \frac{-3}{2} - \frac{\sqrt{5}}{2}\right] \cup \left[\frac{-3}{2} + \frac{\sqrt{5}}{2}, \infty\right)$

$\left(x+\frac{3}{2}\right)^2 - \frac{5}{4} \le 1 \Rightarrow x \in [-3, 0]$

So the answer $x \in \{0, -3\}$

Sol 10: (A) $\alpha = \sin^{-1}\cos\sin^{-1}x$

$\beta = \cos^{-1}\sin\cos^{-1}x$

$\tan\alpha = \tan\sin^{-1}\cos\sin^{-1}x$

$= \tan\sin^{-1}\sqrt{1-x^2} = \frac{\sqrt{1-x^2}}{x}$

$\tan\beta = \tan\cos^{-1}\sin\cos^{-1}x = \tan\cos^{-1}\sqrt{1-x^2} = \frac{x}{\sqrt{1-x^2}}$

$\cot\beta = \tan\alpha$

Sol 11: (A) $x = 2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) + \tan^{-1}\sqrt{3}$

$x = \frac{2\pi}{3} + \frac{\pi}{4} + \frac{\pi}{3}$; $\quad x = \frac{5\pi}{4}$

$y = \cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right) = \cos\frac{1}{2}\sin^{-1}\sin\frac{5\pi}{8}$

$= \cos\frac{1}{2}\left[\pi - \frac{5\pi}{8}\right] = \cos\frac{3\pi}{16}$

Sol 12: (C) $[\tan(\sin^{-1}x)]^2 = \left[\frac{x}{\sqrt{1-x^2}}\right]^2 > 1$

$\Rightarrow \frac{x^2}{1-x^2} > 1$

$\Rightarrow \frac{-1+2x^2}{1-x^2} > 0$

$\Rightarrow \frac{(\sqrt{2}x-1)(\sqrt{2}x+1)}{(x-1)(x+1)} > 0$

$x \in (-1, 1) - \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

Sol 13: (D) $\sin^{-1}x = 2\sin^{-1}a$

$\Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4}$

$\Rightarrow -\frac{1}{\sqrt{2}} \Rightarrow a \le \frac{1}{\sqrt{2}}$

Sol 14: (C) $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^z$ for this to satisfy

$x = w = y = z = 1$

or $x = w = y = z = -1$

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ or $\begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} = 0$

independent of N_1, N_2, N_3, N_4

Sol 15: (C) $\theta = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$

$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{18}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{11}\right) + \tan^{-1}\frac{1}{18}$

$\Rightarrow \theta = \tan^{-1}\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11\times 8}} = \tan^{-1}\frac{65}{195} = \tan^{-1}\frac{1}{3}$

$\Rightarrow \cot\theta = 3$

Sol 16 : (A) (A) f(x) = {x} (Periodic)

(B) $g(x) = x\sin\frac{1}{x}$ (not periodic)

(C) h(x) = x cos x (not periodic)

(D) $\sin(\sin^{-1}x)$ (not periodic)

Sol 17: (A) $2\tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{\frac{1}{7}+\frac{1}{7}}{1-\frac{1}{49}} = \tan^{-1}\frac{7}{24}$

$4\cot^{-1}3 = 4\tan^{-1}\frac{1}{3}$

$= 2\tan^{-1}\frac{3}{4} = \tan^{-1}\frac{6/4}{1-\frac{9}{16}} = \tan^{-1}\frac{24}{7}$

$3\cot^{-1}4 = 3\tan^{-1}\frac{1}{4}$

$= \tan^{-1}\frac{8}{15}+\tan^{-1}\frac{1}{4} = \tan^{-1}\frac{47}{52}$

$4\cot^{-1}4 = \tan^{-1}\frac{47}{52} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{240}{173}$

Checking all options one by one

Sol 18: (B) $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$

Put x = sin y

$\sin^{-1}\sin 2y = 2\sin^{-1}\sin y\cos y$

$\Rightarrow -1 \le \sin 2y \le 1$

$\Rightarrow -\frac{\pi}{2} \le 2y \le \frac{\pi}{4}$

$\Rightarrow -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

So it is true if $x \in \left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

Sol 19: (B) $\sum_{k=1}^{k=n}\tan^{-1}\frac{2k}{2+k^2+k^4} = \tan^{-1}\frac{6}{7}$

L. H. S. = $\sum\tan^{-1}\frac{2k}{1+(k^4+k^2+1)}$

$= \sum\tan^{-1}\frac{k^2+k+1-(k^2-k+1)}{1+(k^2-k+1)(k^2+k+1)}$

$= \sum_{k=1}^{\infty}(\tan^{-1}(k^2+k+1)-\tan^{-1}(k^2-k+1))$

$= \tan^{-1}(n^2+n+1) - \tan^{-1}(1) = \tan^{-1}\frac{6}{7}$

$\Rightarrow \tan^{-1}\left[\left(n+\frac{1}{2}\right)^2+\frac{3}{4}\right] = \tan^{-1}\frac{13}{1}$

$\left(n+\frac{1}{2}\right)^2 = \frac{49}{4} \Rightarrow n = 3$

Sol 20: (A) $f(x) = \cos^{-1}\left[\left(\frac{3x^2-7x+8}{1+x^2}\right)\right]$

$1 + x^2 \ge 1$

$3x^2 - 7x + 8 = 3(x^2+1) - 7x + 5$

$= 3\left[\left(x-\frac{7}{6}\right)^2+8-\frac{49}{36}\right] = 3\left[\left(x-\frac{7}{6}\right)^2+\frac{23}{36}\right]$

$\Rightarrow \frac{3x^2-7x+8}{x+1} = 3 - \frac{7x-5}{x^2+1}$

$\Rightarrow -1 \le 3 - \frac{7x-5}{x^2+1} < 2$

$\Rightarrow -4 \le \frac{-(7x-5)}{(x^2+1)} < -1$

$\Rightarrow 4 \ge \frac{(7x-5)}{x^2+1} > 1$

$4x^2 - 7x + 9 \ge 0$ & $x^2 - 7x + 6 < 0$

always true & $(x-6)(x-1) < 0 \Rightarrow x \in (1, 6)$

Sol 21: (B) $a_r = 2^{r-1} = \frac{1}{b_r}$

$2a_r + \frac{1}{b_r} = 2^r + 2^{1-r} = 2^2 + \frac{2}{2^r}$

$t_r = \cot^{-1}(2a_r + b_r) = \tan^{-1}\frac{2^r}{2^{2r}+2}$

$= \tan^{-1}\frac{2^{r-1}}{1+2^{2r-1}} = \tan^{-1}\frac{2^r-2^{r-1}}{1+2^r\,2^{2r-1}}$

$= \tan^{-1}(2^r) - \tan^{-1}(2^{r-1})$

$\sum_{r=1}^{\infty}t_r = \tan^{-1}(2^{\infty}) - \tan^{-1}(2^{1-1}) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$

Sol 22: (B) $f(x) = \sin^{-1}(3x-4x^3)$

Let's put $x = \sin y = \sin^{-1}\sin 3y$

$-\frac{\pi}{2} \le 3y \le \frac{\pi}{2}$ $f(x) = 3\sin^{-1}x$

$\frac{3\pi}{2} \ge 3y \ge \frac{\pi}{2}$ $f(x) = \pi - 3\sin^{-1}x$

$-\frac{3\pi}{2} \Rightarrow 3y \le -\frac{\pi}{2}$ f(x) = $-\pi - 3\sin^{-1}x$

It's not differentiable 2 times.

Sol 23: (D) $\sec^{-1}x = \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\frac{5}{3\sqrt{3}}$

$\sec^{-1}(x) = \pi - \cos^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\frac{\sqrt{2}}{3\sqrt{3}}$

$= \pi + \cos^{-1}\left(\frac{\sqrt{2}}{3\sqrt{3}}\frac{1}{2} + \frac{5}{3\sqrt{3}}\frac{\sqrt{3}}{2}\right)$

$\sec^{-1}x = \pi + \cos^{-1}\left(\frac{5\sqrt{3}+\sqrt{2}}{6\sqrt{3}}\right)$

$x = \sec\left(\pi + \cos^{-1}\frac{15+\sqrt{6}}{18}\right) = -\frac{18}{15+\sqrt{6}}$

Sol 24: (C) $\left[\tan\left(\frac{x}{4} + \frac{1}{2}\sin^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{a}{b}\right)\right]^{-1}$

$= \left[\frac{1+\tan\frac{1}{2}\sin^{-1}\frac{a}{b}}{1-\tan\frac{1}{2}\sin^{-1}\frac{a}{b}} + \frac{1-\tan\frac{1}{2}\sin^{-1}\frac{a}{b}}{1+\tan\frac{1}{2}\sin^{-1}\frac{a}{b}}\right]^{-1}$

$= \left[\frac{2\left(1+\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}\right)}{\left(1-\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}\right)}\right]^{-1}$

$= \frac{1-\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}}{2\left(1+\tan^2\frac{1}{2}\sin^{-1}\frac{a}{b}\right)} = \frac{1}{2}\cos\sin^{-1}\frac{a}{b} = \frac{1}{2}\frac{\sqrt{b^2-a^2}}{b}$

Sol 25: (C) $x = \tan^{-1}(1) - \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{4} - \frac{2\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{4}$

$y = \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right) = \sqrt{\frac{1+\cos\cos^{-1}\left(\frac{1}{8}\right)}{2}}$

$= \sqrt{\frac{9}{16}} = \frac{3}{4}$

$y = -\frac{3}{4}\tan x.$

Sol 26: (B) RHS = $\cot^{-1}\frac{2x^2-1}{\sqrt{4x^2-4x^4}}$

$= \cos^{-1}\frac{\sqrt{4x^2-4x^4}}{4x^4+1-4x^2+4x^2-4x^4} = \cos^{-1}\sqrt{4x^2-4x^4}$

Put x = cos y

$f(x) = \cos^{-1}|\sin 2y|$

RHS = $\frac{\pi}{2} - \sin^{-1}|\sin 2y|$

Since $|\sin 2y| \ge 0$, so RHS will always be greater than zero.

Then x can be (0, 1)

Sol 27: (A) $\sin^{-1}x = \cos^{-1}x + \sin^{-1}(3x-2)$

$x \in [-1, 1]$

$(3x-2) \in [-1, 1]$

$\Rightarrow x \in \left[\frac{1}{3}, 1\right]$

$\Rightarrow \sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$

$\Rightarrow 2\sin^{-1}x = \frac{\pi}{2} + \sin^{-1}(3x-2)$

Taking cosine of both sides

$\Rightarrow \cos(2\sin^{-1}x) = -(3x-2)$

$\Rightarrow 1 - 2x^2 = -3x + 2$

$\Rightarrow 2x^2 - 3x + 1 = 0$

$\Rightarrow (x-1)\left(x-\frac{1}{2}\right) = 0$

$x = 1, \frac{1}{2}$

Sol 28: (A) $f(x) = \sin^{-1}\left|\frac{1-x^2}{1+x^2}\right|$

$g(x) = \cot^{-1}x - \tan^{-1}x = \frac{\pi}{2} - 2\tan^{-1}x$

Put x = tan y in f(x)

$f(x) = \sin^{-1}|\cos 2y|$

$\frac{\pi}{2} - \cos^{-1}|\cos 2y|$

$f(x) = g(x)$ when $x \in [0, 1]$

Sol 29: (D) $\tan[\cos^{-1}\{\sin(2\tan^{-1}2)\}]$

$= \tan[\cos^{-1}\{2\sin(\tan^{-1}2)\cos(\tan^{-1}2)\}]$

$= \tan\left[\cos^{-1}\left(\frac{2\times 2}{\sqrt{5}}\times\frac{1}{\sqrt{5}}\right)\right] = \tan\cos^{-1}\frac{4}{5} = \frac{3}{4}$

Sol 30: (A) $\sum_{n=1}^{\infty}\left|\frac{\sin^{-1}x-\cos^{-1}x}{\pi r}\right|$ is finite

$= \sum_{n=1}^{\infty}\left|\frac{\frac{\pi}{2}-2\cos^{-1}x}{\pi r}\right|^n$

$\frac{\pi}{2} \geq \frac{\pi}{2}-2\cos^{-1}x \geq -\frac{3\pi}{2}$

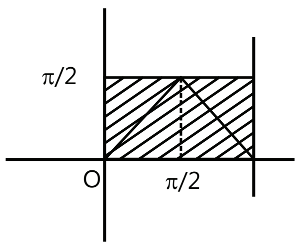
$\pi r > \frac{\pi}{2}$

$\Rightarrow r > \frac{1}{2}$

Sol 31: (C) $y = \sin^{-1}(\sin x)$, $x \in [0, \pi]$

$0 < x \leq \frac{\pi}{2}\, y = x$

$\frac{\pi}{2} < x \Rightarrow \pi y = \pi - x$



$A_1 \leq A_2 \leq A_3$

$A_1 = A_2 = \frac{A_3}{2}$

$A_1 = \frac{1}{2}\times\frac{\pi}{2}\times\frac{\pi}{2} = \frac{\pi^2}{8}$

$A_3 = \frac{\pi^2}{4}$

Sol 32: (A) $\sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{n^4-2n^2+2}$

$= \sum_{n=1}^{\infty}\tan^{-1}\frac{4n}{1+(n^2-1)(n^2-1)}$

$= \sum_{n=1}^{\infty}\tan^{-1}\frac{(n+1)^2-(n-1)^2}{1+(n-1)^2(n+1)^2}$

$= \sum_{n=1}^{\infty}\tan^{-1}(n+1)^2-\tan^{-1}(n-1)^2$

$= 2[\tan^{-1}(\infty)] - \tan^{-1}(1) - \tan^{-1}0 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\tan^{-1}(2) + \tan^{-1}(3) = \pi + \tan^{-1}(-1) = \frac{3\pi}{4}$

$4\tan^{-1}(1) - \pi$

$\sec^{-1}(1-\sqrt{2}) = \cos^{-1}\left(\frac{1}{1-\sqrt{2}}\right) = -\cos^{-1}(\sqrt{2}+1)$

Previous Years' Questions

Sol 1: (B) $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$= \tan\left[\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$\left[\because \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)\right]$

$= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right] = \tan\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{17}{6}$

Sol 2: (C) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

Sol 3: (C) Given function is $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

Function is defined, if

(i) $x(x + 1) \geq 0 \Rightarrow$ Domain of square root function.

(ii) $x^2 + x + 1 \geq 0 \Rightarrow$ Domain of square root function.

(iii) $\sqrt{x^2+x+1} \leq 1 \Rightarrow$ Domain of $\sin^{-1}$ function.

From (ii) and (iii)

$0 \leq x^2 + x + 1 \leq 1 \cap x^2 + x \geq 0$

$\Rightarrow 0 \leq x^2 + x + 1 \leq 1 \cap x^2 + x + 1 \geq 1$

$\Rightarrow x^2 + x + 1 = 1$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\Rightarrow x = 0, x = -1$

Sol 4: (B) We know that, $\sin^{-1}(\alpha) + \cos^{-1}(\alpha) = \frac{\pi}{2}$

Therefore, α should be equal in both functions.

$\therefore x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots = x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots$

$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x^2}{1+\frac{x^2}{2}}$

$$\Rightarrow \frac{\frac{x}{2+x}}{2} = \frac{\frac{x^2}{2+x^2}}{2}$$

$$\Rightarrow \frac{2x}{2+x} = \frac{2x^2}{2+x^2}$$

$\Rightarrow 2x(2 + x^2) = 2x^2(2 + x)$

$\Rightarrow 4x + 2x^3 = 4x^2 + 2x^3$

$\Rightarrow x(4 + 2x^2 - 4x - 2x^2) = 0$

$\Rightarrow$ Either $x = 0$ or $4 - 4x = 0$

$\Rightarrow x = 0$ or $x = 1$

$\because 0 < |x| < \sqrt{2}$,

$\therefore x = 1$ and $x \neq 0$

Sol 5: (D) Given, $\sin[\cot^{-1}(1+x)] = \cos(\tan^{-1}x)$...(i)

and we know,

$$\cot^{-1}\theta = \sin^{-1}\left(\frac{1}{\sqrt{1+\theta^2}}\right),$$

and $\tan^{-1}\theta = \cos^{-1}\left(\frac{1}{\sqrt{1+\theta^2}}\right)$

$\therefore$ From Eq. (i),

$$\sin\left(\sin^{-1}\frac{1}{\sqrt{1+(1+x)^2}}\right) = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

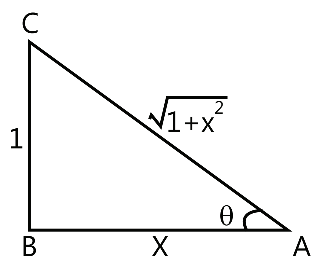
$$\Rightarrow \frac{1}{\sqrt{1+(1+x)^2}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow 1 + x^2 + 2x + 1 = x^2 + 1$

$$\Rightarrow x = -\frac{1}{2}$$

Sol 6: (C) We have, $0 < x < 1$

Let $\cot^{-1}x = \theta$



$\Rightarrow \cot\theta = x$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{1+x^2}} = \sin(\cot^{-1}x)$$

and $\cos\theta = \frac{x}{\sqrt{1+x^2}} = \cos(\cot^{-1}x)$

Now

$$\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[\left(x\frac{x}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right)^2-1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[\left(\frac{1+x^2}{\sqrt{1+x^2}}\right)^2-1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\,[1+x^2-1]^{\frac{1}{2}} = x\sqrt{1+x^2}$$

Sol 7: Given,

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ac}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)\right]$$

$$= \tan^{-1}\left(\frac{\sqrt{a+b+c}\left(\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ca}}+\sqrt{\frac{c}{ab}}\right)-(a+b+c)\sqrt{\frac{a+b+c}{abc}}}{1-(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\frac{a+b+c}{abc}}(a+b+c)-(a+b+c)\sqrt{\frac{a+b+c}{abc}}}{1-\frac{(a+b+c)\,(ab+bc+ca)}{abc}}\right)$$

$\Rightarrow \theta = \tan^{-1}0 \Rightarrow \tan\theta = 0$

Sol 8 : $$\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right] = \tan\left[\tan^{-1}\left(\frac{2.\frac{1}{5}}{1-\frac{1}{25}}\right)-\frac{\pi}{4}\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{5}{12}\right)-\frac{\pi}{4}\right]$$

$$= \frac{\tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right]-\tan\left(\frac{\pi}{4}\right)}{1+\tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right]\tan\frac{\pi}{4}} = \frac{\frac{5}{12}-1}{1+\frac{5}{12}\cdot 1} = -\frac{7}{17}$$

Sol 9: (A) Given, $A = 2\tan^{-1}(2\sqrt{2}-1)$ and

$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

Here, $A = 2\tan^{-1}(2\sqrt{2}-1)$

$= 2\tan^{-1}(2 \times 1.414 - 1) = 2\tan^{-1}(1.828)$

$\therefore\ A > 2\tan^{-1}(\sqrt{3}) = 2.\frac{\pi}{3} = \frac{2\pi}{3}$

To find the value of B, we first say

$\sin^{-1}\frac{1}{3} < \sin^{-1}\cdot\frac{1}{2} = \frac{\pi}{6}$

So that, $0 < 3\sin^{-1}\frac{1}{3} < \frac{\pi}{2}$

Now, $3\sin^{-1}\frac{1}{3} = \sin^{-1}\left(3\cdot\frac{1}{3} - 4\cdot\frac{1}{27}\right) = \sin^{-1}\left(\frac{23}{27}\right)$

$= \sin^{-1}(0.851) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

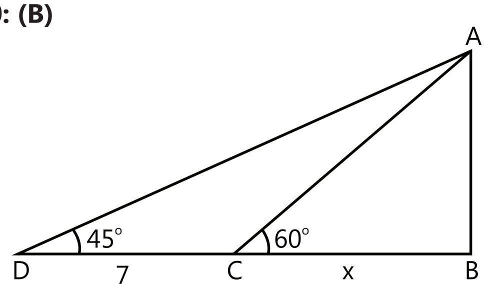
$\sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}(0.6) < \sin^{-1}\left(\frac{\sqrt{3}}{5}\right) = \frac{\pi}{3}$

$\therefore\ B < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Thus, $A > \frac{2\pi}{3}$ and $B < \frac{2\pi}{3}$

Hence, greater angle is A.

Sol 10: (B)



BD = AB = 7 + x

Also $AB = x\tan 60^\circ = x\sqrt{3}$

$\therefore\quad x\sqrt{3} = 7 + x$

$x = \frac{7}{\sqrt{3}-1}$

$AB = \frac{7\sqrt{3}}{2}(\sqrt{3}+1)$

Sol 11: (A) Let $\cot\left(\text{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$

$\Rightarrow\quad E = \cot\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$\Rightarrow\quad E = \cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$

$\Rightarrow\ E = \cot\left(\tan^{-1}\frac{17}{6}\right) = \frac{6}{17}$

Sol 12: (A) $\cos(\alpha+\beta) = \frac{4}{5} \quad\Rightarrow\quad \tan(\alpha+\beta) = \frac{3}{4}$

$\sin(\alpha-\beta) = \frac{5}{13} \quad\Rightarrow\quad \tan(\alpha-\beta) = \frac{5}{12}$

$\tan 2\alpha = \tan(\alpha+\beta+\alpha-\beta) = \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\frac{5}{12}} = \frac{56}{33}$

Sol 13: (B) $r = \frac{a}{2}\cot\frac{\pi}{n}$

'a' is side of polygon.

$R = \frac{a}{2}\text{cosec}\frac{\pi}{n}$

$\frac{r}{R} = \frac{\cot\frac{\pi}{n}}{\text{cosec}\frac{\pi}{n}} = \cos\frac{\pi}{n}$

$\cos\frac{\pi}{n} \neq \frac{2}{3}$ for any $n \in N$

Sol 14: (B) $\ell = \cos 45^\circ = \frac{1}{\sqrt{2}}$

$m = \cos 120^\circ = -\frac{1}{2}$

$n = \cos\theta$

Where θ is the angle which line makes with positive z-axis.

Now $\ell + m^2 + n^2 = 1 \quad\Rightarrow\quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$

$\cos^2\theta = \frac{1}{4} \quad\Rightarrow\quad \cos\theta = \frac{1}{2}$ (θ being acute)

$\Rightarrow\quad \theta = \frac{\pi}{3}.$

Sol 15: (A) $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

$x = \tan\theta$

$\frac{-\pi}{6} < \theta < \frac{\pi}{6}$

$\tan^{-1} y = \theta + \tan^{-1}\tan 2\theta = \theta + 2\theta = 3\theta$

$y = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$y = \frac{3x - x^3}{1 - 3x^2}$

Sol 16: (B) $0 \le x < 2\pi$

$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$(\cos x + \cos 4x) + (\cos 2x + \cos 3x) = 0$

$2\cos\frac{5x}{2}\cos\frac{3x}{2} + 2\cos\frac{5x}{2}\cos\frac{x}{2} = 0$

$2\cos\frac{5x}{2}\left[2\cos x\cos\frac{x}{2}\right] = 0$

$\cos\frac{5x}{2} = 0$ or $\cos x = 0$ or $\cos\frac{x}{2} = 0$

$x = \frac{(2n+1)\pi}{5}$ or $x = (2n+1)\frac{\pi}{2}$ or $x = (2n+1)\pi$

$x = \left\{\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

Number of solution is 7

Sol 17: (A) At $x = \frac{\pi}{6} \Rightarrow y = \frac{\pi}{3}$

$f(x) = \tan^{-1}\left(\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|\right) \quad \because x \in \left(0, \frac{\pi}{2}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

$f(x) = \frac{\pi}{4} + \frac{x}{2} \qquad f'(x) = \frac{1}{2}$

Slope of normal = -2

Equation of normal $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$

$y = -2x + \frac{2\pi}{3}$

JEE Advanced/Boards

Exercise 1

Sol 1: $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$; $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$,

Put x = tan y

$\alpha = 2\tan^{-1}\tan\left(y + \frac{\pi}{4}\right)$

$\beta = \sin^{-1}\cos 2y = \frac{\pi}{2} - \cos^{-1}\cos 2y$

If $0 < \tan x < 1$; $0 < y < \frac{\pi}{4}$

$0 < 2y < \frac{\pi}{2} \Rightarrow \frac{\pi}{4} < y + \frac{\pi}{4} < \frac{\pi}{2}$

$\alpha = 2\left(y + \frac{\pi}{4}\right) = 2y + \frac{\pi}{2}$

$\beta = \frac{\pi}{2} - 2y$

$\Rightarrow \alpha + \beta = \pi$

If $x > 1 \Rightarrow \tan y > 1$

$\Rightarrow \frac{\pi}{2} > y > \frac{\pi}{4} \Rightarrow \pi > 2y > \frac{\pi}{2}$

$\Rightarrow \frac{3\pi}{4} > y + \frac{\pi}{4} > \frac{\pi}{2}$

$\alpha = 2\left[-\pi + \frac{\pi}{4} + \tan^{-1}x\right] = \frac{-3\pi}{2} + 2\tan^{-1}x$

$\beta = \frac{\pi}{2} - [2 + 2\tan^{-1}] = +\frac{3\pi}{2} - 2\tan^{-1}x$

$\Rightarrow \alpha + \beta = 0$

Sol 2: $y = \tan^{-1}\left[\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$

$\tan y = \left[\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$

$\Rightarrow \frac{1 - \tan y}{1 + \tan y} = \sqrt{\frac{1-x^2}{1+x^2}}$

$\Rightarrow \frac{1 + \tan^2 y - 2\tan y}{1 + \tan^2 y + 2\tan y} = \frac{1-x^2}{1+x^2}$

$\Rightarrow x^2 = \frac{2\tan y}{1 + \tan^2 y} = \sin 2y$

Sol 3: (i) n^{th} term = $\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$

$= \tan^{-1}\frac{2^n - 2^{n-1}}{1+2^n(2^{2x-1})}$

n^{th} term = $\tan^{-1}(2n) - \tan^{-1}(2^{n-1})$

Sum of infinite series

$= \tan^{-1}(\infty) - \tan^{-1}(1) = \frac{\pi}{4}$

(ii) $\tan^{-1}\frac{(x+1)-x}{1+x(x+1)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+2)(x+1)} + \ldots .$

$=\tan^{-1}(x+1)-\tan^{-1}(x)+\tan^{-1}(x+2)-\tan^{-1}(x+1) + \ldots$

$= \tan^{-1}(x + 2) - \tan^{-1}(x) + \ldots.$

$= \tan^{-1}(x + n) - \tan^{-1}(x)$

Sol 4: $x \in \left[-1, \frac{-1}{2}\right]$

$f(x) = \sin^{-1}(3x - 4x^3) + \cos^{-1}(4x^3 - 3x)$

$f(x) = g(x) + h(x)$

$g(x) = \sin^{-1}\sin 3y$ where $y = \sin^{-1}x$

$h(x) = \cos^{-1}\cos 3z$ where $z = \cos^{-1}x$

$x \in \left[-1, \frac{-1}{2}\right]$

$y = \sin^{-1}x \in \left[-\frac{\pi}{2}, -\frac{\pi}{6}\right] \Rightarrow 3y \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$

$z = \cos^{-1}x \in \left[\frac{2\pi}{3}, \pi\right] \Rightarrow 3z \in [2\pi, 3\pi]$

$g(x) = -\pi - 3\sin^{-1}x$

$= -\pi - 3\left(\frac{\pi}{2} - \cos^{-1}x\right) = -\frac{5\pi}{2} + 3\cos^{-1}x$

$h(x) = -2\pi + 3\cos^{-1}x$

$f(x) = 6\cos^{-1}x - \frac{9\pi}{2}$

$\therefore a = 6, b = -\frac{9}{2}$

Sol 5: (i) $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$

$\Rightarrow \sin^{-1}x = \frac{\pi}{3} - \sin^{-1}2x$

$\Rightarrow x = \sin\left(\frac{\pi}{3} - \sin^{-1}2x\right)$

$\Rightarrow x = \frac{\sqrt{3}}{2}\cos\sin^{-1}2x - \frac{1}{2}(2x)$

$\Rightarrow 2x = \frac{\sqrt{3}}{2}\sqrt{1-4x^2} \Rightarrow \frac{16x^2}{3} = 1 - 4x^2$

$\frac{28x^2}{3} = 1 \Rightarrow x = \frac{\sqrt{3}}{\sqrt{28}} = \frac{1}{2}\sqrt{\frac{3}{7}}$

(ii) $\tan^{-1}(x - 1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}3x$

$\Rightarrow \tan^{-1}(x - 1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1}(x)$

$\Rightarrow \frac{(x-1)+(x+1)}{1-(x^2-1)} = \frac{3x-x}{1+3x^2}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2} \Rightarrow 4x^2 = 1 ; x = 0$

(iii) $\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-x}{2x+x}\right)$

$= \tan^{-1}\frac{\frac{x-1}{x+1}+\frac{2x-1}{2x+1}}{1-\frac{(x-1)(2x-1)}{(x+1)(2x+1)}}$

$= \tan^{-1}\frac{2x^2-1-x+2x^2-1+x}{6x}$

$= \tan^{-1}\left(\frac{4x^2-2}{6x}\right) = \tan^{-1}\left(\frac{2x^2-1}{3x}\right)$

$\Rightarrow \frac{2x^2-1}{x} = \frac{23}{12}$

$\Rightarrow 24x^2 - 12 = 23x$

$\Rightarrow 24x^2 - 23x - 12 = 0$

$x = \frac{23 \pm \sqrt{529+1152}}{48} \Rightarrow x = \frac{4}{3}$

(iv) $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$

LHS = $\tan^{-1}\frac{\sqrt{(x^2+1)^2-(x^2-1)^2}}{(x^2-1)} + \tan^{-1}\frac{2x}{x^2-1}$

$= \tan^{-1}\frac{|2x|}{x^2-1} + \tan^{-1}\frac{2x}{x^2-1}$

If $x < 0$ LHS = $= 2\tan^{-1}\frac{2x}{x^2-1}$

If $x > 0 = 2\tan^{-1}\frac{2x}{x^2-1}$

$\frac{2x}{x^2-1} = \tan\frac{\pi}{3} \Rightarrow 2x = \sqrt{3}(x^2-1)$

$4x^2 = 3(x^4 + 1 - 2x^2) \Rightarrow 3x^4 - 10x^2 + 3 = 0$

$\Rightarrow (x^2-3)(3x^2-1) = 0 \Rightarrow x = \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}$

$x = \pm\frac{1}{\sqrt{3}}$ does not satisfy.

Sol 6: $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

$\text{LHS} = \tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}\frac{\left(x+\frac{1}{y}\right)}{1-\frac{x}{y}} = \tan^{-1}(3)$

$\frac{x+\frac{1}{y}}{1-\frac{x}{y}} = 3 \Rightarrow x + \frac{1}{y} = 3 - \frac{3x}{y}$

$\frac{1}{y}(1+3x) = 3 - x \Rightarrow y = \frac{1+3x}{3-x}$

At $x = 1$; $y = 2$

At $x = 2$; $y = 7$

Sol 7: $x^2 - 4x + 1 = 0$

$(x-2)^2 = 3$

$\alpha = 2+\sqrt{3}$; $\beta = 2-\sqrt{3}$; $\alpha+\beta = 4$; $\alpha\beta = 1$

$$f(\alpha,\beta) = \frac{(2-\sqrt{3})^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)$$

$$+ \frac{(2+\sqrt{3})^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

$$= \frac{(2-\sqrt{3})^3}{2}\frac{1}{[\sin\theta_1/2]^2} + \frac{(2+\sqrt{3})^3}{2}\frac{1}{[\cos\theta_2/2]^2}$$

$$\frac{(2-\sqrt{3})^3}{1-\cos\tan^{-1}\left[\frac{(2-\sqrt{3})}{(2-\sqrt{3})}\right]} + \frac{(2+\sqrt{3})^3}{1-\cos\tan^{-1}\left[\frac{(2+\sqrt{3})}{(2-\sqrt{3})}\right]}$$

$$= \frac{(2-\sqrt{3})^3}{1-\frac{2+\sqrt{3}}{\sqrt{14}}} + \frac{(2+\sqrt{3})^3}{1+\frac{(2-\sqrt{3})}{2\sqrt{7}}}$$

$$= \frac{\beta^3}{1-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}} + \frac{\alpha^3}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}$$

$$= \sqrt{\alpha^2+\beta^2}\left[\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha} + \frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta}\right]$$

By putting all value, we get = 56

Sol 8: $f(x) = \sin^{-1}\frac{2x}{1+x^2}$; $g(x) = \cos^{-1}\frac{1-x^2}{1+x^2}$

$h(x) = \tan^{-1}\frac{2x}{1-x^2}$

put $x = \tan y$

$f(x) = \sin^{-1}\sin 2y$; $g(x) = \cos^{-1}\cos 2y$

$h(x) = \tan^{-1}\tan 2y$

(i) $x \in (-1, 1)$

$-\frac{\pi}{4} \le y \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le 2y \le \frac{\pi}{2}$

$f(x) = 2y = 2\tan^{-1}x$

$$g(x) = \begin{cases} 2\tan^{-1}x & ; x \ge 0 \\ -2\tan^{-1}x & ; x \le 0 \end{cases}$$

$h(x) = 2\tan^{-1}x$

$$f(x) + g(x) + h(x) = \begin{cases} 2\tan^{-1}x & ; x \le 0 \\ 6\tan^{-1}x & ; x \ge 0 \end{cases}$$

$x = \tan\frac{\pi}{12} = 2-\sqrt{3}$

(ii) $f(2) + g(2) + h(2)$

$f(2) = \sin^{-1}\left(\frac{4}{5}\right)$

$g(2) = \cos^{-1}\left(\frac{-3}{5}\right)$

$h(2) = \tan^{-1}\left(\frac{4}{-3}\right) = \cot^{-1}\left(\frac{-3}{4}\right)$

$f(2) = -g(2)$

$f(2) + g(2) + h(2) = \cot^{-1}\left(\frac{-3}{4}\right)$

Sol 9: (i) $(\cot^{-1}x)^2 - 5(\cot^{-1}x) + 6 > 0$

$(\cot^{-1}x - 3)(\cot^{-1}x - 2) > 0$

$\cot^{-1}x \in (-\infty, 2) \cup (3, \infty)$

$\cot^{-1}x \in (0, 2) \cup (3, \pi)$

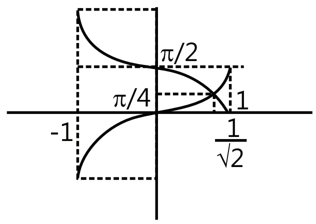
$x \in (\cot 2, \infty) \cup (-\infty, \cot 3)$

(ii) $\sin^{-1}x > \cos^{-1}x$

$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$0 \le \cos^{-1}x \Rightarrow \pi$

$x \in \left[\frac{1}{\sqrt{2}}, 1\right]$



(iii) $\tan^2(\sin^{-1}x) > 1$

$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \frac{x^2}{1-x^2} > 1$

$\Rightarrow \frac{2x^2-1}{x^2-1} < 0$

$x \in \left(-1, \frac{-1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, 1\right)$

Sol 10: $x(x - 2)(3x - 7) = 2$ are real and positive at $x = 0, +2, \frac{7}{3}$ it has (–2) value.

At $x = 4f(4) = 38$

At $x = \frac{1}{2} f\left(\frac{1}{2}\right) = \frac{17}{8}$

One root between 0 to $\frac{1}{2}$, one between $\frac{1}{2}$ to 2, one between $\frac{7}{3}$ to 4.

$\tan^{-1}(r) + \tan^{-1}(s) + \tan^{-1}(t)$

$= \tan^{-1}\left[\frac{r+s+t-rst}{1-(rs+st+tr)}\right]$ ..... (i)

equation is

$\Rightarrow 3x^2 - 13x^2 + 14x - 2 = 0$

$r + s + t = \frac{13}{3}$

$rst = \frac{-(-2)}{3} = \frac{2}{3}$

$rs + st + tr = \frac{+14}{3} = \tan^{-1}(-1) = \frac{-\pi}{4}$

since r, s, t are always positive so value will be $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

Sol 11: $f(x) = \frac{\pi}{4} + \cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) - \tan^{-1} x$

sgn(f(x)) = 1 when f(x) > 0

for any x > 0

$f(x) = \frac{\pi}{4} + \tan^{-1}\frac{1}{x} - \tan^{-1}x$

$f(x) = \frac{3\pi}{4} - 2\tan^{-1}x$

$\frac{3\pi}{4} - 2\tan^{-1}x > 0$

$0 < \tan^{-1}x < \frac{3\pi}{4}$

$0 \le x < \tan\left(\frac{3\pi/4}{2}\right)$

$\Rightarrow 0 < x < \sqrt{2} + 1$

$0 \le x < \sqrt{2} + 1$

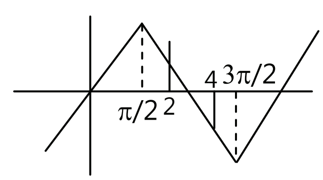
$x = 0, 1, 2$

Sol 12: $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$

$\frac{2x^4+4}{1+x^2} = 2 + \frac{2}{1+x^2}$

$4 \ge \frac{2x^2+4}{1+x^2} > 2$

$-\left(\frac{2x^2+4}{1+x^2}\right) + \pi < \pi - 3$



$\frac{2x^2+4}{1+x^2} > 3$

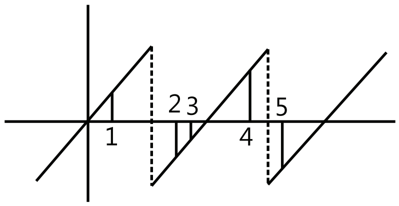
$\frac{1-x^2}{1+x^2} > 0$

$x \in (-1, 1)$

Sol 13: $f(x) = \tan^{-1}(\cot x - 2\cot 2x)$

$\sum_{r=1}^{5} f(r) = a - b\pi$

$f(x) = \tan^{-1}\left[\frac{1}{\tan x} - \frac{2(1-\tan^2 x)}{2\tan x}\right] = \tan^{-1}(\tan x)$



$\Sigma f(r) = 1 + (-\pi + 2) + (\pi + 3) + (\pi + 4) + (2\pi + 5)$

$= 15 - 5\pi$

$a = 15, b = 5, a + b = 20$

Sol 14: $f(x) = (2a + b)\cos^{-1}x + (a + 2b)\sin^{-1}x$

Domain $-1 \le x \le 1$

Then range should be $-1 \le f(x) \le 1$

$f(x) = a[2\cos^{-1}x + \sin^{-1}x] + b[\cos^{-1}x + 2\sin^{-1}x]$

$= a\left[\frac{\pi}{2} + \cos^{-1}x\right] + b\left[\frac{\pi}{2} + \sin^{-1}x\right]$

$= \frac{\pi}{2}(a + b) + (a\cos^{-1}x + b\sin^{-1}x)$

$= \frac{\pi}{2}(a + b) + a(\cos^{-1}x + \sin^{-1}x) + (b - a)\sin^{-1}x$

$= \frac{\pi}{2}(2a + b) + (b - a)\sin^{-1}x$

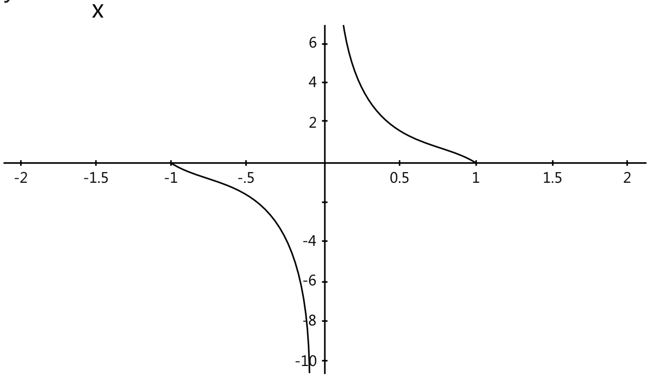
$\frac{\pi}{2}(3a) < f(x) < \frac{\pi}{2}(a + 2b)$

$\frac{\pi}{2}(3a) = -1 \Rightarrow a = \frac{-2}{3\pi}$; $b = \frac{4}{3\pi}$

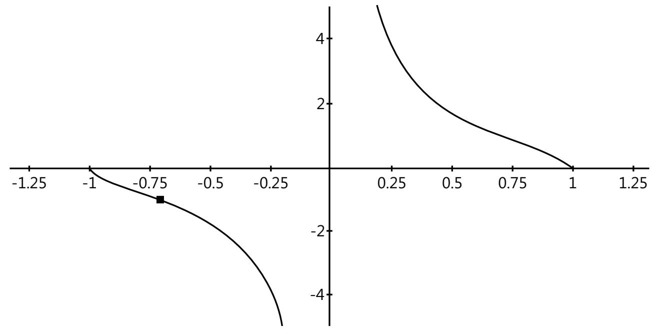
$\pi(a - b) = -2$

Sol 15: (i) $y = \tan(\cos^{-1}x) = \frac{\sqrt{1-x^2}}{x}$ except $x = 0$

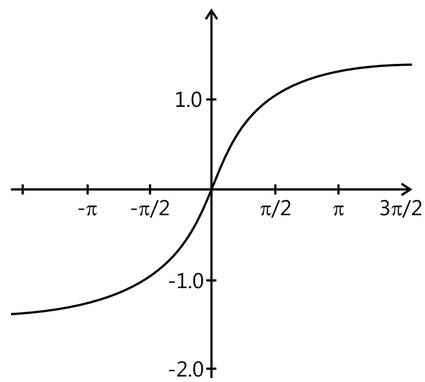
$y = \frac{\sqrt{1-x^2}}{x}$ identical



(ii) $y = \tan(\cot^{-1}x) = \frac{1}{x}$ except $x = 0$ identical



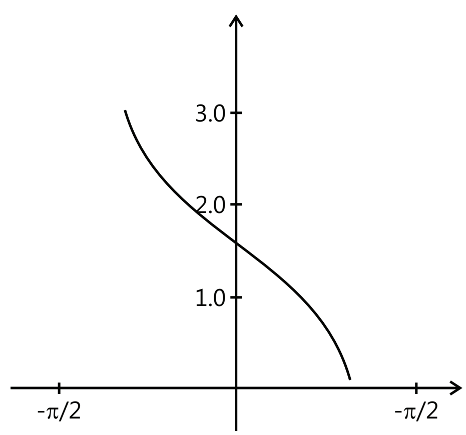
(iii) $y = \sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}$ identical



(iv) $y = \cos(\tan^{-1}x)$

$= \cos\left(\frac{\pi}{2} - \cot^{-1}x\right) = -\sin(-\cot^{-1}x)$

$= \sin(\cot^{-1}x)$ identical



Sol 16: (i) $f(x) = \cot^{-1}(2x - x^2)$

$2x - x^2 = x(2 - x)$

Domain $x \in R$

Range $2x - x^2 < 1 \Rightarrow x \in \left[\frac{\pi}{4}, \pi\right)$

(ii) $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$

Domain $\tan x > 0$, $\tan x \neq 1$

$$\log_3 \tan x + \frac{1}{\log_3 \tan x} > 2$$

$$\text{or } \log_3 \tan x + \frac{1}{\log_3 \tan x} < -2$$

$$x \in \left[\left(n\pi, n\pi + \frac{\pi}{2}\right) - \left\{\pi + \frac{\pi}{4}\right\}\right]$$

$$\text{Range} \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left\{\frac{\pi}{2}\right\}$$

(iii) $f(x) = \cos^{-1}\dfrac{\sqrt{2x^2+1}}{x^2+1}$

Domain $\dfrac{\sqrt{2x^2+1}}{x^2+1} \leq 1$

$2x^2 + 1 \leq x^4 + 2x^2 + 1$

$x^4 \geq 0$

Always true $x \in R$

(iv) $f(x) = \tan^{-1}(\log_{4/5}(5x^2 - 8x + 4))$

$5x^2 - 8x + 4 > 0$

$$\left(x - \frac{8}{10}\right)^2 + \frac{4}{5} - \frac{64}{100} > 0$$

$$\left(x - \frac{8}{10}\right)^2 + \frac{16}{100} > 0$$

Domain $x \in R$

Range $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{4}\right]$

Sol 17: $y = \sin^{-1}\sin 8 - \tan^{-1}\tan 1$

$+ \cos^{-1}\cos 12 - \sec^{-1}\sec 9 + \cot^{-1}\cot 6 - \operatorname{cosec}^{-1}\operatorname{cosec} 7$

$8 \sim \dfrac{5\pi}{2} + h$; $12 \sim 4\pi - h$; $6 \sim 2\pi - h$

$1 \sim \dfrac{\pi}{2} - h$; $9 \sim 3\pi - h$; $7 \sim \dfrac{5\pi}{2} - h$

$y = (3\pi - 8) - 1 + 4\pi - 12 - (9 - 2\pi)$

$+ (6 - \pi) - (7 - 2\pi) = -31 + 10\pi$

Sol 18: $\alpha = \sin^{-1}\left(\dfrac{36}{85}\right)$ $\beta = \cos^{-1}\dfrac{4}{5}$

$\gamma = \tan^{-1}\left(\dfrac{8}{15}\right)$

$$\alpha + \beta + \gamma = \sin^{-1}\left(\frac{36}{85}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{8}{15}\right)$$

$$= \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{8}{15}\right)$$

$$= \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\frac{\frac{3}{4} + \frac{8}{15}}{1 - \frac{24}{60}}$$

$$= \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\left(\frac{77}{36}\right) = \frac{\pi}{2}$$

(i) $\Sigma\cot\alpha = \cot\alpha + \cot\beta + \cot\gamma$

since $\alpha + \beta + \gamma = \dfrac{\pi}{2}$

$1 = \tan\alpha . \tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha$

$\cot\alpha\cot\beta\cot\gamma = \cot\alpha + \cot\beta + \cot\gamma$

Hence prove.

(ii) Since $\alpha + \beta + \gamma = \pi/2$

Hence, $\Sigma(\tan\alpha\tan\beta) = 1$

Sol 19: $\sin^{-1}\left(\sin\dfrac{33\pi}{7}\right) + \cos^{-1}\cos\dfrac{46\pi}{7}$

$+ \tan^{-1}\left(-\tan\dfrac{13\pi}{7}\right) + \cot^{-1}\cot\left(\dfrac{-19\pi}{8}\right)$

$\text{LHS} = \dfrac{33\pi}{7} - 5\pi + 7\pi - \dfrac{46\pi}{7}$

$- \left(\dfrac{13\pi}{8} - 2\pi\right) + \pi + \left(\dfrac{19\pi}{8} - 2\pi\right)$

$= \dfrac{-13\pi}{7} + 2\pi + \pi - \dfrac{13\pi}{4} = \dfrac{13\pi}{4}$

Sol 20: (i) $\cos^{-1}\dfrac{5}{13} + \cos^{-1}\left(\dfrac{-7}{25}\right) + \sin^{-1}\left(\dfrac{36}{325}\right)$

$$= \tan^{-1}\left(\frac{12}{5}\right) - \tan^{-1}\frac{24}{7} + \sin^{-1}\left(\frac{36}{325}\right)$$

$$= \tan^{-1}\frac{\left(\frac{12}{5} - \frac{24}{7}\right)}{1 + \frac{12}{5} \times \frac{24}{7}} + \sin^{-1}\frac{36}{325}$$

$= \tan^{-1}\left(\frac{-36}{323}\right) + \tan^{-1}\left(\frac{36}{323}\right) = \pi$

(ii) LHS $= \cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$

$= \cos^{-1}\left[\frac{\sqrt{2}}{\sqrt{3}}\left(\frac{\sqrt{6}+1}{2\sqrt{3}}\right) + \sqrt{\frac{1}{3}}\sqrt{1-\frac{7+2\sqrt{6}}{12}}\right]$

$= \cos^{-1}\left[\left(\frac{\sqrt{6}+1}{3\sqrt{2}}\right) + \frac{\sqrt{5-2\sqrt{6}}}{3\sqrt{2}\sqrt{2}}\right]$

$= \cos^{-1}\left(\frac{\sqrt{12}+\sqrt{2}+\sqrt{5-2\sqrt{6}}}{6}\right)$

$= \cos^{-1}\left(\frac{\sqrt{12}+\sqrt{3}}{6}\right) = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

Sol 21:

$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$

$= \tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$

$= \tan^{-1}(a) - \tan^{-1}(b) + \tan^{-1}(b)$

$- \tan^{-1}(c) + \tan^{-1}(c) - \tan^{-1}(a) = 0$

Sol 22: $x^2 + 5x - 49 = 0 \Rightarrow \alpha, \beta$

$\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$

$= \frac{(\cot\cot^{-1}\alpha)(\cot\cot^{-1}\beta)-1}{(\cot\cot^{-1}\alpha)+(\cot\cot^{-1}\beta)}$

$= \frac{\alpha\beta-1}{\alpha+\beta} = \frac{-1-49}{-5} = 10$

Sol 23: $\theta_1 + \theta_2 + \theta_3 = \pi$

$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}+k\right) + \tan^{-1}\left(\frac{1}{2}+2k\right) = \pi$

$\Rightarrow$ Use the formula

$\tan^{-1}x + \tan^{-1}y + \tan z = \tan^{-1}\left(\frac{x+y+z}{1-xy-yz-zx}\right)$

$\frac{1}{2} + \left(\frac{1}{2}+k\right) + \left(\frac{1}{2}+2k\right) = \frac{1}{2}\left(\frac{1}{2}+k\right)\left(\frac{1}{2}+2k\right)$

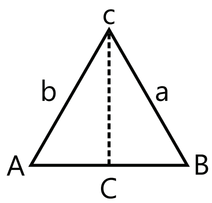
$\Rightarrow 1 + 2k + 1 + 4k + 1 = \frac{(2k+1)(4k+1)}{4}$

$\Rightarrow 24k + 12 = 8k^2 + 1 + 6k$

$\Rightarrow 8k^2 - 18k - 11 = 0$

$\Rightarrow k = \frac{11}{4}$

Sol 24 :



Area (ΔABC) $= \frac{1}{2} \times c \times (b \sin A)$

$\angle A = \angle B =$

$\frac{1}{2}\left[\sin^{-1}\left(\sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{3}}\left(\frac{-1}{2}\right)\right) + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$= \frac{1}{2}\left[\sin^{-1}\sin\left(\frac{2\pi}{3} - \sin^{-1}\frac{1}{\sqrt{3}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$\angle A = \angle B = \frac{\pi}{3}, \qquad \angle C = \frac{\pi}{3}$

Area $\Delta ABC = \frac{C}{2} \times \frac{C}{\sin C} \sin B \sin A$

$= \frac{C^2}{2} \times \frac{\sin^2 A}{\sin C} = \frac{C^2}{2} \times \frac{\sin^2 A}{\sin(180^\circ - 2A)} = \frac{C^2}{4}\tan A$

$= \frac{(6)^2(3)^{1/2}}{4}\sqrt{3} = 27$

Sol 25: (i) $f(x) = \cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right), x \in \left(\frac{1}{2}, 1\right)$

$f(x) = \cos^{-1}x + \cos^{-1}\left[\frac{1}{2}(x) + \frac{\sqrt{3}}{2}\sqrt{(1-x^2)}\right]$

$= \cos^{-1}x + \left|\cos^{-1}x - \cos^{-1}\frac{1}{2}\right|$

$x \in \left(\frac{1}{2}, 1\right) \cos^{-1}x < \cos^{-1}\frac{1}{2}$

$= \cos^{-1}x + \cos^{-1}\frac{1}{2} - \cos^{-1}x = \cos^{-1}\frac{1}{2} = \frac{\pi}{6}$

(ii) $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put x = tan y

$f(x) = \tan^{-1}\frac{1-\cos y}{\sin y} = \tan^{-1}\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}$

$= \tan\frac{y}{2} = \frac{1}{2}\tan^{-1}x$

Sol 26 : $f(x) = \cot^{-1}(x^2 + 4x + \alpha^2 - \alpha)$

$\Rightarrow$ f(x) is onto function so

$\Rightarrow x^2 + 4x + \alpha^2 - \alpha \geq 0$

$\Rightarrow (x + 2)^2 + \alpha^2 - \alpha - 4 \geq 0$

$\Rightarrow (\alpha^2 - \alpha - 4)$ should be zero

$\Rightarrow \alpha^2 - \alpha - 4 = 0$

$\Rightarrow \alpha = \frac{1\pm\sqrt{17}}{2}$

Sol 27 : LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$

$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{2}{36}}\right) = \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{y}{2}\right)$

$= \frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right) = \frac{1}{2}\tan^{-1}\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{4}}$

$= \frac{1}{2}\tan^{-1}\frac{4}{3} = \frac{1}{2}\cos^{-1}\frac{3}{5}$

$= \frac{1}{2}\sin^{-1}\frac{4}{5}$ = RHS

Sol 28 : LHS =

$$\cot^{-1}\frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$= \cot^{-1}\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}$

$= \cot^{-1}\cot\frac{x}{2} = \frac{x}{2}$

Sol 29 : (i) $f(x) = \cos^{-1}\frac{2x}{1+x}$

We know that $-1 \leq \frac{2x}{1+x} \leq 1$

$\frac{2x}{1+x} + 1 \geq 0$ and $\frac{2x}{1+x} - 1 \leq 0$

$\frac{3x+1}{x+1} \geq 0$ and $\frac{x-1}{x+1} \leq 0$

$x \in (-\infty,-1) \cup \left(\frac{-1}{3},\infty\right]$ and $x \in (-1, 1]$

So $x \in \left[\frac{-1}{3},1\right]$

(ii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\frac{1+x^2}{2x}$

$\cos(1) \leq \cos(\sin x) \leq 1$

and $\frac{1+x^2}{2x} = \frac{1}{2}\left(\frac{1}{x}+x\right) \geq 1$

for x > 0

or $= \frac{1}{2}\left(\frac{1}{x}+x\right) \leq 1$ for x < 0

(iii) $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4 - x)$

then x = 1, –1.

$-1 \leq \frac{x-3}{2} \leq 1 \Rightarrow 1 \leq x \leq 5;\ 4 - x > 0$

$\Rightarrow x < 4$ so $x \in [1, 4]$

(iv) $f(x) = \sin^{-1}[x(x + 2)]$

We can write here that $-1 \leq x^2 + 2x \leq 1$

$x \in [-(1 + \sqrt{2}), (\sqrt{2} - 1)]$

(v) $f(x) = \frac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}[1 - \{x\}]$

$1 - \sin x \geq 0 \Rightarrow \sin x \leq 1$

and $1 - 4x^2 \neq 1$. Also $x \neq 0$ and $1 - 4x^2 > 0$

$\Rightarrow \left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) < 0 \Rightarrow x \in \left(-\frac{1}{2},\frac{1}{2}\right)$

So domain $x \in \left(-\frac{1}{2},\frac{1}{2}\right) - \{0\}$

(vi) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right)$

$+ \log_6(2|x| - 3) + \sin^{-1}(\log_2 x)$

$\Rightarrow -1 \le \log_2 x \le 1 \Rightarrow \frac{1}{2} \le x \le 2$

and $2|x| - 3 > 0 \Rightarrow |x| > \frac{3}{2}$

So now $x \in \left(-\infty, \frac{-3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ and $-1 \le \frac{3-2x}{5} \le 1$

$\Rightarrow -8 \le -2x \le 2 \Rightarrow 4 \ge x \ge 1$

Now we have $x \in [1, 4]$ and $3 - x \ge 0$

$\Rightarrow x \le 3$

So domain will be $x \in \left(\frac{3}{2}, 2\right]$

(vii) $f(x) = \log_{10}(1 - \log_7(x^2 - 5x + 13) + \cos^{-1}\left(\frac{3}{2 + \sin\frac{9\pi}{2}x}\right)$

We can write here $-1 \le \frac{3}{2 + \sin\frac{9\pi}{2}x} \le +1$

$\Rightarrow \sin\frac{9\pi}{2}x = +1$

$\Rightarrow \frac{9\pi}{2}x = 2n\pi + \frac{\pi}{2} \Rightarrow x = \frac{4n+1}{9}$

$\Rightarrow x^2 - 5x + 13 = \left(x - \frac{5}{2}\right)^2 + 13 - \frac{25}{4} = \left(x - \frac{5}{2}\right)^2 + \frac{27}{4} < 7$

This gives $x \in (2, 3)$

So the domain would be $x = \frac{21}{9}, \frac{25}{9}$

(viii) $f(x) = e^{\sin^{-1}\left(\frac{x}{2}\right)} + \tan^{-1}\left(\frac{x}{2} - 1\right) + \ln\sqrt{x - [x]}$

Now $x - [x] \ne 0 \Rightarrow x \notin I$

and $-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2$

So the Domain will be $(-2, 2) - \{-1, 0, 1\}$

Exercise 2

Single Correct Choice Type

Sol 1: (C) $x^2 - 4x + 5 > \sin^{-1}(\sin 3) + 2\cos^{-1}(\cos 2) - \pi$

$(x - 2)^2 + 1 > \pi - 3 + 4 - \pi$

$(x - 2)^2 + 1 \ge 1$

Always true except {2}

Sol 2: (C) We have

$S_1 = \Sigma x_1 = \sin 2\beta$

$S_2 = \Sigma x_1x_2 = \cos 2\beta$

$S_3 = \Sigma x_1x_2x_3 = \cos \beta$

$S_4 = x_1x_2x_3x_4 = -\sin \beta$

So that $\sum_{i=1}^{4} \tan^{-1} x_i = \tan{-1} \frac{S_1 - S_3}{1 - S_2 + S_4}$

$= \tan^{-1}\frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \tan^{-1}\frac{\cos\beta\,(2\sin\beta - 1)}{\sin\beta\,(2\sin\beta - 1)}$

$= \tan^{-1}(\cot\beta) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \beta\right)\right)$

$\Rightarrow \frac{\pi}{2} - \beta$

Sol 3: (B) $f(x) = \cot^{-1}\log_{4/5}(5x^2 - 8x + 4)$

$5x^2 - 8x + 4 \ge \frac{4}{5}$

$\log_{4/5}(5x^2 - 8x + 4) \le 1$

$f(x) \in \left[\frac{\pi}{4}, \pi\right)$

Sol 4: (C) $(1 + x)\cos y - x^2 = 0$

$y = \cos^{-1}\frac{x^2}{1+x}$

$\Rightarrow -1 \le \frac{x^2}{1+x} \le 1$

$\frac{x^2}{1+x} \le -1 \Rightarrow \frac{x^2 + x + 1}{1+x} \le 0$

$\frac{x^2 - x - 1}{1+x} \le 0$

$\frac{\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}}{x+1} \le 0$

$x \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

Sol 5: (B) $4(\tan^{-1}x)^2 - (\tan^{-1}x) - 3 \le 0$

$(\tan^{-1}x)^2 - 2\left(\frac{1}{8}\tan^{-1}x\right) + \frac{1}{64} - \frac{1}{64} - \frac{3}{4} \Rightarrow 0$

$$\left[\tan^{-1}x-\frac{1}{8}\right]^2-\frac{49}{64}\le 0$$

$$(\tan^{-1}x-1)\left(\tan^{-1}x+\frac{3}{4}\right)\le 0$$

$$-\frac{3}{4}\le \tan^{-1}x\le 1$$

$$-\tan^{-1}\left(\frac{3}{4}\right)\le x\le\frac{\pi}{4}$$

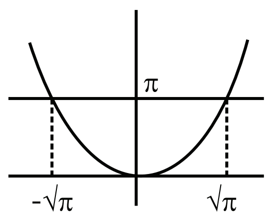
Sol 6: (B) $\sec^{-1}[-\sin^2x]$ is defined only if

$[-\sin^2x] = 1, -1$

$[-\sin^2x] = -1$ when $x \notin n\pi$

$\sec^{-1}(-1) = \pi$

So area bounded



$$=\int_{-\sqrt{\pi}}^{\pi}(\pi-x^2)dx=\left[\pi x-\frac{x^3}{3}\right]_{-\sqrt{\pi}}^{\sqrt{\pi}}$$

$$=\pi\sqrt{\pi}-\frac{\pi\sqrt{\pi}}{3}+\pi\sqrt{\pi}-\frac{\pi\sqrt{\pi}}{3}=\frac{4}{3}\pi\sqrt{\pi}$$

Sol 7: (B) We have from the given equation

$$\tan^{-1}\frac{(a+b)x}{x^2-ab}=\frac{\pi}{2}-\tan^{-1}\frac{(c+d)x}{x^2-cd}$$

$$\Rightarrow\ \tan^{-1}\frac{(a+b)x}{x^2-ab}=\cot^{-1}\frac{(c+d)x}{x^2-cd}$$

$$=\tan^{-1}\frac{x^2-cd}{(c+d)x}$$

$$\Rightarrow\ (x^2-ab)(x^2-cd)=(a+b)(c+d)x^2$$

$$\Rightarrow\ x^4-x^2\sum ab+abcd=0$$

Sol 8: (C) $\sin^{-1}\sqrt{1-x^2}+\cos^{-1}x=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)-\sin^{-1}x$

$$\sin^{-1}\sqrt{1-x^2}+\frac{\pi}{2}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$x\in[-1, 1]-\{0\}$

$$\cos\left(\frac{\pi}{2}+\sin^{-1}\sqrt{1-x^2}\right)=\cos\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$-\sin\sin^{-1}\sqrt{1-x^2}=\cos\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$-\sqrt{1-x^2}=\cos\cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

If $x > 0$ then it won't satisfy except 1.

If $x < 0$ then it will satisfy.

$x\in[-1, 0)\cup\{1\}$

Sol 9: (C) $f(x)=\operatorname{cosec}^{-1}\sqrt{\log_{\frac{3-4\sec x}{1-2\sec x}}2}$

$$2\ge\frac{3-4\sec x}{1-2\sec x}>1$$

$$\Rightarrow x\in\left(2n\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}\right)-\{2n\pi\}$$

Range $\in\left(0,\frac{\pi}{2}\right)$

Sol 10: (A) $[\sin^{-1}x] = [\cos^{-1}x]$

$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$$

$1\sim\frac{\pi}{3}$; $\frac{\pi}{2}\sim 1.6$

$[\sin^{-1}x] = -2, -1, 0, 1$

$0\le\cos^{-1}x\Rightarrow\pi\Rightarrow[\cos^{-1}x] = 0,1,2$

so $[\sin^{-1}x] = [\cos^{-1}x] = 0$ or 1

$x\in[\cos 1, \sin 1]$

Multiple Correct Choice Type

Sol 11: (A, B) (A) $\cos^{-1}\left(\ln\frac{2e+4}{3}\right)$

$\frac{2e+4}{3}\sim 3 > e\Rightarrow\ln\left(\frac{2e+4}{3}\right)>1$

meaning less because

$\cos^{-1}\left(\ln\frac{2e+4}{3}\right)$ is not defined.

(B) In $\operatorname{cosec}^{-1}\left(\frac{\pi}{4}\right)$, $\frac{\pi}{4}<1$

$\operatorname{cosec}^{-1}\left(\frac{\pi}{4}\right)$ not defined

(C) $\cot^{-1}\left(\frac{\pi}{2}\right)$ defined

(D) $\sec^{-1}(\pi)$ defined

Sol 12: (A, B, C) Let $\cos^{-1}\left(\frac{4}{5}\right) = \alpha$, that is, $\cos\alpha = \frac{4}{5}$,

so that $\tan\alpha = \sqrt{\left(\frac{5}{4}\right)^2 - 1} = \frac{3}{4}$ ($\because 0 < \alpha < \pi$ and $\cos\alpha > 0$)

And $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \dfrac{\tan\alpha + \frac{2}{3}}{1 - \tan\alpha\cdot\frac{2}{3}}$

$= \dfrac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{2}{3}\cdot\frac{3}{4}} = \frac{17}{6} = \frac{a}{b}$ (given)

So, a = 17, b = 6, a +b = 23, a - b = 11 and 3b = a + 1

Sol 13: (B, C) (A) $y^2 = \sqrt{1 - x^2}$

$\Rightarrow y^4 + x^2 = 1$

Not circle

(B) $y = \sin(\cos^{-1}(1 - x))$

$y = \sqrt{1 - (1 - x)^2}$

Half circle for y > 0

(C) $y^2 = (\sin\cos^{-1}x)^2$

$y^2 = (1 - x^2) \Rightarrow y^2 + x^2 = 1$

Which is a circle

(D) $y = \sin^{-1}\cos^2 y$

Not a circle

Sol 14: (A, B) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}(xy - \sqrt{1 - x^2}\sqrt{1 - y^2}) = \pi - \cos^{-1}z$

Taking cosine of both sides

$xy - \sqrt{1 - x^2}\sqrt{1 - y^2} = -z$

$(xy + z)^2 = (1 - x^2)(1 - y^2)$

$x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$x^2 + y^2 + z^2 = 1 - 2xyz$

(B) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{\pi}{2}$

L. H. S.

$= \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y + \frac{\pi}{2} - \cos^{-1}z$

$= \frac{3\pi}{2} - (\cos^{-1}x + \cos^{-1}y + \cos^{-1}z) = \frac{\pi}{2}$

Match the Columns

Sol 15: $A \to s; B \to p; C \to r; D \to q$

(A) $f(x) = \sin^{-1}\left(\frac{x}{1 + |x|}\right)$

Range $f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(B) $g(x) = \cos^{-1}.\ \frac{x}{1 + |x|}$

$\frac{x}{1 + |x|}$ if $x \geq 0 \Rightarrow 0 \leq \frac{x}{1 + x} < 1$

$x \leq 0 \Rightarrow 0 \geq \frac{x}{1 - x} > -1$

the Range $f(x) \in (0, x)$

(C) $h(x) = \tan^{-1}\frac{x}{1 + |x|}$

Range $f(x) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(D) $k(x) = \cot^{-1}\left(\frac{x}{1 + |x|}\right)$

Range $f(x) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

Previous Years' Questions

Sol 1: $A \to p; B \to q; C \to p; D \to s$

(A) If a = 1, b = 0, then $\sin^{-1}x + \cos^{-1}y = 0$

$\Rightarrow \sin^{-1}x = -\cos^{-1}y$

$\Rightarrow x^2 + y^2 = 1$

(B) If a = 1 and b = 1, then

$\sin^{-1}x + \cos^{-1}y + \cos^{-1}xy = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x - \cos^{-1}y = \cos^{-1}xy$

$\Rightarrow xy + \sqrt{1 - x^2}\sqrt{1 - y^2} = xy$

$\Rightarrow (x^2 - 1)(y^2 - 1) = 0$

(C) If a = 1, b = 2, then

$\sin^{-1}x + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x - \cos^{-1}y = \cos^{-1}(2xy)$

$\Rightarrow xy + \sqrt{1-x^2}\,\sqrt{1-y^2} = 2xy$

$\Rightarrow x^2 + y^2 = 1$

(D) If a = 2, b = 2, then

$\sin^{-1}(2x) + \cos^{-1}(y) + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}(2x) - \cos^{-1}(y) = \cos^{-1}(2xy)$

$\Rightarrow 2xy + \sqrt{1-4x^2}\,\sqrt{1-y^2} = 2xy$

$\Rightarrow (4x^2 - 1)\,(y^2 - 1) = 0$

Sol 2: Given than, $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2} = 1 \Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow (x + 1)\,(6x - 1) = 0 \Rightarrow x = -1$ or $\frac{1}{6}$

But x = –1 does not satisfy the given equation.

$\therefore$ We take $x = \frac{1}{6}$

Sol 3: Let $f(x) \cos(2\cos^{-1}x + \sin^{-1}x)$

$= \cos\left(\cos^{-1}x + \frac{\pi}{2}\right)\left[\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right]$

$= -\sin(\cos^{-1}x)$

$\Rightarrow f(x) = -\sin\left(\sin^{-1}\sqrt{1-x^2}\right)$

$\Rightarrow f\left(\frac{1}{5}\right) = -\sin\left(\sin^{-1}\sqrt{1-\frac{1}{5^2}}\right)$

$= -\sin\left(\sin^{-1}\frac{2\sqrt{6}}{5}\right) = -\frac{2\sqrt{6}}{5}$

Sol 4: LHS = $\cos\tan^{-1}[\sin(\cot^{-1}x)]$

$= \cos\tan^{-1}\left[\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right]$

$= \cos\left(\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \sqrt{\frac{x^2+1}{x^2+2}}$ = RHS

Sol 5: $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \frac{2}{2R}$ (a cos C + c cos A)

$= \frac{b}{R} = 2\sin B = 2\sin 60^{\circ} = \sqrt{3}$

Sol 6: (B, C, D) $\frac{\pi}{2} < \alpha < \pi,\ \pi < \beta < \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} < \alpha + \beta < \frac{5\pi}{2}$

$\Rightarrow \sin\beta > 0;\ \cos\alpha < 0$

$\Rightarrow \cos(\alpha + \beta) > 0$