Motion Of Charged Particles In Electric & Magnetic Fields (Part - 1)

Q. 372. At the moment t = 0 an electron leaves one plate of a parallel-plate capacitor with a negligible velocity. An accelerating voltage, varying as V = at, where a = 100 V/s, is applied between the plates. The separation between the plates is l = 5.0 cm. What is the velocity of the electron at the moment it reaches the opposite plate?

Solution. 372. Let the electron leave the negative plate of the capacitor at time t = 0

As , $E_x = -\frac{d\varphi}{dx}$, $E = \frac{\varphi}{l} = \frac{at}{l}$,

And, therefore, the acceleration of the electron,

$$w = \frac{eE}{m} = \frac{eat}{ml} \quad \text{or,} \quad \frac{dv}{dt} = \frac{eat}{ml}$$

Or, $\int_0^v dv = \frac{ea}{ml}\int_0^t t\,dt, \quad \text{or,} \quad v = \frac{1}{2}\frac{ea}{ml}t^2$ (1)

But, from $s = \int v\,dt$,

$$l = \frac{1}{2}\frac{ea}{ml}\int_0^t t^2\,dt = \frac{eat^3}{6\,ml} \quad \text{or,} \quad t = \left(\frac{6\,ml^2}{ea}\right)^{\frac{1}{3}}$$

Putting the value of t in (1),

$$v = \frac{1}{2}\frac{ea}{ml}\left(\frac{6\,ml^2}{ea}\right)^{\frac{2}{3}} = \left(\frac{9}{2}\frac{ale}{m}\right)^{\frac{1}{3}} = 16 \text{ km/s.}$$

Q. 373. A proton accelerated by a potential difference V gets into the uniform electric field of a parallel-plate capacitor whose plates extend over a length 1 in the motion direction. The field strength varies with time as E = at, where a is a constant. Assuming the proton to be non-relativistic, find the angle between the motion directions of the proton before and after its flight through the capacitor; the proton gets in the field at the moment t = 0. The edge effects are to be neglected.

Solution. 373. The electric field inside the capacitor varies with time as,
E = at.

Hence, electric force on the proton,
F = eat

And subsequently, acceleration of the proton,

$$w = \frac{eat}{m}$$

Now, if t is the time elapsed during the motion of the proton between the plates,

then $t = \frac{l}{v_{\parallel}}$, as no acceleration is effective in this direction. (Here $v_{\parallel}$ is velocity along the length of the plate.)

From kinematics, $\frac{dv_{\perp}}{dt} = w$

So, $\int_0^{v_{\perp}} dv_{\perp} = \int_0^{t} w\, dt,$

(as initially, the component of velocity in the direction, ⊥ to plates, was zero.)

or $,v_{\perp} = \int_0^{t} \frac{ea}{m} \frac{t^2}{2m} = \frac{ea}{2m} \frac{l^2}{v_{\parallel}^2}$

Now, $\tan \alpha = \frac{v_{\perp}}{v_{\parallel}} = \frac{e\,al^2}{2\,m\,v_{\parallel}^3}$

$$= \frac{eal^2}{2m\left(\frac{2\,eV}{m}\right)^{\frac{3}{2}}}, \text{ as } v_{\parallel} = \left(\frac{2\,eV}{m}\right)^{\frac{1}{2}},$$ From energy conservation.

$$= \frac{al^2}{4}\sqrt{\frac{m}{2\,eV^3}}$$

Q. 374. A particle with specific charge qlm moves rectilinearly due to an electric

field E = E_0 — ax, where a is a positive constant, x is the distance from the point where the particle was initially at rest. Find:
(a) the distance covered by the particle till the moment it came to a standstill;
(b) the acceleration of the particle at that moment.

Solution. 374. The equation of motion is,

$$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{q}{m}(E_0 - ax)$$

Integrating

$$\frac{1}{2}v^2 - \frac{q}{m}\left(E_0 x - \frac{1}{2}ax^2\right) = \text{constant.}$$

But initially v = 0 when x = 0, so "constant" = 0

Thus, $v^2 = \frac{2q}{m}\left(E_0 x - \frac{1}{2}ax^2\right)$

Thus. v = 0, again for $x = x_m = \frac{2E_0}{a}$

The corresponding acceleration is,

$$\left(\frac{dv}{dt}\right)_{x_m} = \frac{q}{m}(E_0 - 2E_0) = -\frac{qE_0}{m}$$

Q. 375. An electron starts moving in a uniform electric field of strength E = 10 kV/cm. How soon after the start will the kinetic energy of the electron become equal to its rest energy?

Solution. 375. from the law of relativistic conservation of energy

$$\frac{m_0 c^2}{\sqrt{1-(v^2/c^2)}} - eEx = m_0 c^2.$$

As the electron is at rest (v = 0 for x = 0) initially.

Thus clearly T = eEx.

On the other hand, $\sqrt{1-(v^2/c^2)} = \frac{m_0 c^2}{m_0 c^2 + eEx}$

Or, $\frac{v}{c} = \frac{\sqrt{(m_0 c^2 + e Ex)^2 - m_0^2 c^4}}{m_0 c^2 + eEx}$

Or, $ct = \int c\, dt = \int \frac{(m_0 c^2 + eEx)\, dx}{\sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4}}$

$$= \frac{1}{2\, eE} \int \frac{dy}{\sqrt{y - m_0^2 c^4}} = \frac{1}{eE} \sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4} + \text{constant}$$

The "constant" = 0, at t = 0, for x = 0,

So, $ct = \frac{1}{eE} \sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4}.$

Finally, using T = eEx,

$$c\, e\, E\ t_0 = \sqrt{T\,(T + 2\, m_0\, c^2)} \quad \text{or,} \quad t_0 = \frac{\sqrt{T\,(T + 2\, m_0\, c^2)}}{e\, Ec}$$

Q. 376. Determine the acceleration of a relativistic electron moving along a uniform electric field of strength E at the moment when its kinetic energy becomes equal to T.

Solution. 376. As before, T = Ex

Now in linear motion,

$$\frac{d}{dt} \frac{m_0\, v}{\sqrt{1 - v^2/c^2}} = \frac{m_0\, w}{\sqrt{1 - v^2/c^2}} + \frac{m_0\, w}{(1 - v^2/c^2)^{3/2}} \frac{v}{c^2} w$$

$$= \frac{m_0}{(1 - v^2/c^2)^{3/2}} w = \frac{(T + m_0\, c^2)^3}{m_0^2\, c^6} w = e\, E,$$

So, $w = \frac{e\, E\, m_0^2\, c^6}{(T + m_0\, c^2)^3} = \frac{eE}{m_0} \left(1 + \frac{T}{m_0\, c^2}\right)^{-3}$

Q. 377. At the moment t = 0 a relativistic proton flies with a velocity v, into the region where there is a uniform transverse electric field of strength E, with $v_0 \perp$ E. Find the time dependence of

(a) the angle θ between the proton's velocity vector v and the initial direction of its

motion;
(b) the projection v_x of the vector v on the initial direction of motion.

Solution. 377. The equations are,

$$\frac{d}{dt}\left(\frac{m_0 v_x}{\sqrt{1-(v^2/c^2)}}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{m_0 v_y}{\sqrt{1-v^2/c^2}}\right) = eE$$

Hence, $\dfrac{v_x}{\sqrt{1-v^2/c^2}} = \text{constant} = \dfrac{v_0}{\sqrt{1-(v_0^2/c^2)}}$

Also, by energy conservation,

$$\frac{m_0 c^2}{\sqrt{1-(v^2/c^2)}} = \frac{m_0 c^2}{\sqrt{1-(v_0^2/c^2)}} + eEy$$

Dividing $v_x = \dfrac{v_0 \varepsilon_0}{\varepsilon_0 + eEy}, \ \varepsilon_0 = \dfrac{m_0 c^2}{\sqrt{1-(v_0^2/c^2)}}$

Also, $\dfrac{m_0}{\sqrt{1-(v^2/c^2)}} = \dfrac{\varepsilon_0 + eEy}{c^2}$

Thus, $(\varepsilon_0 + eEy)\, v_y = c^2 eEt + \text{constant.}$

"constant" $= 0$ as $v_y = 0$ at $t = 0$.

Integrating again,

$$\varepsilon_0 y + \frac{1}{2} eEy^2 = \frac{1}{2} c^2 E t^2 + \text{constant.}$$

"constant" $= 0$, as $y = 0$, at $t = 0$.
Thus, $(ceEt)^2 = (eyE)^2 + 2\varepsilon_0 eEy + \varepsilon_0^2 - \varepsilon_0^2$

Or, $ceEt = \sqrt{(\varepsilon_0 + eEy)^2 - \varepsilon_0^2}$

Or, $\varepsilon_0 + eEy = \sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}$

Hence, $v_x = \frac{v_0 \varepsilon_0}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}}$ also, $v_y = \frac{c^2 e E t}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}}$

And $\tan\theta = \frac{v_y}{v_x} = \frac{eEt}{m_0 v_0}\sqrt{1 - (v_0^2/c^2)}$.

Q. 378. A proton accelerated by a potential difference V = 500 kV flies through a uniform transverse magnetic field with induction B = 0.51 T. The field occupies a region of space d =10 cm in thickness (Fig. 3.99). Find the angle α through which the proton deviates from the initial direction of its motion.

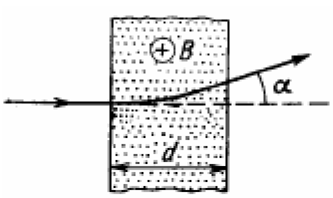


Fig. 3.99.

Solution. 378. from the figure,

$$\sin\alpha = \frac{d}{R} = \frac{d\,q\,B}{m\,v},$$

As radius of the arc $R = \frac{m\,v}{q\,B}$, where v is the velocity of the particle, when it enters into the field. From initial condition of the problem,

$$qV = \frac{1}{2}mv^2 \quad \text{or,} \quad v = \sqrt{\frac{2qV}{m}}$$

Hence, $$\sin\alpha = \frac{dqB}{m\sqrt{\frac{2qV}{m}}} = dB\sqrt{\frac{q}{2\,m\,V}}$$

$$\text{and} \quad \alpha = \sin^{-1}\left(dB\sqrt{\frac{q}{2\,mV}}\right) = 30°,$$ on putting the values.

Q. 379. A charged particle moves along a circle of radius r = 100 mm in a uniform magnetic field with induction B = 10.0 mT. Find its velocity and period of revolution if that particle is

(a) a non-relativistic proton;
(b) a relativistic electron.

Solution. 379. (a) For motion along a circle, the magnetic force acted on the particle, will provide the centripetal force, necessary for its circular motion.

i.e. $\frac{mv^2}{R} = evB$ or, $v = \frac{eBR}{m}$

And the period of revolution $T = \frac{2\pi}{\omega} = \frac{2\pi R}{v} = \frac{2\pi m}{eB}$

(b) Generally, $\frac{d\vec{p}}{dt} = \vec{F}$

But, $\frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m_0 \vec{v}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0 \dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} + \frac{m_0}{\left(1-(v^2/c^2)\right)^{3/2}} \frac{\vec{v}(\vec{v}\cdot\dot{\vec{v}})}{c^2}$

For transverse motion, $\vec{v}\cdot\dot{\vec{v}} = 0$ so,

$\frac{d\vec{p}}{dt} = \frac{m_0 \dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0}{\sqrt{1-(v^2/c^2)}} \frac{v^2}{r}$, here.

Thus, $\frac{m_0 v^2}{r\sqrt{1-(v^2/c^2)}} = B\,ev$ or, $\frac{v/c}{\sqrt{1-(v^2/c^2)}} = \frac{Ber}{m_0 c}$

Or, $\frac{v}{c} = \frac{Ber}{\sqrt{B^2 e^2 r^2 + m_0^2 c^2}}$

Finally, $T = \frac{2\pi r}{v} = \frac{2\pi m_0}{eB\sqrt{1-v^2/c^2}} = \frac{2\pi}{cBe}\sqrt{B^2 e^2 r^2 + m_0^2 c^2}$

Q. 380. A relativistic particle with charge q and rest mass m_0 moves along a circle of radius r in a uniform magnetic field of induction B. Find:
(a) the modulus of the particle's momentum vector;
(b) the kinetic energy of the particle;
(c) the acceleration of the particle.

Solution. 380. (a) As before, p = B qr.

(b) $T = \sqrt{c^2 p^2 + m_0^2 c^4} = \sqrt{c^2 B^2 q^2 r^2 + m_0^2 c^4}$

(c) $w = \frac{v^2}{r} = \frac{c^2}{r\left[1 + (m_0 c / Bqr)^2\right]}$

Using the result for v from the previous problem.

Q. 381. Up to what values of kinetic energy does the period of revolution of an electron and a proton in a uniform magnetic field exceed that at non-relativistic velocities by η = 1.0 % ?

Solution. 381. from (Q.279),

$$T = \frac{2\pi\varepsilon}{c^2 eB} \text{ (relativistic)}, \quad T_0 = \frac{2\pi m_0 c^2}{c^2 eB} \text{ (nonrelativistic)},$$

Here, $m_0 c^2 / \sqrt{1 - v^2/c^2} = E$

Here, $\delta T = \frac{2\pi T}{c^2 eB}, \ (T = K.E.)$

Now, $\frac{\delta T}{T_0} = \eta = \frac{T}{m_0 c^2}, \quad \text{so,} \quad T = \eta m_0 c^2$

Q. 382. An electron accelerated by a potential difference V = 1.0 kV moves in a uniform magnetic field at an angle α = 30° to the vector B whose modulus is B = 29 mT. Find the pitch of the helical trajectory of the electron.

Solution. 382.

$T = eV = \frac{1}{2} m v^2$

(The given potential difference is not large enough to cause significant deviations from the nonrelativistic formula).

Thus, $v = \sqrt{\frac{2eV}{m}}$

So, $v_{\parallel} = \sqrt{\frac{2eV}{m}} \cos\alpha, \ v_{\perp} = \sqrt{\frac{2eV}{m}} \sin\alpha$

Now, $\frac{m v_\perp^2}{r} = Bev_\perp$ or, $r = \frac{mv_\perp}{Be}$,

And $T = \frac{2\pi r}{v_\perp} = \frac{2\pi m}{Be}$

Pitch $p = v_\parallel T = \frac{2\pi m}{Be}\sqrt{\frac{2eV}{m}}\cos\alpha = 2\pi\sqrt{\frac{2mV}{eB^2}}\cos\alpha$

Q. 383. A slightly divergent beam of non-relativistic charged particles accelerated by a potential difference V propagates from a point A along the axis of a straight solenoid. The beam is brought into focus at a distance l from the point A at two successive values of magnetic induction B_1 and B_2. Find the specific charge q/m of the particles.

Solution. 383. The charged particles will traverse a helical trajectory and will be focussed on the axis after traversing a number of turms. Thus

$$\frac{l}{v_0} = n\cdot\frac{2\pi m}{qB_1} = (n+1)\frac{2\pi m}{qB_2}$$

So, $\frac{n}{B_1} = \frac{n+1}{B_2} = \frac{1}{B_2 - B_1}$

Hence, $\frac{l}{v_0} = \frac{2\pi m}{q(B_2 - B_1)}$

or, $\frac{l^2}{2qV/m} = \frac{(2\pi)^2}{(B_2 - B_1)^2}\times\frac{1}{(q/m)^2}$

or, $\frac{q}{m} = \frac{8\pi^2 V}{l^2(B_2 - B_1)^2}$

Q. 384. A non-relativistic electron originates at a point A lying on the axis of a straight solenoid and moves with velocity v at an angle α to the axis. The magnetic induction of the field is equal to B. Find the distance r from the axis to the point on the screen into which the electron strikes. The screen is oriented at right angles to the axis and is located at a distance l from the point A.

Solution. 384. Let us take the point A as the origin O and the axis of the solenoid as z-axis. At an arbitrary moment of time let us resolve the velocity of electron into its two rectangular components, $\vec{v_\parallel}$ along the axis and $\vec{v_\perp}$ to the axis of solenoid. We know the magnetic force does no work, so the kinetic energy as well as the speed of the electron $|\vec{v_\perp}|$ will remain constant in the x-y plane. Thus $\vec{v_\perp}$ can change only its direction as shown in the Fig... $\vec{v_\parallel}$ will Remain constant as it is parallel to $\vec{B}$.

Thus at t = t

$$v_x = v_\perp \cos \omega t = v \sin \alpha \cos \omega t,$$
$$v_y = v_\perp \sin \omega t = v \sin \alpha \sin \omega t$$

And $v_z = v \cos \alpha, \quad \text{where } \omega = \frac{eB}{m}$

As at r = 0, we have x = y = z = 0, so the motion law of the electron is.

$$\left.\begin{aligned} z &= v \cos \alpha\, t \\ x &= \frac{v \sin \alpha}{\omega} \sin \omega t \\ y &= \frac{v \sin \alpha}{\omega} (\cos \omega t - 1) \end{aligned}\right]$$

(The equation of the helix)

On the screen, $z = l, \text{ so } t = \frac{l}{v \cos \alpha}.$

Then, $r^2 = x^2 + y^2 = \frac{2 v^2 \sin^2 \alpha}{\omega^2}\left(1 - \cos \frac{\omega l}{v \cos \alpha}\right)$

$$r = \frac{2 v \sin \alpha}{\omega} \left| \sin \frac{\omega l}{2 v \cos \alpha} \right| = 2 \frac{mv}{eB} \sin \alpha \left| \sin \frac{leB}{2 mv \cos \alpha} \right|$$

Q. 385. From the surface of a round wire of radius a carrying a direct current I an electron escapes with a velocity vo perpendicular to the surface. Find what will be the maximum distance of the electron from the axis of the wire before it turns back due to the action of the magnetic field generated by the current.

Solution. 385. Choose the wire along the z-axis, and the initial direction of the electron, along the x-axis. Then the magnetic field in the x - z plane is along the y - axis and outside the wire it is,

$$B = B_y = \frac{\mu_0 I}{2\pi x}, \ (B_x = B_z = 0, \text{ if } y = 0)$$

The motion must be confined to the x - z plane. Then the equations of motion are,

$$\frac{d}{dt} m v_x = -e v_z B_y$$

$$\frac{d(m v_z)}{dt} = +e v_x B_y$$

Multiplying the first equation by v_x and the second by vz and then adding,

$$v_x \frac{dv_x}{dt} + v_z \frac{dv_z}{dt} = 0$$

or, $v_x^2 + v_z^2 = v_0^2$, say, or, $v_z = \sqrt{v_0^2 - v_x^2}$

Then, $v_x \frac{dv_x}{dx} = -\frac{e}{m}\sqrt{v_0^2 - v_x^2}\,\frac{\mu_0 I}{2\pi x}$

or, $-\frac{v_x\, dv_x}{\sqrt{v_0^2 - v_x^2}} = \frac{\mu_0 I e}{2\pi m}\frac{dx}{x}$

Integrating, $\sqrt{v_0^2 - v_x^2} = \frac{\mu_0 I e}{2\pi m}\ln\frac{x}{a}$

on using, $v_x = v_0$, if x = a (i.e. initially).

Now, $v_x = 0$, when $x = x_m$,

So, $x_m = a\, e^{v_0/b}$, where $b = \frac{\mu_0 I e}{2\pi m}$.

Q. 386. A non-relativistic charged particle flies through the electric field of a cylindrical capacitor and gets into a uniform transverse magnetic field with induction B (Fig. 3.100). In the capacitor the particle moves along the arc of a circle, in the magnetic field, along a semi-circle of radius r. The potential

difference applied to the capacitor is equal to V, the radii of the electrodes are equal to a and b, with a < b. Find the velocity of the particle and its specific charge q/m.

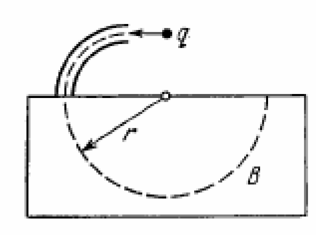


Fig. 3.100.

Solution. 386. Inside the capacitor, the electric field follows $a\,\frac{1}{r}$ law, and so the potential can be written as

$$\varphi = \frac{V \ln r/a}{\ln b/a},\quad E = \frac{-V}{\ln b/a}\frac{1}{r}.$$

Here r is the distance from the axis of the capacitor.

Also, $$\frac{mv^2}{r} = \frac{qV}{\ln b/a\, r}\frac{1}{r} \quad \text{or} \quad mv^2 = \frac{qV}{\ln b/a}$$

On the other hand,

mv = q B r in the magnetic field.

Thus, $$v = \frac{V}{B\, r \ln b/a} \quad \text{and} \quad \frac{q}{m} = \frac{v}{Br} = \frac{V}{B^2 r^2 \ln (b/a)}$$

Q. 387. Uniform electric and magnetic fields with strength E and induction B respectively are directed along the y axis (Fig. 3.101). A particle with specific charge q/m leaves the origin O in the direction of the x axis with an initial non-relativistic velocity v_0. Find:
(a) the coordinate y_n of the particle when it crosses the y axis for the nth time;
(b) the angle α between the particle's velocity vector and the y axis at that moment.

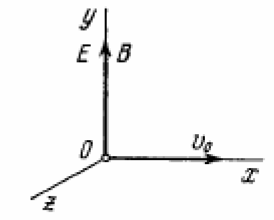


Fig. 3.101.

Solution. 387. The equations of motion are,

$$m\frac{dv_x}{dt} = -qBv_z\,,\ m\frac{dv_y}{dt} = qE \text{ and } m\frac{dv_z}{dt} = q\,v_x B$$

These equations can be solved easily.

First, $v_y = \frac{qE}{m}t,\ y = \frac{qE}{2m}t^{\prime 2}$

Then, $v_x^2 + v_z^2 = \text{constant} = v_0^2$ as before.

In fact, $v_x = v_0 \cos \omega t$ and v_z - v_0 sin cor as one can check.

Integrating again and using x = z = 0, at t = 0

$$x = \frac{v_0}{\omega}\sin \omega t,\ z = \frac{v_0}{\omega}(1 - \cos \omega t)$$

Thus, $x = z = 0 \text{ for } t = t_n = n\frac{2\pi}{\omega}$

At that instant, $y_n = \frac{qE}{2m} \times \frac{2\pi}{qB/m} \times n^2 \times \frac{2\pi}{qB/m} = \frac{2\pi^2 m E n^2}{qB^2}$

Also, $\tan \alpha_n = \frac{v_x}{v_y}$, ($v_z = 0$ at this moment)

$$= \frac{mv_0}{qEt_n} = \frac{mv_0}{qE} \times \frac{qB}{m} \times \frac{1}{2\pi n} = \frac{Bv_0}{2\pi E n}.$$

Q. 388. A narrow beam of identical ions with specific charge q/m, possessing different velocities, enters the region of space, where there are uniform parallel electric and magnetic fields with strength E and induction B, at the point O (see Fig. 3.101). The beam direction coincides with the x axis at the point O. A plane

screen oriented at right angles to the x axis is located at a distance l from the point O. Find the equation of the trace that the ions leave on the screen. Demonstrate that at z ≪ l it is the equation of a parabola.

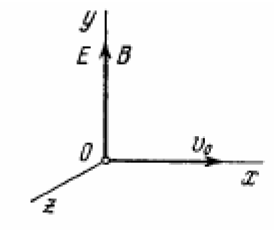


Fig. 3.101.

Solution. 388. The equation of the trajectory is,

$$x = \frac{v_0}{\omega}\sin\omega t,\ z = \frac{v_0}{\omega}(1-\cos\omega t),\ y = \frac{qE}{2m}t^2$$ as before see (Q.384).

Now on the screen x = l, so

$$\sin\omega t = \frac{\omega l}{v_0} \quad \text{or,} \quad \omega t = \sin^{-1}\frac{\omega l}{v_0}$$

At that moment,

$$y = \frac{qE}{2m\omega^2}\left(\sin^{-1}\frac{\omega l}{v_0}\right)^2$$

so, $$\frac{\omega l}{v_0} = \sin\sqrt{\frac{2m\omega^2 y}{qE}} = \sin\sqrt{\frac{2qB^2y}{Em}}$$

and $$z = \frac{v_0}{\omega}2\sin^2\frac{\omega t}{2} = l\tan\frac{\omega t}{2}$$

$$= l\tan\frac{1}{2}\left[\sin^{-1}\frac{\omega l}{v_0}\right] = l\tan\sqrt{\frac{qB^2y}{2mE}}$$

For small

z, $$\frac{qB^2y}{2mE} = \left(\tan^{-1}\frac{z}{l}\right)^2 \approx \frac{z^2}{l^2}$$

or, $$y = \frac{2mE}{qB^2l^2}\cdot z^2 \text{ is a parabola.}$$

Motion Of Charged Particles In Electric & Magnetic Fields (Part - 2)

Q. 389. A non-relativistic proton beam passes without deviation through the region of space where there are uniform transverse mutually perpendicular electric and magnetic fields with E = 120 kV/m and B = 50 mT. Then the beam strikes a grounded target. Find the force with which the beam acts on the target if the beam current is equal to I = 0.80 mA.

Solution. 389. In crossed field,

$eE = evB, \text{ so } v = \frac{E}{B}$

Then, F = force exerted on the plate $= \frac{I}{e} \times m \frac{E}{B} = \frac{m I E}{eB}$

Q. 390. Non-relativistic protons move rectilinearly in the region of space where there are uniform mutually perpendicular electric and magnetic fields with E = 4.0 kV/m and B = 50 mT. The trajectory of the protons lies in the plane xz (Fig. 3.102) and forms an angle φ = 30° with the x axis. Find the pitch of the helical trajectory along which the protons will move after the electric field is switched off.

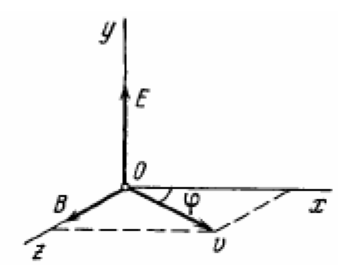


Fig. 3.102.

Solution. 390. When the electric field is switched off, the path followed by the particle

will be helical, and pitch, $\Delta l = v_{\parallel} T$, (where $v_{\parallel}$ is the velocity of the particle, parallel

to $\vec{B}$, and T, the time period of revolution.)

$$= v \cos(90 - \varphi) T = v \sin \varphi \, T$$
$$= v \sin \varphi \frac{2\pi m}{qB} \left(\text{as } T = \frac{2\pi}{qB}\right) \quad (1)$$

Now, when both the fields were present, $qE = qvB \sin(90 - \varphi)$, as no net force was effective or, $v = \frac{E}{B \cos \varphi}$ (2)

From (1) and (2), $\Delta l = \frac{E}{B}\frac{2\pi m}{qB}\tan\varphi = 6$ cm.

Q. 391. A beam of non-relativistic charged particles moves without deviation through the region of space A (Fig. 3.103) where there are transverse mutually perpendicular electric and magnetic fields with strength E and induction B. When the magnetic field is switched off, the trace of the beam on the screen S shifts by Δx. Knowing the distances a and b, find the specific charge q/m of the particles.

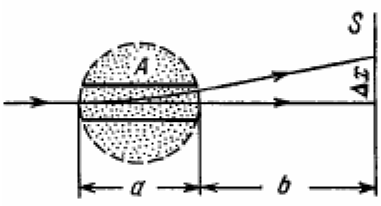


Fig. 3.103.

Solution. 391. When there is no deviation, $-q\vec{E} = q(\vec{v}\times\vec{B})$

or, in scalar from, $E = vB$ (as $\vec{v}\perp\vec{B}$) or, $v = \frac{E}{B}$ (1)

Now, when the magnetic field is switched on, let the deviation in the field be x. Then,

$x = \frac{l}{2}\left(\frac{q\,vB}{m}\right)t^2,$

where t is the time required to pass through this region,

also, $t = \frac{a}{v}$

Thus $x = \frac{1}{2}\left(\frac{qvB}{m}\right)\left(\frac{a}{v}\right)^2 = \frac{1}{2}\frac{q}{m}\frac{a^2B^2}{E}$ (2)

For the region where the field is absent, velocity in upward direction

$= \left(\frac{qvB}{m}\right)t = \frac{q}{m}aB$ (3)

Now, $\Delta x - x = \frac{qaB}{m}t'$

$$= \frac{q}{m}\frac{aB^2 b}{E} \text{ when } t' = \frac{b}{v} = \frac{bB}{E} \qquad (4)$$

From (2) and (4),

$$\Delta x - \frac{1}{2}\frac{q}{m}\frac{a^2 B^2}{E} = \frac{q}{m}\frac{a B^2 b}{E}$$

Or, $$\frac{q}{m} = \frac{2 E \Delta x}{a B^2 (a + 2b)}$$

Q. 392. A particle with specific charge qim moves in the region of space where there are uniform mutually perpendicular electric and magnetic fields with strength E and induc- tion B (Fig. 3.104). At the moment t = 0 the particle was located at the point O and had zero velocity. For the non-relativistic case find:
(a) the law of motion x (t) and y (t) of the particle; the shape of the trajectory;
(b) the length of the segment of the trajectory between two nearest points at which the velocity of the particle turns into zero;
(c) the mean value of the particle's velocity vector projection on the x axis (the drift velocity).

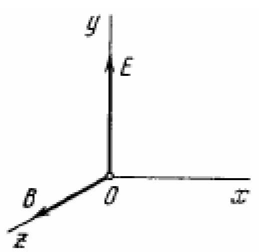


Fig. 3.104.

Solution. 392. (a) The equation of motion is,

$$m\frac{d^2\vec{r}}{dt^2} = q(\vec{E} + \vec{v}\times\vec{B})$$

Now, $$\vec{v}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = \vec{i}B\dot{y} - \vec{j}B\dot{x}$$

So, the equation becomes,

$$\frac{dv_x}{dt} = \frac{qBv_y}{m}, \ \frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{m}v_x, \text{ and } \frac{dv_z}{dt} = 0$$

Here, $v_x = \dot{x}\ ,\ v_y = \dot{y}\ ,\ v_z = \dot{z}.$ The last equation is easy to integrate;

v_z = constant = 0,

since v_z is zero initially. Thus integrating again,

z = constant = 0,

and motion is confined to the x - y plane. We now multiply the second equation by i

and add to the first equation.

$$\xi = v_x + iv_y$$

we get the equation,

$$\frac{d\xi}{dt} = i\omega\frac{E}{B} - i\,\omega\,\xi,\ \omega = \frac{qB}{m}.$$

This equation after being multiplied by $e^{i\omega t}$can be rewritten as,

$$\frac{d}{dt}(\xi\, e^{i\omega t}) = i\,\omega\, e^{i\omega t}\frac{E}{B}$$

and integrated at once to give,

$$\xi = \frac{E}{B} + C\, e^{-i\omega t - i\alpha},$$

where C and α are two real constants. Taking real and imaginary parts.

$$v_x = \frac{E}{B} + C\cos(\omega\, t + \alpha)\ \text{ and }\ v_y = -C\sin(\omega t + \alpha)$$

Since $v_y = 0$, when t = 0, we can take α = 0, then $v_x = 0$ at t = 0 gives, $C = -\frac{E}{B}$ and we get,

$$v_x = \frac{E}{B}(1 - \cos\omega t)\ \text{ and }\ v_y = \frac{E}{B}\sin\omega t.$$

Integrating again and using x = y = 0, at t = 0, we get

$$x(t) = \frac{E}{B}\left(t - \frac{\sin\omega t}{\omega}\right),\ y(t) = \frac{E}{\omega B}(1 - \cos\omega t).$$

This is the equation of a cycloid.

(b) The velocity is zero, when ωt - 2nπ. We see that

$$v^2 = v_x^2 + v_y^2 = \left(\frac{E}{B}\right)^2 (2 - 2\cos\omega t)$$

or, $$v = \frac{ds}{dt} = \frac{2E}{B}\left|\sin\frac{\omega t}{2}\right|$$

The quantity inside the modulus is positive for $0 < \omega t < 2\pi$. Thus we can drop the modulus and write for the distance traversed between two successive zeroes of velocity.

$$S = \frac{4E}{\omega B}\left(1 - \cos\frac{\omega t}{2}\right)$$

Putting $\omega t = 2\pi$, we get

$$S = \frac{8E}{\omega B} = \frac{8\,mE}{q\,B^2}$$

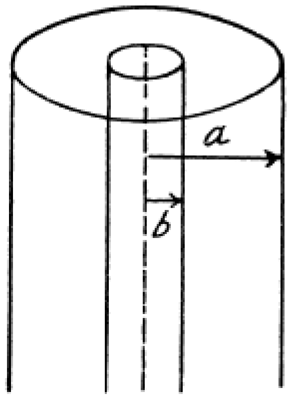
(c) The drift velocity is in the x-direction and has the magnitude,

$$< v_x > = \left< \frac{E}{B}(1 - \cos\omega t) \right> = \frac{E}{B}.$$

Q. 393. A system consists of a long cylindrical anode of radius a and a coaxial cylindrical cathode of radius b (b < a). A filament located along the axis of the system carries a heating current I producing a magnetic field in the surrounding space. Find the least potential difference between the cathode and anode at which the thermal electrons leaving the cathode without initial velocity start reaching the anode.

Solution. 393. When a current I flows along the a x is, a magnetic field $B_\varphi = \frac{\mu_0 I}{2\pi\rho}$ is set up where $\rho^2 = x^2 + y^2.$ In terms of components,

$$B_x = -\frac{\mu_0 I y}{2\pi\rho^2},\ B_y = \frac{\mu_0 I x}{2\pi\rho^2} \text{ and } B_z = 0$$



Suppose a p.d. V is set up between the inner cathode and the outer anode. This means a potential function of the form

$$\varphi = V\,\frac{\ln \rho/b}{\ln a/b}, \quad a > \rho > b,$$

as one can check by solving Laplace equation. The electric field corresponding to this is,

$$E_x = -\frac{Vx}{\rho^2 \ln a/b}, \quad E_y = -\frac{Vy}{\rho^2 \ln a/b}, \quad E_z = 0.$$

The equations of motion are,

$$\frac{d}{dt} m v_x = +\frac{|e|\,Vz}{\rho^2 \ln a/b} + \frac{|e|\,\mu_0 I}{2\pi\rho^2} x\,\dot{z}$$

$$\frac{d}{dt} m v_y = +\frac{|e|\,Vy}{\rho^2 \ln a/b} + \frac{|e|\,\mu_0 I}{2\pi\rho^2} y\,\dot{z}$$

And $$\frac{d}{dt} m v_z = -|e|\frac{\mu_0 I}{2\pi\rho^2}(x\dot{x} + y\dot{y}) = -|e|\frac{\mu_0 I}{2\pi}\frac{d}{dt}\ln \rho$$

$(-|e|)$ is the charge on the electron,

Integrating the last equation,

$$m v_z = -|e|\,\frac{\mu_0 I}{2\pi}\ln \rho/a = m\dot{z}.$$

since $v_z = 0$ where p = a. We now substitute this this $\dot{z}$ in the other two equations to get

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2\right)$$

$$= \left[\frac{|e|V}{\ln a/b}-\frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \rho/b\right]\cdot\frac{x\dot{x}+y\dot{y}}{\rho^2}$$

$$= \left[\frac{|e|V}{\ln\frac{a}{b}}-\frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{\rho}{b}\right]\cdot\frac{1}{2\rho^2}\frac{d}{dt}\rho^2$$

$$= \left[\frac{|e|V}{\ln\frac{a}{b}}-\frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{\rho}{b}\right]\frac{d}{dt}\ln\frac{\rho}{b}$$

Integrating and using $v^2 = 0$, at p = b, we get,

$$\frac{1}{2}mv^2 = -\left[\frac{|e|V}{\ln\frac{a}{b}}\ln\frac{\rho}{b}-\frac{1}{2m}|e|^2\left(\frac{\mu_0 I}{2\pi}\right)^2\left(\ln\frac{\rho}{b}\right)\right]$$

The RHS must be positive, for all a > p > b. The condition for this is,

$$V \geq \frac{1}{2}\frac{|e|}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{a}{b}$$

Q. 394. Magnetron is a device consisting of a filament of radius a and a coaxial cylindrical anode of radius b which are located in a uniform magnetic field parallel to the filament. An accelerating potential difference V is applied between the filament and the anode. Find the value of magnetic induction at which the electrons leaving the filament with zero velocity reach the anode.

Solution. 394. This differs from the previous problem in $(a \leftrightarrow b)$ and the magnetic field is along the z-direction. Thus $B_x = B_y = 0$, $B_z = B$

Assuming as usual the charge of the electron to be - | e |, we write the equation of motion

$$\frac{d}{dt}mv_x = \frac{|e|V_x}{\rho^2\ln\frac{b}{a}}-|e|B\dot{y},\quad \frac{d}{dt}mv_y = \frac{|e|V_y}{\rho^2\ln\frac{b}{a}}+|e|B\dot{x}$$

and $\frac{d}{dt} m v_z = 0 \Rightarrow z = 0$

The motion is confined to the plane z = 0. Eliminating B from the first two equations,

$$\frac{d}{dt}\left(\frac{1}{2} m v^2\right) = \frac{|e| V}{\ln b/a} \frac{x\dot{x} + y\dot{y}}{\rho^2}$$

or, $\frac{1}{2} m v^2 = |e| V \frac{\ln \rho/a}{\ln b/a}$

so, as expected, since magnetic forces do not work,

$$v = \sqrt{\frac{2|e|V}{m}}, \text{ at } \rho = b.$$

On the other hand, eliminating V, we also get,

$$\frac{d}{dt} m (x v_y - y v_x) = |e| B (x\dot{x} + y\dot{y})$$

i.e. $(x v_y - y v_x) = \frac{|e| B}{2m} \rho^2 + \text{constant}$

The constant is easily evaluated, since v is zero at p = a. Thus,

$$(x v_y - y v_x) = \frac{|e| B}{2m} (\rho^2 - a^2) > 0$$

At $\rho = b, \ (x v_y - y v_x) \leq v b$

Thus, $v b \geq \frac{|e| B}{2m} (b^2 - a^2)$

or, $B \leq \frac{2 m b}{b^2 - a^2} \sqrt{\frac{2|e|V}{m}} \times \frac{1}{|e|}$

or, $B \leq \frac{2b}{b^2 - a^2} \sqrt{\frac{2m B}{|e|}}$

Q. 395. A charged particle with specific charge qim starts moving in the region of space where there are uniform mutually perpendicular electric and magnetic fields. The magnetic field is constant and has an induction B while the strength of

the electric field varies with time as E = E_m cos ωt, where ω = qB/m. For the non-relativistic case find the law of motion x (t) and y (t) of the particle if at the moment t = 0 it was located at the point O (see Fig. 3.104). What is the approximate shape of the trajectory of the particle?

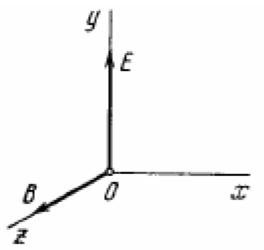


Fig. 3.104.

Solution. 395. The equations are as in Q.392.

with $\frac{dv_x}{dt} = \frac{qB}{m} v_y$, $\frac{dv_y}{dt} = \frac{qE_m}{m} \cos \omega t - \frac{qB}{m} v_x$ and $\frac{dv_z}{dt}$

$\omega = \frac{qB}{m}$, $\xi = v_x + i v_y$, we get,

$$\frac{d\xi}{dt} = i \frac{E_m}{B} \omega \cos \omega t - i \omega \xi$$

or multiplying by $e^{i\omega t}$,

$$\frac{d}{dt}(\xi e^{i\omega t}) = i \frac{E_m}{2B} \omega (e^{2i\omega t} + 1)$$

or integrating, $\xi e^{i\omega t} = \frac{E_m}{4B} e^{2i\omega t} + \frac{E_m}{2B} i \omega t$

or, $\xi = \frac{E_m}{4B} (e^{i\omega t} + 2 i \omega t e^{i\omega t}) + C e^{i\omega t}$

since $\xi = 0$ at $t = 0$, $C = -\frac{E_m}{4B}$.

Thus, $\xi = i \frac{E_m}{2B} \sin \omega t + i \frac{E_m}{2B} \omega t e^{i\omega t}$

or, $v_x = \frac{E_m}{2B} \omega t \sin \omega t$ and $v_y = \frac{E_m}{2B} \sin \omega t + \frac{E_m}{2B} \omega t \cos \omega t$

Integrating again,

$$x = \frac{a}{2\omega^2}(\sin \omega t - \omega t \cos \omega t), \ y = \frac{a}{2\omega} t \sin \omega t.$$

Where $a = \frac{qE_m}{m}$, and we have used x = y = 0, at t = 0.

The trajectory is an unwinding spiral.

Q. 396. The cyclotron's oscillator frequency is equal to v = 10 MHz. Find the effective accelerating voltage applied across the dees of that cyclotron if the distance between the neighbouring trajectories of protons is not less than Δr = 1.0 cm, with the trajectory radius being equal to r = 0.5 m.

Solution. 396. We know that for a charged particle (proton) in a magnetic field,

$$\frac{mv^2}{r} = B\,ev \text{ or } mv = Ber$$

But, $\omega = \frac{eB}{m}$,

Thus $E = \frac{1}{2}mv^2 = \frac{1}{2}m\,\omega^2 r^2.$

So, $\Delta E = m\omega^2 r\,\Delta r = 4\pi^2 \nu^2 mr\,\Delta r$

On the other hand ΔE = 2 eV, where V is the effective acceleration voltage, across the Dees, there being two crossings per revolution. So,

$$V \geq 2\pi^2 \nu^2 mr\,\Delta r/e$$

Q. 397. Protons are accelerated in a cyclotron so that the maximum curvature radius of their trajectory is equal to r = 50 cm. Find:
(a) the kinetic energy of the protons when the acceleration is completed if the magnetic induction in the cyclotron is B = 1.0 T;
(b) the minimum frequency of the cyclotron's oscillator at which the kinetic energy of the protons amounts to T = 20 MeV by the end of acceleration.

Solution. 397. (a) From $\frac{mv^2}{r} =$ Bev, or, mv = Ber

And $T = \frac{(Ber)^2}{2m} = \frac{1}{2} mv^2 = 12 \text{ MeV}$

(b) From $\frac{2\pi}{\omega} = \frac{2\pi r}{v}$

We get, $f_{min} = \frac{v}{2\pi r} = \frac{1}{\pi r}\sqrt{\frac{T}{2m}} = 15 \text{ MHz}$

Q. 398. Singly charged ions He^+ are accelerated in a cyclotron so that their maximum orbital radius is r = 60 cm. The frequency of a cyclotron's oscillator is equal to v = 10.0 MHz, the effective accelerating voltage across the dees is V = 50 kV. Neglecting the gap between the dees, find:
(a) the total time of acceleration of the ion;
(b) the approximate distance covered by the ion in the process of its acceleration.

Solution. 398. (a) The total time of acceleration is,
$t = \frac{1}{2\nu} \cdot n,$

Where n is the number of passages of the Dees.

But, $T = neV = \frac{B^2 e^2 r^2}{2m}$

Or, $n = \frac{B^2 e r^2}{2mV}$

So, $t = \frac{\pi}{eB/m} \times \frac{B^2 er^2}{2mV} = \frac{\pi B r^2}{2V} = \frac{\pi^2 m\nu r^2}{eV} = 30 \, \mu\text{s}$

(b) The distance covered is, $s = \sum v_n \cdot \frac{1}{2\nu}$

But, $v_n = \sqrt{\frac{2eV}{m}} \sqrt{n}$,

So, $s = \sqrt{\frac{eV}{2m\nu^2}} \sum \sqrt{n} = \sqrt{\frac{eV}{2m\nu^2}} \int \sqrt{n}\, dn = \sqrt{\frac{eV}{2m\nu^2}} \frac{2}{3} n^{3/2}$

But, $n = \frac{B^2 e^2 r^2}{2\,eVm} = \frac{2\pi^2 m \nu^2 r^2}{eV}$

Thus, $s = \frac{4\pi^3 \nu^2 m r^2}{3eV} = 1{\cdot}24$ km

Q. 399. Since the period of revolution of electrons in a uniform magnetic field rapidly increases with the growth of energy, a cyclotron is unsuitable for their acceleration. This drawback is rectified in a microtone (Fig. 3.105) in which a change ΔT in the period of revolution of an electron is made multiple with the period of accelerating field T_0. How many times has an electron to cross the accelerating gap of a microtone to acquire an energy W = 4.6 MeV if $\Delta T = T_0$, the magnetic induction is equal to B = 107 mT, and the frequency of accelerating field to v = 3000 MHz?

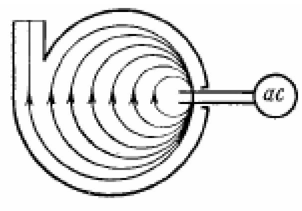


Fig. 3.105.

Solution. 399. In the nth orbit, $\frac{2\pi r_n}{v_n} = n\,T_0 = \frac{n}{\nu}$. We ignore the rest mass of the electron and write

$v_n \approx c$. Also $W \approx cp = cBer_n$.

Thus, $\frac{2\pi W}{Bec^2} = \frac{n}{\nu}$

or, $n = \frac{2\pi W\nu}{Bec^2} = 9$

Q. 400. The ill effects associated with the variation of the period of revolution of the particle in a cyclotron due to the increase of its energy are eliminated by slow monitoring (modulating) the frequency of accelerating field. According to what law ω (t) should this frequency be monitored if the magnetic induction is equal to B and the particle acquires an energy ΔW per revolution? The charge of the particle is q and its mass is m.

Solution. 400. The basic condition is the relativistic equation,

$$\frac{mv^2}{r} = Bqv, \quad \text{or,} \quad mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Bqr.$$

Or calling, $\omega = \frac{Bq}{m}$,

We get, $$\omega = \frac{\omega_0}{\sqrt{1 + \frac{\omega_0^2 r^2}{c^2}}}, \quad \omega_0 = \frac{Bq}{m_0} r$$

is the radius of the instantaneous orbit.

The time of acceleration is,

$$t = \sum_{n=1}^{N} \frac{1}{2\nu_n} = \sum_{n=1}^{N} \frac{\pi}{\omega_n} = \sum_{n}^{N} \frac{\pi W_n}{q B c^2}.$$

N is the number of crossing of either Dee.

But, $W_n = m_0 c^2 + \frac{n \Delta W}{2}$, there being two crossings of the Dees per revolution.

So, $$t = \sum \frac{\pi m_0 c^2}{qB c^2} + \sum \frac{\pi \Delta W_n}{2q B c^2}$$

$$= N \frac{\pi}{\omega_0} + \frac{N(N+1)}{4} \frac{\pi \Delta W}{qB c^2} \approx N^2 \frac{\pi \Delta W}{4 q B c^2} \; (N >> 1)$$

Also, $$r = r_N \frac{v_N}{\omega_N} \approx \frac{c}{\pi} \frac{\partial t}{\partial N} \approx \frac{\Delta W}{2q Bc} N$$

Hence finally,

$$\omega = \frac{\omega_0}{\sqrt{1 + \frac{q^2 B^2}{m_0^2 c^2} \times \frac{\Delta W^2}{4 q^2 B^2 c^2}} N^2}$$

$$= \frac{\omega_0}{\sqrt{1 + \frac{(\Delta W)^2}{4 m_0^2 c^4} \times \frac{4 qBc^2}{\pi \Delta W} t}} = \frac{\omega_0}{\sqrt{1 + at}};$$

$$a = \frac{q B \Delta W}{\pi m_0^2 c^2}$$

Q. 401. A particle with specific charge q/m is located inside a round solenoid at a distance r from its axis. With the current switched into the winding, the magnetic induction of the field generated by the solenoid amounts to B. Find the velocity of the particle and the curvature radius of its trajectory, assuming that during the increase of current flowing in the solenoid the particle shifts by a negligible distance.

Solution. 401. When the magnetic field is being set up in the solenoid, and electric field will be induced in it, this will accelerate the charged particle. If B is the rate, at which the magnetic field is increasing, then.

$$\pi r^2 \dot{B} = 2 \pi r E \quad \text{or} \quad E = \frac{1}{2} r \dot{B}$$

Thus, $m \frac{dv}{dt} = \frac{1}{2} r \dot{B} q, \quad \text{or} \quad v = \frac{qBr}{2m},$

After the field is set up, the particle will execute a circular motion of radius p, where

$$mv = B q \rho, \quad \text{or} \quad \rho = \frac{1}{2} r$$

Q. 402. In a beta Tron the magnetic flux across an equilibrium orbit of radius r = 25 cm grows during the acceleration time at practically constant rate $\dot{\Phi} = 5.0$ Wb/s. In the process, the electrons acquire an energy W = 25 MeV. Find the number of revolutions made by the electron during the acceleration time and the corresponding distance covered by it.

Solution. 402. The increment in energy per revolution is $e \dot{\Phi}$, so the number of revolutions is,

$$N = \frac{W}{e \dot{\Phi}}$$

The distance traversed is, $s = 2 \pi r N$

Q. 403. Demonstrate that electrons move in a beta Tron along a round orbit of constant radius provided the magnetic induction on the orbit is equal to half the mean value of that inside the orbit (the beta Tron condition).

Solution. 403. On the one hand,

$$\frac{dp}{dt} = eE = \frac{e}{2\pi r}\frac{d\Phi}{dt} = \frac{e}{2\pi r}\frac{d}{dt}\int_0^r 2\pi r' B(r')\,dr'$$

On the other

p = B (r) er, r = constant.

so, $\frac{dp}{dt} = er\frac{d}{dt}B(r) = er\dot{B}(r)$

Hence, $er\dot{B}(r) = \frac{e}{2\pi r}\pi r^2\frac{d}{dt}<B>$

So, $\dot{B}(r) = \frac{1}{2}\frac{d}{dt}<B>$

This equations is most easily satisfied by taking $B(r_0) = \frac{1}{2}<B>.$

Q. 404. Using the beta Tron condition, find the radius of a round orbit of an electron if the magnetic induction is known as a function of distance r from the axis of the field. Examine this problem for the specific case $B = B_0 - ar^2$, where B_0 and a are positive constants.

Solution. 404. The condition $B(r_0) = \frac{1}{2}<B> = \frac{1}{2}\int_0^{r_0} B\cdot 2\pi r\,dr/\pi r_0^2$

Or, $B(r_0) = \frac{1}{r_0^2}\int_0^{r_0} Br\,dr$

This gives r_0.

In the present case,

$$B_0 - ar_0^2 = \frac{1}{r_0^2}\int_0^{r_0}(B - ar^2)\,r\,dr = \frac{1}{2}\left(B_0 - \frac{1}{2}ar_0^2\right)$$

Or, $\frac{3}{4} a r_0^2 = \frac{1}{2} B_0 \quad \text{or} \quad r_0 = \sqrt{\frac{2B_0}{3a}}$.

Q. 405. Using the beta Tron condition, demonstrate that the strength of the eddy-current field has the extremum magnitude on an equilibrium orbit.

Solution. 405. The induced electric field (or eddy current field) is given by,

$$E(r) = \frac{1}{2\pi r} \frac{d}{dt} \int_0^r 2\pi r' (r') B(r') dr'$$

Hence,

$$\frac{dE}{dr} = -\frac{1}{2\pi r^2} \frac{d}{dt} \int_0^r 2\pi r' B(r') dr' + \frac{dB(r)}{dt}$$

$$= -\frac{1}{2} \frac{d}{dt} < B > + \frac{dB(r)}{dt}$$

This vanishes for r = r_0 by the beta Tron condition, where r_0 is the radius of the equilibrium orbit

Q. 406. In a beta Tron the magnetic induction on an equilibrium orbit with radius r = 20 cm varies during a time interval Δt = 1.0 ms at practically constant rate from zero to B = 0.40 T. Find the energy acquired by the electron per revolution.

Solution. 406. From the beta Tron condition,

$$\frac{1}{2} \frac{d}{dt} < B > = \frac{dB}{dt}(r_0) = \frac{B}{\Delta t}$$

Thus, $\frac{d}{dt} < B > = \frac{2B}{\Delta t}$

And $\frac{d\Phi}{dt} = \pi r^2 \frac{d < B >}{dt} = \frac{2\pi r^2 B}{\Delta t}$

So, energy increment per revolution is,

$$e \frac{d\Phi}{dt} = \frac{2\pi r^2 e B}{\Delta t}$$

Q. 407. The magnetic induction in a beta Tron on an equilibrium orbit of radius r varies during the acceleration time at practically constant rate from zero to B. Assuming the initial velocity of the electron to be equal to zero, find:
(a) the energy acquired by the electron during the acceleration time;
(b) the corresponding distance covered by the electron if the acceleration time is equal to Δt.

Solution. 407. (a) Even in the relativistic case, we know that : p = Her

Thus, $W = \sqrt{c^2 p^2 + m_0^2 c^4} - m_0 c^2 = m_0 c^2 \left(\sqrt{1 + (Ber / m_0 c)^2} - 1\right)$

(b) The distance traversed is,

$$2\pi r \frac{W}{e\Phi} = 2\pi r \frac{W}{2\pi r^2 eB / \Delta t} = \frac{W \Delta t}{Ber},$$

On using the result of the previous problem.