Hyperbola Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

1. P is any point on the hyperbola $x^2 - y^2 = a^2$. If F_1 and F_2 are the foci of the hyperbola and $PF_1 \cdot PF_2 = \lambda(OP)^2$, where O is the origin, then λ is equal to

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 3

2. If the sum of the slopes of the normals from a point P to the hyperbola $xy = c^2$ is equal to $\lambda\,(\lambda \in R^+)$, then locus of point P is

(a) $x^2 - y^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$
(c) $xy = \lambda c^2$ (d) $x^2 = \lambda c^2$

3. If $xy = \lambda^2 - 9$ be a rectangular hyperbola whose branches lie only in the second and fourth quadrant, then

(a) $|\lambda| \geq 3$ (b) $|\lambda| < 3$
(c) $\lambda \in R - \{-3, 3\}$ (d) None of these

4. If there are two points A and B on rectangular hyperbola $xy = c^2$ such that abscissa of A = ordinate of B, then the locus of point of intersection of tangents at A and B is

(a) $y^2 = x^2 + 2c^2$ (b) $y^2 = x^2 + \frac{c^2}{2}$
(c) $y = x$ (d) $y = 3x$

5. A series of hyperbola is drawn having a common transverse axis of length $2a$. Then the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is

(a) $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$
(b) $(x^2 - y^2)^2 = x^2(x^2 - a^2)$
(c) $(x^2 - y^2)^2 = 4y^2(x^2 - a^2)$
(d) $(x^2 - y^2)^2 = y^2(x^2 - a^2)$

6. If a rectangular hyperbola $(x-1)(y-2) = 4$ cuts a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at points (3, 4) (5, 3), (2, 6) and $(-1, 0)$, then the value of $(g + f)$ is equal to

(a) -8 (b) -9 (c) 8 (d) 9

7. If $f(x) = ax^3 + bx^2 + cx + d$, ($a, b, c, d$ are rational numbers) and roots of $f(x) = 0$ are eccentricities of a parabola and a rectangular hyperbola, then $a + b + c + d$ equals

(a) -1 (b) 0 (c) 1 (d) data inadequate

8. From a point on the line $y = x + c$, c (parameter), tangents are drawn to the hyperbola $\frac{x^2}{2} - \frac{y^2}{1} = 1$ such that chords of contact pass through a fixed point (x_1, y_1). Then, $\frac{x_1}{y_1}$ is equal to

(a) 2 (b) 3
(c) 4 (d) None of these

9. Two conics $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x^2 = -\frac{a}{b}y$ intersect, if

(a) $0 < b \leq \frac{1}{2}$ (b) $0 < a < \frac{1}{2}$
(c) $a^2 < b^2$ (d) $a^2 > b^2$

10. The number of points outside the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ from where two perpendicular tangents can be drawn to the hyperbola are

(a) 0 (b) 1
(c) 2 (d) None of these

11. Let $A \equiv (-3, 4)$ and $B \equiv (2, -1)$ be two fixed points. A point C moves such that

$$\tan\left(\frac{1}{2}\angle ABC\right) : \tan\left(\frac{1}{2}\angle BAC\right) = 3:1$$

Thus, locus of C is a hyperbola, distance between whose foci is

(a) 5 (b) $5\sqrt{2}$ (c) $\frac{5}{2}$ (d) $\frac{5}{\sqrt{2}}$

12. A point P is taken on the right half of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having its foci as S_1 and S_2. If the internal angle bisector of the angle $\angle S_1PS_2$ cuts the x-axis at point $Q(\alpha, 0)$, then range of α is

(a) $[-a, a]$ (b) $[0, a]$
(c) $(0, a]$ (d) $[-a, 0)$

13. If angle between asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 120° and product of perpendiculars drawn from foci upon its any tangent is 9, then locus of point of intersection of perpendicular tangents of the hyperbola can be

(a) $x^2 + y^2 = 3$ (b) $x^2 + y^2 = 6$
(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 18$

14. If $\alpha+\beta=3\pi$, then the chord joining the points α and β for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ passes through

(a) focus
(b) centre
(c) one of the end point of the transverse axis
(d) one of the end points of the conjugate axis

15. If the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ and $x^2-y^2=c^2$ cut at right angles, then

(a) $a^2+b^2=2c^2$ (b) $b^2-a^2=2c^2$
(c) $a^2-b^2=2c^2$ (d) $a^2b^2=2c^2$

16. If chords of the hyperbola $x^2-y^2=a^2$ touch the parabola $y^2=4ax$, then the locus of the middle points of these chords is the curve

(a) $y^2(x+a)=x^3$ (b) $y^2(x-a)=x^3$
(c) $y^2(x+2a)=3x^3$ (d) $y^2(x-2a)=2x^3$

17. An ellipse has eccentricity 1/2 and one focus at the point $P(1/2, 1)$. Its one directrix is the common tangent nearer to the point P, to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. The equation of the ellipse is standard form is

(a) $9x^2+12y^2=108$
(b) $9(x-1/3)^2+12(y-1)^2=1$
(c) $9(x-1/3)^2+4(y-1)^2=36$
(d) None of the above

18. The equation of the line passing through the centre of a rectangular hyperbola is $x-y-1=0$.
If one of its asymptote is $3x-4y-6=0$, the equation of the other asymptote is

(a) $4x-3y+8=0$ (b) $4x+3y+17=0$
(c) $3x-2y+15=0$ (d) None of these

19. The condition that a straight line with slope m will be normal to parabola $y^2=4ax$ as well as a tangent to rectangular hyperbola $x^2-y^2=a^2$ is

(a) $m^6-4m^2+2m-1=0$ (b) $m^4+3m^3+2m+1=0$
(c) $m^6-2m=0$ (d) $m^6+4m^4+3m^2+1=0$

20. The locus of the middle points of chords of hyperbola $3x^2-2y^2+4x-6y=0$ parallel to $y=2x$ is

(a) $3x-4y=4$ (b) $3y-4x+4=0$
(c) $4x-4y=3$ (d) $3x-4y=2$

21. The coordinates of the centre of the hyperbola $x^2+3xy+2y^2+2x+3y+2=0$ is

(a) $(-1, 0)$ (b) $(1, 0)$ (c) $(-1, 1)$ (d) $(1, -1)$

22. Let F_1, F_2 are foci of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ and F_3, F_4 are the foci of its conjugate hyperbola.
If e_H and e_C are their eccentricities respectively, then the statement which holds true is

(a) their equations of their asymptotes are different
(b) $e_H>e_C$
(c) area of the quadrilateral formed by their foci is 50 sq units
(d) their auxiliary circles will have the same equation

23. Locus of the point of intersection of the tangents at the points with eccentric angles ϕ and $\frac{\pi}{2}-\phi$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(a) $x=a$ (b) $y=b$
(c) $x=ab$ (d) $y=ab$

24. Latusrectum of the conic satisfying the differential equation $xdy+ydx=0$ and passing through the point $(2, 8)$ is

(a) $4\sqrt{2}$ (b) 8
(c) $8\sqrt{2}$ (d) 16

25. The points of the intersection of the curves whose parametric equations are $x=t^2+1, y=2t$ and $x=2s, y=\frac{2}{s}$ is given by

(a) $(1, -3)$ (b) $(2, 2)$
(c) $(-2, 4)$ (d) $(1, 2)$

26. If the tangent and normal to a rectangular hyperbola cut off intercepts x_1 and x_2 on one axis and y_1 and y_2 on the other axis, then

(a) $x_1y_1+x_2y_2=0$ (b) $x_1y_2+x_2y_1=0$
(c) $x_1x_2+y_1y_2=0$ (d) None of these

27. The focus of rectangular hyperbola $(x-h)(y-k)=p^2$ is

(a) $(h-p, k-p)$
(b) $(h-p, k+p)$
(c) $(h+p, k-p)$
(d) None of the above

28. The equation of a hyperbola, conjugate to the hyperbola $x^2+3xy+2y^2+2x+3y=0$ is

(a) $x^2+3xy+2y^2+2x+3y+1=0$
(b) $x^2+3xy+2y^2+2x+3y+2=0$
(c) $x^2+3xy+2y^2+2x+3y+3=0$
(d) $x^2+3xy+2y^2+2x+3y+4=0$

29. If the values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of the equation $x^2 - (a+b)x - 4 = 0$, then the value of $a + b$ is

(a) -2 (b) 0
(c) 2 (d) 4

30. Let C be a curve which is the locus of the point of intersection of lines $x = 2 + m$ and $my = 4 - m$. A circle $s = (x-2)^2 + (y+1)^2 = 25$ intersects the curve C at four points P, Q, R and S. If O is the centre of the curve C, then $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2$ is

(a) 25 (b) 50 (c) 100 (d) 200

Hyperbola Exercise 2 :
More than One Correct Option Type Questions

- This section contains **15 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. Equation of common tangent to the parabola $y^2 = 8x$ and hyperbola $x^2 - \dfrac{y^2}{3} = 1$ is

(a) $2x - y + 1 = 0$ (b) $2x - y - 1 = 0$
(c) $2x + y + 1 = 0$ (d) $2x + y - 1 = 0$

32. If the foci of the ellipse $\dfrac{x^2}{k^2a^2} + \dfrac{y^2}{a^2} = 1$ and the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ coincide, then k is equal to

(a) $-\sqrt{2}$ (b) $\sqrt{2}$ (c) $-\sqrt{3}$ (d) $\sqrt{3}$

33. If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then $\tan\left(\dfrac{\theta}{2}\right)\tan\left(\dfrac{\phi}{2}\right)$ is equal to

(a) $\dfrac{e-1}{e+1}$ (b) $\dfrac{1-e}{1+e}$ (c) $\dfrac{1+e}{1-e}$ (d) $\dfrac{e+1}{e-1}$

34. If foci of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ coincide with the foci of $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ and eccentricity of the hyperbola is 3, then

(a) $a^2 + b^2 = 9$
(b) there is no director circle to the hyperbola
(c) centre of the director circle is $(0, 0)$
(d) length of the latusrectum of the hyperbola = 16

35. The equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola with

(a) length of the transverse axis = $2\sqrt{3}$
(b) length of the conjugate axis = 8
(c) centre at $(1, -2)$
(d) eccentricity = $\sqrt{19}$

36. If the line $ax + by + c = 0$ is normal to the hyperbola $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

37. If $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ and $S(x_4, y_4)$ are 4 concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of the orthocentre of the ΔPQR are

(a) $(x_4, -y_4)$ (b) (x_4, y_4)
(c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$

38. The line $y = x + 5$ touches

(a) the parabola $y^2 = 20x$
(b) the ellipse $9x^2 + 16y^2 = 144$
(c) the hyperbola $\dfrac{x^2}{29} - \dfrac{y^2}{4} = 1$
(d) the circle $x^2 + y^2 = 25$

39. The coordinates of a point common to a directrix and an asymptote of the hyperbola $x^2/25 - y^2/16 = 1$ are

(a) $(25/\sqrt{41}, 20/\sqrt{41})$ (b) $(-25/\sqrt{41}, -20/\sqrt{41})$
(c) $(25/3, 20/3)$ (d) $(-25/3, -20/3)$

40. If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin, then

(a) $e = \dfrac{\sqrt{386}}{12}$ (b) $e = \dfrac{\sqrt{386}}{13}$
(c) latusrectum $= \dfrac{121}{3}$ (d) latusrectum $= \dfrac{121}{6}$

41. For the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn

(a) if $n = 1$, then $e = \sqrt{2}$
(b) if $n > 1$, then $0 < e < \sqrt{2}$
(c) if $n = 0$, then $e > \sqrt{2}$
(d) None of the above

42. Which of the following equations in parametric form can represent a hyperbola, where 't' is a parameter?

(a) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(b) $\frac{tx}{a} - \frac{y}{b} + t = 0$ and $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(c) $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$

(d) $x^2 - 6 = 2\cos t$ and $y^2 + 2 = 4\cos^2\left(\frac{t}{2}\right)$

43. Equation of common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is

(a) $y = x + \sqrt{(a^2 - b^2)}$

(b) $y = x - \sqrt{(a^2 - b^2)}$

(c) $y = -x + \sqrt{(a^2 - b^2)}$

(d) $y = -x - \sqrt{(a^2 - b^2)}$

44. Given ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$, if the ordinate of one of the points of intersection is produced to cut an asymptote at P, then which of the following is true?

(a) They have the same foci

(b) Square of the ordinate of point of intersection is $\frac{63}{25}$

(c) Sum of the squares of coordinate of P is 16

(d) P lies on the auxiliary circle formed by ellipse

45. Solution of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$, where $a \in R$, is

(a) a conic which is an ellipse

(b) centre of the conic is $(0, a)$

(c) length of one of the principal axes is 1

(d) length of one of the principal axes is equal to 2

Hyperbola Exercise 3 :
Paragraph Based Questions

- This section contains **5 paragraphs** based upon each of the paragraph 3 **multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I

(Q. Nos. 46 to 48)

The graph of the conic $x^2 - (y-1)^2 = 1$ has one tangent line with positive slope that passes through the origin. The point of tangency being (a, b)`.

46. The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(a) $\frac{5\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

47. Length of the latusrectum of the conic is

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4

48. If e be the eccentricity of the conic, then the value of $(1 + e^2 + e^4)$ is

(a) 3 (b) 7 (c) $\frac{7}{4}$ (d) 21

Paragraph II

(Q. Nos. 49 to 51)

A point P moves such that the sum of the slopes of the normals drawn from it to the hyperbola $xy = 4$ is equal to the sum of the ordinates of feet of normals. The locus of P is a curve C.

49. The equation of the curve C is

(a) $x^2 = 2y$ (b) $x^2 = 4y$

(c) $x^2 = 6y$ (d) $x^2 = 8y$

50. If the tangent to the curve C cuts the coordinate axes at A and B, then, the locus of the middle-point of AB is

(a) $x^2 + 2y = 0$ (b) $x^2 = y$

(c) $2x^2 + y = 0$ (d) $x^2 = 2y$

51. The area of the equilateral triangle inscribed in the curve C having one vertex as the vertex of curve C is

(a) $8\sqrt{3}$ sq units (b) $12\sqrt{3}$ sq units

(c) $27\sqrt{3}$ sq units (d) $48\sqrt{3}$ sq units

Paragraph III

(Q. Nos. 52 to 54)

Let $P(x, y)$ be a variable point such that

$$\left|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5)^2 + (y-5)^2}\right| = 4$$

which represents a hyperbola.

52. The eccentricity of the corresponding conjugate hyperbola is

(a) $\frac{5}{4}$ (b) $\frac{4}{3}$

(c) $\frac{5}{3}$ (d) $\frac{3}{2}$

53. Locus of point of intersection of two perpendicular tangents to the hyperbola is

(a) $(x-3)^2+(y-7/2)^2=\frac{1}{4}$

(b) $(x-3)^2+(y-7/2)^2=\frac{3}{4}$

(c) $(x-3)^2+(y-7/2)^2=\frac{5}{4}$

(d) $(x-3)^2+(y-7/2)^2=\frac{7}{4}$

54. If origin is shifted to point (3, 7/2) and axes are rotated in anticlockwise sense through an angle θ, so that the equation of hyperbola reduces to its standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then θ equals

(a) $\tan^{-1}\left(\frac{4}{3}\right)$ (b) $\tan^{-1}\left(\frac{3}{4}\right)$

(c) $\tan^{-1}\left(\frac{5}{4}\right)$ (d) $\tan^{-1}\left(\frac{4}{5}\right)$

Paragraph IV

(Q. Nos. 55 to 57)

Let $P(\theta_1)$ and $Q(\theta_2)$ are the extremities of any focal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose eccentricity is e. Let θ be the angle between its asymptotes. Tangents are drawn to the hyperbola at some arbitrary point R. These tangent meet the coordinate axes at the points A and B respectively. The rectangle OACB (O being the origin) is completed, then

55. Locus of point C is

(a) $\frac{b^2}{x^2}-\frac{a^2}{y^2}=1$ (b) $\frac{b^2}{x^2}+\frac{a^2}{y^2}=1$

(c) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ (d) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$

56. If $\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)=\lambda\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)$, then λ is equal to

(a) $\left(\frac{a^2+b^2}{a^2}\right)$ (b) $\left(\frac{a^2+b^2}{b^2}\right)$

(c) $\left(\frac{a^2+b^2}{ab}\right)$ (d) $\left(\frac{a^2+b^2}{2ab}\right)$

57. The value of $\cos\left(\frac{\theta}{2}\right)$ is

(a) $\frac{1}{2e}$ (b) $\frac{1}{e}$

(c) $\frac{1}{e^2}$ (d) $\frac{1}{2e^2}$

Paragraph V

(Q. Nos. 58 to 60)

The vertices of ΔABC lie on a rectangular hyperbola such that the orthocentre of the triangle is (2, 3) and the asymptotes of the rectangular hyperbola are parallel to the coordinate axes. The two perpendicular tangents of the hyperbola intersect at the point (1, 1).

58. The equation of the asymptotes is

(a) $xy-1=y-x$ (b) $xy+1=x+y$

(c) $xy-1=x-y$ (d) $xy+1=-x-y$

59. The equation of the rectangular hyperbola is

(a) $xy-5=y-x$

(b) $xy-1=x+y$

(c) $xy=x+y+1$

(d) $xy-11=-x-y$

60. The number of real tangents that can be drawn from the point (1, 1) to the rectangular hyperbola is

(a) 0 (b) 2

(c) 3 (d) 4

Hyperbola Exercise 4 :
Single Integer Answer Type Questions

- This section contains **10 questions.** The answer to each question is **a single digit integer** ranging from 0 to 9 (both inclusive)

61. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ are given to be confocal and length of minor axis of the ellipse is same as the conjugate axis of the hyperbola. If e_1 and e_2 represents the eccentricities of ellipse and hyperbola respectively, then the value of $e_1^{-2}+e_2^{-2}$ is

62. If abscissa of orthocentre of a triangle inscribed in a rectangular hyperbola $xy=4$ is $\frac{1}{2}$, then the ordinate of orthocentre of triangle is

63. Normal drawn to the hyperbola $xy=2$ at the point $P(t_1)$ meets the hyperbola again at $Q(t_2)$, then minimum distance between the point P and Q is

64. The normal at P to a hyperbola of eccentricity $\frac{3}{2\sqrt{2}}$ intersects the transverse and conjugate axes at M and N respectively. The locus of mid-point of MN is a hyperbola, then its eccentricity

65. If radii of director circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b_1^2}=1$ are in the ratio 1 : 3 and $4e_1^2-e_2^2=\lambda$, where e_1 and e_2 are the eccentricities of ellipse and hyperbola respectively, then the value of λ is

66. The shortest distance between the curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $4x^2+4y^2=a^2\ (b>a)$ is $f(a,b)$, then the value of $f(4,6)+f(2,3)$ is

67. ABC is a triangle such that $\angle ABC=2\angle BAC$.
If AB is fixed and locus of C is a hyperbola, then the eccentricity of the hyperbola is

68. Point P lie on $2xy=1$. A triangle is constructed by P, S and S' (where S and S' are foci). The locus of ex-centre opposite S (S and P lie in first quadrant) is $(x+py)^2=(\sqrt{2}-1)^2(x-y)^2+q$, then the value of $p+q$ is

69. Chords of the circle $x^2+y^2=4$, touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{16}=1$. The locus of their middle-points is the curve $(x^2+y^2)^2=\lambda x^2-16y^2$, then the value of λ is

70. Tangents are drawn from the point (α,β) to the hyperbola $3x^2-2y^2=6$ and are inclined at angles θ and ϕ to the X-axis. If $\tan\theta\,.\tan\phi=2$, then the value of $2\alpha^2-\beta^2$ is

Hyperbola Exercise 5 :
Matching Type Questions

- This section contains 3 **questions.** Each question has four statements (A), (B), (C) and (D) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or move statements (*s*) given in **Column II.**

71. Match the following.

	Column I		Column II
(A)	If λ be the length of the latusrectum of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, then 3λ is divisible by	(p)	4
(B)	If the chord $x\cos\alpha + y\sin\alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, a circle touches the given chord and concentric with hyperbola, then the diameter of circle is divisible by	(q)	6
(C)	For the hyperbola $xy = 8$ any tangent of it at P meets coordinate axes at Q and R, then the area of triangle CQR is divisible by (where 'C' is centre of the hyperbola)	(r)	8
(D)	For the hyperbola $x^2 - 3y^2 = 9$, acute angle between its asymptotes is $\frac{\pi\lambda}{24}$, then λ is divisible by	(s)	16

72. Match the following.

	Column I		Column II
A.	If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then the ratio of the square of its conjugate axis to the square of its transverse axis is	(p)	A Natural number
B.	With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is	(q)	A Prime number
C.	If S_1 and S_2 are the foci of the hyperbola whose length of the transverse axis is 4 and that of the conjugate axis is 6, and S_3 and S_4 are the foci of the conjugate hyperbola, then the area of quadrilateral $S_1S_2S_3S_4$ is	(r)	A Composit e number
D.	If equation of hyperbola whose conjugate axis is 5 and distance between its foci is 13, is $ax^2 - by^2 = c$, where a and b are co-prime, then $\frac{3ab}{2c}$ is	(s)	A Perfect number

73. If e_1 and e_2 are the roots of the equation $x^2 - \lambda x + 2 = 0$

	Column I		Column II
A.	If e_1 and e_2 are the eccentricities of ellipse and hyperbola respectively, then the values of λ are	(p)	$2\sqrt{2}$
B.	If both e_1 and e_2 are the eccentricities of the hyperbolas, then the values of λ are	(q)	$2\sqrt{3}$
C.	If e_1 and e_2 are the eccentricities of the hyperbola and conjugate hyperbola, then the values of λ are	(r)	$2\sqrt{5}$
D.	If e_1 is the eccentricity of the hyperbola for which there exist infinite points from which perpendicular tangents can be drawn and e_2 is the eccentricity of the hyperbola in which no such points exist, then the values of λ are	(s)	$2\sqrt{6}$

Hyperbola Exercise 6 :
Statement I and II Type Questions

- **Directions** (Q. Nos. 74 to 81) are Assertion-Reason type questions. Each of these questions contains two statements :
Statement I (Assertion) and **Statement II** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below :
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

74. **Statement I** $\frac{5}{3}$ and $\frac{5}{4}$ are the eccentricities of two conjugate hyperbolas.
Statement II If e_1 and e_2 are the eccentricities of two conjugate hyperbolas, then $e_1 e_2 > 1$.

75. **Statement I** A hyperbola and its conjugate hyperbola have the same asymptotes.
Statement II The difference between the second degree curve and pair of asymptotes is constant.

76. **Statement I** The equation of the director circle to the hyperbola $5x^2 - 4y^2 = 20$ is $x^2 + y^2 = 1$.
Statement II Director circle is the locus of the point of intersection of perpendicular tangents.

77. **Statement I** Two tangents are drawn from a point on the circle $x^2 + y^2 = 9$ to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, then angle between tangents is $\pi/2$.
Statement II $x^2 + y^2 = 9$ is the director circle of $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

78. **Statement I** If eccentricity of a hyperbola is 2, then eccentricity of its conjugate hyperbola is $2/\sqrt{3}$.
Statement II If e and e_1 are the eccentricities of two conjugate hyperbolas, then $ee_1 > 1$.

79. **Statement I** The line $4x - 5y = 0$ will not meet the hyperbola $16x^2 - 25y^2 = 400$.
Statement II The line $4x - 5y = 0$ is an asymptote to the hyperbola.

80. **Statement I** The point $(5, -3)$ inside the hyperbola $3y^2 - 5x^2 + 1 = 0$.
Statement II The point (x_1, y_1) inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$.

81. **Statement I** A hyperbola whose asymptotes include $\pi/3$ is said to be equilateral hyperbola.
Statement II The eccentricity of an equilateral hyperbola is $\sqrt{2}$.

Hyperbola Exercise 7 :
Subjective Type Questions

- In this section, there are **12 Subjective questions.**

82. Given the base of a triangle and the ratio of the tangent of half the base angles. Show that the vertex moves on a hyperbola whose foci are the extremities of the base.

83. A, B, C are three points on the rectangular hyperbola $xy = c^2$, find
(i) The area of the triangle ABC.
(ii) The area of the triangle formed by the tangents at A, B and C.

84. If a hyperbola be rectangular and its equation be $xy = c^2$, prove that the locus of the middle points of chords of constant length $2d$ is
$(x^2 + y^2)(xy - c^2) = d^2 xy$.

85. If four points be taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two, and if $\alpha, \beta, \gamma, \delta$ be the inclinations to either asymptote of the straight line joining these points to the centre, prove that $\tan\alpha \tan\beta \tan\gamma \tan\delta = 1$.

86. P and Q are two variable points on the rectangular hyperbola $xy = c^2$ such that tangent at Q passes through the foot of the ordinate of P. Show that the locus of the intersection of tangents at P and Q is a hyperbola with the same asymptotes as of the given hyperbola.

87. A circle cuts two perpendicular lines so that each intercept is of given length. Prove that the locus of the centre of the circle is a rectangular hyperbola.

88. (a) Prove that any line parallel to either of the asymptotes of a hyperbola shall meet it in one point at infinity.

(b) Prove that the asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn parallel to the axes at the vertices of the hyperbola [i.e. at $(\pm a, 0)$ and $(0, \pm b)$].

89. Let the tangent at a point P on the ellipse meet the major axis at B and the ordinate from *it* meet the major axis at A. If Q is a point on the line AP such that $AQ = AB$, prove that the locus of Q is a hyperbola. Find the asymptotes of this hyperbola.

90. From the point (x_1, y_1) and (x_2, y_2), tangents are drawn to the rectangular hyperbola $xy = c^2$. If the conic passing through the two given points and the four points of contact is a circle, then show that $x_1x_2 = y_1y_2$ and $x_1y_2 + x_2y_1 = 4c^2$.

91. A rectangular hyperbola passes through two fixed points and its asymptotes are in given directions. Prove that its vertices lie on an ellipse and hyperbola which intersect orthogonally.

92. Let normals are drawn from (α, β) to the hyperbola $xy = 1$, and (x_i, y_i), $i = 1, 2, 3, 4$ be the feet of the co-normal points. If the algebraic sum of the perpendicular distances drawn from (x_i, y_i), $i = 1, 2, 3, 4$ onto a variable line vanishes, show that the variable line passes through the point $(\alpha/4, \beta/4)$.

93. A series of hyperbolas is drawn having a common transverse axis of length $2a$. Prove that the locus of a point P on each hyperbola such that its distance from the transverse axis is equal to its distance from on asymptote, is the curve $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$.

Hyperbola Exercise 8 :
Questions Asked in Previous 13 Year's Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main** & **JEE Advanced** from year **2005** to **2017**.

94. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is **[AIEEE 2005, 3M]**

(a) an ellipse
(b) a circle
(c) a parabola
(d) a hyperbola

95. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then **[IIT- JEE 2006, 5M]**

(a) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(b) the equations of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
(c) focus of hyperbola is (5, 0)
(d) vertex of hyperbola is $(5\sqrt{3}, 0)$

96. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then, its equation is **[IIT- JEE 2007, 3M]**

(a) $x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$
(b) $x^2\sec^2\theta - y^2\operatorname{cosec}^2\theta = 1$
(c) $x^2\sin^2\theta - y^2\cos^2\theta = 1$
(d) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

97. Two branches of a hyperbola **[IIT- JEE 2007, 1.5M]**

(a) have a common tangent
(b) have a common normal
(c) do not have a common tangent
(d) do not have a common normal

98. For the hyperbola $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$, which of the following remains constant when α varies

[AIEEE 2007, 3M]

(a) abscissae of vertices (b) abscissae of foci
(c) eccentricity (d) directrix

99. Consider a branch of the hyperbola
$x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$
with vertex at the point A. Let B be one of the end points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

[IIT- JEE 2008, 3M]

(a) $1-\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}-1$ (c) $1+\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}}+1$

100. An ellipse intersects the hyperbola $2x^2-2y^2=1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

[IIT- JEE 2009, 4M]

(a) equation of ellipse is $x^2+2y^2=2$
(b) the foci of ellipse are $(\pm 1,0)$
(c) equation of ellipse is $x^2+2y^2=4$
(d) the foci of ellipse are $(\pm\sqrt{2},0)$

Paragraph

(Q. Nos. 100 and 102)

The circle $x^2+y^2-8x=0$ and hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ intersect at the points A and B.

101. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is
(a) $2x-\sqrt{5}y-20=0$ (b) $2x-\sqrt{5}y+4=0$
(c) $3x-4y+8=0$ (d) $4x-3y+4=0$

102. Equation of the circle with AB as its diameter is
(a) $x^2+y^2-12x+24=0$ (b) $x^2+y^2+12x+24=0$
(c) $x^2+y^2+24x-12=0$ (d) $x^2+y^2-24x-12=0$

[IIT- JEE 2010, 3 + 3M]

103. The line $2x+y=1$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the X-axis, then the eccentricity of the hyperbola is

[IIT- JEE 2010, 3M]

104. Let $P(6,3)$ be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the X-axis at (9, 0), then the eccentricity of the hyperbola is

[IIT- JEE 2011, 3M]

(a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$
(c) $\sqrt{2}$ (d) $\sqrt{3}$

105. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2+4y^2=4$. If the hyperbola passes through a focus of the ellipse, then

[IIT- JEE 2011, 4M]

(a) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$
(b) a focus of the hyperbola is $(2,0)$
(c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
(d) the equation of the hyperbola is $x^2-3y^2=3$

106. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line $2x-y=1$. The points of contact of the tangents on the hyperbola are

[IIT- JEE 2012, 4M]

(a) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$
(c) $(3\sqrt{3},-2\sqrt{2})$ (d) $(-3\sqrt{3},2\sqrt{2})$

107. Consider the hyperbola $H: x^2-y^2=1$ and a circle S with centre $N(x_2,0)$. Suppose that H and S touch each other at a point $P(x_1,y_1)$ with $x_1>1$ and $y_1>0$. The common tangent to H and S at P intersects the X-axis at point M. If (l,m) is the centroid of the triangle PMN, then the correct expression(s) is(are)

[JEE Advanced 2015, 4M]

(a) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1>1$ (b) $\frac{dm}{dx_1}=\frac{x_1}{3(\sqrt{(x_1^2-1)})}$ for $x_1>1$
(c) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1>1$ (d) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1>0$

108. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

[JEE Main 2016, 4M]

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$
(c) $\frac{4}{3}$ (d) $\frac{4}{\sqrt{3}}$

109. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the taught to this hyperbola at P also passes through the point

[JEE Main 2017, 4M]

(a) $(-\sqrt{2},-\sqrt{3})$ (b) $(3\sqrt{2},2\sqrt{3})$
(c) $(2\sqrt{2},3\sqrt{3})$ (d) $(\sqrt{3},\sqrt{2})$

110. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following cannot be sides of a right angled triangle? **[JEE Advanced 2017, 4M]**

(a) $2a, 8, 1$ (b) $a, 4, 1$
(c) $a, 4, 2$ (d) $2a, 4, 1$

- **Direction** (Q. No. 111 to 113) Matching the information given in the three columns of the following table. Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.

[JEE Advanced 2017, (3 + 3 + 3) M]

	Column 1		Column 2		Column 3
(I)	$x^2 + y^2 = a^2$	(i)	$my = m^2x + a$	(P)	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II)	$x^2 + a^2y^2 = a^2$	(ii)	$y = mx + a\sqrt{m^2+1}$	(Q)	$\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III)	$y^2 = 4ax$	(iii)	$y = mx + \sqrt{a^2m^2-1}$	(R)	$\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV)	$x^2 - a^2y^2 = a^2$	(iv)	$y = mx + \sqrt{a^2m^2+1}$	(S)	$\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

111. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only correct combination?

(a) (IV) (iii) (S)
(b) (II) (iv) (R)
(c) (IV) (iv) (S)
(d) (II) (iii) (R)

112. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact $(-1, 1)$, then which of the following options is the only correct combination for obtaining its equation?

(a) (III) (i) (P) (b) (I) (i) (P)
(c) (II) (ii) (Q) (d) (I) (ii) (Q)

113. If a tangent of a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only correct combination?

(a) (III) (i) (P) (b) (III) (ii) (Q)
(c) (II) (iv) (R) (d) (I) (ii) (Q)

Answers

Chapter Exercises

1. (a) **2.** (d) **3.** (b) **4.** (c) **5.** (a) **6.** (a)
7. (b) **8.** (a) **9.** (b) **10.** (a) **11.** (b) **12.** (c)
13. (d) **14.** (b) **15.** (c) **16.** (b) **17.** (b) **18.** (b)
19. (d) **20.** (a) **21.** (a) **22.** (c) **23.** (b) **24.** (c)
25. (b) **26.** (c) **27.** (a) **28.** (b) **29.** (b) **30.** (c)
31. (a,c) **32.** (c,d) **33.** (b,c) **34.** (a,b,d) **35.** (a,b,c) **36.** (b,c)
37. (b,c) **38.** (a,b,c) **39.** (a,b) **40.** (a,d) **41.** (a,b,c) **42.** (a,c,d)
43. (a,b,c,d) **44.** (a,b,c,d) **45.** (a,b,d)
46. (c) **47.** (c) **48.** (b) **49.** (b) **50.** (c) **51.** (d)
52. (c) **53.** (d) **54.** (b) **55.** (d) **56.** (a) **57.** (b)
58. (b) **59.** (c) **60.** (b) **61.** (2) **62.** (8) **63.** (4)
64. (3) **65.** (7) **66.** (3) **67.** (2) **68.** (5) **69.** (4)
70. (7)
71. (A) $\rightarrow$ (p,r,s); (B) $\rightarrow$ (p,q,r); (C) $\rightarrow$ (p,r,s); (D) $\rightarrow$ (p,r)
72. (A) $\rightarrow$ (p,q); (B) $\rightarrow$ (p,q); (C) $\rightarrow$ (p,r); (D) $\rightarrow$ (p,r,s)
73. (A) $\rightarrow$ (q,r,s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q,r,s)
74. (b) **75.** (a) **76.** (d) **77.** (a) **78.** (b) **79.** (a)
80. (c) **81.** (d)
83. (i) $\frac{c^2}{2t_1t_2t_3} |(t_1 - t_2)((t_2 - t_3)(t_3 - t_1)|$

(ii) $2c^2 \left| \frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)} \right|$

89. $x = 0$ and $x + y = 0$
94. (d) **95.** (a,c) **96.** (a) **97.** (b,c) **98.** (b) **99.** (b)
100. (a,b) **101.** (b) **102.** (a) **103.** (2) **104.** (b) **105.** (b,d)
106. (a,b) **107.** (a,b,d) **108.** (a) **109.** (c) **110.** (a,b,c)
111. (b) **112.** (d) **113.** (a)

Solutions

1. $\because PF_1 \cdot PF_2 = e\left(x_1 - \frac{a}{e}\right) \cdot e\left(x_1 + \frac{a}{e}\right)$

$$= e^2x_1^2 - a^2$$

$$= 2x_1^2 - a^2 \qquad \text{(for rectangular hyperbola } e = \sqrt{2})$$

$$= x_1^2 + x_1^2 - a^2 \qquad (\because P(x_1, y_1),\ x_1^2 - y_1^2 = a^2)$$

$$= x_1^2 + y_1^2$$

$$= (OP)^2$$

$\therefore \quad \lambda = 1$

2. Let the point of contact of normal from point $P(h, k)$, be $R(t)$ for the hyperbola $xy = c^2$

$\Rightarrow$ The equation of normal is

$ct^4 - xt^3 + yt - c = 0$ which gives

$ct^4 - ht^3 + kt - c = 0 \qquad [\because$ It passes through $(h,k)]$...(i)

$\Rightarrow \quad t_1^2 + t_2^2 + t_3^2 + t_4^2 = \lambda \qquad$ (given) ... (ii)

From Eq. (i),

$$\Sigma t_1 = \frac{h}{c},\ \Sigma t_1 t_2 = 0$$

$\therefore \quad (\Sigma t_1)^2 - 2\Sigma t_1 t_2 = t_1^2 + t_2^2 + t_3^2 + t_4^2$

$\Rightarrow \quad \left(\frac{h}{c}\right)^2 - 0 = \lambda$

or $\quad h^2 = \lambda c^2$

$\therefore$ Locus of P is $x^2 = \lambda c^2$

3. As branches lies in the second and fourth quadrant.

$\therefore$ We have $xy < 0$

$\Rightarrow \quad m^2 - 9 < 0 \Rightarrow |m| < 3$

4. Let A is (α, β), the B is (β, α)

$\because$ A and B an symmetrical about the line $y = x$.

So, tangents at A and B will be mirror images of each other about $y = x$. Thus, point of intersection will lie on $y = x$.

5. Let $P(h,k)$ be any point on any one member of hyperbola family, having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; b is any arbitrary constant, then its asymptotes are given by $y = \pm \frac{b}{a}x$, then according to question

$$|k| = \frac{\left|\pm \frac{b}{a} h - k\right|}{\sqrt{\left(\frac{b^2}{a^2} + 1\right)}}$$

$\Rightarrow \quad k^2 = \left(\pm \frac{b}{a} h - k\right)^2 \left(\frac{a^2}{a^2 + b^2}\right)$

or $\quad k^2 = \frac{(\pm bh - ak)^2}{(a^2 + b^2)} \qquad$ ...(i)

Further (h,k) lies on hyperbola

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1 \Rightarrow b = \sqrt{\left(\frac{a^2k^2}{h^2 - a^2}\right)} \qquad \text{...(ii)}$$

From Eq. (i) and (ii), we get

$$(k^2 - h^2) = \pm 2h\sqrt{(h^2 - a^2)}$$

$\Rightarrow \quad (k^2 - h^2)^2 = 4h^2(h^2 - a^2)$

$\therefore$ Required locus is

$$(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$$

6. We know that if a circle cuts a rectangular hyperbola, then arithmetic mean of points of intersections is the mid-point of centre of hyperbola and circle.

So, $\quad \frac{3 + 5 + 2 + (-1)}{4} = \frac{-g + 1}{2}$

and $\quad \frac{4 + 3 + 6 + 0}{4} = \frac{-f + 2}{2}$

$\therefore \quad g + f = -8$

7. We know that eccentricity of a parabola and rectangular hyperbola are 1 and $\sqrt{2}$ respectively. Also, irrational roots occur in conjugate pair, thus roots of $f(x) = 0$ are 1, $\sqrt{2}$ and $-\sqrt{2}$

$\therefore \quad f(x) = (x - 1)(x - \sqrt{2})(x + \sqrt{2})$

$$= x^3 - x^2 - 2x + 2$$

$\therefore \quad a + b + c + d = 1 - 1 - 2 + 2 = 0$

8. Let the point be (α, β)

$\Rightarrow \quad \beta = \alpha + c$

Chord of contact of hyperbola $T = 0$.

$\therefore \quad \frac{x\alpha}{2} - \frac{y\beta}{1} = 1$

$\Rightarrow \quad \frac{x\alpha}{2} - y(\alpha + c) = 1$

$\Rightarrow \quad \left(\frac{x}{2} - y\right)\alpha - (yc + 1) = 0$

Since, this passes through point (x_1, y_1)

$\therefore \quad x_1 = 2y_1$ and $y_1 c + 1 = 0$

$\therefore \quad y_1 = \frac{x_1}{2}$

Hence, $\quad \frac{x_1}{y_1} = 2$

9. Eliminating x, we have

$$\frac{y^2}{b^2} + \frac{y}{ab} + 1 = 0$$

This equation has real and distinct roots

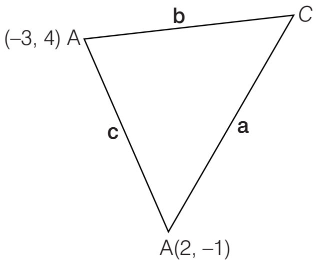
$\therefore \quad \frac{1}{a^2b^2} - \frac{4}{b^2} > 0$

i.e. $\quad \frac{1}{a^2} > 4$ or $a^2 < \frac{1}{4}$

$\Rightarrow \quad a < \frac{1}{2}$ and hence the conics intersect if $0 < a < \frac{1}{2}$.

10. Points from where perpendicular tangents can be drawn to the give hyperbola lie on the director circle $x^2 + y^2 = 9 - 16 = -7$ which is an imaginary circle. Hence, no point exists.

11. $\because \dfrac{\tan\left(\frac{1}{2}\angle ABC\right)}{\tan\left(\frac{1}{2}\angle BAC\right)} = \dfrac{3}{1}$



or $\quad \dfrac{\tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right)} = 3 \quad$ or $\quad \dfrac{\frac{\Delta}{s(s-b)}}{\frac{\Delta}{s(s-a)}} = 3$

or $\quad \dfrac{s-a}{s-b} = 3$

$\Rightarrow \quad s - a = 3s - 3b$

or $\quad 2s = 3b - a$

or $\quad a + b + c = 3b - a$

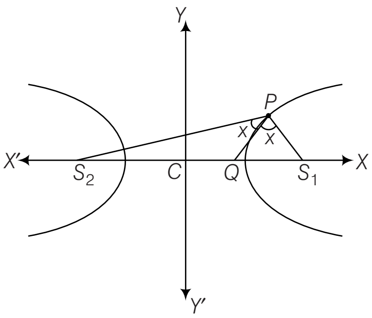
or $\quad b - a = \dfrac{c}{2}$

or $\quad AC - BC = \text{constant}$

$\therefore$ Locus of C is a hyperbola, whose foci are A and B.

$\therefore$ Distance between foci $= |AB| = 5\sqrt{2}$.

12.



Let $\quad P \equiv (x_1, y_1)$

It is clear from figure

$$\alpha > 0 \qquad \text{... (i)}$$

and $Q(\alpha, 0)$ out side on the hyperbola, then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \le 0$$

$\Rightarrow \quad \dfrac{\alpha^2}{a^2} - 1 \le 0$

or $\quad \alpha^2 \le a^2$

$\therefore \quad -a \le \alpha \le a \qquad \text{... (ii)}$

From Eqs. (i) and (ii), we get

$$0 < \alpha \le a$$

$$\alpha \in (0, a]$$

13. $\because$ Angle between asymptotes $= 2\tan^{-1}\left(\dfrac{b}{a}\right) = 120°$

or $\quad \tan^{-1}\left(\dfrac{b}{a}\right) = 60°$

or $\quad \dfrac{b}{a} = \sqrt{3}$

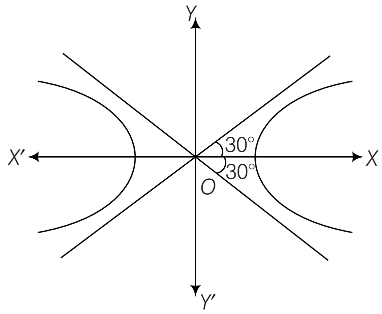
$\therefore \quad b = a\sqrt{3}$ or $b^2 = 3a^2$

$\Rightarrow \quad a^2 = 3 \qquad (\because b^2 = 9)$

Required locus is director circle of the hyperbola and which is $x^2 + y^2 = a^2 - b^2 = 3 - 9 = -6$

which is not possible.

Now, angle between asymptotes $= 2\tan^{-1}\left(\dfrac{b}{a}\right) = 60°$.



or $\quad \tan^{-1}\left(\dfrac{b}{a}\right) = 30°$

or $\quad \dfrac{b}{a} = \dfrac{1}{\sqrt{3}}$

or $\quad a^2 = 3b^2 = 27 \qquad (\because b^2 = 9)$

$\therefore$ Required locus is

$$x^2 + y^2 = a^2 - b^2 = 27 - 9 = 18$$

14. Equation of chord joining α and β is

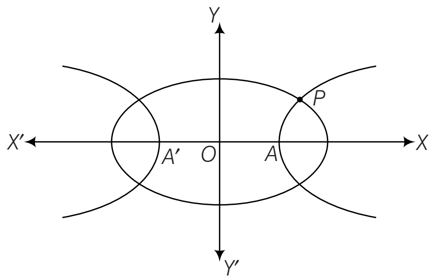
$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

Put $\alpha + \beta = 3\pi$, then

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) + \frac{y}{b} = 0$$

It passes through the centre (0, 0).

15. Let $P(a\cos\theta, b\sin\theta)$ on the ellipse.



$\therefore$ Equation of tangent at P on ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad \text{... (i)}$$

and equation of tangent at P on $x^2 - y^2 = c^2$ is

$$x(a\cos\theta) - y(b\sin\theta) = c^2 \quad \text{...(ii)}$$

Since, curves intersect at right angle, then

$$-\frac{b}{a}\cot\theta \times \frac{a}{b}\cot\theta = -1$$

$\therefore$ $$\tan^2\theta = 1 \quad \text{... (iii)}$$

Since, $P(a\cos\theta, b\sin\theta)$ also lies on hyperbola

$\therefore$ $$(a\cos\theta)^2 - (b\sin\theta)^2 = c^2$$

Dividing both sides by $\cos^2\theta$, then

or $$a^2 - b^2\tan^2\theta = c^2(1 + \tan^2\theta)$$

$$a^2 - b^2 = 2c^2 \quad \text{[from Eq. (iii)]}$$

16. Let mid-point of the chord is (h,k).

$\therefore$Equation of chord of $x^2 - y^2 = a^2$ is

$$T = S_1$$

$$hx - ky = h^2 - k^2$$

or $$y = \frac{h}{k}x - \frac{(h^2 - k^2)}{k}$$

which is tangent of $y^2 = 4ax$

$\therefore$ $$-\frac{(h^2 - k^2)}{k} = \frac{a}{\frac{h}{k}}$$

or $$-(h^2 - k^2) = \frac{ak^2}{h}$$

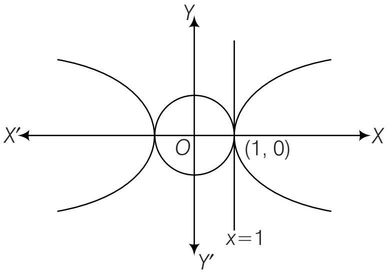
or $$-h^3 + hk^2 = ak^2$$

or $$k^2(h - a) = h^3$$

Hence, locus of mid-point is

$$y^2(x - a) = x^3$$

17. It is clear from the figure the common tangent to the circle $x^2 + y^2 = 1$ and hyperbola $x^2 - y^2 = 1$ is $x = 1$ (which is nearer to P (1/2, 1) and given one focus at P (1/2, 1), so the equation of the directrix is $x = 1$. Hence, the equation of the ellipse is



$$\sqrt{(x - 1/2)^2 + (y - 1)^2} = \frac{1}{2}|x - 1|$$

$\Rightarrow$ $$9(x - 1/3)^2 + 12(y - 1)^2 = 1$$

18. Since, asymptotes of rectangular hyperbola are perpendicular to each other.

$\because$ Given asymptote is $3x - 4y - 6 = 0$

$\therefore$ Other asymptote is $4x + 3y + \lambda = 0$ ... (i)

Given, centre of hyperbola lies on $x - y - 1 = 0$

Since, asymptotes pass through the centre of hyperbola

$\therefore$ Centre is the point of intersection of $x - y - 1 = 0$ and $3x - 4y - 6 = 0$

$\therefore$Centre is $(-2, -3)$, also $(-2, -3)$ lies on Eq. (i)

then $$-8 - 9 + \lambda = 0$$

$\therefore$ $$\lambda = 17$$

Hence, other asymptote is

$$4x + 3y + 17 = 0 \quad \text{[from Eq. (i)]}$$

19. Equation of normal of $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

which is tangent of $x^2 - y^2 = a^2$

$\therefore$ $$(-2am - am^3)^2 = a^2m^2 - a^2$$

or $$4m^2 + m^6 + 4m^4 = m^2 - 1$$

or $$m^6 + 4m^4 + 3m^2 + 1 = 0$$

20. Let the middle-point of the chord is (h,k)

$\therefore$ $$T = S_1$$

$$3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$$

Slope of this chord $= \dfrac{3h + 2}{2k + 3} = 2$ (given)

or $$3h + 2 = 4k + 6$$

$\Rightarrow$ $$3h - 4k = 4$$

Hence, locus of middle-point is

$$3x - 4y = 4$$

21. Let $f(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

$\therefore$ $\dfrac{\partial f}{\partial x} = 2x + 3y + 2$ and $\dfrac{\partial f}{\partial y} = 3x + 4y + 3$

For centre, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

$\therefore$ $$2x + 3y + 2 = 0$$

$$3x + 4y + 3 = 0$$

After solving, we get

$$x = -1, y = 0$$

$\therefore$Coordinates of centre are $(-1, 0)$.

22. $\because$ Hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ ... (i)

Foci are $(\pm 5, 0)$

$$F_1 \equiv (5, 0), F_2 \equiv (-5, 0)$$

Also, $$4e_H = 5$$

$\therefore$ $$e_H = \frac{5}{4}$$

Conjugate hyperbola of Eq. (i) is

$$-\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \text{... (ii)}$$

Foci are $(0, \pm 5)$

$$F_3 \equiv (0, 5), F_4 \equiv (0, -5)$$

Also, $$3e_c = 5$$

$\therefore$ $$e_c = \frac{5}{3}$$

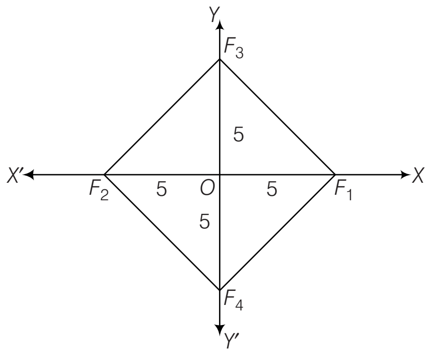
Equation of asymptotes of Eqs. (i) and (ii) are same

$$y = \pm\frac{3}{4}x \text{ and } e_H < e_c$$

Auxiliary circle of Eq. (i) is $x^2 + y^2 = 7$ and Eq. (ii) is $x^2 + y^2 = -7$

and area of quadrilateral formed by their foci $= 4 \times \frac{1}{2} \times 5 \times 5$

$= 50$ sq units



23. Point of intersection of tangents at $(a\sec\alpha, b\tan\alpha)$ and $(a\sec\beta, b\tan\beta)$ is

$$\left(\frac{a\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right)$$

Here, $\alpha = \phi, \beta = \frac{\pi}{2} - \phi$

then, point of intersection is $\left(\frac{a\cos\left(\frac{\pi}{4}-\phi\right)}{\frac{1}{\sqrt{2}}}, b\right)$

$$x = a\sqrt{2}\cos\left(\frac{\pi}{4}-\phi\right)$$

and $y = b$

$\therefore$ Required locus is $y = b$

24. $xdy + ydx = 0$

$\Rightarrow \quad d(xy) = 0$

$\therefore \quad xy = c$

which pass through (2, 8), then $2 \times 8 = c$

$\therefore \quad c = 16$

Equation of conic is $xy = 16$

$\therefore$Length of latusrectum $= 2\sqrt{2}\,(4) = 8\sqrt{2}$

25. $x = t^2 + 1, y = 2t$

$\Rightarrow \quad x = \left(\frac{y}{2}\right)^2 + 1$

$\Rightarrow \quad x = \frac{y^2}{4} + 1$

$\Rightarrow \quad y^2 = 4(x-1)$ ... (i)

Also, $\quad x = 2s, y = \frac{2}{s}$

$\therefore \quad xy = 4$ ...(ii)

From Eqs. (i) and (ii),

$$\left(\frac{4}{x}\right)^2 = 4(x-1)$$

$\Rightarrow \quad \frac{16}{x^2} = 4(x-1)$

$\Rightarrow \quad 4 = x^3 - x^2$

$\Rightarrow \quad x^3 - x^2 - 4 = 0$

$\Rightarrow \quad (x-2)(x^2 + x + 2) = 0$

$\therefore \quad x = 2, x^2 + x + 2 \neq 0$

From Eq. (ii),

$$y = \frac{4}{2} = 2$$

Point of intersection is (2, 2)

26. Let rectangular hyperbola $xy = c^2$

Equation of tangent at 't' is

$$\frac{x}{t} + yt = 2c$$

$\Rightarrow \quad \frac{x}{2ct} + \frac{y}{\left(\frac{2c}{t}\right)} = 1$ ... (i)

and equation of normal at 't' is

$$xt^3 - yt - ct^4 + c = 0$$

$\Rightarrow \quad xt^3 - ty = ct^4 - c$

$\Rightarrow \quad \frac{x}{\left(\frac{ct^4 - c}{t^3}\right)} + \frac{x}{\left(\frac{-ct^4 + c}{t}\right)} = 1$ ... (ii)

From Eqs. (i) and (ii) it is clear that

$$x_1 = 2ct, x_2 = \left(\frac{ct^4 - c}{t^3}\right)$$

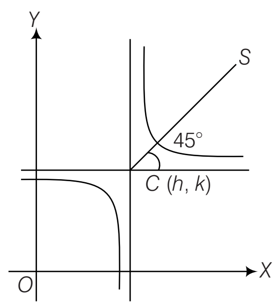
and $\quad y_1 = \frac{2c}{t}, y_2 = \left(\frac{-ct^4 + c}{t}\right)$

$\therefore \quad x_1x_2 + y_1y_2$

$$= \frac{2ct(ct^4 - c)}{t^3} + \frac{2c}{t}\cdot\frac{(-ct^4 + c)}{t}$$

$$= \frac{2c^2(t^4 - 1)}{t^2} + \frac{2c^2(t^4 - 1)}{t^2} = 0$$

27. $CS = p\sqrt{2}$



$\therefore$Coordinate of S is either

$$(h + p, k + p) \text{ or } (h - p, k - p)$$

28. $\because H : x^2 + 3xy + 2y^2 + 2x + 3y = 0$

Let pair of asymptotes is

$$x^2 + 3xy + 2y^2 + 2x + 3y + \lambda = 0$$

$\therefore \quad \Delta = 0$

$$1 \times 2 \times \lambda + 2 \times \frac{3}{2} \times 1 \times \frac{3}{2} - 1 \times \frac{9}{4} - 2 \times 1 - \lambda \times \frac{9}{4} = 0$$

$$\Rightarrow \quad -\frac{\lambda}{4} + \frac{9}{2} - \frac{9}{4} - 2 = 0$$

$$\Rightarrow \quad \frac{\lambda}{4} = \frac{9}{4} - 2 = \frac{1}{4}$$

$$\therefore \quad \lambda = 1$$

$$A : x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$$

$$\because \quad H + C = 2A$$

$$\therefore \quad C = 2A - H$$

$$= x^2 + 3xy + 2y^2 + 2x + 3y + 2$$

$\therefore$ Conjugate hyperbola is $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

29. The equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

The equation of the tangent is

$$y = mx + \sqrt{(9m^2 - 16)}$$

or $\quad \sqrt{(9m^2 - 16)} = 2\sqrt{5}$

or $\quad 9m^2 - 16 = 20$

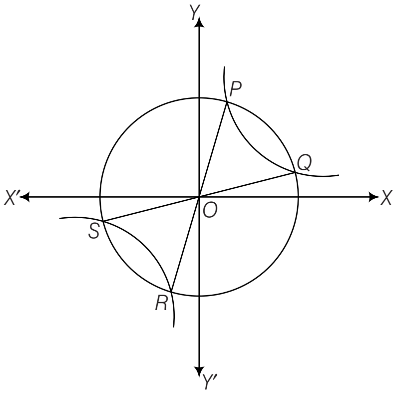
or $\quad m^2 = 4$

$\therefore \quad m = \pm 2$

or $\quad a + b = \text{sum of roots} = 0$

30. $\because x - 2 = m$ and $y + 1 = \frac{4}{m}$

$\therefore \quad (x - 2)(y + 1) = 4$



Let $\quad x - 2 = X, y + 1 = Y$

$\therefore \quad XY = 4 \quad \ldots \text{(i)}$

and $\quad s = (x - 2)^2 + (y + 1)^2 = 25$

or $\quad X^2 + Y^2 = 25$

Curve C and circle are concentric, therefore,

$$(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$$

$$= 4(5)^2$$

$$= 100$$

31. Equation of any tangent to the parabola $y^2 = 8x$

is $\quad y = mx + \frac{2}{m} \quad \ldots \text{(i)}$

which is also touches the hyperbola

$\frac{x^2}{1} - \frac{y^2}{3} = 1$, then $\left(\frac{2}{m}\right)^2 = 1 \times m^2 - 3$

or $\quad m^4 - 3m^2 - 4 = 0$

or $\quad (m^2 - 4)(m^2 + 1) = 0$

$\therefore \quad m^2 - 4 = 0$

or $\quad m = \pm 2$

From Eq. (i), common tangents are

$$y = 2x + 1 \text{ and } y = -2x - 1$$

i.e. $\quad 2x - y + 1 = 0$ and $2x + y + 1 = 0$

32. Foci of the ellipse $\frac{x^2}{k^2a^2} + \frac{y^2}{a^2} = 1$ are $(\pm a\sqrt{(k^2 - 1)}, 0)$ and foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $(\pm ae, 0)$, $e > 1$

$\because$ Foci are coincide, then

$a\sqrt{(k^2 - 1)} = ae \quad$ or $\quad \sqrt{(k^2 - 1)} > 1$

or $\quad k^2 - 1 > 1 \quad$ or $\quad k^2 > 2$

$\therefore \quad k < -\sqrt{2} \quad$ or $\quad k > \sqrt{2}$

33. Equation of chord joining θ and ϕ is

$$\frac{x}{a}\cos\left(\frac{\theta - \phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta + \phi}{2}\right)$$

It passes through $(kae, 0)$, where $k = \pm 1$, then

$$\frac{\cos\left(\frac{\theta - \phi}{2}\right)}{\cos\left(\frac{\theta + \phi}{2}\right)} = \frac{1}{ke}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\theta - \phi}{2}\right) - \cos\left(\frac{\theta + \phi}{2}\right)}{\cos\left(\frac{\theta - \phi}{2}\right) + \cos\left(\frac{\theta + \phi}{2}\right)} = \frac{1 - ke}{1 + ke}$$

$$\therefore \quad \tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right) = \frac{1 - ke}{1 + ke}$$

$$= \begin{cases} \frac{1 - e}{1 + e}, & \text{for } k = 1 \\ \frac{1 + e}{1 - e}, & \text{for } k = -1 \end{cases}$$

34. $\because$ Foci of hyperbola are $(\pm 3a, 0)$ and foci of ellipse are $(\pm\sqrt{(25 - 16)}, 0)$ i.e. $(\pm 3, 0)$ according to question.

$$a = 1$$

and $\quad b^2 = a^2(e^2 - 1) = 1(9 - 1) = 8$

Now, hyperbola is $\frac{x^2}{1} - \frac{y^2}{8} = 1$

Alternate (a) $a^2 + b^2 = 1 + 8 = 9$

Alternate (b) Director circle is $x^2 + y^2 = 1 - 8 = -7$

i.e. there is no director circle.

Alternate (d) Length of latusrectum $= \frac{2b^2}{a} = \frac{2(8)}{1} = 16$

35. $16x^2 - 3y^2 - 32x - 12y - 44 = 0$

$$\Rightarrow \quad 16(x^2 - 2x) - 3(y^2 + 4y) - 44 = 0$$

$$\Rightarrow 16\{(x - 1)^2 - 1\} - 3\{(y + 2)^2 - 4\} - 44 = 0$$

$$\Rightarrow \quad 16(x - 1)^2 - 3(y + 2)^2 = 48$$

$$\frac{(x - 1)^2}{(\sqrt{3})^2} - \frac{(y + 2)^2}{4^2} = 1$$

Alternate (a) : Length of transverse axis

$$= 2 \times \sqrt{3} = 2\sqrt{3}$$

Alternate (b) : Length of conjugate axis

$$= 2 \times 4 = 8$$

Alternate (c) : Centre $x - 1 = 0$

and $\quad y + 2 = 0$, i.e. $(1, -2)$

Alternate (d) : $4^2 = (\sqrt{3})^2 (e^2 - 1)$

or $\quad e^2 - 1 = \dfrac{16}{3}$

$\therefore \quad e^2 = \dfrac{19}{3}$

then $\quad e = \sqrt{\dfrac{19}{3}}$

36. $xy = 1$

Equation of normal at 't' i.e. $\left(t, \dfrac{1}{t}\right)$

is $\quad xt^3 - yt - t^4 + 1 = 0$

Slope is $\quad t^2 = -\dfrac{a}{b}$,

$$-\frac{a}{b} > 0 \Rightarrow \frac{a}{b} < 0$$

$\therefore \quad a > 0, b < 0$

or $\quad a < 0, b > 0$

37. P, Q, R, S lies on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (i)}$$

and also lies on

$$xy = c^2 \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), then

$$x^2 + \left(\frac{c^2}{x}\right)^2 + 2gx + \frac{2fc^2}{x} + c = 0$$

$\Rightarrow \quad x^4 + 2gx^3 + cx^2 + 2fc^2x + c^4 = 0$

$\therefore \quad x_1x_2x_3x_4 = c^4 \quad \text{... (iii)}$

and $\quad P \equiv (x_1, y_1) \equiv \left(x_1, \dfrac{c^2}{x_1}\right)$

$$Q \equiv \left(x_2, \frac{c^2}{x_2}\right)$$

and $\quad R \equiv \left(x_3, \dfrac{c^2}{x_3}\right)$

Let orthocentre $O \equiv (h, k)$

Then, slope of $QR \times$ slope of $OP = -1$

$$\left(\frac{\dfrac{c^2}{x_3} - \dfrac{c^2}{x_2}}{x_3 - x_2}\right) \times \left(\frac{k - \dfrac{c^2}{x_1}}{h - x_1}\right) = -1$$

$\Rightarrow \quad -\dfrac{c^2}{x_2x_3} \times \left(\dfrac{k - \dfrac{c^2}{x_1}}{h - x_1}\right) = -1$

$\Rightarrow \quad k - \dfrac{c^2}{x_1} = \dfrac{hx_2x_3}{c^2} - \dfrac{x_1x_2x_3}{c^2} \quad \text{... (iv)}$

Also, slope of $PQ \times$ slope of $OR = -1$

$$k - \frac{c^2}{x_3} = \frac{h\, x_1x_2}{c^2} - \frac{x_1x_2x_3}{c^2} \quad \text{...(v)}$$

From Eqs. (iii) and (iv),

$\therefore \quad h = -\dfrac{c^4}{x_1x_2x_3}$ and $k = -\dfrac{x_1x_2x_3}{c^2}$

From Eq. (iii),

$$h = -x_4 \text{ and } k = -\frac{c^2}{x_4}$$

$\therefore$ Orthocentre lies on $xy = c^2$

i.e. $\quad (x_4, y_4)$ and $(-x_4, -y_4)$

38. (c) $y = x + 5$

Comparing with $y = mx + c$

$\therefore \quad m = 1, c = 5$

Alternate (a) : Condition of tangency

$$c = \frac{a}{m}$$

$$5 = \frac{5}{1}$$

which is true.

Alternate (b) :

$$9x^2 + 16y^2 = 144$$

$\Rightarrow \quad \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

$\because$ Condition of tangency

$$c^2 = a^2m^2 + b^2$$

$\Rightarrow \quad 25 = 16 \times 1 + 9 = 25$

Which is true

Alternate (c) :

$$\frac{x^2}{29} - \frac{y^2}{4} = 1$$

$\because$ Condition of tangency

$$c^2 = a^2m^2 - b^2$$

$$25 = 29 \times 1 - 4 = 25$$

Which is true.

Alternate (d) : Now length of perpendicular from centre (0, 0) to the line $y = x + 5$ is $\dfrac{|5|}{\sqrt{2}}$ i.e. $\dfrac{5}{\sqrt{2}} \neq$ radius (5).

39. Equation of the directrices of the given hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ are

$$x = \pm \frac{5 \times 5}{\sqrt{(25 + 16)}} = \pm \frac{25}{\sqrt{41}} \quad \text{... (i)}$$

Equations of the asymptotes of the given hyperbola are

$$\frac{x^2}{15} - \frac{y^2}{16} = 0 \quad \text{...(ii)}$$

The points of intersection of Eqs. (i) and (ii) are

$(\pm 25 / \sqrt{41}, \pm 20 / \sqrt{41})$

40. $\because |SP - S'P| = 2a$

$$\Rightarrow \quad 2a = \sqrt{(24-0)^2 + (7-0)^2} - \sqrt{(5-0)^2 + (12-0)^2}$$

$$= 25 - 13 = 12$$

$$\therefore \quad a = 6$$

and $\quad 2ae = \sqrt{(24-5)^2 + (7-12)^2} = \sqrt{(386)}$

$$\therefore \quad e = \frac{\sqrt{386}}{12}$$

and $\quad b^2 = a^2(e^2 - 1) = 36\left(\frac{386}{144} - 1\right) = \frac{121}{2}$

$\therefore$ Latusrectum $= \dfrac{2b^2}{a} = \dfrac{121}{6}$

41. The locus of the point of intersection of perpendicular tangents is director circle

$$x^2 + y^2 = a^2 - b^2$$

Now, $\quad e^2 = 1 + \dfrac{b^2}{a^2}$

if $a^2 = b^2$, there is exactly one point (centre of the hyperbola)

i.e, $\quad e = \sqrt{2}$

if $\quad a^2 > b^2$ or $\dfrac{b^2}{a^2} < 1$

i.e. $\quad e^2 < 2$

or $0 < e < \sqrt{2}$, there are infinite (or more than one) points on the circle.

if $\quad a^2 < b^2$ or $\dfrac{b^2}{a^2} > 1$

i.e. $\quad e^2 > 2$

or $e > \sqrt{2}$, there does not exist any point on the plane.

42. **Alternate (a) :**

$$\left(\frac{2x}{a}\right)^2 - \left(\frac{2y}{b}\right)^2 = \left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2 = 4$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Alternate (b) :

$\because \quad \dfrac{tx}{a} - \dfrac{y}{b} + t = 0 \Rightarrow t = \dfrac{ay}{b(x+a)} \quad$ ... (i)

and $\quad \dfrac{x}{a} + \dfrac{ty}{b} - 1 = 0 \Rightarrow t = \dfrac{(a-x)\,b}{ay} \quad$ ... (ii)

From Eqs. (i) and (ii), we get

$$\frac{ay}{b\,(x+a)} = \frac{(a-x)b}{ay}$$

$$\Rightarrow \quad a^2y^2 = b^2(a^2 - x^2)$$

$$\Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2$$

or $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Alternate (c) : $x^2 - y^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2 = 4$

Alternate (d) : $\because x^2 - 6 = 2\cos t$

$\Rightarrow \quad x^2 - 4 = 2(1 + \cos t) = 4\cos^2\left(\dfrac{t}{2}\right) \quad$ ... (i)

and $\quad y^2 + 2 = 4\cos^2\left(\dfrac{t}{2}\right) \quad$ ... (ii)

From Eqs. (i) and (ii), then

$$x^2 - 4 = y^2 + 2$$

or $\quad x^2 - y^2 = 6$

43. Equation of any tangent to hyperbola is

$$y = mx \pm \sqrt{(a^2m^2 - b^2)} \quad \text{... (i)}$$

which is tangent of $-\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

then $\quad a^2m^2 - b^2 = a^2 - b^2m^2$

$$\Rightarrow \quad (a^2 + b^2)m^2 = (a^2 + b^2)$$

$$\therefore \quad m^2 = 1$$

or $\quad m = \pm 1$

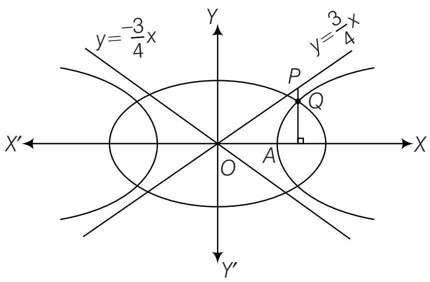
From Eq. (i), common tangent are

$$y = \pm x \pm \sqrt{(a^2 - b^2)}$$

44. Vertices and foci of ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$ are $(\pm 4, 0)$ and $(\pm 3, 0)$ respectively.

Vertices and foci of the hyperbola $\dfrac{x^2}{\left(\frac{144}{25}\right)} - \dfrac{y^1}{\left(\frac{81}{25}\right)} = 1$

are $\left(\pm \dfrac{12}{5}, 0\right)$ and $(\pm 3, 0)$, respectively.



For point of intersection

$$7\left(1 - \frac{x^2}{16}\right) = 81\left(\frac{x^2}{144} - \frac{1}{25}\right)$$

$$\Rightarrow \quad x^2 = \frac{256}{25}$$

and $\quad y^2 = 7\left(1 - \dfrac{16}{25}\right) = \dfrac{63}{25}$

= square of the ordinate of point of intersection

$$y^2 \text{ for } P = \frac{9}{16}x^2 = \frac{9}{16} \times \frac{256}{25} = \frac{144}{25}$$

$\therefore$ Sum of the squares of coordinates of P is $x^2 + y^2 = 16$ which is auxiliary circle formed by ellipse.

45. $(1 - x^2)\dfrac{dy}{dx} + xy = ax$

or $\quad \dfrac{2dy}{(a-y)} = \dfrac{2x\,dx}{(1-x^2)}$

$$\Rightarrow \quad -2\ln(a-y) = -\ln(1-x^2) - \ln c$$

or $\quad (a-y)^2 = c(1-x^2)$

$$\Rightarrow \quad cx^2 + (y-a)^2 = c$$

or $\quad \dfrac{x^2}{1} + \dfrac{(y-a)^2}{c} = 1$

for $c > 0$, ellipse and for $c < 0$, hyperbola centre of the conic is $(0, a)$ and length of one of the principal axes $= 2a = 2$

Sol. (Q. Nos. 46 to 48)

Give conic is

$$x^2 - y^2 + 2y - 2 = 0 \quad \text{... (i)}$$

Equation of tangent at (a,b) is

$$ax - by + (y+b) - 2 = 0$$

$\because$ Tangent line pass through (0, 0), then

$$b = 2$$

Also, (a, b) lies on Eq. (i), then

$$a^2 - b^2 + 2b - 2 = 0$$

or $\quad a^2 - 4 + 4 - 2 = 0$

$$\Rightarrow \quad a^2 = 2$$

$$\therefore \quad a = \pm\sqrt{2}$$

46. $\because$ Slope of tangent $= \dfrac{a}{b}$

$$= \frac{\sqrt{2}}{2} \quad \text{(for positive slope)}$$

$$= \frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin^{-1}\left(\frac{a}{b}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

47. $\because$ Conic is $x^2 - (y-1)^2 = 1$

$\therefore$ Length of latusrectum $= \dfrac{2(1)^2}{1} = 2$

48. $\because$ Given curve is rectangular hyperbola

$$\therefore \quad e = \sqrt{2}$$

Thus, $\quad 1 + e^2 + e^4 = 1 + 2 + 4 = 7$

Sol. (Q. Nos. 49 to 51)

Any point on the hyperbola is $(2t, 2/t)$ normal at this point is

$$xt^3 - yt - 2t^4 + 2 = 0$$

If the normal passes through $P(h,k)$, then

$$2t^4 - ht^3 + kt - 2 = 0$$

The equation has roots t_1, t_2, t_3, t_4, then

$$\Sigma t_1 = \frac{h}{2}, \Sigma t_1 t_2 = 0,$$

$$\Sigma t_1 t_2 t_3 = -\frac{k}{2}, t_1 t_2 t_3 t_4 = -1$$

$\because$ Feet of normals are $\left(2t_1, \dfrac{2}{t_1}\right), \left(2t_2, \dfrac{2}{t_2}\right),$

$$\left(2t_3, \frac{2}{t_3}\right) \text{ and } \left(2t_4, \frac{2}{t_4}\right)$$

$\because$ Sum of slopes of normals = Sum of ordinates of feet of normals

$$\therefore \quad t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{2}{t_1} + \frac{2}{t_2} + \frac{2}{t_3} + \frac{2}{t_4}$$

$$\Rightarrow \quad \Sigma t_1^2 = 2\Sigma\left(\frac{1}{t_1}\right)$$

49. $(\Sigma t_1)^2 - 2\Sigma t_1 t_2 = 2\left(\dfrac{\Sigma t_1 t_2 t_3}{t_1 t_2 t_3 t_4}\right)$

$$\Rightarrow \quad \left(\frac{h}{2}\right)^2 - 0 = 2\left(\frac{-k/2}{-1}\right) \text{ or } h^2 = 4k$$

Hence, the locus of (h, k) is $x^2 = 4y$

50. $\because$ Curve C is $x^2 = 4y$

$\therefore$ Equation of tangent at $(2t, t^2)$ is $tx = y + t^2$

$$\Rightarrow \quad A \equiv (t, 0) \text{ and } B \equiv (0, -t^2)$$

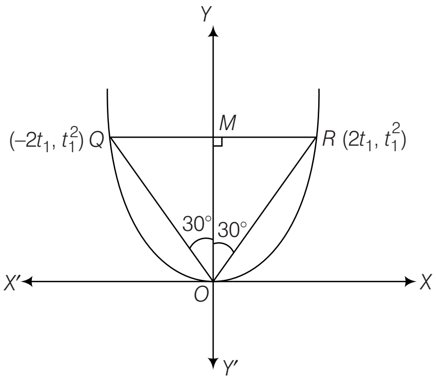
Let mid-point of AB is (h, k).

$$\therefore \quad 2h = t \text{ and } 2k = -t^2$$

$$\Rightarrow \quad 2k = -(2h)^2 \text{ or } 2h^2 + k = 0$$

$\therefore$ Required locus is $2x^2 + y = 0$

51.



$$\therefore \quad \tan 30° = \frac{2t_1}{t_1^2} \text{ or } \frac{1}{\sqrt{3}} = \frac{2}{t_1}$$

$$\therefore \quad t_1 = 2\sqrt{3}$$

$$\Rightarrow \quad QR = 4t_1 = 8\sqrt{3}$$

Area of equilateral ΔOQR

$$= \frac{\sqrt{3}}{4}(8\sqrt{3})^2 = 48\sqrt{3} \text{ sq units}$$

Sol. (Q. Nos. 52 to 54)

$\because S_1(1,2)$ and S_2 (5, 5) are the foci

then $\quad S_1S_2 = 5 = 2ae \quad$ ... (i)

and $\quad |S'P - SP| = 2a$

Here, $\quad 2a = 4 \quad$ ... (ii)

From Eqs. (i) and (ii)

$$e = \frac{5}{4}$$

52. Let e' be the eccentricity of conjugate hyperbola.

$$\therefore \quad \frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\Rightarrow \quad \frac{16}{25} + \frac{1}{e'^2} = 1 \Rightarrow \frac{1}{e'^2} = \frac{9}{25}$$

$$\therefore \quad e' = \frac{5}{3}$$

53. Locus of point of intersection of two mutually perpendicular tangents is the director circle given by

$$(x-h)^2+(y-k)^2=(a^2-b^2);$$

where (h, k) is the centre of hyperbola given by

$$(h,k)=\left(\frac{1+5}{2},\frac{2+5}{2}\right)\equiv\left(3,\frac{7}{2}\right)$$

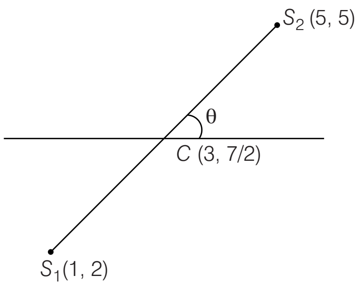
$\therefore$ $$a^2=4, b^2=a^2(e^2-1)$$

$$=4\left(\frac{25}{16}-1\right)=\frac{9}{4}$$

$\therefore$Required locus (director circle) is

$$(x-3)^2+\left(y-\frac{7}{2}\right)^2=4-\frac{9}{4}=\frac{7}{4}$$

54.



$\because$ θ should be the angle between the transverse axis and X-axis, given by

$$\tan\theta=\frac{5-2}{5-1}=\frac{3}{4}$$

$\therefore$ $$\theta=\tan^{-1}\left(\frac{3}{4}\right)$$

Sol. (Q. Nos. 55 to 57)

$\because$ $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ are the extremities of focal chord, then

$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)=\frac{1-e}{1+e} \text{ for focus } S(ae,0) \quad \text{...(i)}$$

and $$\theta=2\tan^{-1}\left(\frac{b}{a}\right) \quad \text{... (ii)}$$

Let $R\equiv(a\sec\phi, b\tan\phi)$

$\therefore$Equation of tangent at R is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1$$

$\therefore$ $A\equiv(a\cos\phi,0)$

and $B\equiv(0,-b\cot\phi)$

55. Let $C=(h,k)$

$\because$ $OACB$ is rectangle

$\therefore$ Mid-point of OC = mid-point of AB

$$\Rightarrow \quad \left(\frac{h}{2},\frac{k}{2}\right)\equiv\left(\frac{a\cos\phi}{2},\frac{-b\cot\phi}{2}\right)$$

or $$h=a\cos\phi \quad\Rightarrow\quad \sec\phi=\frac{a}{h} \quad \text{... (iii)}$$

and $$k=-b\cot\phi \quad\Rightarrow\quad \tan\phi=-\frac{b}{k} \quad \text{... (iv)}$$

From Eqs. (iii) and (iv), we get

$$\frac{a^2}{h^2}-\frac{b^2}{k^2}=1$$

Hence, required locus is

$$\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$$

56. From Eq. (i)

$$1+\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)=\frac{2}{1+e}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)}=\frac{2}{1+e} \quad \text{... (v)}$$

and $$1-\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)=\frac{2e}{1+e}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)}=\frac{2e}{1+e} \quad \text{... (vi)}$$

From Eqs. (v) and (vi), we get

$$\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}=e \text{ or } \frac{\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)}=e^2$$

or $$\lambda=e^2 \quad \text{(given)}$$

$$=\frac{a^2+b^2}{a^2}$$

57. From Eq. (ii),

$$\theta=2\tan^{-1}\left(\frac{b}{a}\right)$$

$\therefore$ $$\tan\left(\frac{\theta}{2}\right)=\frac{b}{a}$$

$$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right)=\frac{1}{\sec\left(\frac{\theta}{2}\right)}=\frac{1}{\sqrt{1+\tan^2\left(\frac{\theta}{2}\right)}}$$

$$=\frac{1}{\sqrt{\left(1+\frac{b^2}{a^2}\right)}}$$

$$=\frac{1}{\sqrt{1+e^2-1}}=\frac{1}{e}$$

Sol. (Q. Nos. 58 to 60)

58. $\because$ Perpendicular tangents intersect at the centre of rectangular hyperbola. Hence, centre of the hyperbola is (1, 1) and the equations of asymptotes are

$$x-1=0 \text{ and } y-1=0$$

$\therefore$Pair of asymptotes is

$$(x-1)(y-1)=0$$

or $$xy-x-y+1=0$$

or $$xy+1=x+y$$

59. Let equation of the hyperbola be

$$xy - x - y + \lambda = 0$$

It passes through (2, 3), then

$$6 - 2 - 3 + \lambda = 0$$

$$\therefore \quad \lambda = -1$$

So, equation of hyperbola is

$$xy = x + y + 1$$

60. From the centre of the hyperbola, we can draw two real tangents to the rectangular hyperbola.

61. For ellipse (if $a > b$)

$$ae_1 = \sqrt{(a^2 - b^2)}$$

or $\quad a^2e_1^2 = a^2 - b^2$

and for hyperbola

$$Ae_2 = \sqrt{(A^2 + B^2)}$$

$\Rightarrow \quad = \sqrt{(A^2 + b^2)} \qquad [\because 2b = 2B \text{ (given)}]$

or $\quad A^2e_2^2 = A^2 + b^2$

$$\Rightarrow \quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{(a^2 - b^2)} + \frac{A^2}{(A^2 + b^2)} \qquad \text{... (i)}$$

Since, both the curves are confocal $\Rightarrow ae_1 = Ae_2$

or $\quad a^2e_1^2 = A^2e_2^2$

$\Rightarrow \quad a^2 - b^2 = A^2 + b^2$

or $\quad A^2 = a^2 - 2b^2 \qquad \text{... (ii)}$

From Eqs. (i) and (ii), we get

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{a^2 - b^2} + \frac{a^2 - 2b^2}{a^2 - b^2}$$

$$= \frac{2(a^2 - b^2)}{(a^2 - b^2)} = 2$$

62. Othrocentre of triangle formed by the points $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ on the rectangular hyperbola.

$$xy = c^2 \text{ is } \left(-\frac{c}{t_1t_2t_3}, -c\, t_1t_2t_3\right)$$

Here, $\quad c = 2$ and $\dfrac{-c}{t_1t_2t_3} = \dfrac{1}{2}$ or $t_1t_2t_3 = -2c$

$\therefore$ Ordinate of orthocentre $= -c\, t_1t_2t_3$

$$= -c \times -2c$$

$$= 2c^2 = 2(2)^2$$

$$= 8$$

63. Since the normal drawn at $P(t_1)$ meets the hyperbola $xy = 2$ again at $Q(t_2)$, then

$$t_1^3\, t_2 = -1 \qquad \text{... (i)}$$

$$\because \quad P \equiv \left(\sqrt{2}\, t_1, \frac{\sqrt{2}}{t_1}\right) \text{ and } Q \equiv \left(\sqrt{2}\, t_2, \frac{\sqrt{2}}{t_2}\right)$$

$$\therefore \quad \text{Distance } PQ = \sqrt{(\sqrt{2}t_1 - \sqrt{2}t_2)^2 + \left(\frac{\sqrt{2}}{t_1} - \frac{\sqrt{2}}{t_2}\right)^2}$$

$$= \sqrt{2}\,|t_1 - t_2| \sqrt{\left(1 + \frac{1}{t_1^2t_2^2}\right)}$$

$$= \sqrt{2}\, \frac{(t_1^4 + 1)^{3/2}}{|t_1^3|} \qquad \text{[from Eq. (i)]}$$

$$= \sqrt{2}\left(t_1^2 + \frac{1}{t_1^2}\right)^{3/2} \geq \sqrt{2}\,(2)^{3/2} = 4 \quad (\because AM \geq GM)$$

$\therefore \quad$ Distance $PQ \geq 4$

Hence, minimum distance between the points P and Q is 4.

64. Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with eccentricity $e = \dfrac{3}{2\sqrt{2}}$

$$\therefore \quad a^2 = 8b^2 \qquad (\because b^2 = a^2(e^2 - 1) \quad \text{...(i)}$$

$\because$ Equation of normal at P is

$$ax\cos\theta + \text{by } \cot\theta = a^2 + b^2$$

$$\therefore \quad M \equiv \left(\left(\frac{a^2 + b^2}{a}\right)\sec\theta, 0\right)$$

$$N \equiv \left(0, \left(\frac{a^2 + b^2}{b}\right)\tan\theta\right)$$

Let mid-point of MN is $R(h,k)$

$$\therefore \quad 2h = \left(\frac{a^2 + b^2}{a}\right)\sec\theta \qquad \text{... (ii)}$$

and $\quad 2k = \left(\dfrac{a^2 + b^2}{b}\right)\tan\theta \qquad \text{... (iii)}$

From Eqs. (ii) and (iii), we get

$$4a^2h^2 - 4b^2k^2 = (a^2 + b^2)^2$$

$\therefore$ Locus of R is

$$\frac{x^2}{\left(\dfrac{a^2 + b^2}{2a}\right)^2} - \frac{y^2}{\left(\dfrac{a^2 + b^2}{2b}\right)^2} = 1$$

which have eccentricity

$$e_1 = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{(1 + 8)} = 3 \qquad \text{[from Eq. (i)]}$$

65. Equations of director circle of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 + b^2$$

$$\therefore \quad \text{Radius } (r_1) = \sqrt{(a^2 + b^2)}$$

$$= \sqrt{a^2 + a^2(1 - e_1^2)}$$

$$= a\sqrt{(2 - e_1^2)} \qquad \text{... (i)}$$

and director circle of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b_1^2} = 1$ is

$$x^2 + y^2 = a^2 - b_1^2$$

$$\therefore \quad \text{Radius } (r_2) = \sqrt{(a^2 - b_1^2)}$$

$$= \sqrt{a^2 - a^2(e_2^2 - 1)} = a\sqrt{(2 - e_2^2)} \qquad \text{... (ii)}$$

Given, $\dfrac{r_1}{r_2} = \dfrac{1}{3}$

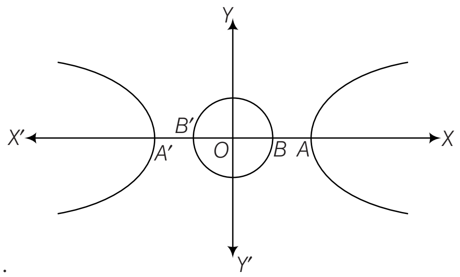
$\Rightarrow \quad \dfrac{a\sqrt{(2-e_1^2)}}{a\sqrt{(2-e_2^2)}} = \dfrac{1}{3}$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad 9 - 9e_1^2 = 2 - e_2^2$

$\therefore \quad 9e_1^2 - e_2^2 = 7 = \lambda$

$\therefore \quad \lambda = 7$

66. $\because \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $x^2 + y^2 = \dfrac{a^2}{4}$ having origin as their common centre.



So, vertex is the nearest point. Hence, shortest distance = BA

$$= a - \frac{a}{2} = \frac{a}{2}$$

$\therefore \quad f(a, b) = \dfrac{a}{2}$

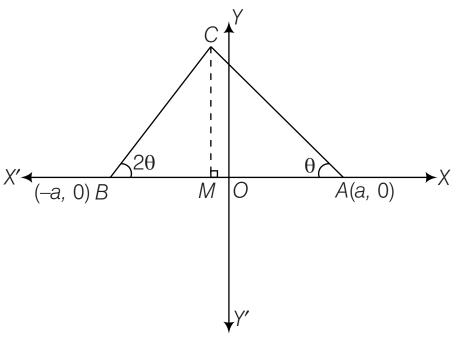
Hence, $\quad f(4,6) + f(2,3) = \dfrac{4}{2} + \dfrac{2}{2} = 3$

67. Let $A \equiv (a, 0)$ and $B \equiv (-a, 0)$

If $\quad C \equiv (x, y)$

$\therefore \quad \tan\theta = \dfrac{y}{a+x}$ ... (i)

and $\quad \tan 2\theta = \dfrac{y}{a-x}$ ... (ii)



From Eq. (ii),

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{y}{a-x}$$

$\Rightarrow \quad \dfrac{\dfrac{2y}{a+x}}{1-\left(\dfrac{y}{a+x}\right)^2} = \dfrac{y}{a-x}$ [from Eq. (i)]

or $\quad \dfrac{2y(a+x)}{a^2+x^2+2ax-y^2} = \dfrac{y}{a-x}$

$\Rightarrow \quad 2(a^2-x^2) = a^2 + x^2 + 2ax - y^2$

or $\quad 3x^2 + 2ax - y^2 - a^2 = 0$

or $\quad 3\left(x+\dfrac{a}{3}\right)^2 - y^2 = \dfrac{2a^2}{3}$

or $\quad \dfrac{\left(x+\dfrac{a}{3}\right)^2}{\left(\dfrac{2a^2}{9}\right)} - \dfrac{y^2}{\left(\dfrac{2a^2}{3}\right)} = 1$

$\therefore \quad$ Eccentricity $(e) = \sqrt{\left(\dfrac{\dfrac{2a^2}{9}+\dfrac{2a^2}{3}}{\dfrac{2a^2}{9}}\right)} = 2$

68. Let $P\left(\dfrac{t}{\sqrt{2}}, \dfrac{1}{t\sqrt{2}}\right)$ be any point on $2xy = 1$ and $S \equiv (1,1)$

and $\quad S' \equiv (-1,-1)$

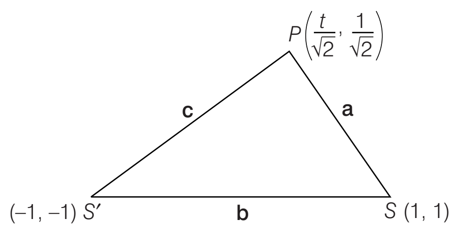
Here $\quad t > 0$

Let $R(h, k)$ be the ex-centre of $\Delta PSS'$ opposite to vertex S.

$\Rightarrow \quad h = \dfrac{-a-c+\dfrac{tb}{\sqrt{2}}}{a+b-c}$

and $\quad k = \dfrac{-a-c+\dfrac{b}{t\sqrt{2}}}{a+b-c}$

Here, $\quad b = 2\sqrt{2}, a = ePM$



$$= \sqrt{2}\,\frac{\left(\dfrac{t}{\sqrt{2}}+\dfrac{1}{t\sqrt{2}}-1\right)}{\sqrt{2}} = \frac{t}{\sqrt{2}} + \frac{1}{t\sqrt{2}} - 1$$

and $\quad c = \sqrt{2}\,\dfrac{\left(\dfrac{t}{\sqrt{2}}+\dfrac{1}{t\sqrt{2}}+1\right)}{\sqrt{2}}$

$= \dfrac{t}{\sqrt{2}} + \dfrac{1}{t\sqrt{2}} + 1 \quad$ ($\because$ Directrices are $x + y = \pm 1$)

$\therefore \quad h = \dfrac{-\sqrt{2}\left(t+\dfrac{1}{t}\right)+2t}{2\sqrt{2}-2}$

and $\quad k = \dfrac{-\sqrt{2}\left(t+\dfrac{1}{t}\right)+\dfrac{2}{t}}{2\sqrt{2}-2}$

$\therefore \quad h + k = -\left(t+\dfrac{1}{t}\right)$ ... (i) and $\quad h - k = \dfrac{t - \frac{1}{t}}{\sqrt{2} - 1}$ ... (ii)

$\because \quad \left(t + \dfrac{1}{t}\right)^2 = \left(t - \dfrac{1}{t}\right)^2 + 4$

$\Rightarrow \quad (h + k)^2 = (\sqrt{2} - 1)^2 (h - k)^2 + 4$

$\therefore$ Locus of R is $(x + y)^2 = (\sqrt{2} - 1)^2 (x - y)^2 + 4$

On comparing, we get $p = 1$ and $q = 4$

$\therefore \quad p + q = 5$

69. Let $P(h,k)$ be the middle-point, then equation of chord whose mid-point $P(h,k)$ is

$$T = S_1$$

$\Rightarrow \quad hx + ky - 4 = h^2 + k^2 - 4$

or $\quad y = -\dfrac{h}{k}x + \dfrac{(h^2 + k^2)}{k}$

It will touch the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$, then

$$\left(\frac{h^2 + k^2}{k}\right)^2 = 4 \times \frac{h^2}{k^2} - 16$$

$\Rightarrow \quad (h^2 + k^2)^2 = 4h^2 - 16k^2$

$\therefore$ Locus of $P(h,k)$ is

$$(x^2 + y^2)^2 = 4x^2 - 16y^2$$

$\therefore \quad \lambda = 4$

70. The given hyperbola is $\dfrac{x^2}{2} - \dfrac{y^2}{3} = 1$

Equation of tangents is $y = mx \pm \sqrt{(2m^2 - 3)}$

$\because$ Tangents from the point (α, β) will be

$$(\beta - m\alpha)^2 = 2m^2 - 3$$

or $\quad m^2(\alpha^2 - 2) - 2m\alpha\beta + \beta^2 + 3 = 0$

$\because \quad m_1 m_2 = \dfrac{\beta^2 + 3}{\alpha^2 - 2} \Rightarrow \tan\theta \tan\phi = \dfrac{\beta^2 + 3}{\alpha^2 - 2}$

$\Rightarrow \quad 2 = \dfrac{\beta^2 + 3}{\alpha^2 - 2}$

or $\quad 2\alpha^2 - \beta^2 = 7$

71. **(A)** $16x^2 - 9y^2 + 32x + 36y - 164 = 0$

$\Rightarrow 16(x^2 + 2x + 1) - 9(y^2 - 4y + 4) - 144 = 0$

or $\quad 16(x + 1)^2 - 9(y - 2)^2 = 144$

or $\quad \dfrac{(x + 1)^2}{9} - \dfrac{(y - 2)^2}{16} = 1$

$\therefore \quad \lambda =$ length latusrectum $= \dfrac{2 \times 16}{3} = \dfrac{32}{3}$

or $\quad 3\lambda = 32$

(B) Making $\dfrac{x^2}{16} - \dfrac{y^2}{18} = 1$ homogeneous with the help of $x\cos\alpha + y\sin\alpha = p$, then

$$\frac{x^2}{16} - \frac{y^2}{18} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$$

Since, these lines are perpendicular to each other, then

$$\left(\frac{1}{16} - \frac{\cos^2\alpha}{p^2}\right) + \left(-\frac{1}{18} - \frac{\sin^2\alpha}{p^2}\right) = 0$$

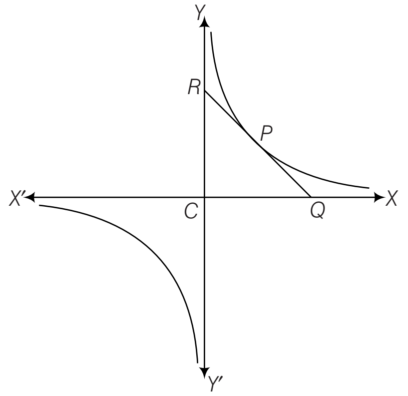
or $\quad \dfrac{1}{16} - \dfrac{1}{18} = \dfrac{1}{p^2}$

or $\quad p = \pm 12$

$\therefore$ Radius of circle $= |p| = 2$

Hence, diameter of the circle $= 24$

(C) Any point on $xy = 8$ is $P\left(\sqrt{8}\,t, \dfrac{\sqrt{8}}{t}\right)$



Equation of tangent at P

$$x \times \frac{\sqrt{8}}{t} + y \times \sqrt{8}\,t = 16 \text{ or } \frac{x}{\left(\frac{16t}{\sqrt{8}}\right)} + \frac{x}{\left(\frac{16}{\sqrt{8}\,t}\right)} = 1$$

$\therefore \quad CQ = \dfrac{16t}{\sqrt{8}}$ and $CR = \dfrac{16}{\sqrt{8}t}$

Hence, area of $\Delta CQR = \dfrac{1}{2} \times CQ \times CR$

$$= \frac{1}{2} \times \frac{16t}{\sqrt{8}} \times \frac{16}{\sqrt{8}t} = 16$$

(D) Hyperbola is $x^2 - 3y^2 = 9$

or $\quad \dfrac{x^2}{9} - \dfrac{y^2}{3} = 1$

$\because$ Angle between asymptotes

$$= 2\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 2 \times \frac{\pi}{6} = \frac{\pi\lambda}{24} \quad \text{(given)}$$

$\therefore \quad \lambda = 8$

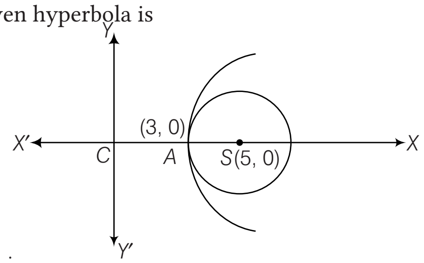
72. **(A)** Let the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

then $\quad 2a = ae$

$\Rightarrow \quad e = 2$

$\therefore \quad \dfrac{(2b)^2}{(2a)^2} = \dfrac{b^2}{a^2} = e^2 - 1 = (2)^2 - 1 = 3$

(B) Given hyperbola is



$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{... (i)}$$

or $\quad e^2 = 1 + \frac{16}{9} = \frac{25}{9}$

or $\quad e = \frac{5}{3}$

Hence, its foci are $(\pm 5, 0)$

The equation of the circle with (5, 0) as centre is

$$(x-5)^2 + (y-0)^2 = r^2 \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we have

$$16x^2 - 9(r^2 - (x-5)^2) = 144$$

or $\quad 25x^2 - 90x - 9r^2 + 81 = 0$

Since, the circle touches the hyperbola, the above equation must have equal roots.

Hence, $\quad (90)^2 - 4(25)(81 - 9r^2) = 0$

or $\quad r = 0$

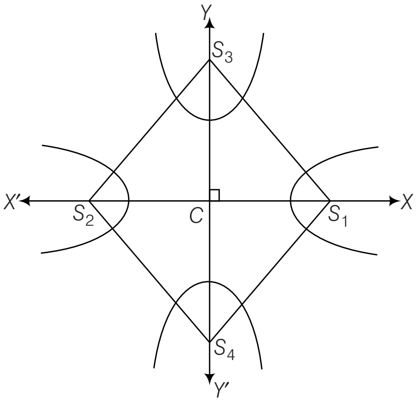
which is not possible.

Hence, the circle can not touch at two points. It can only be tangent at the vertex.

Hence, $\quad r = 5 - 3 = 2$

(C) Equation of hyperbola is

$$\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$$



Its foci $(\pm\sqrt{(2^2+3^2)}, 0)$

or $S_1 \equiv (\sqrt{13}, 0)$ and $S_2 \equiv (-\sqrt{13}, 0)$

and equation of conjugate hyperbola is

$$-\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

Its foci $(0, \pm\sqrt{(3^2+2^2)})$

or $S_3 \equiv (0, \sqrt{13})$ and $S_4 \equiv (0, -\sqrt{13})$

Hence, area of quadrilateral $S_1S_2S_3S_4$

$$= 4 \times \text{Area of } \Delta S_1CS_3$$
$$= 4 \times \frac{1}{2} \times CS_1 \times CS_3$$
$$= 2 \times \sqrt{13} \times \sqrt{13} = 26$$

(D) Given hyperbola is

$$ax^2 - by^2 = c$$

or $\quad \frac{x^2}{(c/a)} - \frac{y^2}{(c/b)} = 1$

$\because \quad \sqrt{\frac{c}{b}} = \frac{5}{2} \quad \text{...(i)}$

and $\quad 2\sqrt{\left(\frac{c}{a} + \frac{c}{b}\right)} = 13 \quad \text{...(ii)}$

From Eqs. (i) and (ii)

$$2\sqrt{\left(\frac{c}{a} + \frac{25}{4}\right)} = 13$$

or $\quad \frac{c}{a} = \frac{169}{4} - \frac{25}{4} = 36 \quad \text{... (iii)}$

and $\quad \frac{c}{b} = \frac{25}{4} \quad \text{... (iv)}$

$\therefore \quad \frac{a}{b} = \frac{\frac{c}{b}}{\frac{c}{a}} = \frac{25}{144}$

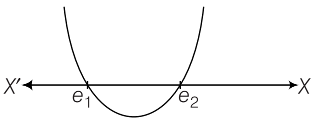
$\because$ a and b are co-prime

$\therefore$ $a = 25, b = 144$

and from Eq. (iii), $c = 900$

Hence, $\quad \frac{3ab}{2c} = \frac{3 \times 25 \times 144}{2 \times 900} = 6$

73. Let $f(x) = x^2 - \lambda x + 2$



$\therefore \quad D = \lambda^2 - 8$

x-coordinate of vertex $= \frac{\lambda}{2}$ and $f(1) = 3 - \lambda$

(A) We must have

$e_1 < 1 < e_2$

then $\quad D > 0$ and $f(1) < 0$

$\Rightarrow \quad \lambda^2 > 8$ and $3 - \lambda < 0$

$\Rightarrow \quad \lambda \in (-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$ and $\lambda > 3$

$\therefore \quad \lambda > 3$

(B) $\because \quad e_1 > 1, e_2 > 1$

then $\quad D \geq 0$ and $f(1) > 0$ and $\frac{\lambda}{2} > 1$

$\Rightarrow \quad \lambda \in (-\infty, -2\sqrt{2})] \cup [2\sqrt{2}, \infty)$

and $\quad \lambda < 3$ and $\lambda > 2$

$\therefore \quad \lambda \in [2\sqrt{2}, 3)$

(C) We must have $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$

or $\qquad e_1^2+e_2^2=(e_1e_2)^2$

$\Rightarrow \quad (e_1+e_2)^2-2e_1e_2=(e_1e_2)^2$

$\Rightarrow \qquad \lambda^2-4=(2)^2$ or $\lambda^2=8$

$\therefore \qquad \lambda=\pm 2\sqrt{2}$

(D) We must have $e_1<\sqrt{2}<e_2$

then $\qquad D>0$ and $f(\sqrt{2})<0$

$\Rightarrow \qquad \lambda^2>8$ and $2-\lambda\sqrt{2}+2<0$

or $\qquad \lambda\in(-\infty,-2\sqrt{2})\cup(2\sqrt{2},\infty)$

and $\qquad \lambda>2\sqrt{2}$

$\therefore \qquad \lambda>2\sqrt{2}$

74. Since, e_1 and e_2 are the eccentricities of two conjugate hyperbolas, so $e_1>1$ and $e_2>1$.

$\Rightarrow \qquad e_2e_2>1$

$\therefore$ Statement II is true.

As for e_1 and e_2 for hyperbola and its conjugate,

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

Let $\qquad e_1=\frac{5}{3}$ and $e_2=\frac{5}{4}$

Now, $\qquad \frac{1}{e_1^2}+\frac{1}{e_2^2}=\frac{9}{25}+\frac{16}{15}=1$

$\therefore$Statement I is true, but Statement II is not correct explanation of Statement I.

75. By definition, if $H(x,y)=0$

then $\qquad A_1(x,y)=H(x,y)+\lambda$

$\therefore$ Statement II is true

Since, $\qquad C(x,y)=H(x,y)+2$

$\qquad\qquad =A_1(x,y)+(2-\lambda)$

Since, for $A_1(x,y)=0$, we have $\Delta=0$ and also for $A_2(x,y)$, $\Delta=0$, So $A_1=A_2$

$\therefore$Statement I is true and Statement II is correct explanation of Statement I.

76. $\because$ Director circle is the locus of the point of intersection of perpendicular tangents.

$\therefore$ Statement II is true.

$\because$ Director circle of $5x^2-4y^2=20$

or $\qquad \frac{x^2}{4}-\frac{y^2}{5}=1$

is $\qquad x^2+y^2=4-5$ or $x^2+y^2=-1$

$\therefore$ Statement I is false.

77. Director circle of $\frac{x^2}{25}-\frac{y^2}{16}=1$ is

$$x^2+y^2=25-16=9$$

Hence, angle between tangents is $\pi/2$.

$\therefore$ Statement I and Statement II are true and Statement II is a correct explanation for Statement I.

78. The conjugate hyperbolas are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

then $\qquad e^2=\frac{a^2+b^2}{a^2}$ and $e_1^2=\frac{a^2+b^2}{b^2}$

$\therefore \qquad \frac{1}{e^2}+\frac{1}{e_1^2}=1$

$\because \qquad \frac{1}{2^2}+\frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2}=\frac{1}{4}+\frac{3}{4}=1$ and $ee_1=\frac{4}{\sqrt{3}}>1$

$\therefore$ Statement I and Statement II are both true and Statement II is not a correct explanation of Statement I.

79. If we solve $4x-5y=0$

or $\qquad y=\frac{4}{5}x$

and $\qquad 16x^2-25y^2=400$

or $\qquad \frac{x^2}{25}-\frac{y^2}{16}=1$

we get, $0=1\Rightarrow$ No solution

$\Rightarrow$ The line $y=\frac{4}{5}x$ does not meet but the line satisfies the condition of being a tangent.

$$c^2-a^2m^2+b^2=0-25\times\frac{16}{25}+16=0$$

$\Rightarrow$ It must touch the curve at infinity.

$\therefore$Statement I and Statement II are both true but Statement II is a correct explanation for Statement I.

80. The point (x_1,y_1) inside the hyperbola.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0$

$\therefore$ Statement II is false.

Also, $\qquad 3y^2-5x^2+1=0$

or $\qquad 5x^2-3y^2-1=0$

$\therefore$ Value of $5x^2-3y^2-1$ at $(5,-3)$ is

$\qquad 5(5)^2-3(-3)^2-1$

$\qquad =125-27-1=97>0$

$\therefore$ Statement I is true.

81. If hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\therefore$Angle between asymptotes $=2\tan^{-1}\left(\frac{b}{a}\right)$

For equilateral hyperbola $a=b$

then, angle between asymptotes

$$=2\tan^{-1}(1)=2\,.\,\frac{\pi}{4}=\frac{\pi}{2}$$

$\therefore$Statement I is false.

and eccentricity $e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}=\sqrt{\left(\frac{a^2+a^2}{a^2}\right)}=\sqrt{2}$

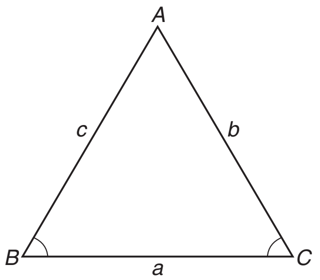
$\therefore$Statement 2 is true.

82. $BC =$ base of the triangle $= a$ (constant) and A is the vertex.

$$\frac{\tan(B/2)}{\tan(C/2)} = \frac{\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}}{\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}}$$

$$= \frac{s-c}{s-b} = \frac{2s-2c}{2s-2b}$$

$$= \frac{a+b-c}{a-b+c} = k \qquad \text{(let)}$$



By componendo and dividendo

$$\Rightarrow \qquad \frac{k-1}{k+1} = \frac{b-c}{a}$$

$$\Rightarrow \qquad b - c = a\left(\frac{k-1}{k+1}\right) = \text{ constant}$$

$$\Rightarrow \qquad CA - BA = \text{constant}$$

By the focal property the locus of A is a hyperbola whose foci are B and C.

83. Let coordinates of A, B and C on the hyperbola $xy = c^2$ are $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

(i) $\therefore$ Area of triangle $ABC = \dfrac{1}{2}\left| \begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix} \right|$

$$= \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\left|t_2^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\left|(t_1-t_2)(t_2-t_3)(t_3-t_1)\right|$$

(ii) Equations of tangents at A, B, C are

$$x + yt_1^2 - 2ct_1 = 0$$

$$x + yt_2^2 - 2ct_2 = 0$$

and $$x + yt_3^2 - 2ct_3 = 0$$

$$\therefore \qquad \text{Required area } = \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2 \qquad \text{... (i)}$$

where $C_1 = \begin{vmatrix} 1 & t_2^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2 \end{vmatrix}$

$$\therefore \qquad C_1 = t_3^2 - t_2^2, C_2 = t_1^2 - t_3^2 \quad \text{and} \quad C_3 = t_2^2 - t_1^2$$

From Eq. (i)

$$= \frac{1}{2|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)|} \; 4c^2 \cdot (t_1 - t_2)^2 (t_2 - t_3)^2 (t_3 - t_1)^2$$

$$= 2c^2\left|\frac{(t_1-t_2)(t_2-t_3)(t_3-t_1)}{(t_1+t_2)(t_2+t_3)(t_3+t_1)}\right|$$

$\therefore$ Required area is, $2c^2\left|\dfrac{(t_1-t_2)(t_2-t_3)(t_3-t_1)}{(t_1+t_2)(t_2+t_3)(t_3+t_1)}\right|$

84. Let chord be PQ and coordinates of P and Q are $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ respectively.

Now let mid-point of PQ is $M(h, k)$

$$\therefore \qquad h = \frac{1}{2}(ct_1 + ct_2)$$

$$\therefore \qquad t_1 + t_2 = \frac{2h}{c} \quad \text{and} \quad k = \frac{1}{2}\left\{\frac{c}{t_1} + \frac{c}{t_2}\right\}$$

$$\text{or} \qquad \frac{2k}{c} = \frac{t_1 + t_2}{t_1t_2}$$

$$\text{or} \qquad \frac{2k}{c} = \frac{\frac{2h}{c}}{t_1t_2} \qquad \left(\because t_1 + t_2 = \frac{2h}{c}\right)$$

$$\text{or} \qquad t_1t_2 = \frac{h}{k}$$

$$\therefore \qquad (t_1 - t_2)^2 = (t_1 + t_2)^2 - 4t_1t_2 = \frac{4h^2}{c^2} - \frac{4h}{k}$$

$$= \frac{4h}{c^2k}(hk - c^2) \qquad \text{...(i)}$$

Since, $|PQ| = 2d$

$$\therefore \qquad \sqrt{(ct_1 - ct_2)^2 + \left(\frac{c}{t_1} - \frac{c}{t_2}\right)^2} = 2d$$

$$\text{or} \qquad c^2(t_1 - t_2)^2\left\{1 + \frac{1}{t_1^2t_2^2}\right\} = (2d)^2$$

$$\text{or} \qquad c^2 \times \frac{4h}{c^2k}(hk - c^2)\left\{1 + \frac{k^2}{h^2}\right\} = 4d^2 \qquad \text{[from Eq. (i)]}$$

$$\text{or} \qquad (h^2 + k^2)(hk - c^2) = d^2hk$$

Hence, locus of the middle-point of PQ is

$$(x^2 + y^2)(xy - c^2) = d^2xy.$$

85. Let the rectangular hyperbola is

$$xy = c^2 \qquad \text{...(i)}$$

Since, the centre of hyperbola (i) is origin (0, 0) and equation of asymptotes are $x = 0$ and $y = 0$.

The equation of a line through (0, 0) and makes an angle θ with asymptote (X-axis) is $y = x\tan\theta$.

It will meet the hyperbola, where $x\,(x \tan \theta) = c^2$

i.e. $\quad x = c\sqrt{\cot \theta}$

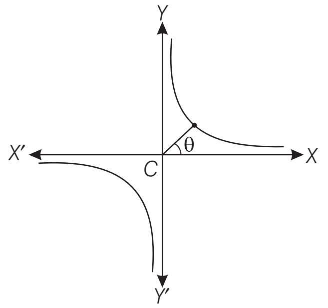
Putting $\quad x = c\sqrt{\cot \theta}$ in (1)

then $\quad y = c\sqrt{(\tan \theta)}$

$\therefore$ The four points are $(c\sqrt{\cot \theta}, c\sqrt{\tan \theta})$ where

$$Q = \alpha, \beta, \gamma, \delta.$$

The line joining the points α and β is perpendicular to the line joining the points γ and δ.



Therefore, the product of their slopes $= -1$

i.e. $$\frac{c\sqrt{\tan \beta} - c\sqrt{\tan \alpha}}{c\sqrt{\cot \beta} - c\sqrt{\cot \alpha}} \times \frac{c\sqrt{\tan \delta} - c\sqrt{\tan \gamma}}{c\sqrt{\cot \delta} - c\sqrt{\cot \gamma}} = -1$$

$\Rightarrow \quad (-\sqrt{\tan \alpha}\sqrt{\tan \beta}) \times (-\sqrt{\tan \gamma}\sqrt{\tan \delta}) = -1$

or $\quad \tan \alpha \tan \beta \tan \gamma \tan \delta = 1$

86. Let $P\left(ct_1, \dfrac{c}{t_1}\right)$ and $Q\left(ct_2, \dfrac{c}{t_2}\right)$ be any two points on $xy = c^2$.

Then tangents at P and Q are

$$x + yt_1^2 = 2ct_1 \qquad \text{...(i)}$$

and $$x + yt_2^2 = 2ct_2 \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), the point of intersection, say (h, k), is given by

$$h = \frac{2ct_1t_2}{t_1 + t_2} \quad \text{and} \quad k = \frac{2c}{t_1 + t_2} \qquad \text{...(iii)}$$

The foot of the ordinate of P is $(ct_1, 0)$ and it lies on Eq. (ii) then

$$ct_1 + 0 = 2ct_2$$

$\therefore$ $$t_1 = 2t_2 \qquad \text{...(iv)}$$

Then, from Eqs. (iii) and (iv),

$$h = \frac{2c \cdot 2t_2 \cdot t_2}{2t_2 + t_2} \quad \text{and} \quad k = \frac{2c}{2t_2 + t_2}$$

$\therefore$ $$h = \frac{4c}{3}t_2 \quad \text{and} \quad k = \frac{2c}{3t_2}$$

$\therefore$ $$h \cdot k = \frac{4c}{3}t_2 \times \frac{2c}{3t_2}$$

$\therefore$ $$hk = \frac{8}{9}c^2$$

$\therefore$ Locus of (h, k) is $xy = \dfrac{8}{9}c^2$, which is a rectangular hyperbola with the same asymptotes $x = 0$ and $y = 0$ as those of $xy = c^2$.

87. Take two given perpendicular straight lines as the coordinate axes and let the equation of variable circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Suppose circle (i) make an intercept of length a on X-axis and an intercept of length b on Y-axis.

$\therefore$ $$a = 2\sqrt{(g^2 - c)}$$

and $$b = 2\sqrt{(f^2 - c)}$$

Squaring and subtracting these, we get

$$4(g^2 - c) - 4(f^2 - c) = a^2 - b^2$$

or $$g^2 - f^2 = \frac{1}{4}(a^2 - b^2)$$

or $$(-g)^2 - (-f)^2 = \frac{1}{4}(a^2 - b^2)$$

Hence locus of the centre of circle $(-g, -f)$ is

$$x^2 - y^2 = \frac{1}{4}(a^2 - b^2)$$

which is a rectangular hyperbola.

88. Let the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

Its asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0$$

(a) Let us take the asymptote

$$\frac{x}{a} - \frac{y}{b} = 0 \Rightarrow y = \frac{b}{a}x \qquad \text{...(ii)}$$

Any line parallel to Eq. (ii) is

$$y = \frac{b}{a}x + c \qquad \text{...(iii)}$$

Eliminate y from Eqs. (i) and (iii), then

$$\frac{x^2}{a^2} - \frac{1}{b^2}\left(\frac{b}{a}x + c\right)^2 = 1$$

$\Rightarrow$ $$\frac{x^2}{a^2} - \frac{1}{b^2}\left(\frac{b^2x^2}{a^2} + c^2 + \frac{2bc}{a}x\right) = 1$$

$\Rightarrow$ $$0 \cdot x^2 - \frac{2c}{ab}x - \frac{(c^2 + b^2)}{b^2} = 0$$

One root of its equation is infinite since coefficient of x^2 in it is zero. Also from Eq. (iii), when $y \to \infty$ as $x \to \infty$.

$\therefore$ Eqs. (i) and (iii) meet in one point at infinity.

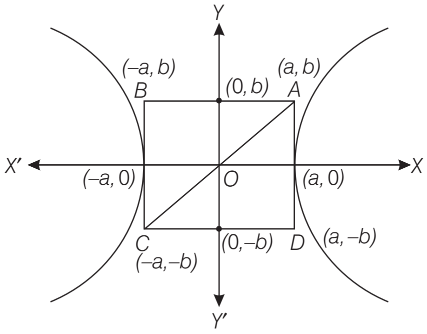
(b) Lines through $(a, 0)$; $(-a, 0)$; $(0, b)$ and $(0, -b)$ are parallel to the principal axes, enclose a rectangle whose vertices are $A\,(a, b)$, $B\,(-a, b)$, $C\,(-a, -b)$ and $D\,(a, -b)$. Now equation of diagonal AC is

$$y - b = \frac{b - (-b)}{a - (-a)}(x - a)$$

$\Rightarrow$ $$y - b = \frac{b}{a}(x - a)$$

$\Rightarrow$ $$y = \frac{b}{a}x$$

$\Rightarrow$ $$\frac{x}{a} - \frac{y}{b} = 0$$



i.e. The diagonal AC is one of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Similarly the diagonal BD is the other asymptote.

89. Let the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Tangent to Eq. (i) at $P\,(a\cos\phi,\, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

Coordinate of point B is

$$B \equiv (a\sec\phi,\, 0) \text{ and } AB = a\{\sec\phi - \cos\phi\}$$

Let coordinate of Q be (x_1, y_1) then

$$x_1 = a\cos\phi \text{ and } y_1 = a(\sec\phi - \cos\phi)$$

So $$y_1 = a\left(\frac{a}{x_1} - \frac{x_1}{a}\right)$$

$\Rightarrow$ $$x_1 y_1 = a^2 - x_1^2$$

Hence, locus of Q is $xy = a^2 - x^2$ ...(ii)

which is clearly a hyperbola.

Since, the equation of a hyperbola and its asymptotes differ in constant terms only, asymptotes of Eq. (ii) are given by $x^2 + xy - a^2 + k = 0$, k is any constant.

It represents two straight lines. The required condition for this is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

then $$k = a^2$$

$\therefore$ Asymptotes are

$$x = 0 \quad \text{and} \quad x + y = 0$$

90. The given rectangular hyperbola is

$$xy = c^2 \quad \text{...(i)}$$

Equations of chords of contact of (x_1, y_1) and (x_2, y_2) w.r.t., $xy = c^2$ are

$$xy_1 + x_1 y = 2c^2 \quad \text{...(ii)}$$

and $$xy_2 + x_2 y = 2c^2 \quad \text{...(iii)}$$

The equation of the conic passing through Eq. (i) with Eqs. (ii) and (iii) is

$$(xy_1 + x_1 y - 2c^2)(xy_2 + x_2 y - 2c^2) + \lambda$$
$$(xy - c^2) = 0 \quad \text{...(iv)}$$

Now, Eq. (iv) represents a circle

$\therefore$ coefficient of x^2 = coefficient of y^2

$\therefore$ $$y_1 y_2 = x_1 x_2$$

and coefficient of $xy = 0$

$\Rightarrow$ $$x_1 y_2 + x_2 y_1 + \lambda = 0 \quad \text{...(v)}$$

Again the conic (iv) passes through (x_1, y_1) and (x_2, y_2) then

$$(2x_1 y_1 - 2c^2)(x_1 y_2 + x_2 y_1 - 2c^2) + \lambda\,(x_1 y_1 - c^2) = 0$$

or $$2(x_1 y_2 + x_2 y_1 - 2c^2) + \lambda = 0 \quad (\because x_1 y_1 \ne c^2) \text{...(vi)}$$

and $$(x_2 y_1 + y_2 x_1 - 2c^2)(y_2 x_2 + x_2 y_2 - 2c^2) + \lambda\,(x_2 y_2 - c^2) = 0$$

$\Rightarrow$ $$2(x_2 y_1 + x_1 y_2 - 2c^2) + \lambda = 0 \quad (\because x_2 y_2 \ne c^2) \text{... (vii)}$$

From Eqs. (v) and (vi), we get

$$2(-\lambda - 2c^2) + \lambda = 0$$
$$-\lambda - 4c^2 = 0$$

$\therefore$ $$\lambda = -4c^2$$

Then from Eq. (v)

$$x_1 y_2 + x_2 y_1 = 4c^2$$

which is the other condition.

91. Let two perpendicular asymptotes of a rectangular hyperbola are CD and CE.

Let the coordinates of A, B and C are $(a, 0)$, $(0, b)$, (α, β). OX and OY are parallel to CD and CE.

Then equations of CD and CE are

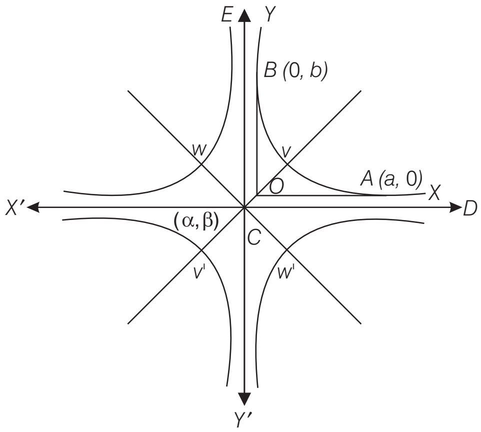
$$y - \beta = 0 \quad \text{and} \quad x - \alpha = 0$$

Thus combined equation of CD and CE is

$$(x - \alpha)(y - \beta) = 0$$

or $$xy - \alpha y - \beta x + \alpha\beta = 0$$

is the equation of asymptotes of rectangular hyperbola.



Hence, equation of rectangular hyperbola is

$$xy - \alpha y - \beta x + k = 0,$$

where k is any constant.

It passes through $A\,(a, 0)$ and $B\,(0, b)$

$\therefore$ $$0 - 0 - \beta a + k = 0$$

and $$0 - \alpha b - 0 + k = 0$$

$\therefore$ $$\beta = \frac{k}{a} \quad \text{and} \quad \alpha = \frac{k}{b}$$

Hence equation of rectangular hyperbola becomes

$$xy - k\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0 \quad \text{...(i)}$$

Eq. (i) represents two sets of rectangular hyperbolas whose vertices lie on the straight lines VCV' and WCW' both passing through (α, β) inclined at 45° and 135° to OX.

Then $\alpha = \dfrac{k}{b}, \beta = \dfrac{k}{a}$ and their equations are

$$y - \beta = \pm (x - \alpha)$$

$$\Rightarrow \quad y - \frac{k}{a} = \pm \left(x - \frac{k}{b}\right)$$

$$\text{or} \quad x + y = k\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$\text{and} \quad x - y = k\left(\frac{1}{b} - \frac{1}{a}\right) \quad \text{...(ii)}$$

$$\text{or} \quad k = \frac{ab\,(x+y)}{(a+b)}$$

$$\text{and} \quad k = \frac{ab\,(x-y)}{(a-b)}$$

Putting in Eq. (i) then

$$\frac{x^2}{a} + \frac{y^2}{b} - x - y = 0$$

$$\text{and} \quad \frac{x^2}{b} - \frac{y^2}{a} - x + y = 0$$

Differentiating these equations, we get

$$\frac{2x}{a} + \frac{2y}{b}\frac{dy}{dx} - 1 - \frac{dy}{dx} = 0$$

$$\text{and} \quad \frac{2x}{b} - \frac{2y}{a}\frac{dy}{dx} - 1 + \frac{dy}{dx} = 0$$

$\therefore$ Slopes of lines are

$$m_1 = \frac{dy}{dx} = \frac{b\,(a-2x)}{a\,(2y-b)}$$

$$\text{and} \quad m_2 = \frac{dy}{dx} = \frac{a\,(2x-b)}{b\,(2y-a)}$$

At (0, 0) $\quad m_1 = -1 \quad$ and $\quad m_2 = 1$

$$\therefore \quad m_1 m_2 = -1$$

which shows that curves intersect orthogonally.

92. Equation of the normal to the hyperbola $xy = 1$ at t is

$$xt^3 - yt - t^4 + 1 = 0 \quad \text{...(i)}$$

$\because$ It passes through (α, β)

$$\therefore \quad t^4 - \alpha t^3 + \beta t - 1 = 0$$

If foot of the co-normal points are

$$\left(t_1, \frac{1}{t_1}\right), \left(t_2, \frac{1}{t_2}\right), \left(t_3, \frac{1}{t_3}\right) \text{and} \left(t_4, \frac{1}{t_4}\right)$$

$$\therefore \quad t_1 + t_2 + t_3 + t_4 = \alpha$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = \alpha$$

$$\text{and} \quad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \beta$$

$$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = \beta$$

Let the variable line be

$$px + qy + r = 0 \quad \text{...(ii)}$$

Since, algebraic sum of perpendicular distances from $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) to Eq. (ii) is zero.

$$\left(\frac{px_1 + qy_1 + r}{\sqrt{(p^2+q^2)}}\right) + \left(\frac{px_2 + qy_2 + r}{\sqrt{p^2+q^2}}\right) + \left(\frac{px_3 + qy_3 + r}{\sqrt{(p^2+q^2)}}\right) + \left(\frac{px_4 + qy_4 + r}{\sqrt{(p^2+q^2)}}\right) = 0$$

$$\Rightarrow \quad p\,(x_1 + x_2 + x_3 + x_4) + q\,(y_1 + y_2 + y_3 + y_4) + 4r = 0$$

$$\Rightarrow \quad p\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) + q\left(\frac{y_1 + y_2 + y_3 + y_4}{4}\right) + r = 0$$

$$\Rightarrow \quad p \cdot \frac{\alpha}{4} + q \cdot \frac{\beta}{4} + r = 0$$

$\therefore$ Eq. (ii) passes through $\left(\dfrac{\alpha}{4}, \dfrac{\beta}{4}\right)$.

93. Let the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

whose asymptotes are

$$\frac{x}{a} + \frac{y}{b} = 0$$

$$\text{and} \quad \frac{x}{a} - \frac{y}{b} = 0$$

Let there be any point $P\,(h, k)$ on the hyperbola

$k =$ distance of P from transverse axis

$h =$ distance of P from any asymptotes

$$\text{say}\left(\frac{x}{a} - \frac{y}{b} = 0\right)$$

$$\therefore \quad k = \frac{bh - ak}{\sqrt{(a^2+b^2)}}$$

Squaring $(a^2 + b^2)\,k^2 = b^2h^2 + a^2k^2 - 2ab\,hk$

$$\Rightarrow \quad b\,(h^2 - k^2) = 2ahk$$

$$\Rightarrow \quad (h^2 - k^2) = \frac{2ahk}{b}$$

Squaring $\quad (h^2 - k^2)^2 = 4a^2h^2\left(\dfrac{k^2}{b^2}\right)$

$$= 4a^2h^2\left(\frac{h^2}{a^2} - 1\right) \qquad \left\{\because \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1\right\}$$

$$\Rightarrow \quad (h^2 - k^2)^2 = 4h^2(h^2 - a^2)$$

Hence locus of $P\,(h, k)$ is

$$(x^2 - y^2)^2 = 4x^2(x^2 - a^2).$$

94. Tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Given that $y = \alpha x + \beta$ is the tangent of hyperbola

$$\Rightarrow \quad m = \alpha \text{ and } a^2m^2 - b^2 = \beta^2$$

$$\therefore \quad a^2\alpha^2 - b^2 = \beta^2$$

Locus is $a^2x^2 - y^2 = b^2$ which is hyperbola.

95. For the given ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

$\Rightarrow \qquad e=\sqrt{\left(1-\dfrac{16}{25}\right)}=\dfrac{3}{5}$

$\Rightarrow$ Eccentricity of hyperbola $=\dfrac{5}{3}$

Let the hyperbola be $\dfrac{x^2}{A^2}-\dfrac{y^2}{B^2}=1$ then

$$B^2=A^2\left(\frac{25}{9}-1\right)=\frac{16}{9}A^2$$

$\therefore\ \dfrac{x^2}{A^2}-\dfrac{9y^2}{16A^2}=1$, As it passes through focus of ellipse i.e. $(3,0)$

$\therefore$ We get $A^2=9\Rightarrow B^2=16$

$\therefore$ Equation of hyperbola is $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$, focus of hyperbola is $(5, 0)$, vertex of hyperbola is $(3, 0)$.

96. The length of transverse axis $=2\sin\theta=2a$

$\Rightarrow \qquad a=\sin\theta$

Also for ellipse $3x^2+4y^2=12$

or $\qquad \dfrac{x^2}{4}+\dfrac{y^2}{3}=1,\ a^2=4,\ b^2=3$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$$

$\therefore$ Focus of ellipse $=\left(2\times\dfrac{1}{2},\ 0\right)\Rightarrow(1,0)$

As hyperbola is confocal with ellipse, focus of hyperbola $=(1,0)$

$\Rightarrow \qquad ae=1\ \Rightarrow\ \sin\theta\times e=1$

$\Rightarrow \qquad e=\operatorname{cosec}\theta$

$\therefore \qquad b^2=a^2(e^2-1)$

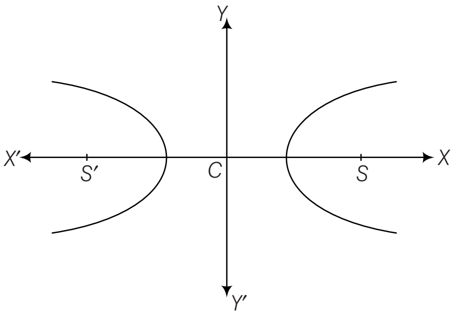
$\qquad\qquad =\sin^2\theta(\operatorname{cosec}^2\theta-1)=\cos^2\theta$

$\therefore$ Equation of hyperbola is

$$\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$$

or, $\qquad x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$

97. Two branches of hyperbola have no common tangent but have a common normal joining SS'.



98. Given, equation of hyperbola is

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$$

Here, $a^2=\cos^2\alpha$ and $b^2=\sin^2\alpha$

$\because \qquad b^2=a^2(e^2-1)$

$\therefore \qquad \sin^2\alpha=\cos^2\alpha\,(e^2-1)$

or $\sin^2\alpha+\cos^2\alpha=\cos^2\alpha\,.\,e^2$

or $e^2=1+\tan^2\alpha=\sec^2\alpha\Rightarrow e=\sec\alpha$

$\therefore \qquad ae=\cos\alpha\,.\,\dfrac{1}{\cos\alpha}=1$

Coordinates of foci are $(\pm ae,\ 0)$ i.e. $(\pm 1, 0)$

Hence, abscissae of foci remain constant, when α varies.

99. The given hyperbola is

$$x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$$

$\Rightarrow\ (x^2-2\sqrt{2}x+2)-2(y^2+2\sqrt{2}y+2)=6+2-4$

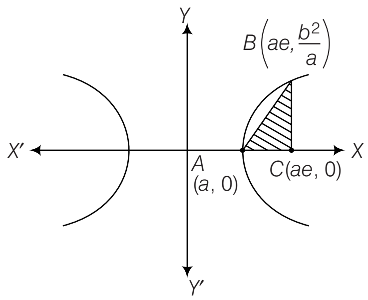
$\Rightarrow \qquad (x-\sqrt{2})^2-2(y+\sqrt{2})^2=4$

$\Rightarrow \qquad \dfrac{(x-\sqrt{2})^2}{2^2}-\dfrac{(y+\sqrt{2})^2}{(\sqrt{2})^2}=1$

$\therefore \qquad a=2,\ b=\sqrt{2}\ \ \Rightarrow\ e=\sqrt{\left(1+\dfrac{2}{4}\right)}=\sqrt{\dfrac{3}{2}}$

Clearly, ΔABC is a right triangle.

$$B\left(ae,\frac{b^2}{a}\right)$$



$\therefore$ Area of $(\Delta ABC)=\dfrac{1}{2}\times AC\times BC$

$$=\frac{1}{2}\times(ae-a)\times\frac{b^2}{a}$$

$$=\frac{1}{2}(e-1)\,b^2=\frac{1}{2}\left(\sqrt{\frac{3}{2}}-1\right).\,2$$

$$=\left(\sqrt{\frac{3}{2}}-1\right)$$

100. The given hyperbola is

$$x^2-y^2=\frac{1}{2} \qquad \text{... (i)}$$

which is rectangular hyperbola

$\therefore \qquad e=\sqrt{2}$

Let the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Its eccentricity $=\dfrac{1}{\sqrt{2}}$

$\therefore \qquad b^2=a^2(1-e^2)=a^2\left(1-\dfrac{1}{2}\right)=\dfrac{a^2}{2}$

So, the equation of ellipse becomes

$$x^2 + 2y^2 = a^2 \quad \text{... (ii)}$$

Let the hyperbola (i) and ellipse (ii) intersect each other at $P(x_1, y_1)$.

$\therefore$(Slope of hyperbola (i) at (x_1, y_1)) $\times$ (Slope of ellipse (ii) at (x_1, y_1)) $= -1$

$$\Rightarrow \quad \frac{x_1}{y_1} \times \frac{-x_1}{2y_1} = -1$$

$$\text{or} \quad x_1^2 = 2y_1^2 \quad \text{... (iii)}$$

Also (x_1, y_1) lies on $x^2 - y^2 = \dfrac{1}{2}$

$$\therefore \quad x_1^2 - y_1^2 = \frac{1}{2} \quad \text{... (iv)}$$

From Eqs. (iii) and (iv), we get $y_1^2 = \dfrac{1}{2}$ and $x_1^2 = 1$

and (x_1, y_1) lies on ellipse $x^2 + 2y^2 = a^2$

$$\therefore \quad x_1^2 + 2y_1^2 = a^2$$

$$\Rightarrow \quad 1 + 1 = a^2 \text{ or } a^2 = 2$$

$\therefore$ Equation of ellipse is $x^2 + 2y^2 = 2$

whose foci $(\pm 1, 0)$.

Sol. (Q. Nos. 101 and 102)

The intersection points of given circle

$$x^2 + y^2 - 8x = 0 \quad \text{... (i)}$$

and hyperbola

$$4x^2 - 9y^2 = 36 \quad \text{... (ii)}$$

can be obtained by solving these equations substituting value of y^2 from Eq. (i) in Eq. (ii), we get

$$4x^2 - 9(8x - x^2) = 36$$

$$\Rightarrow \quad 13x^2 - 72x - 36 = 0$$

$$\Rightarrow \quad x = 6, -\frac{6}{13}$$

$$\Rightarrow \quad y^2 = 12, -\frac{660}{169} \quad \text{(not possible)}$$

$\therefore A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$ are points of intersection.

101. Equation of tangent to hyperbola having slope m is

$$y = mx + \sqrt{(9m^2 - 4)} \quad \text{... (i)}$$

Equation of tangent to circle is

$$y = m(x - 4) + \sqrt{(16m^2 + 16)} \quad \text{... (ii)}$$

Eqs. (i) and (ii) will be identical, then

$$m = \frac{2}{\sqrt{5}}$$

So, equation of common tangents

$$y = \frac{2x}{\sqrt{5}} + \sqrt{\left(\frac{36}{5} - 4\right)}$$

$$\Rightarrow \quad y = \frac{2x}{\sqrt{5}} + \frac{4}{\sqrt{5}}$$

or $\quad 2x - \sqrt{5}y + 4 = 0$

102. Equation of circle with AB as its diameter is

$$(x - 6)(x - 6) + (y - 2\sqrt{3})(y + 2\sqrt{3}) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 12x + 24 = 0$$

103. $\because$ line $2x + y = 1$ is tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\therefore \quad (1)^2 = a^2(-2)^2 - b^2$$

$$\Rightarrow \quad 4a^2 - b^2 = 1 \quad \text{... (i)}$$

Intersection point of nearest directrix $x = \dfrac{a}{e}$ and X-axis is $\left(\dfrac{a}{e}, 0\right)$

As, $2x + y = 1$ passes through $\left(\dfrac{a}{e}, 0\right)$

$$\therefore \quad \frac{2a}{e} + 0 = 1 \Rightarrow a = \frac{e}{2} \quad \text{... (ii)}$$

$$\text{and} \quad b^2 = a^2(e^2 - 1) = \frac{e^2}{4}(e^2 - 1) \quad \text{... (iii)}$$

Substituting the values of a and b from Eqs. (ii) and (iii) in Eq. (i), then

$$e^2 - \frac{e^2}{4}(e^2 - 1) = 1$$

$$\Rightarrow \quad (e^2 - 4)(e^2 - 1) = 0$$

$$\therefore \quad e^2 = 4, e^2 \neq 1 \quad (\because e > 1)$$

$$\text{Hence,} \quad e = 2$$

104. Equation of normal at $P(6, 3)$ is

$$\frac{a^2x}{6} + \frac{b^2y}{3} = a^2 + b^2$$

$\because$ Normal intersects the X-axis at $(9, 0)$, then

$$\frac{9a^2}{6} + 0 = a^2 + b^2 \Rightarrow 3a^2 = 2a^2 + 2b^2$$

$$\text{or} \quad a^2 = 2b^2$$

$$\text{or} \quad a^2 = 2a^2(e^2 - 1)$$

$$\therefore \quad e^2 = \frac{3}{2}$$

$$\text{Hence,} \quad e = \sqrt{\frac{3}{2}}$$

105. Given, ellipse is $x^2 + 4y^2 = 4$

$$\text{or} \quad \frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$\therefore \quad e = \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2}$$

and foci are $(\pm\sqrt{3}, 0)$

$\therefore$ Eccentricity of hyperbola $= \dfrac{2}{\sqrt{3}} = e_1$

$$\text{and} \quad b^2 = a^2(e_1^2 - 1) = a^2\left(\frac{4}{3} - 1\right) = \frac{a^2}{3}$$

then equation of hyperbola becomes

$$x^2 - 3y^2 = a^2$$

which pass through $(\pm\sqrt{3}, 0)$

$$3 - 0 = a^2$$

$$\Rightarrow \quad a^2 = 3$$

$\therefore$Equation of hyperbola is

$$x^2 - 3y^2 = 3$$

and foci of hyperbola are $\left(\pm\sqrt{3} \times \dfrac{2}{\sqrt{3}}, 0\right)$ i.e., $(\pm 2, 0)$

106. Equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{9} - \frac{yy_1}{4} = 1 \quad \text{... (i)}$$

Equation of line parallel to

$$2x - y = \lambda \quad \text{... (ii)}$$

$\because$ Line (ii) is tangent of $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then

$$\lambda^2 = 9 \times 2^2 - 4$$

$\therefore$ $\lambda = \pm 4\sqrt{2}$

From Eq. (ii), Equation of tangent is

$$2x - y = \pm 4\sqrt{2} \quad \text{... (iii)}$$

Comparing Eqs. (i) and (iii), we get $\frac{x_1}{18} = \frac{y_1}{4} = \pm \frac{1}{4\sqrt{2}}$

or $x_1 = \pm \frac{9}{2\sqrt{2}}$ and $y_1 = \pm \frac{1}{\sqrt{2}}$

Hence, points of contact of the tangents on the hyperbola are $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

107. $H : x^2 - y^2 = 1$

S : Circle with centre $N(x_2, 0)$

Common tangent to H and S at $P(x_1, y_1)$ is

$$xx_1 - yy_1 = 1 \Rightarrow m_1 = \frac{x_1}{y_1}$$

Also radius of circle S with centre $N\ (x_2, 0)$ through point of contact (x_1, y_1) is perpendicular to tangent.

$\therefore$ $m_1 m_2 = -1 \Rightarrow \frac{x_1}{y_1} \times \frac{0 - y_1}{x_2 - x_1} = -1$

$\Rightarrow$ $x_1 = x_2 - x_1$ or $x_2 = 2x_1$

M is the point of intersection of tangent at P and X-axis

$\therefore$ $M\left(\frac{1}{x_1}, 0\right)$

$\because$ Centroid of ΔPMN is (ℓ, m)

$\therefore$ $x_1 + \frac{1}{x_1} + x_2 = 3\ell$ and $y_1 = 3$ m

Using $x_2 = 2x_1$,

$\Rightarrow$ $\frac{1}{3}\left(3x_1 + \frac{1}{x_1}\right) = l$ and $\frac{y_1}{3} = m$

$\therefore$ $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \frac{dm}{dy_1} = \frac{1}{3}$

Also, (x_1, y_1) lies on H,

$\therefore$ $x_1^2 - y_1^2 = 1$ or $y_1 = \sqrt{(x_1^2 - 1)}$

$\therefore$ $m = \frac{1}{3}\sqrt{(x_1^2 - 1)}$

$\therefore$ $\frac{dm}{dx_1} = \frac{x_1}{3\sqrt{(x_1^2 - 1)}}$

108. $\because$ $\frac{2b^2}{a} = 8$ and $2b = \frac{1}{2}(2ae)$

$\Rightarrow$ $b^2 = 4a$ and $b = \frac{1}{2}ae$ or $b^2 = \frac{1}{4}a^2e^2$

$\Rightarrow$ $a^2(e^2 - 1) = \frac{1}{4}a^2e^2 \Rightarrow 4e^2 - 4 = e^2$ or $e^2 = \frac{4}{3}$

$\therefore$ $e = \frac{2}{\sqrt{3}}$

$\Rightarrow$ $b^2 = 4a$ and $b = \frac{1}{2}ae$ or $b^2 = \frac{1}{4}a^2e^2$

109. Let Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

When it passes through $P(\sqrt{2}, \sqrt{3})$, then $\frac{2}{a^2} - \frac{3}{b^2} = 1$

$\Rightarrow$ $2b^2 - 3a^2 = a^2b^2$ ...(i)

and $ae = 2 \Rightarrow a^2 + b^2 = 4$ ...(ii)

From Eqs. (i) and (ii), we get $2(4 - a^2) - 3a^2 = a^2(4 - a^2)$

$\Rightarrow$ $a^4 - 9a^2 + 8 = 0$ $\Rightarrow (a^2 - 1)(a^2 - 8) = 0$

$\therefore$ $a^2 = 1$ $(\because a^2 \neq 8)$

From Eq. (ii), $b^2 = 3$

$\therefore$ Equation of Hyperbola is $\frac{x^2}{1} - \frac{y^2}{3} = 1$

Equation of tangent at $P(\sqrt{2}, \sqrt{3})$ is $\frac{x\sqrt{2}}{1} - \frac{y\sqrt{3}}{3} = 1$

Which is passes through $(2\sqrt{2}, 3\sqrt{2})$

110. (a,b,c) $\because$ $2x - y + 1 = 0$ is a tangent of $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then

$$(1)^2 = a^2(2)^2 - 16 \text{ or } (2a)^2 = (1)^2 + (4)^2$$

$\therefore$ $2a$, 4, 1 are the sides of a right angled triangle.

Sol. (Q. Nos. 111-113)

I. $x^2 + y^2 = a^2$

Equations of tangents in terms of slopes are

$$y = mx + a\sqrt{(m^2 + 1)} \text{ (ii)}$$

and points of contact in terms of slopes are

$$\left(\frac{-ma}{\sqrt{(m^2 + 1)}}, \frac{a}{\sqrt{(m^2 + 1)}}\right) \text{(Q)}$$

$\therefore$ I (ii) (Q) (Ans. Q.No. 112) (Here, $a = \sqrt{2}, m = \pm 1$)

II. $x^2 + a^2y^2 = a^2$ or $\frac{x^2}{a^2} + \frac{y^2}{1^2} = 1$

Equations of tangents in terms of slopes are

$$y = mx + \sqrt{(a^2m^2 + 1)} \text{ (iv)}$$

and points of contact in terms of slopes are

$$\left(\frac{-a^2m}{\sqrt{(a^2m^2 + 1)}}, \frac{1}{\sqrt{(a^2m^2 + 1)}}\right) \text{(R)}$$

$\therefore$ II (iv) (R) (Ans. Q.No. 111) (Here, $a = 2, m = \frac{-\sqrt{3}}{2}$)

III. $y^2 = 4ax$

Equation of tangent in terms of slopes are

$$y = mx + \frac{a}{m} \text{ or } my = m^2x + a \text{ (i)}$$

and points of contact in terms of slopes are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) \text{(P)}$$

$\therefore$ III (i) (P) (Ans. Q.No. 113) (Here, $a = 8, m = 1$)