Trigonometric Functions and Identities Exercise 1 :
Single Option Correct Type Questions

1. The value of $\sum_{n=1}^{10}\left(\sin\frac{2n\pi}{11}-\cos\frac{2n\pi}{11}\right)$ is equal to

(a) 2 (b) 1
(c) 0 (d) -1

2. Given, $a^2+2a+\text{cosec}^2\left(\frac{\pi}{2}(a+x)\right)=0$, then which of the following holds good?

(a) $a=1;\ \frac{x}{2}\in 1$
(b) $a=-1;\ \frac{x}{2}\in I$
(c) $a\in R;\ x\in\phi$
(d) a, x are finite but not possible to find

3. The minimum value of the function $f(x)=(3\sin x-4\cos x-10)(3\sin x+4\cos x-10)$, is

(a) 49 (b) $\frac{195-60\sqrt{2}}{2}$
(c) 84 (d) 48

4. The value of expression $\sum_{\theta=0}^{8}\frac{1}{1+\tan^3(10\theta)^\circ}$ equal is to

(a) 5 (b) $\frac{21}{4}$
(c) $\frac{14}{3}$ (d) $\frac{9}{2}$

5. The value of $\sqrt{1-\sin^2 110^\circ}\cdot\sec 110^\circ$ is equal to

(a) 2 (b) -1
(c) -2 (d) 1

6. If $\tan\alpha$, $\tan\beta$ are the roots of the equation $x^2+px+q=0$, then the value of $\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)$ is

(a) independent of p but dependent on q
(b) independent of q but dependent on p
(c) independent of both p and q
(d) dependent on both p and q

7. The value of the product

$$\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2007}}\right)\cos\left(\frac{\pi}{2^{2006}}\right)\ldots\cos\left(\frac{\pi}{2^3}\right)\cos\left(\frac{\pi}{2^2}\right),$$ is

(a) $\frac{1}{2^{2007}}$ (b) $\frac{1}{2^{2008}}$
(c) $\frac{1}{2^{2009}}$ (d) $\frac{1}{2^{2010}}$

8. If $\tan B=\frac{n\sin A\cos A}{1-n\cos^2 A}$, then $\tan(A+B)$ equals to

(a) $\frac{\sin A}{(1-n)\cos A}$ (b) $\frac{(n-1)\cos A}{\sin A}$
(c) $\frac{\sin A}{(n-1)\cos A}$ (d) $\frac{\sin A}{(n+1)\cos A}$

9. If $P=(\tan(3^{n+1}\theta)-\tan\theta)$ and $Q=\sum_{r=0}^{n}\frac{\sin(3^r\theta)}{\cos(3^{r+1}\theta)}$, then

(a) $P=2Q$ (b) $P=3Q$
(c) $2P=Q$ (d) $3P=Q$

10. The value of $(\cos^4 1^\circ+\cos^4 2^\circ+\cos^4 3^\circ+\ldots+\cos^4 179^\circ)-(\sin^4 1^\circ+\sin^4 2^\circ+\sin^4 3^\circ+\ldots+\sin^4 179^\circ)$ equals to

(a) $2\cos 1^\circ$ (b) -1
(c) $2\sin 1^\circ$ (d) 0

11. Suppose that 'a' is a non-zero real number for which $\sin x+\sin y=a$ and $\cos x+\cos y=2a$. The value of $\cos(x-y)$, is

(a) $\frac{3a^2-2}{2}$ (b) $\frac{7a^2-2}{2}$
(c) $\frac{9a^2-2}{2}$ (d) $\frac{5a^2-2}{2}$

12. Let $P(x)=\sqrt{(\cos x+\cos 2x+\cos 3x)^2+(\sin x+\sin 2x+\sin 3x)^2}$, then $P(x)$ is equal to

(a) $1+2\cos x$ (b) $1+\sin 2x$
(c) $1-2\cos x$ (d) None of these

13. If the maximum value of the expression $\frac{1}{5\sec^2\theta-\tan^2\theta+4\,\text{cosec}^2\theta}$ is equal to $\frac{p}{q}$ (where, p and q are coprime), then the value of $(p+q)$ is

(a) 14 (b) 15
(c) 16 (d) 18

14. Let $f_n(a)=\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\ldots+\sin(2n-1)\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\ldots+\cos(2n-1)\alpha}$.

Then, the value of $f_4\left(\frac{\pi}{32}\right)$ is equal to

(a) $\sqrt{2}+1$ (b) $\sqrt{2}-1$
(c) $2+\sqrt{3}$ (d) $2-\sqrt{3}$

15. The minimum value of $\left|\sin x+\cos x+\frac{\cos x+\sin x}{\cos^4 x-\sin^4 x}\right|$ is

(a) 2 (b) $\frac{3}{2}$ (c) $\sqrt{2}$ (d) 1

16. If $a = \cos(2012\pi)$, $b = \sec(2013\pi)$ and $c = \tan(2014\pi)$, then

(a) $a < b < c$ (b) $b < c < a$
(c) $c < b < a$ (d) $a = b < c$

17. In a ΔABC, the minimum value of $\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}$ is equal to

(a) 3 (b) 4
(c) 5 (d) 6

18. The number of ordered pairs (x, y) of real numbers satisfying $4x^2 - 4x + 2 = \sin^2 y$ and $x^2 + y^2 \le 3$, is equal to

(a) 0 (b) 2
(c) 4 (d) 8

19. In a ΔABC, $3\sin A + 4\cos B = 6$ and $3\cos A + 4\sin B = 1$, then $\angle C$ can be

(a) 30° (b) 60°
(c) 90° (d) 150°

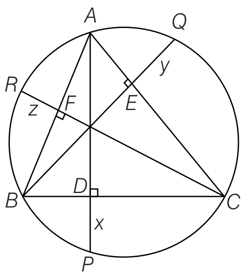
20. An equilateral triangle has side length 8. The area of the region containing all points outside the triangle but not more than 3 units from a point on the triangle is :

(a) $9(8 + \pi)$
(b) $8(9 + \pi)$
(c) $9\left(8 + \frac{\pi}{2}\right)$
(d) $8\left(9 + \frac{\pi}{2}\right)$

21. If $a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$. Then, $(m+n)^{2/3} + (m-n)^{2/3}$ is equal to

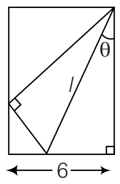
(a) $2a^2$ (b) $2a^{1/3}$
(c) $2a^{2/3}$ (d) $2a^3$

22. As shown in the figure, AD is the altitude on BC and AD produced meets the circumcircle of ΔABC at P where $DP = x$. Similarly, $EQ = y$ and $FR = z$. If a, b, c respectively denotes the sides BC, CA and AB, then $\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z}$ has the value equal to



(a) $\tan A + \tan B + \tan C$
(b) $\cot A + \cos B + \cot C$
(c) $\cos A + \cos B + \cos C$
(d) $\text{cosec}\, A + \text{cosec}\, B + \text{cosec}\, C$

23. One side of a rectangular piece of paper is 6 cm, the adjacent sides being longer than 6 cm. One corner of the paper is folded so that is sets on the opposite longer side. If the length of the crease is l cm and it makes an angle θ with the long side as shown, then l is



(a) $\frac{3}{\sin\theta\cos^2\theta}$ (b) $\frac{6}{\sin^2\theta\cos\theta}$
(c) $\frac{3}{\sin\theta\cos\theta}$ (d) $\frac{3}{\sin^2\theta}$

24. The average of the numbers $n\sin n°$ for $n = 2, 4, 6, \dots 180$

(a) 1 (b) $\cot 1°$
(c) $\tan 1°$ (d) $\frac{1}{2}$

25. A circle is inscribed inside a regular pentagon and another circle is circumscribed about this pentagon. Similarly, a circle is inscribed in a regular heptagon and another circumscribed about the heptagon. The area of the regions between the two circles in two cases are A_1 and A_2, respectively. If each polygon has a side length of 2 units, then which one of the following is true ?

(a) $A_1 = \frac{5}{7}A_2$ (b) $A_1 = \frac{25}{49}A_2$
(c) $A_1 = \frac{49}{25}A_2$ (d) $A_1 = A_2$

26. The value of $\sum_{r=1}^{18}\cos^2(5r)°$, where $x°$ denotes the x degrees, is equal to

(a) 0 (b) $\frac{7}{2}$
(c) $\frac{17}{2}$ (d) $\frac{25}{2}$

27. Minimum value of $4x^2 - 4x|\sin x| - \cos^2\theta$ is equal to

(a) -2 (b) -1
(c) $-\frac{1}{2}$ (d) 0

28. If in a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) π (d) $\frac{4\pi}{3}$

29. If $|\tan A| < 1$ and $|A|$ is acute, then $\frac{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}}{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}}$ is equal to

(a) $\tan A$ (b) $-\tan A$
(c) $\cot A$ (d) $-\cot A$

30. For any real θ, the maximum value of $\cos^2(\cos\theta)+\sin^2(\sin\theta)$ is

(a) 1 (b) $1+\sin^2 1$
(c) $1+\cos^2 1$ (d) does not exist

31. Minimum value of $27^{\cos 2x}\cdot 81^{\sin 2x}$ is

(a) -5 (b) $\frac{1}{5}$
(c) $\frac{1}{243}$ (d) $\frac{1}{27}$

32. $ABCD$ is a trapezium, such that AB and CD are parallel $BC\perp CD$. If $\angle ADB=\theta$, $BC=p$ and $CD=q$, then AB is equal to

(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$
(c) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

33. If $4n\alpha=\pi$, then $\cot\alpha\cot 2\alpha\cot 3\alpha\ldots\cot(2n-1)\alpha$ is equal to

(a) 0 (b) 1
(c) n (d) None of these

34. If in a triangle $ABC(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)=3\sin A\sin B$, then angle C is equal to

(a) 30° (b) 45°
(c) 60° (d) 75°

35. If α,β,γ are acute angles and $\cos\theta=\frac{\sin\beta}{\sin\alpha}$, $\cos\phi=\frac{\sin\gamma}{\sin\alpha}$ and $\cos(\theta-\phi)=\sin\beta\sin\gamma$, then $\tan^2\alpha-\tan^2\beta-\tan^2\gamma$ is equal to

(a) -1 (b) 0
(c) 1 (d) None of these

36. If $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, then $\tan(\alpha-\beta)$ is equal to

(a) $n\tan\alpha$ (b) $(1-n)\tan\alpha$
(c) $(1+n)\tan\alpha$ (d) None of these

37. If $\frac{\cos\theta}{a}=\frac{\sin\theta}{b}$, then $\frac{a}{\sec 2\theta}+\frac{b}{\operatorname{cosec} 2\theta}$ is equal to

(a) a (b) b
(c) $\frac{a}{b}$ (d) $a+b$

38. The graph of the function $\cos x\cos(x+2)-\cos^2(x+1)$ is

(a) a straight line passing through $(0,-\sin^2\theta)$ with slope 2
(b) a straight line passing through (0, 0)
(c) a parabola with vertex $(1,-\sin^2 1)$
(d) a straight line passing through the point $\left(\frac{\pi}{2},-\sin^2 1\right)$ and parallel to the X-axis

39. $f(\theta)=|\sin\theta|+|\cos\theta|, \theta\in R$, then

(a) $f(\theta)\in[0,2]$ (b) $f(\theta)\in[0,\sqrt{2}]$
(c) $f(\theta)\in[0,1]$ (d) $f(\theta)\in[1,\sqrt{2}]$

40. If $A=\cos(\cos x)+\sin(\cos x)$ the least and greatest value of A are

(a) 0 and 2 (b) -1 and 1
(c) $-\sqrt{2}$ and $\sqrt{2}$ (d) 0 and $\sqrt{2}$

41. If $U_n=\sin n\theta\sec^n\theta$, $V_n=\cos n\theta\sec^n\theta\neq 1$, then $\frac{V_n-V_{n-1}}{U_{n-1}}+\frac{1}{n}\frac{U_n}{V_n}$ is equal to

(a) 0 (b) $\tan\theta$
(c) $-\tan\theta+\frac{\tan n\theta}{n}$ (d) $\tan\theta+\frac{\tan n\theta}{n}$

42. If $0\le x\le\frac{\pi}{3}$ then range of $f(x)=\sec\left(\frac{\pi}{6}-x\right)+\sec\left(\frac{\pi}{6}+x\right)$ is

(a) $\left(\frac{4}{\sqrt{3}},\infty\right)$ (b) $\left[\frac{4}{\sqrt{3}},\infty\right)$
(c) $\left(0,\frac{4}{\sqrt{3}}\right]$ (d) $\left(0,\frac{4}{\sqrt{3}}\right)$

43. If $A=\sin^8\theta+\cos^{14}\theta$, then for all values of θ,

(a) $A\ge 1$ (b) $0<A\le 1$
(c) $1<2a\le 3$ (d) None of these

44. The expression $3\left\{\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right\}-2\left\{\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right\}$ is equal to

(a) 0 (b) -1
(c) 1 (d) 3

45. The maximum value of $\sin\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)$ in the interval $\left(0,\frac{\pi}{2}\right)$ is attained at

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

46. If $\cot^2 x=\cot(x-y)\cdot\cot(x-z)$, then $\cot 2x$ is equal to $\left(x\neq\pm\frac{\pi}{4}\right)$

(a) $\frac{1}{2}(\tan y+\tan z)$ (b) $\frac{1}{2}(\cot y+\cot z)$
(c) $\frac{1}{2}(\sin y+\sin z)$ (d) None of these

47. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where α,β,γ are real numbers satisfying $\alpha+\beta+\gamma=\pi$, is

(a) positive (b) zero
(c) negative (d) None of these

48. If $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$, then $\tan\frac{x}{2}$ is equal to

(a) $\cot\frac{\alpha}{2}\tan\frac{\beta}{2}$ (b) $\cot\frac{\beta}{2}\tan\frac{\alpha}{2}$
(c) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ (d) None of these

49. If $\cos^4\theta\sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta\,\text{cosec}^2\alpha$ are in AP, then $\cos^8\theta\sec^6\alpha, \frac{1}{2}$ and $\sin^8\theta\cdot\text{cosec}^6\alpha$ are in

(a) AP (b) GP
(c) HP (d) None of these

50. The maximum value of $\cos\alpha_1\cdot\cos\alpha_2\cdot\cos\alpha_3\cdot\ldots\cdot\cos\alpha_n$ under the restriction $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \frac{\pi}{2}$ and $\cot\alpha_1\cdot\cot\alpha_2\cdot\ldots\cdot\cot\alpha_n = 1$ is

(a) $\frac{1}{2^{\frac{n}{2}}}$ (b) $\frac{1}{2^n}$
(c) $\frac{-1}{2^n}$ (d) 1

51. If $x \in (0, \pi)$ and $\sin x\cdot\cos^3 x > \cos x\cdot\sin^3 x$, then complete set of values of x is

(a) $x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$
(b) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$
(c) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(d) None of the above

52. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by

(a) $2(a^2 + b^2)$ (b) $2\sqrt{a^2 + b^2}$
(c) $(a + b)^2$ (d) $(a - b)^2$

53. For a positive integer n, let $f_n(\theta) = \frac{\tan\theta}{2}(1 + \sec\theta)(1 + \sec 2\theta)\ldots(1 + \sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right) = 0$ (b) $f_3\left(\frac{\pi}{32}\right) = -1$
(c) $f_4\left(\frac{\pi}{64}\right) = -1$ (d) $f_5\left(\frac{\pi}{128}\right) = 1$

Trigonometric Functions and Identities Exercise 2 :
More than One Option Correct Type Questions

54. Suppose $\cos x = 0$ and $\cos(x + z) = \frac{1}{2}$. Then, the possible value(s) of z is (are).

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$
(c) $\frac{7\pi}{6}$ (d) $\frac{11\pi}{6}$

55. Let $f_n(\theta) = 2\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + 2\sin\frac{\theta}{2}\sin\frac{5\theta}{2} + 2\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \ldots + 2\sin\frac{\theta}{2}\sin(2n+1)\frac{\theta}{2}, n \in N$, then which of the following is/are correct?

(a) $f_9\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ (b) $f_n\left(\frac{2\pi}{n}\right) = 0, n \in N$
(c) $f_5\left(\frac{2\pi}{7}\right) = 0$ (d) $f_9\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

56. Let $P = \sin 25^\circ \sin 35^\circ \sin 60^\circ \sin 85^\circ$ and $Q = \sin 20^\circ \sin 40^\circ \sin 75^\circ \sin 80^\circ$. Which of the following relation(s) is (are) correct ?

(a) $P + Q = 0$ (b) $P - Q = 0$
(c) $P^2 + Q^2 = 1$ (d) $P^2 - Q^2 = 0$

57. For $0 < \theta < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$,

$$z = \sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta, \text{ then}$$

(a) $xyz = xz + y$
(b) $xyz = xy + z$
(c) $xyz = x + y + z$
(d) $xyz = yz + x$

58. Let $P(x) = \cot^2 x\left(\frac{1 + \tan x + \tan^2 x}{1 + \cot x + \cot^2 x}\right) + \left(\frac{\cos x - \cos 3x + \sin 3x - \sin x}{2(\sin 2x + \cos 2x)}\right)^2$. Then, which of the following is (are) correct?

(a) The value of $P(18^\circ) + P(72^\circ)$ is 2.
(b) The value of $P(18^\circ) + P(72^\circ)$ is 3.
(c) The value of $P\left(\frac{4\pi}{3}\right) + P\left(\frac{7\pi}{6}\right)$ is 3.
(d) The value of $P\left(\frac{4\pi}{3}\right) + P\left(\frac{7\pi}{6}\right)$ is 2.

59. It is known that $\sin\beta = \frac{4}{5}$ and $0 < \beta < \pi$, then thes value

of $\dfrac{\sqrt{3}\sin(\alpha+\beta) - \dfrac{2}{\cos\frac{11\pi}{6}}\cos(\alpha+\beta)}{\sin\alpha}$ is

(a) independent of α for all β in $(0, \pi)$

(b) $\frac{5}{\sqrt{3}}$ for $\tan\beta > 0$

(c) $\frac{\sqrt{3}(7 + 24\cot\alpha)}{15}$ for $\tan\beta < 0$

(d) zero for $\tan\beta > 0$

60. In cyclic quadrilateral $ABCD$, if $\cot A = \frac{3}{4}$ and $\tan B = \frac{-12}{5}$, then which of the following is (are) correct?

(a) $\sin D = \frac{12}{13}$ (b) $\sin(A + B) = \frac{16}{65}$

(c) $\cos D = \frac{-15}{13}$ (d) $\sin(C + D) = \frac{-16}{65}$

61. If the equation $2\cos^2 x + \cos x - a = 0$ has solutions, then a can be

(a) $\frac{-1}{4}$ (b) $\frac{-1}{8}$

(c) 2 (d) 5

62. If $A = \sin 44^\circ + \cos 44^\circ$, $B = \sin 45^\circ + \cos 45^\circ$ and $C = \sin 46^\circ + \cos 46^\circ$. Then, correct option(s) is/are

(a) $A < B < C$ (b) $C < B < A$

(c) $B > A$ (d) $A = C$

63. If $\tan(2\alpha + \beta) = x$ & $\tan(\alpha + 2\beta) = y$, then $[\tan 3(\alpha + \beta)]. [\tan(\alpha - \beta)]$ is equal to (wherever defined)

(a) $\frac{x^2 + y^2}{1 - x^2y^2}$ (b) $\frac{x^2 - y^2}{1 + x^2y^2}$

(c) $\frac{x^2 + y^2}{1 + x^2y^2}$ (d) $\frac{x^2 - y^2}{1 - x^2y^2}$

64. If $x = \sec\phi - \tan\phi$ and $y = \operatorname{cosec}\phi + \cot\phi$, then

(a) $x = \frac{y+1}{y-1}$ (b) $x = \frac{y-1}{y+1}$

(c) $y = \frac{1+x}{1-x}$ (d) $xy + x - y + 1 = 0$

65. If $\tan\left(\frac{x}{2}\right) = \operatorname{cosec} x - \sin x$, then $\tan^2\left(\frac{x}{2}\right)$ is equal to

(a) $2 - \sqrt{5}$ (b) $\sqrt{5} - 2$

(c) $(9 - 4\sqrt{5})(2 + \sqrt{5})$ (d) $(9 + 4\sqrt{5})(2 - \sqrt{5})$

66 If $y = \frac{\sqrt{1 - \sin 4A} + 1}{\sqrt{1 + \sin 4A} - 1}$, then one of the value of y is

(a) $-\tan A$ (b) $\cot A$

(c) $\tan\left(\frac{\pi}{4} + A\right)$ (d) $-\cot\left(\frac{\pi}{4} + A\right)$

67. If $3\sin\beta = \sin(2\alpha + \beta)$, then

(a) $[\cot\alpha + \cot(\alpha+\beta)][\cot\beta - 3\cot(2\alpha+\beta)] = 6$

(b) $\sin\beta = \cos(\alpha+\beta)\sin\alpha$

(c) $2\sin\beta = \sin(\alpha+\beta)\cos\alpha$

(d) $\tan(\alpha+\beta) = 2\tan\alpha$

68. Let $P_n(u)$ be a polynomial is u of degree n. Then, for every positive integer n, $\sin 2nx$ is expressible is

(a) $P_{2n}(\sin x)$ (b) $P_{2n}(\cos x)$

(c) $\cos x\, P_{2n-1}(\sin x)$ (d) $\sin x\, P_{2n-1}(\cos x)$

69. If $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then

(a) $\sin\alpha - \cos\alpha = \pm\sqrt{2}\sin\theta$

(b) $\sin\alpha + \cos\alpha = \pm\sqrt{2}\cos\theta$

(c) $\cos 2\theta = \sin 2\alpha$

(d) $\sin 2\theta + \cos 2\alpha = 0$

70. If $\cos 5\theta = a\cos\theta + b\cos^3\theta + c\cos^5\theta + d$, then

(a) $a = 20$

(b) $b = -20$

(c) $c = 16$

(d) $d = 5$

71. $x = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$ then $x^2 = a^2 + b^2 + 2\sqrt{p(a^2+b^2) - p^2}$, where p is equal to

(a) $a^2\cos^2\alpha + b^2\sin^2\alpha$

(b) $a^2\sin^2\alpha + b^2\cos^2\alpha$

(c) $\frac{1}{2}[a^2 + b^2 + (a^2 - b^2)\cos 2\alpha]$

(d) $\frac{1}{2}[a^2 + b^2 - (a^2 - b^2)\cos 2\alpha]$

72. $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ (n, even or odd) is equal to

(a) $2\tan^n\left(\frac{A-B}{2}\right)$ (b) $2\cot^n\left(\frac{A-B}{2}\right)$

(c) 0 (d) None of these

73. Let $P(k) = \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \cos\frac{(2k-1)\pi}{4k}\right)\left(1 + \cos\frac{(2k+1)\pi}{4k}\right)\left(1 + \cos\frac{(4k-1)\pi}{4k}\right)$. Then

(a) $P(3) = \frac{1}{16}$ (b) $P(4) = \frac{2-\sqrt{2}}{16}$

(c) $P(5) = \frac{3-\sqrt{5}}{32}$ (d) $P(6) = \frac{2-\sqrt{3}}{16}$

74. If $x = a\cos^3\theta\sin^2\theta$, $y = a\sin^3\theta\cos^2\theta$ and $\frac{(x^2+y^2)^p}{(xy)^q}$ $(p, q \in N)$ is independent of θ, then

(a) $p = 4$ (b) $p = 5$

(c) $q = 4$ (d) $q = 5$

Trigonometric Functions and Identities Exercise 3:
Statement I and II Type Questions

- This section contains 11 questions. Each question contains **Statement I** (Assertion) and **Statement II** (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are
  (a) Both Statement I and Statement II are individually true and R is the correct explanation of Statement I.
  (b) Both Statement I and Statement II are individually true but Statement II is not the correct explanation of Statement I.
  (c) Statement I is true but Statement II is false.
  (d) Statement I is false but Statement II is true.

75. **Statement I** $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha = \cot\alpha$

Statement II $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

76. **Statement I** If $xy + yz + zx = 1$, then $\Sigma \dfrac{x}{(1+x^2)} = \dfrac{2}{\sqrt{\Pi(1+x^2)}}$.

Statement II In a ΔABC $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$

77. **Statement I** If α and β are two distinct solutions of the equation $a\cos x + b\sin x = c$, then $\tan\left(\dfrac{\alpha+\beta}{2}\right)$ is independent of c.

Statement II Solution of $a\cos x + b\sin x = c$ is possible, if $-\sqrt{(a^2+b^2)} \le c \le \sqrt{(a^2+b^2)}$

78. **Statement I** If A is obtuse angle in ΔABC, then $\tan B\tan C > 1$.

Statement II In ΔABC, $\tan A = \dfrac{\tan B + \tan C}{\tan B\tan C - 1}$

79. **Statement I** $\sin\left(\dfrac{2\pi}{7}\right) + \sin\left(\dfrac{4\pi}{7}\right) + \sin\left(\dfrac{8\pi}{7}\right) = -\dfrac{1}{2}$.

Statement II $\cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$ is complex 7th root of unity.

80. **Statement I** The curve $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$ intersects X-axis at eight points in the region $-\pi \le x \le \pi$.

Statement II The curve $y = \sin x$ or $y = \cos x$ intersects the X-axis at infinitely many points.

81. **Statement I** The numbers $\sin 18^\circ$ and $-\sin 54^\circ$ are the roots of a quadratic equation with integer coefficients.

Statement II If $x = 18^\circ$, $\cos 3x = \sin 2x$ and if $y = -54^\circ$ $\sin 2y = \cos 3y$.

82. **Statement I** The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$ where α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$ is negative.

Statement II If $\alpha + \beta + \gamma = \pi$, then α, β, γ are the angles of a triangle.

83. **Statement I** If $2\sin\left(\dfrac{\theta}{2}\right) = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$ then $\dfrac{\theta}{2}$ lies between $2n\pi + \dfrac{\pi}{4}$ and $2n\pi + \dfrac{3\pi}{4}$.

Statement II If $\dfrac{\pi}{4} \le 0 \le \dfrac{3\pi}{4}$ then $\sin\dfrac{\theta}{2} > 0$.

84. **Statement I** If $2\cos\theta + \sin\theta = 1\left(\theta \ne \dfrac{\pi}{2}\right)$ then the value of $7\cos\theta + 6\sin\theta$ is 2.

Statement II If $\cos 2\theta - \sin\theta = \dfrac{1}{2}, 0 < \theta < \dfrac{\pi}{2}$, then $\sin\theta + \cos 6\theta = 0$.

85. **Statement I** If $A > 0$, $B > 0$ and $A + B = \dfrac{\pi}{3}$, then the maximum value of $\tan A\tan B$ is $\dfrac{1}{3}$.

Statement II If $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant), then the value $a_1 a_2 a_3 \ldots a_n$ is greatest when $a_1 = a_2 = a_3 = \ldots = a_n$

Trigonometric Functions and Identities Exercise 4 :
Passage Based Questions

Passage I

(Q. Nos. 86 and 87)

If a, b, c are the sides of $\Delta\, ABC$ such that $3^{2a^2} - 2\cdot 3^{a^2+b^2+c^2} + 3^{2b^2+2c^2} = 0$, then

86. Triangle ABC is

(a) equilateral (b) right angled
(c) isosceles right angled (d) obtuse angled

87. If sides of ΔPQR are a, $b \sec C$, $c \operatorname{cosec} C$. Then, area of ΔPQR is

(a) $\frac{\sqrt{3}}{4}a^2$ (b) $\frac{\sqrt{3}}{4}b^2$ (c) $\frac{\sqrt{3}}{4}c^2$ (d) $\frac{1}{2}abc$

Passage II

(Q. Nos. 88 to 90)

For $0 < x < \frac{\pi}{2}$, let $P_{mn}(x) = m \log_{\cos x}(\sin x) + n \log_{\cos x}(\cot x)$; where $m, n \in \{1, 2, \dots, 9\}$

[For example :

$$P_{29}(x) = 2\log_{\cos x}(\sin x) + 9\log_{\cos x}(\cot x) \text{ and}$$
$$P_{77}(x) = 7\log_{\cos x}(\sin x) + 7\log_{\cos x}(\cot x)]$$

On the basis of above information , answer the following questions :

88. Which of the following is always correct?

(a) $P_{mn}(x) \ge m \;\forall\; m \ge n$ (b) $P_{mn}(x) \ge n \;\forall\; m \ge n$
(c) $2P_{mn}(x) \le n \;\forall\; m \le n$ (d) $2P_{mn}(x) \le m \;\forall\; m \le n$

89. The mean proportional of numbers $P_{49}\left(\frac{\pi}{4}\right)$ and $P_{94}\left(\frac{\pi}{4}\right)$ is equal to

(a) 4 (b) 6
(c) 9 (d) 10

90. If $P_{34}(x) = P_{22}(x)$, then the value of $\sin x$ is expressed as $\left(\frac{\sqrt{q}-1}{p}\right)$, then $(p+q)$ equals

(a) 3 (b) 4
(c) 7 (d) 9

Note Mean proportional of a and b $(a > 0, b > 0)$ is $\sqrt{ab}$]

Passage III

(Q. Nos. 91 to 93)

If $7\theta = (2n+1)\pi$, where $n = 0, 1, 2, 3, 4, 5, 6$, then answer the following questions.

91. The equations whose roots are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ is

(a) $8x^3 + 4x^2 + 4x + 1 = 0$
(b) $8x^3 - 4x^2 - 4x - 1 = 0$
(c) $8x^3 - 4x^2 - 4x - 1 = 0$
(d) $8x^3 + 4x^2 + 4x - 1 = 0$

92. The value of $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$ is

(a) 4 (b) -4
(c) 3 (d) -3

93. The value of $\sec^2\frac{\pi}{7} + \sec^2\frac{3\pi}{7} + \sec^2\frac{5\pi}{7}$ is

(a) -24 (b) 80 (c) 24 (d) -80

Passage IV

(Q. Nos. 94 to 96)

If $1 + 2\sin x + 3\sin^2 x + 4\sin^3 x + \dots$ upto infinite terms = 4 and number of solutions of the equation in $\left[\frac{-3\pi}{2}, 4\pi\right]$ is k.

94. The value of k is equal to

(a) 4 (b) 5 (c) 6 (d) 7

95. The value of $\left|\frac{\cos 2x - 1}{\sin 2x}\right|$ is equal to

(a) 1 (b) $\sqrt{3}$
(c) $2-\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

96. Sum of all internal angles of a k-sided regular polygon is

(a) 5π (b) 4π
(c) 3π (d) 2π

Passage V

(Q. Nos. 97 to 98)

Let α is a root of the equation $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$, β is a root of the equation $3\cos^2 x - 10\cos x + 3 = 0$ and γ is a root of the equation $1 - \sin 2x = \cos x - \sin x$, $0 \le \alpha, \beta, \gamma \le \frac{\pi}{2}$.

97. $\cos\alpha + \cos\beta + \cos\gamma$ can be equal to

(a) $\frac{3\sqrt{6}+2\sqrt{2}+6}{6\sqrt{2}}$ (b) $\frac{3\sqrt{3}+8}{6}$
(c) $\frac{3\sqrt{3}+2}{6}$ (d) None of these

98. $\sin(\alpha - \beta)$ is equal to

(a) 1 (b) 0
(c) $\frac{1-2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3}-2\sqrt{2}}{6}$

Trigonometric Functions and Identities Exercise 5:
Matching Type Questions

99. Match the statement of Column I with values of Column II.

Column I		Column II
(A) If $\theta + \phi = \frac{\pi}{2}$, where θ and ϕ are positive, then $(\sin\theta + \sin\phi)\sin\left(\frac{\pi}{4}\right)$ is always less than	(p)	1
(B) If $\sin\theta - \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then $a^2 + b^2$ cannot exceed	(q)	2
(C) If $3\sin\theta + 5\cos\theta = 5$, $(\theta \neq 0)$ then the value of $5\sin\theta - 3\cos\theta$ is	(r)	3
	(s)	4
	(t)	5

100. Match the statement of Column I with values of Column II.

Column I		Column II
(A) If maximum and minimum values of $\frac{7 + 6\tan\theta - \tan^2\theta}{(1 + \tan^2\theta)}$ for all real values of $\theta \sim \frac{\pi}{2}$ are λ and μ respectively, then	(p)	$\lambda + \mu = 2$
(B) If maximum and minimum values of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ for all real values of θ are λ and μ respectively, then	(q)	$\lambda - \mu = 6$
(C) If maximum and minimum values of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ for all real values of θ and λ and μ respectively, then	(r)	$\lambda + \mu = 6$
	(s)	$\lambda - \mu = 10$
	(t)	$\lambda - \mu = 14$

101. Match the statement of Column I with values of Column II.

Column I		Column II
(A) The number of solutions of the equation $\lvert\cot x\rvert = \cot x + \frac{1}{\sin x}$ $(0 < x < \pi)$ is	(p)	no solution
(B) If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then value of $\cot\left(\frac{\theta + \phi}{2}\right)$ is	(q)	$\frac{1}{3}$
(C) The value of $\sin^2\alpha + \sin\left(\frac{\pi}{3} - \alpha\right)\sin\left(\frac{\pi}{3} + \alpha\right)$ is	(r)	1
(D) If $\tan\theta = 3\tan\phi$, then maximum value of $\tan^2(\theta - \phi)$ is	(s)	2
	(t)	4

102. Match the statement of Column I with values of Column II.

Column I		Column II
(A) In a ΔABC, $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) =$	(p)	$-1 + 4\sin\left(\frac{\pi + A}{4}\right)\sin\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi + C}{4}\right)$
(B) In a ΔABC, $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right) =$	(q)	$4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$
(C) In a ΔABC, $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) - \cos\left(\frac{C}{2}\right)$	(r)	$1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$
	(s)	$-1 + 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$
	(t)	$1 + 4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$

Trigonometric Functions and Identities Exercise 6:
Single Integer Answer Type Questions

103. In a ΔABC, $\frac{1}{1+\tan^2\frac{A}{2}}+\frac{1}{1+\tan^2\frac{B}{2}}+\frac{1}{1+\tan^2\frac{C}{2}}=k\left[1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$, then the value of k is

104. If $\frac{\sin\alpha}{\sin\beta}=\frac{\cos\gamma}{\cos\delta}$, then $\frac{\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\delta}{\sin\left(\frac{\delta-\gamma}{2}\right)\cdot\sin\left(\frac{\delta+\gamma}{2}\right)\cdot\sin\beta}$ is equal to

105. Find the exact value of the expression
$\tan\frac{\pi}{20}-\tan\frac{3\pi}{20}+\tan\frac{5\pi}{20}-\tan\frac{7\pi}{20}+\tan\frac{9\pi}{20}$.

106. Let $x=\frac{\sum_{n=1}^{44}\cos n^\circ}{\sum_{n=1}^{44}\sin n^\circ}$, find the greatest integer that does not exceed.

107. Find θ (in degree) satisfying the equation,
$\tan 15^\circ\cdot\tan 25^\circ\cdot\tan 35^\circ=\tan\theta$, where $\theta\in(0, 45^\circ)$

108. Find the exact value of $\operatorname{cosec}10^\circ+\operatorname{cosec}50^\circ-\operatorname{cosec}70^\circ$.

109. If $\cos 5\alpha=\cos^5\alpha$, where $\alpha\in\left(0,\frac{\pi}{2}\right)$, then find the possible values of $(\sec^2\alpha+\operatorname{cosec}^2\alpha+\cot^2\alpha)$.

110. Compute the value of the expression
$\tan^2\frac{\pi}{16}+\tan^2\frac{2\pi}{16}+\tan^2\frac{3\pi}{16}+\ldots+\tan^2\frac{7\pi}{16}$.

111. Compute the square of the value of the expression
$\frac{4+\sec 20^\circ}{\operatorname{cosec}20^\circ}$

112. In ΔABC, if $\frac{\sin A}{3}=\frac{\cos B}{3}=\frac{\tan C}{2}$, then the value of $\left(\frac{\sin A}{\cot 2A}+\frac{\cos B}{\cot 2B}+\frac{\tan C}{\cot 2C}\right)$ is

113. Let f and g be function defined by $f(\theta)=\cos^2\theta$ and $g(\theta)=\tan^2\theta$, suppose α and β satisfy $2f(\alpha)-g(\beta)=1$, then value of $2f(\beta)-g(\alpha)$ is

114. If sum of the series $1+x\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left(\frac{1+\sin x}{\cos x}\right)^{1/2}+x^2\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left(\frac{1+\sin x}{\cos x}\right)^{1/4}+\ldots\infty$ (wherever defined) is equal to $\frac{k(1-x)}{(2-x)}$, then k is equal to

115. If $\frac{9x}{\cos\theta}+\frac{5y}{\sin\theta}=56$ and $\frac{9x\sin\theta}{\cos^2\theta}-\frac{5y\cos\theta}{\sin^2\theta}=0$ then the value of $\left[(9x)^{\frac{2}{3}}+(5y)^{\frac{2}{3}}\right]^3$ is

116. The angle A of the ΔABC is obtuse.
$x=2635-\tan B\tan C$, if $[x]$ denotes the greatest integer function, the value of $[x]$ is

117. If $4\cos 36^\circ+\cot\left(7\frac{1}{2}^\circ\right)=\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3}+\sqrt{n_4}+\sqrt{n_5}+\sqrt{n_6}$, then the value of $\sum_{i=1}^{6}n_i^2$ must be

118. If $\sin^2 A=x$ and $\prod_{r=1}^{4}\sin(rA)=ax^2+bx^3+cx^4+dx^5$, then the value of $10a-7b+15c-5d$ must be

119. If $x, y\in R$ satisfies $(x+5)^2+(y-12)^2=(14)^2$, then the minimum value of $\sqrt{x^2+y^2}$ is ....

120. The least degree of a polynomial with integer coefficient whose one of the roots may be $\cos 12^\circ$ is

121. If $A+B+C=180^\circ$, $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=k\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ then the value of $3k^3+2k^2+k+1$ is equal to

122. The value of $f(x)=x^4+4x^3+2x^2-4x+7$, when $x=\cot\frac{11\pi}{8}$ is ....

123. In any ΔABC, then minimum value of
$2020\sum\frac{\sqrt{(\sin A)}}{(\sqrt{(\sin B)}+\sqrt{(\sin C)}-\sqrt{(\sin A)}}$ must be

124. If $\sin\theta+\sin^2\theta+\sin^3\theta=1$, then the value of $\cos^6\theta-4\cos^4\theta+8\cos^2\theta$ must be

125. $16\left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta - \cos\frac{5\pi}{8}\right)\left(\cos\theta - \cos\frac{7\pi}{8}\right) = \lambda\cos 4\theta$, then the value of λ is ....

126. If $\frac{1}{\sin 20^\circ} + \frac{1}{\sqrt{3}\cos 20^\circ} = 2k\cos 40^\circ$, then $18k^4 + 162k^2 + 369$ is equal to

Trigonometric Functions and Identities Exercise 7 :

Subjective Type Questions

127. Prove that $\tan 82\frac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

or $\quad \cot 7\frac{1}{2}^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

128. If $m\sin(\alpha+\beta) = \cos(\alpha-\beta)$, prove that

$$\frac{1}{1-m\sin 2\alpha} + \frac{1}{1-m\sin 2\beta} = \frac{2}{1-m^2}.$$

129. If $\alpha+\beta+\gamma = \pi$ and

$\tan\frac{1}{4}(\beta+\gamma-\alpha)\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma) = 1$,

then prove that $1+\cos\alpha+\cos\beta+\cos\gamma = 0$.

130. Find the value of a for which the equation $\sin^4 x + \cos^4 x = a$ has real solutions.

131. If a and b are positive quantities and $a \ge b$, then find the minimum positive values of $a\sec\theta - b\tan\theta$.

132. If a, b, c and k are constant quantities and α, β, γ are variable subjects to the relation $a\tan\alpha + b\tan\beta + c\tan\gamma = k$, then find the minimum value of $\tan^2\alpha + \tan^2\beta + \tan^2\gamma$.

133. If $\frac{x}{\tan(\theta+\alpha)} = \frac{y}{\tan(\theta+\beta)} = \frac{z}{\tan(\theta+\gamma)}$, prove that :

$\Sigma\frac{x+y}{x-y}\sin^2(\alpha-\beta) = 0$.

134. Let $a_1, a_2, \ldots, a_n$ be real constants, x be a real variable and $f(x) = \cos(a_1+x) + \frac{1}{2}\cos(a_2+x) + \frac{1}{4}\cos(a_3+x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n+x)$

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

135. Eliminate θ from the equations
$\tan(n\theta+\alpha) - \tan(n\theta+\beta) = x$ and
$\cot(n\theta+\alpha) - \cot(n\theta+\beta) = y$.

136. If $\{\sin(\alpha-\beta)+\cos(\alpha+2\beta)\sin\beta\}^2 = 4\cos\alpha\sin\beta\sin(\alpha+\beta)$. Then, prove that $\tan\alpha+\tan\beta = \frac{\tan\beta}{(\sqrt{2}\cos\beta-1)^2}$; $\alpha, \beta \in (0, \pi/4)$.

137. If A, B, C are the angle of a triangle and $\begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \cos^3 A & \cos^3 B & \cos^3 C \end{vmatrix} = 0$, then show that ΔABC is an isosceles.

138. In any ΔABC, prove that

$$\Sigma\frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}} \ge 3$$

and the equality holds if and only if triangle is equilateral.

139. If $2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+3 = 0$, prove that $\frac{d\alpha}{\sin(\beta+\theta)\sin(\gamma+\theta)} + \frac{d\beta}{\sin(\alpha+\theta)\sin(\gamma+\theta)} + \frac{d\gamma}{\sin(\alpha+\theta)\sin(\beta+\theta)} = 0$,

where, 'θ' is any real angle such that $\alpha+\theta, \beta+\theta, \gamma+\theta$ are not the multiple of π.

140. If the quadratic equation $4^{\sec^2\alpha}x^2 + 2x + \left(\beta^2 - \beta + \frac{1}{2}\right) = 0$ have real roots, then find all the possible values of $\cos\alpha + \cos^{-1}\beta$.

141. Four real constants a, b, A, B are given and $f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta$. Prove that if $f(\theta) \ge 0, \forall\, \theta \in R$, then $a^2+b^2 \le 2$ and $A^2+B^2 \le 1$.

142. If $\frac{\cos\theta_1}{\cos\theta_2} + \frac{\sin\theta_1}{\sin\theta_2} = \frac{\cos\theta_0}{\cos\theta_2} + \frac{\sin\theta_0}{\sin\theta_2} = 1$, where θ_1 and θ_0 do not differ by an even multiple of π, prove that

$$\frac{\cos\theta_1\cdot\cos\theta_0}{\cos^2\theta_2} + \frac{\sin\theta_1\cdot\sin\theta_0}{\sin^2\theta_2} = -1$$

143. Prove that

$$\sum_{k=1}^{n-1} {}^nC_k [\cos kx . \cos(n+k)x + \sin(n-k)x . \sin(2n-k)x] = (2^n - 2)\cos nx.$$

144. Determine all the values of x in the interval $x \in [0, 2\pi]$ which satisfy the inequality
$2\cos x \le |\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}| \le \sqrt{2}$.

145. Find all the solutions of this equation
$x^2 - 3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 3$, where [.] represents the greatest integer function.

146. In a ΔABC, prove that

$$\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos(rB - (n-r)A) = c^n.$$

147. Resolve $z^5 + 1$ into linear and quadratic factors with real coefficients. Hence, or otherwise deduce that,
$4\sin\left(\frac{\pi}{10}\right) \cdot \cos\left(\frac{\pi}{5}\right) = 1.$

148. Prove that the roots of the equation
$8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ and hence, show that $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$ and deduce the equation whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$.

Trigonometric Functions and Identities Exercise 8 :
Questions Asked in Previous 10 Years Exam

149. Let α and β be non-zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true?

[More than one correct option 2017 Adv.]

(a) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(b) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(c) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

150. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$, and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals to

[Single correct option 2016 Adv.]

(a) $2(\sec\theta - \tan\theta)$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

151. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to

[Single correct option 2016 Adv.]

(a) $3 - \sqrt{3}$ (b) $2(3 - \sqrt{3})$ (c) $2(\sqrt{3} - 1)$ (d) $2(2 + \sqrt{3})$

152. Let $f : (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then, the value(s) of $f\left(\frac{1}{3}\right)$ is/are

[More than one correct option 2012]

(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$ (c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$

153. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations
$(y+z)\cos 3\theta = (xyz)\sin 3\theta$
$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$,
and $(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$ have a solution (x_0, y_0, z_0) with $y_0 z_0 \ne 0$, is ......

[Integer Answer Type 2010]

154. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2} \text{ is/are}$$

[More than one correct option 2009]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

155. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then

[Single correct option 2009]

(a) $\tan^2 x = \frac{2}{3}$ (b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(c) $\tan^2 x = \frac{1}{3}$ (d) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

156. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then

[Single correct option 2006]

(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$
(c) $t_3 > t_1 > t_2 > t_4$ (d) $t_2 > t_3 > t_1 > t_4$

157. The number of ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ is **[Single correct option 2005]**

(a) 0 (b) 1
(c) 2 (d) 4

II. JEE Mains and AIEEE

158. $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is **[2017 JEE Main]**

(a) $-\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $-\frac{7}{9}$

159. If $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R, k \geq 1$, then $f_4(x) - f_6(x)$ is equal to **[2014 JEE Main]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

160. The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as **[2013 JEE Main]**

(a) $\sin A \cos A + 1$ (b) $\sec A \operatorname{cosec} A + 1$
(c) $\tan A + \cot A$ (d) $\sec A + \operatorname{cosec} A$

161. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to **[2012 AIEEE]**

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

162. If $A = \sin^2 x + \cos^4 x$, then for all real x **[2011 AIEEE]**

(a) $\frac{13}{16} \leq A \leq 1$ (b) $1 \leq A \leq 2$
(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

163. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \leq \alpha, \beta \leq \frac{\pi}{4}$. Then, $\tan 2\alpha$ is equal to **[2010 AIEEE]**

(a) $\frac{25}{16}$ (b) $\frac{56}{33}$ (c) $\frac{19}{12}$ (d) $\frac{20}{7}$

164. Let A and B denote the statements

$A : \cos\alpha + \cos\beta + \cos\gamma = 0$

$B : \sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then **[2009 AIEEE]**

(a) A is true and B is false
(b) A is false and B is true
(c) Both A and B are true
(d) Both A and B are false

165. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is **[2006 AIEEE]**

(a) $\sqrt{\frac{x^3}{8}}$ (b) $\frac{1}{2}x^2$ (c) πx^2 (d) $\frac{3}{2}x^2$

166. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is **[2006 AIEEE]**

(a) $\frac{(4 - \sqrt{7})}{3}$ (b) $-\frac{(4 + \sqrt{7})}{3}$
(c) $\frac{(1 + \sqrt{7})}{4}$ (d) $\frac{(1 - \sqrt{7})}{4}$

167. In a ΔPQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0, a \neq 0$, then **[2005 AIEEE]**

(a) $b = a + c$ (b) $b = c$
(c) $c = a + b$ (d) $a = b + c$

Answers

Chapter Exercises

1. (b)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (a)
7. (b)	**8.** (a)	**9.** (a)	**10.** (b)	**11.** (d)	**12.** (d)
13. (d)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (b)
19. (a)	**20.** (a)	**21.** (c)	**22.** (a)	**23.** (a)	**24.** (b)
25. (d)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (b)
31. (c)	**32.** (a)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (b)
37. (a)	**38.** (d)	**39.** (d)	**40.** (c)	**41.** (c)	
42. (b)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (c)
48. (b)	**49.** (a)	**50.** (a)	**51.** (a)	**52.** (d)	**53.** (a)
54. (a,b,c,d)		**55.** (a,b,c)	**56.** (b,d)	**57.** (b,c)	**58.** (b,c)
59. (b,c)	**60.** (a,b,d)	**61.** (b,c)	**62.** (c,d)	**63.** (d)	**64.** (b, c, d)
65. (b, c)	**66.** (a, b, c, d)		**67.** (a, b, c, d)		**68.** (c, d)
69. (a, b, c, d)		**70.** (b, c)	**71.** (a, b, c, d)		**72.** (b, c)
73. (a, b, c, d)		**74.** (b, c)	**75.** (a)	**76.** (b)	**77.** (b)
78. (d)	**79.** (d)	**80.** (a)	**81.** (a)	**82.** (c)	**83.** (b)
84. (b)	**85.** (b)	**86.** (b)	**87.** (a)	**88.** (b)	**89.** (b)
90. (c)	**91.** (b)	**92.** (a)	**93.** (c)	**94.** (b)	**95.** (d)
96. (c)	**97.** (b)	**98.** (c)			

99. A—(p, q, r, s, t); B—(s, t); C—(r)

100. A—(r, s); B—(r, t); C—(p, q)

101. A—(r); B—(p); C—(p); D—(q)

102. A—(r, t); B—(p, s); C—q

103. (2)	**104.** (1)	**105.** (5)	**106.** (2)	**107.** (5)	**108.** (6)
109. (5)	**110.** (35)	**111.** (3)	**112.** (2)	**113.** (1)	**114.** (2)
115. (3136)		**116.** (2634)		**117.** (91)	**118.** (3448)
119. (1)	**120.** (4)	**121.** (1673)		**122.** (6)	**123.** (6060)
124. (4)	**125.** (2)	**126.** (1745)			

130. $\frac{1}{2} \le a \le 1$ **131.** Minimum value is $\sqrt{a^2 - b^2}$

132. Minimum value is $\left(\frac{k^2}{a^2 + b^2 + c^2}\right)$

135. $\cot(\alpha + \beta) = \frac{1}{x} - \frac{1}{y}$

140. $\cos\alpha - \cos^{-1}\beta = \begin{cases} \frac{\pi}{3} - 1, \text{ when } n \text{ is an even integer} \\ \frac{\pi}{3} + 1, \text{ when } n \text{ is an even integer} \end{cases}$

144. $x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

145. Only two solutions, $x = 0, \sqrt{3}$

148. Required equation is, $z^3 - 21z^2 + 35z - 7 = 0$ whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$

149. (b, c)		**150.** (c)	**151.** (c)	**152.** (a,b)	**153.** (3)
154. (c,d)	**155.** (b)	**156.** (b)	**157.** (d)	**158.** (d)	**159.** (d)
160. (b)	**161.** (b)	**162.** (d)	**163.** (b)	**164.** (c)	**165.** (b)
166. (b)	**167.** (c)				

Solutions

1. Given series

$$= \left(\sin\frac{2\pi}{11} - \cos\frac{2\pi}{11}\right) + \left(\sin\frac{4\pi}{11} - \cos\frac{4\pi}{11}\right)$$
$$+ \left(\sin\frac{6\pi}{11} - \cos\frac{6\pi}{11}\right) + \ldots + \left(\sin\frac{20\pi}{11} - \cos\frac{20\pi}{11}\right)$$
$$= \left(\sin\frac{2\pi}{11} + \sin\frac{4\pi}{11} + \ldots + \sin\frac{20\pi}{11}\right)$$
$$-\left(\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \ldots + \cos\frac{20\pi}{11}\right)$$
$$= \frac{\sin\pi\cdot\sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}} - \frac{\cos\pi\cdot\sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}}$$
$$= 0 + \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{\sin\frac{\pi}{11}} = 1$$

2. $(a+1)^2 + \operatorname{cosec}^2\left(\frac{\pi a}{2} + \frac{\pi x}{2}\right) - 1 = 0$

or $\quad (a+1)^2 + \cot^2\left(\frac{\pi a}{2} + \frac{\pi x}{2}\right) = 0$

From option [b], if $a = -1$ and $\cot^2\left(\frac{-\pi}{2} + \frac{\pi x}{2}\right) = 0$

$$\Rightarrow \quad \tan^2\left(\frac{\pi x}{2}\right) = 0$$
$$\Rightarrow \quad \frac{\pi}{2} = 1$$

3. $f(x) = 9\sin^2 x - 16\cos^2 x - 10(3\sin x - 4\cos x)$

$$-10(3\sin x + 4\cos x) + 100$$
$$= 25\sin^2 x - 60\sin x + 84$$
$$= (5\sin x - 6)^2 + 48$$

$\therefore f(x)_{\min}$ occurs when $\sin x = 1$

Minimum value = 49

4. $S = \frac{1}{1+\tan^3 0°} + \frac{1}{1+\tan^3 10°} + \ldots + \frac{1}{1+\tan^3 80°}$

Now, $\frac{1}{1+\tan^3\theta} + \frac{1}{1+\tan^3(90-\theta)}$

$$= \frac{1}{1+\tan^3\theta} + \frac{1}{1+\cot^3\theta}$$
$$= \frac{1}{1+\tan^3\theta} + \frac{\tan^3\theta}{1+\tan^3\theta}$$
$$= \frac{1+\tan^3\theta}{1+\tan^3\theta} = 1$$

Hence, $S = 1 + (1+1+1+1) = 5$

5. Clearly, $\sqrt{1-\sin^2 110°}\cdot\sec 110°$

$$= |\cos 110°|\sec 110°$$
$$= -\cos 110°\sec 110° = -1$$

6. $\tan\alpha + \tan\beta = -p$

$\tan\alpha\ \tan\beta = q$

$\tan(\alpha+\beta) = \frac{-p}{1-q} = \frac{p}{q-1}$

$$\frac{1}{1+\tan^2(\alpha+\beta)}[\tan^2(\alpha+\beta) + p\tan(\alpha+\beta) + q]$$
$$= \frac{1}{1+\frac{p^2}{(q-1)^2}}\left[\frac{p^2}{(q-1)^2} + \frac{p^2}{(q-1)} + q\right]$$
$$= \frac{1}{(q-1)^2+p^2}[p^2 + p^2(q-1) + q(q-1)^2]$$
$$= \frac{1}{p^2+(q-1)^2}[p^2 q + q(q-1)^2]$$
$$= q\left[\frac{p^2+(q-1)^2}{p^2+(q-1)^2}\right] = q$$

7. Let A be the expression. Multiplying A by 2^{2008} and using $2\sin\theta\cos\theta = \sin 2\theta$,

we have $\quad 2^{2008}A = \sin\frac{\pi}{2} = 1.\ A = \frac{1}{2^{2008}}$

Alternatively $\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right) = \frac{1}{2}\sin\left(\frac{\pi}{2^{2008}}\right)$

$$= \frac{1}{2^2}\cdot 2\sin\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)$$
$$= \frac{1}{2^2}\sin\left(\frac{\pi}{2^{2007}}\right)$$

Similarly, continued product upto,

$$\cos\left(\frac{\pi}{2^2}\right) = \frac{1}{2^{2008}}\sin\left(\frac{\pi}{2}\right) = \frac{1}{2^{2008}}$$

8. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$$= \frac{\tan A + \frac{n\sin A\cos A}{1-n\cos^2 A}}{1 - \tan A\cdot\frac{n\sin A\cos A}{1-n\cos^2 A}}$$
$$= \frac{\sin A(1-n\cos^2 A) + n\sin A\cos^2 A}{\cos A(1-n\cos^2 A) - n\sin^2 A\cos A}$$
$$= \frac{\sin A - 0}{\cos A(1-n\cos^2 A - n\sin^2 A)}$$
$$= \frac{\sin A}{(1-n)\cos A}$$

9. We have, $\quad Q = \sum_{r=0}^{n}\frac{\sin(3^r\theta)}{\cos(3^{r+1}\theta)}$

$$= \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta} + \ldots\ldots + \frac{\sin(3^n\theta)}{\cos(3^{n+1}\theta)}$$

As, $\dfrac{\sin\theta}{\cos 3\theta} = \dfrac{2\sin\theta\cos\theta}{2\cos\theta\cos 3\theta} = \dfrac{\sin 2\theta}{2\cos\theta\cos 3\theta}$

$$= \frac{1}{2}\left[\frac{\sin(3\theta - \theta)}{\cos\theta\cos 3\theta}\right]$$

$$= \frac{1}{2}(\tan 3\theta - \tan\theta)$$

$\therefore$ $Q = \frac{1}{2}[(\tan 3\theta - \tan\theta) + (\tan 9\theta - \tan 3\theta) + \ldots + \tan 3^{n+1}\theta - \tan 3^n\theta)]$

$\Rightarrow$ $Q = \frac{P}{2} \Rightarrow P = 2Q$

10. Expression

$$(\cos^4 1° + \cos^4 2° + \cos^4 3° + \ldots + \cos^4 179°)$$
$$- (\sin^4 1° + \sin^4 2° + \sin^4 3° + \ldots + \sin^4 179°)$$
$$= \cos 2° + \cos 4° + \cos 6° + \ldots + \cos(358°)$$
$$= \cos\frac{\left(\frac{2° + 358°}{2}\right)\cdot\sin(179 \times 1°)}{\sin 1°}$$
$$= \cos(180°) = -1.$$

11. $\sin x + \sin y = a$ ...(i)

$\cos x + \cos y = 2a$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$2 + 2\cos(x - y) = 5a^2$$
$$\cos(x - y) = \frac{5a^2 - 2}{2}$$

12. $P(x) = \sqrt{3 + 2(\cos x + \cos x + \cos 2x)}$

$$= \sqrt{3 + 2(2\cos x + 2\cos x^2 x - 1}$$
$$= \sqrt{4\cos^2 x + 4\cos^2 x + 1}$$
$$= |2\cos x + 1|$$

13. Consider $y = 5\,\text{secc}^2\theta - \tan^2\theta + 4\,\text{cosec}^2\theta$

$\therefore$ $y = 5 + 5\tan^2\theta - \tan^2\theta + 4 + \cot^2\theta$

$$y = 9 + 4(\tan^2 + \cot^2)$$
$$= 9 + 4[(\tan\theta - \cot\theta)^2 + 2]$$

$\therefore$ $y_{\min} = 9 + 8 = 17$

$\Rightarrow$ Maximum value of the expression is $\frac{1}{17} = \frac{p}{q}$

$\Rightarrow$ $p + q = 1 + 17 = 18$

14. $f_n(\alpha) = \tan n\alpha$ and $f_n\left(\frac{\pi}{32}\right) = \tan\frac{\pi}{8} = \sqrt{2} - 1$

15. Let $\sin x + \cos x = t$

$\therefore$ $y = \left|t + \frac{1}{t}\right|$

Hence, minimum value of y is 2.

16. $a = \cos(2012\,\pi) = 1$

$b = \sec(2013\,\pi) = -1$

$c = \tan(2014\,\pi) = 0$

$\therefore b < c < a$

17. In ΔABC, $\sum \tan\frac{A}{2}\cdot\tan\frac{B}{2} = 1$

$\therefore$ $\sum \tan^2\frac{A}{2} \geq \sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$

$[\because a^2 + b^2 + c^2 - ab - bc - ca \geq 0, \forall\, a, b, c \in R]$

$\therefore$ $3 + \sum \tan^2\frac{A}{2} \geq 4$

$\Rightarrow$ $3 + \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1 + 3$

$\Rightarrow$ $\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \geq 4$

18. The given equation can be rewritten as

$(2x - 1)^2 + 1 = \sin^2 y$, which is possible only when $x = \frac{1}{2}$,

$\sin^2 y = 1$

$\Rightarrow y = \frac{-\pi}{2}, \frac{\pi}{2}$ [as $x^2 + y^2 \leq 3$]

Thus, there are only two pairs (x, y) satisfying the given equation. They are $\left(\frac{1}{2}, \frac{-\pi}{2}\right)$ and $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.]

19. Given,

$3\sin A + 4\cos B = 6$ ...(i)

$3\cos A + 4\sin B = 1$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$9 + 16 + 24\sin(A + B) = 37$$
$$24\sin(A + B) = 12$$
$$\sin(A + B) = \frac{1}{2}$$

$\Rightarrow$ $\sin C = \frac{1}{2}$

$$C = 30° \text{ or } 150°$$

If $C = 150°$, then even of $B = 0$ and $A = 30°$.

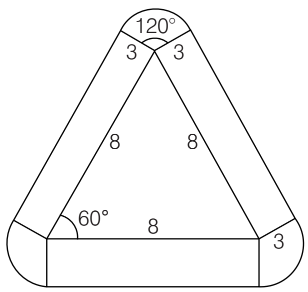
The quantity $3\sin A + 4\cos B$

$$3\cdot\frac{1}{2} + 4 = 5\frac{1}{2} < 6$$

Hence, $C = 150°$ is not possible

$\Rightarrow$ $\angle C = 30°$ only

20. Area $= 3\cdot(8\cdot 3) + 3\cdot\frac{1}{2}r^2\theta$



$$= 72 + \frac{3}{2}\cdot 9\cdot\frac{2\pi}{3}$$
$$= 72 + 9\pi$$
$$= 9(8 + \pi)$$

21. $m+n=a\{(\cos^3\alpha+\sin^3\alpha)+3\cos\alpha\sin\alpha(\cos\alpha+\sin\alpha)\}$

$m+n=a\{\cos\alpha+\sin\alpha\}^3$

Similarly, $\quad m-n=a\{\cos\alpha-\sin\alpha\}^3$

$$(m+n)^{2/3}=a^{2/3}(\cos\alpha+\sin\alpha)^2 \quad \text{...(i)}$$

Similarly, $(m-n)^{2/3}=a^{2/3}(\cos\alpha-\sin\alpha)^2$ ...(ii)

On adding Eqs. (i) and (ii), we get

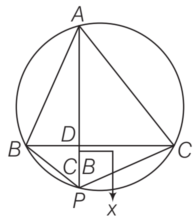
$(m+n)^{2/3}+(m-n)^{2/3}=a^{2/3}\,(2)$

$\Rightarrow \qquad =2a^{2/3}$

22. $BD=x\tan C$ in ΔPDB

and $DC=x\tan B$ for ΔPDC

$\therefore \quad BD+DC=a=x(\tan B+\tan C)$



$$\frac{a}{x}=\tan B+\tan C$$

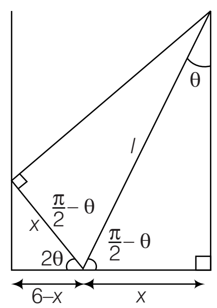
Similarly, $\quad \dfrac{b}{y}=\tan A+\tan C$

$$\frac{c}{z}=\tan A+\tan B$$

$\therefore \quad \dfrac{a}{2x}+\dfrac{b}{2y}+\dfrac{c}{2z}=\dfrac{1}{2}\left(\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}\right)\tan A+\tan B+\tan C$

23. $\sin\theta=\dfrac{x}{l}$ ...(i)

Also, $\qquad \cos 2\theta=\dfrac{6-x}{x}$



$$1+\cos 2\theta=\frac{6}{x};$$

$$2\cos^2\theta=\frac{6}{l\sin\theta}$$

[substituting $x=l\sin\theta$ from Eq. (i)]

$$l=\frac{3}{\sin\theta\cos^2\theta}$$

24. $S=2\sin 2^\circ+4\sin 4^\circ+\ldots+178\sin 178+180^\circ\sin 180^\circ$

$S=2[\sin 2^\circ+2\sin 4^\circ+3\sin 6^\circ+\ldots+89\sin 178^\circ]$ ...(i)

$S=2[89\sin 178^\circ+88\sin 176^\circ+\ldots+1\cdot\sin 2^\circ]$ ...(ii)

On adding Eqs. (i) and (ii), we get

[converting in reverse order]

$2S=2[90\,(\sin 2^\circ+\sin 4^\circ+\sin 6^\circ+\ldots+\sin 178^\circ)]$

$$S=90\cdot\frac{\sin\left(\dfrac{n\theta}{2}\right)}{\sin\dfrac{\theta}{2}}\sin\left(\frac{(n+1)\theta}{2}\right)$$

$$=\frac{90\sin(89^\circ)}{\sin 1^\circ}\cdot\sin 90^\circ \qquad [\theta=2^\circ]$$

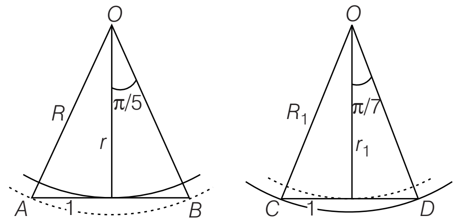
$S=90\cot 1^\circ$

Average value $=\dfrac{90\cot 1^\circ}{90}=\cot 1^\circ$

25. In 1st case, $r=\cot\dfrac{\pi}{5};R=\operatorname{cosec}\dfrac{\pi}{5}$

2nd case, $\quad r_1=\cot\dfrac{\pi}{7};R_1=\operatorname{cosec}\dfrac{\pi}{7}$

$\therefore \qquad A_1=\pi(R^2-r^2)=\pi\left(\operatorname{cosec}^2\dfrac{\pi}{5}-\cot^2\dfrac{\pi}{5}\right)=\pi$



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M $\qquad A_2=\pi(R_1^2-r_1^2)$

$$=\pi\left(\operatorname{cosec}^2\frac{\pi}{7}-\cot^2\frac{\pi}{7}\right)=\pi$$

$\Rightarrow \qquad A_1=A_2$

26. $\displaystyle\sum_{r=1}^{18}\cos^2(5r)^\circ=\cos^2 5^\circ+\cos^2 10^\circ$

$+\cos^2 15^\circ+\ldots+\cos^2 85^\circ+\cos^2 90^\circ$

$=(\cos^2 5^\circ+\cos^2 85^\circ)+(\cos^2 10^\circ+\cos^2 80^\circ)$

$+(\cos^2 15^\circ+\cos^2 75^\circ)+\ldots+(\cos^2 40^\circ+\cos^2 50^\circ)+\cos^2 45^\circ$

$=(\cos^2 5^\circ+\sin^2 5^\circ)+(\cos^2 10^\circ+\sin^2 10^\circ)$

$+(\cos^2 15^\circ+\sin^2 15^\circ)+\ldots+(\cos^2 40^\circ+\sin^2 40^\circ)+\cos^2 45^\circ$

$=1+1+1+\ldots+1+\dfrac{1}{2}=8+\dfrac{1}{2}=\dfrac{17}{2}$

27. $4x^2-4x\,|\sin\theta|-(1-\sin^2\theta)$

$=-1+(2x-|\sin\theta|)^2$

$\therefore$ Minimum value $=-1$

28. $\because \cos 3A+\cos 3B+\cos 3C=1$

$\Rightarrow \qquad \cos 3A+\cos 3B+\cos 3C-1=0$

$\Rightarrow \qquad \cos 3A+\cos 3B+\cos 3C+\cos 3\pi=0$

$\Rightarrow 2\cos\left(\dfrac{3A+3B}{2}\right)\cos\left(\dfrac{3A-3B}{2}\right)+2\cos\left(\dfrac{3\pi+3C}{2}\right)$

$$\cos\left(\frac{3\pi-3C}{2}\right)=0$$

$\Rightarrow 2\cos\left(\dfrac{3\pi-3C}{2}\right)\left\{\cos\left(\dfrac{3A-3B}{2}\right)+\cos\left(\dfrac{3\pi+3C}{2}\right)\right\}=0$

$\Rightarrow \;2\cos\left(\dfrac{3\pi}{2}-\dfrac{3C}{2}\right)\cdot 2\cos\left(\dfrac{3\pi+3C+3A-3B}{4}\right)$

$$\cdot\cos\left(\frac{3\pi+3C-3A+3B}{4}\right)=0$$

$$\Rightarrow\ 2\cos\left(\frac{3\pi}{2}-\frac{3C}{2}\right)\cdot 2\cos\left(\frac{3\pi}{2}-\frac{3B}{2}\right)\cdot\cos\left(\frac{3\pi}{2}-\frac{3A}{2}\right)=0$$

$$\Rightarrow\quad -4\sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right)=0$$

$$\Rightarrow\quad \sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right)=0$$

$$\therefore\quad \frac{3A}{2}=\pi\ \text{ or }\ \frac{3B}{2}=\pi\ \text{ or }\ \frac{3C}{2}=\pi$$

$$\therefore\quad A=\frac{2\pi}{3}\ \text{or}\ B=\frac{2\pi}{3}\ \text{or}\ C=\frac{2\pi}{3}$$

29. $\because |\tan A| < 1$

$$\Rightarrow\quad -1<\tan A<1\ \text{ and }\ 0\le |A| < \frac{\pi}{2}$$

$$\Rightarrow\quad -\frac{\pi}{2}<A<\frac{\pi}{2}$$

$$\because\quad \sqrt{(1+\sin 2A)}=\sqrt{1+\frac{2\tan A}{1+\tan^2 A}}$$

$$=\frac{|1+\tan A|}{\sqrt{(1+\tan^2 A)}}=\frac{(1+\tan A)}{\sqrt{(1+\tan^2 A)}}$$

and $\sqrt{(1-\sin 2A)}=\sqrt{1-\left(\dfrac{2\tan A}{1+\tan^2 A}\right)}$

$$=\frac{|1-\tan A|}{\sqrt{(1+\tan^2 A)}}$$

$$=\frac{(1-\tan A)}{\sqrt{(1-\tan^2 A)}}$$

$$\therefore\quad \frac{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}}{\sqrt{(1+\sin 2A)}-\sqrt{(1-\sin 2A)}}=\frac{2}{2\tan A}=\cot A$$

30. Let $f(\theta)=\cos^2(\cos\theta)+\sin^2(\sin\theta)$

$\because\quad -1\le\cos\theta\le 1$ and $-1\le\sin\theta\le 1$

$\therefore\quad \cos 1\le\cos(\cos\theta)\le 1$ and $-\sin 1\le\sin(\sin\theta)\le\sin 1$

$\therefore\quad \cos^2 1\le\cos^2(\cos\theta)\le 1$ and $0\le\sin^2(\sin\theta)\le\sin^2 1$

$\therefore$ Maximum value of $f(\theta)=1+\sin^2 1$

31. Let $f(x)=27^{\cos 2x}\,81^{\sin 2x}=3^{3\cos 2x+4\sin 2x}$

$$=3^{5\left(\frac{3}{5}\cos 2x+\frac{4}{5}\sin 2x\right)}$$

Let $\dfrac{3}{5}=\sin\phi$ and $\dfrac{4}{5}=\cos\phi$

Thus, $\quad f(x)=3^{5(\sin\phi\,\cos 2x+\cos\phi\,\sin 2x)}=3^{5[(\sin(\phi+2x)]}$

For minimum value of given function, $\sin(\phi+2x)$ will be minimum.

i.e. $\quad \sin(\phi+2x)=-1$

$$\therefore\quad f(x)=3^{5(-1)}=\frac{1}{243}$$

Alternate Method

Let $\quad f(x)=27^{\cos 2x}\,81^{\sin 2x}=3^{4\sin 2x}=3^{3\cos 2x+4\sin 2x}$

For minimum value of given function, $3\cos 2x+4\sin 2x$ will be minimum.

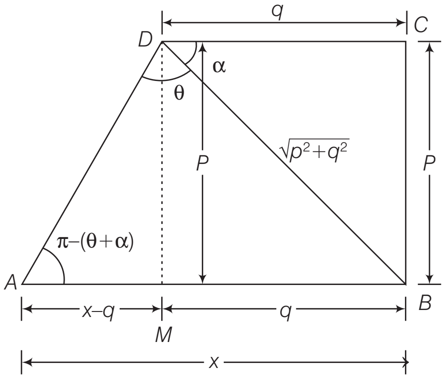
$$\therefore\quad -\sqrt{3^2+4^2}\le 3\cos 2x+4\sin 2x\le\sqrt{3^2+4^2}$$

$$\Rightarrow\quad -5\le 3\cos 2x+4\sin 2x\le 5$$

$\therefore$ Minimum of $3\cos 2x+4\sin 2x=-5$

So, $\quad \min f(x)=3^{-5}=\dfrac{1}{243}$

32. Let $AB=x$ and $\angle BDC=\alpha$



In ΔDAM, $\tan(\pi-\theta-\alpha)=\dfrac{p}{x-q}$

$$\Rightarrow\quad \tan(\theta+\alpha)=\frac{p}{q-x}\Rightarrow q-x=p\cos(\theta+\alpha)$$

$$\Rightarrow\ x=q-p\cot(\theta+\alpha)=q-p\left(\frac{\cot\theta\cot\alpha-1}{\cot\alpha+\cot\theta}\right)$$

$$=\frac{q(\cot\alpha+\cot\theta)+p(\cot\theta\cot\alpha)+p}{\cot\alpha+\cot\theta}$$

$$=\frac{q\left(\dfrac{q}{p}+\dfrac{\cos\theta}{\sin\theta}\right)-p\left(\dfrac{q}{p}\,\dfrac{\cos\theta}{\sin\theta}\right)+p}{\dfrac{q}{p}+\dfrac{\cos\theta}{\sin\theta}}\qquad\left[\because \cot\alpha=\frac{q}{p}\right]$$

$$=\frac{\dfrac{q^2}{p}+\dfrac{q\cos\theta}{\sin\theta}-q\,\dfrac{\cos\theta}{\sin\theta}+p}{\dfrac{q\sin\theta+p\cos\theta}{p\sin\theta}}=\frac{(q^2+p^2)\sin\theta}{q\sin\theta+p\sin\theta}$$

33. Given $4n\alpha=\pi\ \Rightarrow\ 2n\alpha=\dfrac{\pi}{2}$

Now $\cot\alpha\cdot\cot(2n-1)\alpha=\cot\alpha\,\cot\left(\dfrac{\pi}{2}-\alpha\right)$

$$=\cot\alpha\cdot\tan\alpha=1$$

Similarly, $\cot 2\alpha\,\cos(2n-2)\alpha=1$,

$\cot 3\alpha\cdot\cot(2n-3)\alpha=1,\ \ldots,\ \cot(n-1)\alpha\,\cot(n+1)\alpha=1$

Thus $\cot\alpha\,\cot 2\alpha\,\cot 3\alpha\,\ldots\,\cot(2n-1)\alpha$

$=\{\cot\alpha\,\cot(2n-1)\alpha\}\{\cot 2\alpha\,\cot(2n-2)\alpha\}$

$\ldots\{\cot(n-1)\alpha\,\cot(n+1)\alpha\}\cdot\cot n\alpha$

$=1\cdot 1\cdot 1\ldots 1\cdot 1\qquad\left[\because \cot n\alpha=\cot\dfrac{\pi}{4}=1\right]$

$=1$

34. We have $(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)$

$$=3\sin A\sin B$$

$$\Rightarrow\quad (\sin A+\sin B)^2-\sin^2 C=3\sin A\sin B$$

$$\Rightarrow\quad \sin^2 A+\sin^2 B-\sin^2 C=\sin A\sin B$$

$$\Rightarrow \quad \sin^2 A + \sin(B+C)\sin(B-C) = \sin A \sin B$$

$$\Rightarrow \quad \sin A\,[\sin(B+C) + \sin(B-C)] = \sin A \sin B$$

$$[\because A + B + C = \pi]$$

$$\Rightarrow \quad \sin A(2\sin B\cos C) = \sin A \sin B$$

$$\therefore \quad \cos C = \frac{1}{2} \Rightarrow C = 60^\circ$$

35. From the third relation we get

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \sin\beta\sin\gamma$$

$$\Rightarrow \sin^2\theta\sin^2\phi = (\cos\theta\cos\phi - \sin\beta\sin\gamma)^2$$

$$\Rightarrow \left(1 - \frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1 - \frac{\sin^2\gamma}{\sin^2\alpha}\right) = \left(\frac{\sin\beta\sin\gamma}{\sin^2\alpha} - \sin\beta\sin\gamma\right)^2$$

[from the first and second relations]

$$\Rightarrow (\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma) = \sin^2\beta\sin^2\gamma(1-\sin^2\alpha)^2$$

$$\Rightarrow \sin^4\alpha(1 - \sin^2\beta\sin^2\gamma) - \sin^2\alpha(\sin^2\beta + \sin^2\gamma - 2\sin^2\beta\sin^2\gamma) = 0$$

$$\therefore \quad \sin^2\alpha = \frac{\sin^2\beta - \sin^2\gamma - 2\sin^2\beta\sin^2\gamma}{1 - \sin^2\beta\sin^2\gamma} \quad [\because \sin\alpha \neq 0]$$

$$\text{and } \cos^2\alpha = \frac{1 - \sin^2\beta - \sin^2\gamma + \sin^2\beta\sin^2\gamma}{1 - \sin^2\beta\sin^2\gamma}$$

$$\Rightarrow \quad \tan^2\alpha = \frac{\sin^2\beta - \sin^2\beta - \sin^2\gamma + \sin^2\gamma - \sin^2\beta\sin^2\gamma}{\cos^2\beta - \sin^2\gamma(1-\sin^2\beta)}$$

$$= \frac{\sin^2\beta\cos^2\gamma + \cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$= \tan^2\beta + \tan^2\gamma$$

$$\Rightarrow \tan^2\alpha - \tan^2\beta - \tan^2\gamma = 0$$

36. $\tan\beta = \dfrac{n\tan\alpha}{1 + \tan^2\alpha - n\tan^2\alpha}$

$$= \frac{n\tan\alpha}{1 + (1-n)\tan^2\alpha}$$

$$\Rightarrow \tan(\alpha - \beta) = \frac{\tan\alpha - \dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1 + \dfrac{\tan\alpha\cdot n\tan\alpha}{1+(1-n)\tan^2\alpha}}$$

$$= \frac{\tan\alpha + (1-n)\tan^3\alpha - n\tan\alpha}{1 + (1-n)\tan^2\alpha + n\tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha(1+\tan^2\alpha)}{1+\tan^2\alpha} = (1-n)\tan\alpha$$

37. Let $\dfrac{(\cos\theta)}{a} = \dfrac{(\sin\theta)}{b} = k$, so that $\cos\theta = ak$ and $\sin\theta = bk$. Then

$$a\cos 2\theta + b\sin 2\theta = a(1 - 2\sin^2\theta) + 2b\sin\theta\cos\theta$$

$$= a - 2ab^2k^2 + 2b\cdot bk\cdot ak$$

$$= a - 2ab^2k^2 + 2ab^2k^2 = a$$

38. Let $y = \cos x\cos(x+2) - \cos^2(x+1)$

$$= \frac{1}{2}[\cos(2x+2) + \cos 2] - \cos^2(x+1)$$

$$= \frac{1}{2}[2\cos^2(x+1) - 1 + \cos 2] - \cos^2(x+1)$$

$$= -\frac{1}{2}(1 - \cos 2) = -\frac{1}{2}(2\sin^2 1) = -\sin^2 1$$

This shows that $y = -\sin^2 1$ is a straight line which is parallel to X-axis and clearly passes through the point $\left(\dfrac{\pi}{2}, -\sin^2 1\right)$.

39. $f(\theta) = |\sin\theta| + |\cos\theta|, \forall\, \theta \in R$ Clearly, $f(\theta) > 0$.

Also, $\quad f^2(\theta) = \sin^2\theta + \cos^2\theta + |2\sin\theta\cdot\cos\theta|$

$$= 1 + |\sin 2\theta|$$

$$0 \le |\sin 2\theta| \le 1$$

$$\Rightarrow \quad 1 \le f^2(\theta) \le 2 \Rightarrow 1 \le f(\theta) \le \sqrt{2}$$

40. $A = \cos(\cos x) + \sin(\cos x)$

$$= \sqrt{2}\left\{\cos(\cos x)\cos\frac{\pi}{4} + \sin(\cos x)\sin\frac{\pi}{4}\right\}$$

$$= \sqrt{2}\left\{\cos\left(\cos x - \frac{\pi}{4}\right)\right\}$$

$$\because \quad -1 \le \cos\left(\cos x - \frac{\pi}{4}\right) \le 1$$

$$\therefore \quad -\sqrt{2} \le A \le \sqrt{2}$$

41. We have, $\dfrac{U_n}{V_n} = \tan n\theta$

$$\text{and } \frac{V_n - V_{n-1}}{U_{n-1}} = \frac{\cos n\theta\sec^n\theta - \cos(n-1)\theta\sec^{n-1}\theta}{\sin(n-1)\theta\sec^{n-1}\theta}$$

$$= \frac{\cos n\theta\sec\theta - \cos(n-1)\theta}{\sin(n-1)\theta}$$

$$= \frac{\cos n\theta - \cos(n-1)\theta\cos\theta}{\cos\theta\sin(n-1)\theta}$$

$$= \frac{\cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta - \cos(n-1)\theta\cos\theta}{\cos\theta\sin(n-1)\theta}$$

$$= -\tan\theta$$

So, that $\dfrac{V_n - V_{n-1}}{U_{n-1}} + \dfrac{1}{n}\dfrac{U_n}{V_n} = -\tan\theta + \dfrac{\tan n\theta}{n} \neq 0$

42. If $a, b > 0$

Using A.M. $\ge$ G.M., we get

$$\frac{1}{a} + \frac{1}{b} \ge \frac{2}{\sqrt{ab}}$$

$$\Rightarrow \quad f(x) \ge \frac{2}{\sqrt{\cos\left(\frac{\pi}{6} - x\right)\cos\left(\frac{\pi}{6} + x\right)}}$$

$$= \frac{2}{\sqrt{\cos^2\frac{\pi}{6} - \sin^2 x}} = \frac{2}{\sqrt{\frac{3}{4} - \frac{1 - \cos 2x}{2}}}$$

$$= \frac{2}{\sqrt{\frac{1}{4} + \frac{\cos 2x}{2}}}$$

Now for $0 \le x \le \frac{\pi}{3}$, $\frac{-1}{2} \le \cos 2x \le 1$

$$\Rightarrow \quad 0 \le \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \le \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad f(x) \ge \frac{4}{\sqrt{3}}$$

Since 'f' is continuous range of 'f' is $\left[\frac{4}{\sqrt{3}}, \infty\right)$.

43. $\because 0 \le \sin^2 \theta \le 1$ and $0 \le \cos^2 \theta \le 1$

$$\Rightarrow \quad 0 \le \sin^8 \theta \le \sin^2 \theta \text{ and } 0 \le \cos^{14} \theta \le \cos^2 \theta$$

$$\therefore \quad 0 < \sin^8 \theta + \cos^{14} \theta \le \sin^2 \theta + \cos^2 \theta$$

Hence, $\quad 0 < A \le 1$

44. $\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos\alpha, \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$

$$\sin(3\pi + \alpha) = -\sin\alpha$$
$$\sin(5\pi - \alpha) = -\sin\alpha$$

$$\therefore \quad 3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\} - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$

$$= 3\{\cos^4 \alpha + \sin^4 \alpha\} - 2\{\cos^6 \alpha + \sin^6 \alpha\}$$
$$= 3\{1 - 2\sin^2 \alpha \cos^2 \alpha\} - 2\{1 - 3\sin^2 \alpha \cos^2 \alpha\} = 1$$

45. $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$$= \sqrt{2}\left[\sin\left(x + \frac{\pi}{6} + \frac{\pi}{4}\right)\right]$$
$$= \sqrt{2}\sin\left(x + \frac{5\pi}{12}\right) \le \sqrt{2}$$

Equality holds when $x + \frac{5\pi}{12} = \frac{\pi}{2}$ ie, $x = \frac{\pi}{12}$

Therefore, maximum value of given expression is attained at

$$x = \frac{\pi}{12}$$

46. $\cot^2 x = \cot(x - y) \cdot \cot(x - z)$

$$\Rightarrow \quad \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x}\right)\left(\frac{\cot x \cdot \cot z + 1}{\cot z - \cot x}\right)$$

$$\Rightarrow \cot^2 x \cdot \cot y \cdot \cot z - \cot^3 x \cdot \cot y - \cot^3 x \cot z + \cot^4 x$$
$$= \cot^2 x \cdot \cot y \cdot \cot z + \cot x \cdot \cot y + \cot x \cdot \cot z + 1$$

$$\Rightarrow \cot x \cot y(1 + \cot^2 x) + \cot x \cot z(1 + \cot^2 x) + 1 - \cot^4 x = 0$$

$$\Rightarrow \quad \cot x(\cot y + \cot z)(1 + \cot^2 x) + (1 - \cot^2 x)(1 + \cot^2 x) = 0$$

$$\Rightarrow \quad \cot x(\cot y + \cot z) + (1 - \cot^2 x) = 0$$

$$\Rightarrow \quad \frac{\cot^2 x - 1}{2\cot x} = \frac{1}{2}(\cot y + \cot z)$$

$$\Rightarrow \quad \frac{1}{2}(\cot y + \cot z) = \cot 2x$$

47. Given that, $\alpha + \beta + \gamma = \pi$

Taking $\alpha = -\frac{\pi}{2}; \beta = -\frac{\pi}{2}$ and $\gamma = 2\pi$

$$\therefore \quad \sin\alpha + \sin\beta + \sin\gamma = -1 - 1 + 0 = -2$$

but $\sin\alpha + \sin\beta + \sin\gamma \ge -3$ for any α, β, γ

Hence, minimum value of $\sin\alpha + \sin\beta + \sin\gamma$ is negative.

48. $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$

$$\Rightarrow \quad \sin\beta\cos x - \sin\alpha\cos\beta\sin x = \cos\alpha\sin\beta$$

$$\Rightarrow \sin\beta\left(1 - \tan^2\frac{x}{2}\right) - \sin\alpha\cos\beta \cdot 2\tan\frac{x}{2} = \cos\alpha\sin\beta\left(1 + \tan^2\frac{x}{2}\right)$$

$$\Rightarrow \tan^2\frac{x}{2}(-\sin\beta - \cos\alpha\sin\beta) - \sin\alpha\cos\beta \cdot 2\tan\frac{x}{2} + \sin\beta(1 - \cos\alpha) = 0$$

$$\Rightarrow \quad \tan\frac{x}{2} = \frac{-2\sin\alpha\cos\beta \pm \sqrt{4[\sin^2\alpha\cos^2\beta + \sin^2\beta(1 + \cos\alpha)(1 - \cos\alpha)]}}{2\sin\beta(1 + \cos\alpha)}$$

$$= \frac{-\sin\alpha\cos\beta \pm \sqrt{\sin^2\alpha(\sin^2\beta + \cos^2\beta)}}{\sin\beta(1 + \cos\alpha)}$$

$$= \frac{-\sin\alpha\cos\beta \pm \sin\alpha}{\sin\beta(1 + \cos\alpha)} = \frac{\sin\alpha(1 - \cos\beta \pm 1)}{\sin\beta(1 + \cos\alpha)}$$

$$= \tan\frac{\beta}{2}\tan\frac{\alpha}{2} \text{ or } -\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

49. $\because \cos^4\theta\sec^4\alpha, \frac{1}{2}$ and $\sin^4\theta\operatorname{cosec}^2\alpha$ are in AP

$$1 = \cos^4\theta\sec^2\alpha + \sin^4\theta\operatorname{cosec}^2\alpha$$

$$\Rightarrow \quad 1 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow \quad (\sin^2\theta + \cos^2\theta)^2 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow \quad \cos^4\theta\left(\frac{1}{\cos^2\alpha} - 1\right) + \sin^4\theta\left(\frac{1}{\sin^2\alpha} - 1\right) - 2\sin^2\theta\cos^2\theta = 0$$

$$\Rightarrow \quad \sin^4\alpha\cos^4\theta + \sin^4\theta\cos^4\alpha - 2\sin^2\theta\cos^2\theta\sin^2\alpha\cos^2\alpha = 0$$

$$\Rightarrow \quad (\sin^2\alpha\cos^2\theta - \cos^2\alpha\sin^2\theta)^2 = 0$$

$$\Rightarrow \quad \tan^2\theta = \tan^2\alpha$$

$$\therefore \quad \theta = n\pi \pm \alpha, n \in I$$

Now, $\quad \cos^8\theta\sec^6\alpha = \cos^8\alpha\sec^6\alpha = \cos^2\alpha$

and $\quad \sin^8\theta\operatorname{cosec}^6\alpha = \sin^8\alpha \cdot \operatorname{cosec}^6\alpha = \sin^2\alpha$

Hence, $\cos^8\theta\sec^6\alpha, \frac{1}{2}, \sin^8\theta\operatorname{cosec}^6\alpha$

ie, $\cos^2\alpha, \frac{1}{2}, \sin^2\alpha$ are in AP.

50. Given, $(\cot \alpha_1) \cdot (\cot \alpha_2) \ldots (\cot \alpha_n) = 1$

$$\Rightarrow \quad \prod_{i=1}^{n} \cos \alpha_1 = \prod_{i=1}^{n} \sin \alpha_1$$

$$\Rightarrow \quad \prod_{i=1}^{n} \cos^2 \alpha_1 = \prod_{i=1}^{n} \sin \alpha_1 \cos \alpha_1 = \prod_{i=1}^{n} \frac{\sin 2\alpha_1}{2} \le \frac{1}{2^n}$$

$$\Rightarrow \quad \prod_{i=1}^{n} \cos \alpha_1 \le \frac{1}{2^{\frac{n}{2}}}$$

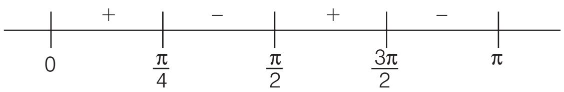
Hence, maximum value of $\prod_{i=1}^{n} \cos \alpha_1$ is $\frac{1}{2^{\frac{n}{2}}}$.

51. $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$

$$\Rightarrow \quad \sin x \cdot \cos x(\cos^2 x - \sin^2 x) > 0$$

$$\Rightarrow \quad \sin x \cdot \cos x \cdot \cos 2x > 0$$

$$\Rightarrow \quad \cos x \cdot \cos 2x > 0$$



$$\therefore \quad x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

52. $u^2 = a^2 + b^2$

$$+ 2\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \times \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$$

$$= a^2 + b^2 + 2\sqrt{\sin^2 \theta \cos^2 \theta(a^4 + b^4) + a^2b^2(\sin^4 \theta + \cos^4 \theta)}$$

$$= a^2 + b^2 + 2\sqrt{a^2b^2(1 - 2\sin^2\theta \cos^2\theta) + (a^4 + b^4)\sin^2\theta \cos^2\theta}$$

$$= (a^2 + b^2) + 2\sqrt{a^2b^2 + (a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta}$$

$$= (a^2 + b^2) + 2\sqrt{a^2b^2 + \frac{(a^2 - b^2)^2}{4} \sin^2 2\theta}$$

Max. $u^2 = (a^2 + b^2) + 2\sqrt{a^2b^2 + \frac{(a^2 - b^2)^2}{4}}$

Min. $u^2 = (a^2 + b^2) + 2ab$

$$\Rightarrow \text{ Difference } 2\sqrt{a^2b^2 + \frac{(a^2 - b^2)^2}{4}} - 2ab$$

$$= \sqrt{4a^2b^2 + a^4 + b^4 - 2a^2b^2} - 2ab$$

$$= \sqrt{(a^2 + b^2)^2} - 2ab$$

$$= a^2 + b^2 - 2ab = (a - b)^2$$

53. $\because \left(\tan \frac{\theta}{2}\right)(1 + \sec \theta) = \tan\left(\frac{\theta}{2}\right)\left(\frac{1 + \cos \theta}{\cos \theta}\right)$

$$= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{2 \cos^2 \frac{\theta}{2}}{\cos \theta} = \frac{2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{\cos \theta}$$

$$= \frac{\sin \theta}{\cos \theta} = \tan \theta \quad \ldots\text{(i)}$$

$\therefore$ By repeated use of Eq. (i), we have

$$f_n(\theta) = \tan \theta\,(1 + \sec 2\theta)\,(1 + \sec 4\theta) \ldots (1 + \sec 2^n \theta)$$

$$= \tan 2\theta\,(1 + \sec 4\theta) \ldots (1 + \sec 2^n \theta)$$

$$= \tan 4\theta\,(1 + \sec 8\theta) \ldots (1 + \sec 2^n \theta)$$

$$= \ldots. = \tan 2^n \theta$$

Now, $f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2 \frac{\pi}{16}\right) = \tan \frac{\pi}{4} = 1$

$$f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3 \frac{\pi}{32}\right) = \tan \frac{\pi}{4} = 1$$

$$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4 \frac{\pi}{64}\right) = \tan \frac{\pi}{4} = 1$$

and $f_5\left(\frac{\pi}{128}\right) = \tan \frac{\pi}{4} = 1$

54. $\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$

$$\Rightarrow \quad \cos\left((2n + 1)\frac{\pi}{2} + z\right) = \frac{1}{2}$$

$$\Rightarrow \quad \sin z = \frac{1}{2} \text{ or } \sin z = -\frac{1}{2}$$

$$\Rightarrow \quad z = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

55. $f_n(\theta) = \cos\theta - \cos 2\theta + \cos 2\theta - \cos 3\theta + \ldots + \cos(n)\theta - \cos(n + 1)\theta$

$$f_n(\theta) = \cos\theta - \cos(n + 1)\theta$$

Now, check options.

56. $P = \sin 25° \sin 35° \sin 60° \sin 85°$

$$= \sin 25° \sin(60° - 25°) \ \sin 60° \sin(60° + 25°)$$

$$= \sin 60° \sin 25° \sin(60° - 25°) \sin(60° + 25°)$$

$$\therefore \quad P = \sin 60° \times \frac{1}{4} \sin 75° \quad \ldots\text{(i)}$$

$$Q = \sin 20° \sin 40° \sin 75° \sin 80°$$

$$= \sin 20° \sin(60° - 20°) \sin 75° \sin(60° + 20°)$$

$$= \sin 75° \sin 20° \sin(60° - 20°) \sin(60° + 20°)$$

$$\therefore \quad Q = \sin 75° \times \frac{1}{4} \times \sin 60° \quad \ldots\text{(ii)}$$

Hence, $P = Q$

57. $x = \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta}, y = \frac{1}{\cos^2\theta},$

$$z = \frac{1}{1 - \sin^2\theta \cos^2\theta}$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} = 1$$

$$\Rightarrow \quad xy = x + y \ \Rightarrow \ \frac{1}{z} = 1 - \frac{1}{xy}$$

$$\Rightarrow \quad xyz = xy + z = x + y + z$$

58. Given $P(x) = \cot^2 x\left(\frac{1 + \tan x + \tan^2 x}{1 + \cot x + \cot^2 x}\right) + \left(\frac{\cos x - \cos 3x + \sin 3x - \sin x}{2(\sin 2x + \cos 2x)}\right)^2$

$$= \frac{\cot^2 + \cot x + 1}{1 + \cot x \ \cot^2 x} + \left(\frac{2 \sin x\,(\sin 2x + \cos 2x)}{2(\sin 2x + \cos 2x)}\right)^2$$

$$= 1 + \sin^2 x$$

$\therefore \quad P(18°) = P(72°) = (1 + \sin^2 18°) + (1 + \sin^2 72°)$

$$= 1 + 1 + (\sin^2 18° + \cos^2 18°) = 3$$

59. $E = \dfrac{3\sin(\alpha+\beta) - 4\cos(\alpha+\beta)}{\sqrt{3}\sin\alpha}$

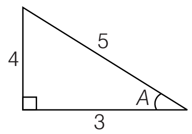
$$= \frac{3(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}{\sqrt{3}\sin\alpha}$$

$$= \frac{5}{\sqrt{3}} \text{ for } 0 < \beta < \frac{\pi}{2}$$

and $E = \dfrac{\sqrt{3}\,(7 + 24\cot\alpha)}{15}$ for $\dfrac{\pi}{2} < \beta < \pi$.

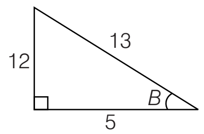
60. $\cot A = \dfrac{3}{4}$

$\Rightarrow \quad \cot C = \dfrac{-3}{4}$



$\Rightarrow$ C is obtuse angle.

$\therefore \quad \sin C = \dfrac{4}{5}, \cos C = -\dfrac{3}{5}$



$\tan B = \dfrac{-12}{5}$

$\Rightarrow \quad \tan D = \dfrac{12}{5}$

$\Rightarrow$ D is an acute angle

$\therefore \quad \sin D = \dfrac{12}{13} \quad \cos D = \dfrac{5}{13}$

Hence, $\sin(C + D) = \sin C \cdot \cos D + \cos C \cdot \sin D$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right)$$

$$= \frac{20 - 36}{65} = \frac{-16}{65}$$

Also, $\sin(A + B) = \sin(2\pi - (C + D))$

$$= -\sin(C + D) = \frac{16}{65}.$$

61. $2\left(\cos^2 x + \dfrac{1}{2}\cos x\right) = a$

$$2\left(\cos x + \frac{1}{4}\right)^2 = a + \frac{1}{8}$$

$\therefore \quad \left(\cos x + \dfrac{1}{4}\right)^2 = \dfrac{a}{2} + \dfrac{1}{16}$

$$\left(\cos x + \frac{1}{4}\right)^2 \in \left[0, \frac{25}{16}\right]$$

$\therefore \quad \dfrac{8a + 1}{16} \in \left[0, \dfrac{25}{16}\right]$

$\Rightarrow \quad 8a + 1 \in [0, 25]$

$\Rightarrow \quad a \in \left[\dfrac{-1}{8}, 3\right].$

62. $A = \sin 44° + \cos 44°$

$$= \cos 46° + \sin 46° = C$$

$$B = \sin 45° + \cos 45° = \sqrt{2}[\sin 90°]$$

$$A = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin 44° + \frac{1}{\sqrt{2}}\cos 44°\right]$$

$$= \sqrt{2}\,[\sin 44° \cdot \cos 45° + \cos 44° \cdot \sin 45°]$$

$$= \sqrt{2}\,\sin 89°$$

$\Rightarrow \quad B > A$

63. $\tan(2\alpha + \beta) = x$

$\tan(\alpha + 2\beta) = y$

$\Rightarrow \quad \tan(3(\alpha + \beta)).\ \tan(\alpha - \beta)$

$$= \tan[(2\alpha + \beta) + (\alpha + 2\beta)]\ .$$

$$\tan[2(\alpha + \beta) - (\alpha + 2\beta)]$$

$$= \frac{\tan(2\alpha + \beta) + \tan(\alpha + 2\beta)}{1 - \tan(2\alpha + \beta).\tan(\alpha + 2\beta)} \cdot \frac{\tan(2\alpha + \beta) - \tan(\alpha + 2\beta)}{1 + \tan(2\alpha + \beta).\tan(\alpha + 2\beta)}$$

$$= \frac{x + y}{1 - xy} \cdot \frac{x - y}{1 + xy} = \frac{x^2 - y^2}{1 - x^2 y^2}$$

64. We have $x = \dfrac{1 - \sin\phi}{\cos\phi}, y = \dfrac{1 + \cos\phi}{\sin\phi}$

Multiplying, we get $xy = \dfrac{(1 - \sin\phi)(1 + \cos\phi)}{\cos\phi\sin\phi}$

$\Rightarrow \quad xy + 1 = \dfrac{1 - \sin\phi + \cos\phi - \sin\phi\cos\phi + \sin\phi\cos\phi}{\cos\phi\sin\phi}$

$$= \frac{1 - \sin\phi + \cos\phi}{\cos\phi\sin\phi}$$

and $\quad x - y = \dfrac{(1 - \sin\phi)\sin\phi - \cos\phi(1 + \cos\phi)}{\cos\phi\sin\phi}$

$$= \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi}{\cos\phi\sin\phi}$$

$$= \frac{\sin\phi - \cos\phi - 1}{\cos\phi\sin\phi} = -(xy + 1)$$

Thus, $xy + x - y + 1 = 0.$

$\Rightarrow \quad x = \dfrac{y - 1}{y + 1}$ and $y = \dfrac{1 + x}{1 - x}.$

65. The given relation can be written as

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x}$$

$$\Rightarrow \quad 2\sin^2\left(\frac{x}{2}\right)=\left[\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)\right]^2$$

$$\Rightarrow \quad 2\tan^2\left(\frac{x}{2}\right)=\left[1-\tan^2\left(\frac{x}{2}\right)\right]^2\Big/\left[1+\tan^2\left(\frac{x}{2}\right)\right]$$

$$\Rightarrow \quad 2y(1+y)=(1-y)^2 \qquad \left[\text{where } y=\tan^2\frac{x}{2}\right]$$

$$\Rightarrow \quad y^2+4y-1=0$$

$$\Rightarrow \quad y=\frac{-4\pm\sqrt{16+4}}{2}=-2\pm\sqrt{5}$$

Since $y > 0$, we get

$$y=\sqrt{5}-2=\frac{(\sqrt{5}-2)^2}{\sqrt{5}+2}\cdot\frac{2+\sqrt{5}}{2+\sqrt{5}}$$

$$=(9-4\sqrt{5})(2+\sqrt{5})$$

66. $y=\dfrac{\sqrt{(\cos 2A-\sin 2A)^2}+1}{\sqrt{(\cos 2A+\sin 2A)^2}-1}$

$$\Rightarrow \quad y=\frac{\pm(\cos 2A-\sin 2A)+1}{\pm(\cos 2A+\sin 2A)-1}$$

which gives us four values of y, say y_1, y_2, y_3 and y_4 We have,

$$y_1=\frac{\cos 2A-\sin 2A+1}{\cos 2A+\sin 2A-1}=\frac{(1+\cos 2A)-\sin 2A}{(\cos 2A-1)+\sin 2A}$$

$$=\frac{2\cos^2 A-2\sin A\cos A}{-2\sin^2 A+2\sin A\cos A}$$

$$=\frac{\cos A(\cos A-\sin A)}{\sin A(\cos A-\sin A)}=\cot A$$

$$y_2=\frac{-(\cos 2A-\sin 2A)+1}{-(\cos 2A+\sin 2A)-1}$$

$$=\frac{(1-\cos 2A)+\sin 2A}{-(1+\cos 2A)-\sin 2A}$$

$$=\frac{2\sin^2 A+2\sin A\cos A}{-2\cos^2 A-2\sin A\cos A}=-\tan A$$

$$y_3=\frac{(\cos 2A-\sin 2A)+1}{-(\cos 2A+\sin 2A)-1}$$

$$=\frac{(1+\cos 2A)-\sin 2A}{-(1+\cos 2A)-\sin 2A}$$

$$=\frac{2\cos^2 A-2\sin A\cos A}{-2\cos^2 A-2\sin A\cos A}$$

$$=-\frac{\cos A-\sin A}{\cos A+\sin A}$$

$$=-\frac{1-\tan A}{1+\tan A}=-\tan\left(\frac{\pi}{4}-A\right)=-\cot\left(\frac{\pi}{4}+A\right)$$

$$y_4=\frac{-(\cos 2A-\sin 2A)+1}{(\cos 2A+\sin 2A)-1}$$

$$=\frac{(1-\cos 2A)+\sin 2A}{-(1-\cos 2A)+\sin 2A}$$

$$=\frac{2\sin^2 A-2\sin A\cos A}{-2\sin A+2\sin A\cos A}$$

67. $\because\ 3\sin\beta=\sin(2\alpha+\beta)$

$$\Rightarrow \quad 2\sin\beta=\sin(2\alpha+\beta)-\sin\beta$$

$$=2\cos(\alpha+\beta)\sin\alpha$$

$$\therefore \quad \sin\beta=\cos(\alpha+\beta)\sin\alpha \qquad \text{...(i)}$$

$\therefore$ **Alternate** (b) is correct

Also, $\quad \sin\beta=\sin(\alpha+\beta)-\alpha)$

$$=\sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha \qquad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$\sin\beta=\sin(\alpha+\beta)\cos\alpha-\sin\beta$$

$$\therefore \quad 2\sin\beta=\sin(\alpha+\beta)\cos\alpha$$

($\therefore$ Alternate (c) is correct)

Alternate (a)

LHS $=(\cot\alpha+\cos(\alpha+\beta))(\cot\beta-3\cot(2\alpha+\beta))$

$$=\left(\frac{\sin(2\alpha+\beta)}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{\cos\beta}{\sin\beta}-\frac{3\cos(2\alpha+\beta)}{\sin(2\alpha+\beta)}\right)$$

$$=\left(\frac{3\sin\beta}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{\cos\beta}{\sin\beta}-\frac{3\cos(2\alpha+\beta)}{3\sin\beta}\right)$$

$(\because 3\sin\beta=\sin(2\alpha+\beta))$

$$=\left(\frac{3\sin\beta}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{\cos\beta-\cos(2\alpha+\beta)}{\sin\beta}\right)$$

$$=\left(\frac{3\sin\beta}{\sin\alpha\cdot\sin(\alpha+\beta)}\right)\left(\frac{2\sin(\alpha+\beta)\sin\alpha}{\sin\beta}\right)$$

$$=6$$

Alternate (d)

$$\because \quad \tan(\alpha+\beta)=2\tan\alpha$$

$$\Rightarrow \quad \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{2\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \quad \sin(\alpha+\beta)\cos\alpha=2\cos(\alpha+\beta)\sin\alpha$$

$$\Rightarrow \quad \sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha=\cos(\alpha+\beta)\sin\alpha$$

$$\sin\beta=\cos(\alpha+\beta)\sin\alpha$$

[Alternate (b)]

68. $P_n(u)$ be a polynomial in u of degree n.

$$\therefore \quad \sin 2nx=2\sin nx\cos nx$$

$$=\sin x\,P_{2n-1}(\cos x)\text{ or }\cos x\,P_{2n-1}(\sin x)$$

69. $\tan\theta=\dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$

$$=\frac{\tan\alpha-1}{\tan\alpha+1}$$

$$\tan\theta=\tan\left(\alpha-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \theta=n\pi+\alpha-\frac{\pi}{4}, n\in I$$

or $\quad 2\theta=2n\pi+2\alpha-\dfrac{\alpha}{2}$

$$\sin 2\theta=\sin\left(2\alpha-\frac{\pi}{2}\right)=-\cos 2\alpha$$

and $\quad \cos 2\theta=\cos\left(2\alpha-\dfrac{\pi}{2}\right)=\sin 2\alpha$

and $\sin\alpha - \cos\alpha = \sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\{\theta - n\pi\} = \pm\sqrt{2}\sin\theta$

and $\sin\alpha + \cos\alpha = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left\{\frac{\pi}{2} + \theta - n\pi\right\}$

$= \sqrt{2}\cos(\theta - n\pi) = \pm\sqrt{2}\cos\theta$

70. $\cos 5\theta = \cos(4\theta + \theta) = \cos 4\theta\cos\theta - \sin 4\theta\sin\theta$

$= (2\cos^2 2\theta - 1)\cos\theta - 2\sin 2\theta\cos 2\theta\sin\theta$

$= [2(2\cos^2\theta - 1)^2 - 1]\cos\theta - 2\cdot 2\cos\theta \sin^2\theta\,(2\cos^2\theta - 1)$

$= [2(4\cos^4\theta - 4\cos^2\theta + 1) - 1]\cos\theta - 4\cos\theta\,(2\cos^2\theta - 1)(1 - \cos^2\theta)$

$= \cos\theta\,(8\cos^4\theta - 8\cos^2\theta + 1) - 4\cos\theta(3\cos^2\theta - 2\cos^4\theta - 1)$

$= \cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5)$

71. $x = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} + \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$

$\Rightarrow \quad x^2 = a^2 + b^2 + \sqrt{\begin{matrix}2(a^2\cos^2\alpha + b^2\sin^2\alpha)\\(a^2\sin^2\alpha + b^2\cos^2\alpha)\end{matrix}}$

$a^2 + b^2 + 2k,$

where $\quad k = \sqrt{\begin{matrix}[(a^2 + b^2) - (a^2\sin^2\alpha + b^2\cos^2\alpha)]\\ \times(a^2\sin^2\alpha + b^2\cos^2\alpha)\end{matrix}}$

$\therefore \quad x = a^2 + b^2 + 2\sqrt{(a^2 + b^2)p - p^2}$

where $\quad p = a^2\sin^2\alpha + b^2\cos^2\alpha$

$= \frac{a^2}{2}(1 - \cos 2\alpha) + \frac{b^2}{2}(1 + \cos 2\alpha)$

72. $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$

$= \cos^n\left(\frac{A - B}{2}\right) + \cot^n\left(\frac{B - A}{2}\right)$

If n even, $2\cot^n\left(\frac{A - B}{2}\right)$, if n odd, O

73. $P(k) = \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \cos\left(\frac{\pi}{2} - \frac{\pi}{4k}\right)\right)\left(1 + \cos\left(\frac{\pi}{2} + \frac{\pi}{4k}\right)\right)\left(1 + \cos\left(\pi - \frac{\pi}{4k}\right)\right)$

$= \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \sin\frac{\pi}{4k}\right)\left(1 - \sin\frac{\pi}{4k}\right)\left(1 - \cos\frac{\pi}{4k}\right)$

$= \left(1 - \cos^2\frac{\pi}{4k}\right)\left(1 - \sin^2\frac{\pi}{4k}\right)$

$= \frac{4\sin^2\frac{\pi}{4k}\cdot\cos^2\frac{\pi}{4k}}{4}$

$P(k) = \frac{1}{4}\sin^2\left(\frac{\pi}{2k}\right)$

$\Rightarrow \quad P(3) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$

$\Rightarrow \quad P(4) = \frac{\pi}{4}\sin^2\frac{\pi}{2k} = \frac{1}{4}\sin^2\frac{\pi}{8}$

$= \frac{1}{8}\left(1 - \cos\frac{\pi}{4}\right) = \frac{2 - \sqrt{2}}{16}$

$\Rightarrow \quad P(5) = \frac{1}{4}\sin^2\frac{\pi}{10} = \frac{1}{8}\left(2\sin^2\frac{\pi}{10}\right) = \frac{1}{8}(1 - \cos 36°)$

$= \frac{1}{8}\left(1 - \frac{\sqrt{5} + 1}{4}\right) = \frac{3 - \sqrt{5}}{32}$

$\Rightarrow \quad P(6) = \frac{1}{4}\sin^2\frac{\pi}{12} = \frac{1}{8}\left(2\sin^2\frac{\pi}{12}\right) = \frac{1}{8}\left(1 - \cos\frac{\pi}{6}\right)$

$= \frac{1}{8}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2 - \sqrt{3}}{16}$

74. $x^2 + y^2 = a^2\sin^4\theta\cos^4\theta$

$xy = a^2\sin^5\theta\cos^5\theta$

$\therefore \quad \frac{(x^2 + y^2)^p}{(xy)^q} = \frac{a^{2p}(\sin\theta\cos\theta)^{4p}}{a^{2q}(\sin\theta\cos\theta)^{5q}}$

which is independent of θ if $4p = 5q$

i.e. $\quad p = 5, q = 4.$

75. LHS

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$= \cot\alpha - 2(\cot 2\alpha - \tan 2\alpha) + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$

$(\because \cot\alpha - \tan\alpha = 2\cot 2\alpha)$

$= \cot\alpha - 4(\cot 4\alpha - \tan 4\alpha) + 8\tan 8\alpha + 16\cot 16\alpha$

$(\because \cot 2\alpha - \tan 2\alpha = 2\cot 4\alpha)$

$= \cot\alpha - 8(\cot 8\alpha - \tan 8\alpha) + 16\cot 16\alpha$

$= \cot\alpha - 16\cot 16\alpha + 16\cot 16\alpha$

$(\because \cot 8\alpha - \tan 8\alpha = 2\cot 16\alpha)$

$= \cot\alpha =$ RHS

76. Let $x = \cot A, y = \cot B, z = \cot C$

$\therefore \quad \cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

$\because \quad A + B + C = 180°$

$\therefore \quad \Sigma\frac{x}{(1 + x^2)} = \Sigma\frac{\cot A}{(1 + \cot^2 A)}$

$= \frac{1}{2}\Sigma\frac{2\tan A}{(1 + \tan^2 A)} = \frac{1}{2}\Sigma\sin 2A$

$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2}(4\sin A\sin B\sin C) = 2\sin A\sin B\sin C$

$= \frac{2}{\sqrt{(1 + \cot^2 A)(1 + \cot^2 B)(1 + \cot^2 C)}}$

$= \frac{2}{\sqrt{(1 + x^2)(1 + y^2)(1 + z^2)}} = \frac{2}{\sqrt{\Pi(1 + x^2)}}$

and $\sin 2A + \sin 2B - \sin 2C$

$$= 2\sin(A+B)\cos(A-B) - 2\sin C\cos C$$
$$= 2\sin C\{\cos(A-B) - \cos C\}$$
$$= 2\sin C\{\cos(A-C) + \cos(A+B)\}$$
$$= 2\sin C(2\cos A\cos B)$$
$$= 4\cos A\cos B\sin C$$

77. $\because a\cos x + b\sin x = c$

$$\Rightarrow \quad a\left\{\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right\} + b\left\{\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right\} = c$$

$$\Rightarrow \quad (a+c)\tan^2\left(\frac{x}{2}\right) - 2b\tan\left(\frac{x}{2}\right) + (c-a) = 0$$

$$\therefore \quad \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = \frac{2b}{(a+c)}$$

and $$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{c-a}{a+c}$$

Now, $$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}{1 - \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}$$

$$= \frac{\frac{2b}{a+c}}{1 - \left(\frac{c-a}{a+c}\right)} = \frac{b}{a} = \text{independent of } c$$

Also, $$-\sqrt{(a^2+b^2)} \le a\cos x + b\sin x \le \sqrt{(a^2+b^2)}$$

$$\therefore \quad -\sqrt{(a^2+b^2)} \le c \le \sqrt{(a^2+b^2)}$$

78. $\because A + B + C = 180°$

$$\Rightarrow \quad A = 180° - (B+C)$$

$$\therefore \quad \tan A = \tan(180° - (B+C))$$
$$= -\tan(B+C) = -\left\{\frac{\tan B + \tan C}{1 - \tan B\tan C}\right\}$$
$$= \left(\frac{\tan B + \tan C}{\tan B\tan C - 1}\right)$$

Now, $\because A$ is obtuse

$\therefore$ $\tan A < 0$,

then $\tan B + \tan C > 0$

$\therefore$ $\tan B\tan C - 1 < 0$

$\Rightarrow$ $\tan B\tan C < 1$

79. Let $S = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

and $C = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right)$

$$\therefore \quad C + iS = \alpha + \alpha^2 + \alpha^4 \quad \text{...(i)}$$

Where $\alpha = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ is complex 7th root of unity.

Then, $$C - iS = \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^4$$
$$= \alpha^6 + \alpha^5 + \alpha^3 \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), then

$$2C = \alpha + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^5 + \alpha^3 = -1$$

($\because$ sum of 7, 7th roots of unity is zero)

$$\therefore \quad C = -\frac{1}{2}$$

Also, multiplying Eqs. (i) and (ii), then $C^2 + S^2 = 2$

($\because \alpha^7 = 1$ and sum of 7, 7th roots of unity)

$$\Rightarrow \quad S^2 = 2 - \left(\frac{1}{2}\right)^2 = \frac{7}{4}$$

$$\therefore \quad S = \frac{\sqrt{7}}{2}$$

80. We observe that $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30 = 0$

$$\Rightarrow \quad 81^{\sin^2 x} + 81^{1-\sin^2 x} - 30 = 0$$
$$\Rightarrow \quad 81^{2\sin^2 x} - 30.81^{\sin^2 x} + 81 = 0$$
$$\Rightarrow \quad (81^{\sin^2 x} - 3)(81^{\sin^2 x} - 27) = 0$$
$$\Rightarrow \quad \sin^2 x = \frac{1}{4} \text{ or } \frac{3}{4}$$
$$\Rightarrow \quad \sin x = \pm\frac{1}{2} \text{ or } \sin x = \pm\frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}, \text{ or } x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

$\Rightarrow$ The graph $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$

Intersects the X-axis at eight points in $(-\pi \le x \le \pi)$.

$\Rightarrow$ Statement-1 is true.

81. Statement-2 is correct, using it we have $\cos 3x = \sin 2x$

$$\Rightarrow \quad 4\cos^3 x - 3\cos x = 2\sin x\cos x$$

Similarly $4\cos^3 y - 3\cos y = 2\sin y\cos y$

So, $$4(1-\sin^2 x) - 3 = 2\sin x$$

$$\Rightarrow \quad 4\sin^2 x + 2\sin x - 1 = 0$$

and $$4\sin^2 y + 2\sin y - 1 = 0$$

Hence, $\sin x = \sin 18°$ and $\sin y = \sin(-54°) = -\sin 54°$ are the roots of a quadratic equation with integer coefficients.

82. The minimum value of the sum can be -3 provided $\sin\alpha = \sin\beta = \sin\gamma = -1$

$$\Rightarrow \quad \alpha = (4l-1)\frac{\pi}{2},\ \beta = (4m-1)\frac{\pi}{2},\ \gamma = (4n-1)\frac{\pi}{2}$$

Now $$\alpha + \beta + \gamma = \pi \Rightarrow [4(l+m+n) - 3]\frac{\pi}{2} = \pi$$

$\Rightarrow 4(l+m+n) = 5$ which is not possible as l, m, n are integers.

1. minimum value can not be -3.

But for $\alpha = \frac{3\pi}{2}, \beta = \frac{3\pi}{2}, \gamma = -2\pi, \alpha + \beta + \gamma = \pi$

and $$\sin\alpha + \sin\beta + \sin\gamma = 2$$

So, $\sin\alpha + \sin\beta + \sin\gamma$ can have negative values and thus the minimum value of the sum is negative proving that statement-1 is correct. But the statement-2 is false as $\alpha + \beta + \gamma = \pi$ for $\alpha = \beta = \frac{3\pi}{2}, \gamma = -2\pi$ which are not the angles of a triangle.

83. We have $2\sin\left(\frac{\theta}{2}\right)$

$$= \sqrt{\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right)^2} + \sqrt{\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right)^2}$$

$$= \left|\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right| + \left|\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right|$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) > 0 \text{ and } \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) < 0$$

$$\Rightarrow \sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right) > 0 \text{ and } \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) < 0$$

$$\Rightarrow 2n\pi + \frac{\pi}{2} < \frac{\theta}{2} + \frac{\pi}{4} > 2n\pi + \pi$$

$$\Rightarrow 2n\pi + \frac{\pi}{4} < \frac{\theta}{2} < 2n\pi + \frac{3\pi}{4}$$

So statement-1 is true but does not follow from statement-2 which is also true.

84. $2\cos\theta + \sin\theta = 1$

$$\Rightarrow \frac{2\left(1 - \tan^2\left(\frac{\theta}{2}\right)\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} + \frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = 1$$

$$\Rightarrow 3\tan^2\left(\frac{\theta}{2}\right) - 2\tan\left(\frac{\theta}{2}\right) - 1 = 0$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right) = -\frac{1}{3} \text{ as } \theta \neq \frac{\pi}{2}$$

Now $7\cos\theta + 6\sin\theta$

$$= \frac{7\left(1 - \tan^2\left(\frac{\theta}{2}\right)\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} + \frac{6 \times 2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{7 - 7\tan^2\left(\frac{\theta}{2}\right) + 12\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \frac{7 - 7 \times \frac{1}{9} + 12\left(-\frac{1}{3}\right)}{1 + \frac{1}{9}} = 2$$

Showing that statement-1 is true.

In statement-2

$$\cos 2\theta - \sin\theta = \frac{1}{2}$$

$$\Rightarrow 2(1 - 2\sin^2\theta) - \sin\theta = 1$$

$$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\Rightarrow \sin\theta = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{5} - 1}{4} \Rightarrow \theta = 18^\circ$$

$$\Rightarrow \cos 6\theta = \cos 108^\circ = \cos(90^\circ + 18^\circ)$$

$$= -\sin 18^\circ$$

$$\Rightarrow \sin\theta + \cos 6\theta = 0$$

So statement-2 is also true but does not lead to statement-1.

85. $\because A + B = \frac{\pi}{3}$

$$\therefore \tan(A + B) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \sqrt{3}$$

$$\Rightarrow \tan A \tan B = 1 - \frac{1}{\sqrt{3}}(\tan A + \tan B)$$

$\therefore \tan A \tan B$ will be maximum if $\tan A + \tan B$ is minimum.

But the minimum value of $\tan A + \tan B$ is obtained when $\tan A = \tan B$

$$\Rightarrow A = B = \frac{\pi}{6}$$

Hence, the maximum value of $\tan A \tan B$

$$= \tan\frac{\pi}{6} \cdot \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

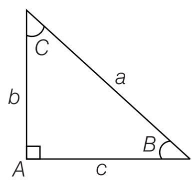
86. Let $3^{a^2} = A$ and $3^{b^2 + c^2} = B$

$$\Rightarrow A^2 - 2A.B + B^2 = 0$$

$$\Rightarrow A = B$$

$$\Rightarrow a^2 = b^2 + c^2$$

87.



From figure, it is clear that $a = b\sec C = c\operatorname{cosec} C$

$\Rightarrow$ Equilateral triangle $\Rightarrow$ Area $= \frac{\sqrt{3}}{4}a^2$

Sol. (Q. Nos. 88 to 90)

$$7\theta = (2n + 1)\pi, n = 0 \text{ to } 6$$

$$4\theta = (2n + 1)\pi - 3\theta$$

$$\cos 4\theta = -\cos 3\theta$$

$$\Rightarrow 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$$

$$\Rightarrow 2(2x^2 - 1) - 1 = -(4x^3 - 3x), \text{ where } x = \cos\theta$$

$$\Rightarrow 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$$

$$(x + 1)(8x^3 - 4x^2 - 4x - 1) = 0$$

88. $P_{mn} = m\log_{\cos x}(\sin x) + n\log_{\cos x}(\cot x)$

$$\geq n(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x)) \ \forall\ m \geq n$$

$$= n(\log_{\cos x}(\sin x \cdot \cot x))$$

$$= n\log_{\cos x}\cos x = n$$

Thus, $P_{mn} \geq n \ \forall\ m \geq n$

89. Clearly, $P_{49}\left(\frac{\pi}{4}\right) = 4\log_{\frac{1}{\sqrt{2}}}\left(\frac{1}{\sqrt{2}}\right) + 9\log_{\frac{1}{\sqrt{2}}}(1) = 4$

Similarly $P_{94} = \left(\frac{\pi}{4}\right) = 9$

Mean proportional of $P_{49}\left(\frac{\pi}{4}\right)$ and $P_{94}\left(\frac{\pi}{4}\right)$ is $\sqrt{9 \times 4} = 6$

90. $P_{34}(x) = P_{22}(x)$

$\Rightarrow$ $3\log_{\cos x}(\sin x) + 4\log_{\cos x}(\cot x)$
$= 2(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x))$

$\Rightarrow$ $3(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x)) + \log_{\cos x}(\cot x) = 2$

$\Rightarrow$ $3 + \log_{\cos x}(\cot x) = 2$

$\Rightarrow$ $\log_{\cos x}(\cot x) = -1$

$\Rightarrow$ $\cot x = (\cos x)^{-1} \Rightarrow \dfrac{\cos x}{\sin x} = \dfrac{1}{\cos x}$

$\Rightarrow$ $\cos^2 x = \sin x$

$1 - \sin^2 x = \sin x \Rightarrow \sin^2 x + \sin x - 1 = 0$

$\Rightarrow$ $\sin x = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow$ $\sin x = \dfrac{\sqrt{5} - 1}{2}$ $\quad (\because \sin x \not< -1)$

Thus, $p + q = 7$

Sol. (Q. Nos. 91 to 93)

$7\theta = (2n+1)\pi,\ n = 0$ to 6

$4\theta = (2n+1)\pi - 3\theta$

$\cos 4\theta = -\cos 3\theta$

$\Rightarrow$ $2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow 2(2x^2 - 1) - 1 = -(4x^3 - 3x)$, where $x = \cos\theta$

$\Rightarrow$ $8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

$(x+1)(8x^3 - 4x^2 - 4x - 1) = 0$

91. The roots are $\cos\dfrac{\pi}{7}, \cos\dfrac{2\pi}{7} \dots, \cos\dfrac{13\pi}{7}$

where $\cos\dfrac{\pi}{7} = \cos\dfrac{13\pi}{7}, \cos\dfrac{3\pi}{7} = \cos\dfrac{11\pi}{7}, \cos\dfrac{5\pi}{7} = \cos\dfrac{9\pi}{7}$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\dfrac{\pi}{7}, \cos\dfrac{3\pi}{7}, \cos\dfrac{5\pi}{7}$.

92. $\sec\dfrac{\pi}{7}, \sec\dfrac{3\pi}{7}, \sec\dfrac{5\pi}{7}$ are roots of $\dfrac{8}{x^3} - \dfrac{4}{x^2} - \dfrac{4}{x} + 1 = 0$

$\Rightarrow$ $x^3 - 4x^2 - 4x + 8 = 0$

$\therefore$ $\sec\dfrac{\pi}{7} + \sec\dfrac{3\pi}{7} + \sec\dfrac{5\pi}{7} = 4$

93. $\sec^2\dfrac{\pi}{7}, \sec^2\dfrac{3\pi}{7}, \sec^2\dfrac{5\pi}{7}$ are roots of $f(\sqrt{x}) = 0$

$\Rightarrow$ $(\sqrt{x})^3 - 4(\sqrt{x})^2 - 4\sqrt{x} + 8 = 0$

$\Rightarrow$ $x^3 - 24x^2 + 80x - 64 = 0$

$\therefore$ $\sec^2\dfrac{\pi}{7} + \sec^2\dfrac{3\pi}{7} + \sec^2\dfrac{5\pi}{7} = 24$

Sol. (Q. Nos. 94 to 96)

Let $S = 1 + 2\sin x + 3\sin^2 x + 4\sin^3 x + \dots$

$\Rightarrow$ $\sin x . S = \sin x + 2\sin^2 x + 3\sin^3 x + \dots$

$\therefore$ $(1 - \sin x)S = 1 \;\; \sin x + \sin^2 x + \dots$

$(1 - \sin x)S = \dfrac{1}{1 - \sin x}$

$\therefore$ $S = \dfrac{1}{(1 - \sin x)^2}$

Given $S = 4 \Rightarrow \dfrac{1}{(1 - \sin x)^2} = 4$

$\Rightarrow$ $\sin x = \dfrac{1}{2}$ or $\dfrac{3}{2}$ (rejected)

Number of solutions in $\left[\dfrac{-3\pi}{2}, 4\pi\right]$ is $k = 5$.

94. $k = 5$

95. $\left|\dfrac{\cos 2x - 1}{\sin 2x}\right| = \left|\dfrac{2\sin^2 x}{2\cos x \sin x}\right| = |\tan x| = \dfrac{1}{\sqrt{3}}$

96. Sum of interior angles $= (k-2)\pi = 3\pi$

97. Now, $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow$ $(1 + \cos x)[2\sin x - \cos x - 1 + \cos x] = 0$

$\Rightarrow$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow$ $\cos x = -1$ or $\sin x = \dfrac{1}{2}$

So, $\sin\alpha = \dfrac{1}{2}$ $\quad \left[\text{as } 0 \le \alpha \le \dfrac{\pi}{2}\right]$

$\therefore$ $\cos\alpha = \dfrac{\sqrt{3}}{2}$

$3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow$ $\cos x = \dfrac{1}{3}, \cos x \ne 3$

$\Rightarrow$ $\cos\beta = \dfrac{1}{3}, \sin\beta = \dfrac{2\sqrt{2}}{3}$

and $1 - \sin 2x = \cos x - \sin x$

$\Rightarrow$ $\sin^2 x + \cos^2 x - 2\sin x\cos x = \cos x - \sin x$

$\Rightarrow (\cos x - \sin x)(\cos x - \sin x - 1) = 0$

$\Rightarrow$ $\sin x = \cos x = \dfrac{1}{\sqrt{2}}$

or $\cos x - \sin x = 1$

$\Rightarrow$ $\cos x = 1, \sin x = 0$

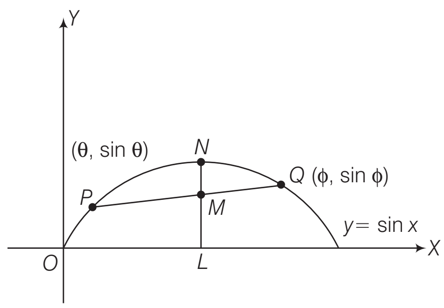
$\Rightarrow$ $\cos\gamma = 1, \sin\gamma = 0$

$\therefore$ $\cos\alpha + \cos\beta + \cos\gamma = \dfrac{\sqrt{3}}{2} + \dfrac{1}{3} + 1 = \dfrac{3\sqrt{3} + 8}{6}$

98. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \dfrac{1}{2} \times \dfrac{1}{3} - \dfrac{\sqrt{3}}{2} \times \dfrac{2\sqrt{2}}{3} = \dfrac{1 - 2\sqrt{6}}{6}$

99. (A) If M is mid point of PQ, then $M = \left(\dfrac{\theta + \phi}{2}, \dfrac{\sin\theta + \sin\phi}{2}\right)$



Also, $N \equiv \left(\frac{\theta+\phi}{2}, \sin\left(\frac{\theta+\phi}{2}\right)\right)$

It is clear from the figure.

$$ML \le NL$$

$$\Rightarrow \quad \frac{\sin\theta + \sin\phi}{2} \le \sin\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \quad \sin\theta + \sin\phi \le 2\sin\left(\frac{\theta+\phi}{2}\right)$$

$$= 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\therefore \quad \sin\theta + \sin\phi \le \sqrt{2}$$

and $\quad (\sin\theta + \sin\phi)\sin\frac{\pi}{4} \le 1$ (p, q, r, s, t)

(B) $\because a^2 + b^2 = (\sin\theta - \sin\phi)^2 + (\cos\theta + \cos\phi)^2$

$$= 2 + 2\cos(\theta+\phi)$$

$$= 4\cos^2\left(\frac{\theta+\phi}{2}\right) \le 4 \text{ (s, t)}$$

(C) $\because 3\sin\theta + 5\cos\theta = 5$

$$\Rightarrow \quad 3\sin\theta = 5(1-\cos\theta)$$

Squaring both sides, then

$$9\sin^2\theta = 25(1-\cos\theta)^2$$

$$\Rightarrow \quad 9(1-\cos\theta)(1+\cos\theta) = 25(1-\cos\theta)^2$$

$$\Rightarrow \quad 9(1+\cos\theta) = 25(1-\cos\theta)\ (1-\cos\theta \ne 0)$$

$$\therefore \quad 34\cos\theta = 16$$

$$\cos\theta = \frac{8}{17}, \text{ then } \sin\theta = \frac{15}{17}$$

$$\therefore \quad 5\sin\theta - 3\cos\theta = \frac{75}{17} - \frac{24}{17} = 3 \text{ (r)}$$

Hence, $5\sin\theta - 3\cos\theta = 3$

100. (A) Let $y = \dfrac{7 + 6\tan\theta - \tan^2\theta}{(1+\tan^2\theta)}$

$$= 7\cos^2\theta + 6\sin\theta\cos\theta - \sin^2\theta$$

$$= 7\left(\frac{1+\cos\theta}{2}\right) + 3\sin 2\theta - \left(\frac{1-\cos 2\theta}{2}\right)$$

$$= 3\sin 2\theta + 4\cos 2\theta + 3$$

$$-\sqrt{(3^2+4^2)} + 3 \le 3\sin 2\theta + 4\cos 2\theta + 3$$

$$\le \sqrt{(3^2+4^2)} + 3$$

$$\therefore \quad -2 \le y \le 8 \Rightarrow \lambda = 8, \mu = -2$$

$$\Rightarrow \quad \lambda + \mu = 6, \lambda - \mu = 10 \text{ (R, S)}$$

(B) Let $y = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

$$= 5\cos\theta + 3\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) + 3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$\therefore \quad 3 - \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$\le 3 + \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad 3 - 7 \le y \le 3 + 7$$

$$\Rightarrow \quad -4 \le y \le 10$$

$$\therefore \quad \lambda = 10, \mu = -4$$

$$\Rightarrow \quad \lambda + \mu = 6, \lambda - \mu = 14 \text{ (R, T)}$$

(C) Let $y = 1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$$= 1 + \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} + \theta\right)\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$$

$$= 1 + \cos\left(\frac{\pi}{4} - \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$$

$$= 1 + 3\cos\left(\frac{\pi}{4} - \theta\right)$$

$$\therefore \quad -1 \le \cos\left(\frac{\pi}{4} - \theta\right) \le 1$$

$$\Rightarrow \quad -3 \le 3\cos\left(\frac{\pi}{4} - \theta\right) \le 3$$

$$\Rightarrow \quad 1 - 3 \le 1 + 3\cos\left(\frac{\pi}{4} - \theta\right) \le 1 + 3$$

$$\therefore \quad -2 \le y \le 4$$

$$\Rightarrow \quad \lambda = 4, \mu = -2$$

$$\therefore \quad \lambda + \mu = 2, \lambda - \mu = 6 \text{ (P, Q)}$$

101. (A) $|\cot x| = \cot x + \dfrac{1}{\sin x}$

If $0 < x < \frac{\pi}{2} \Rightarrow \cot x > 0$

So $\cot x = \cot x + \frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$, no solution

If $\frac{\pi}{2} < \cot x < \pi, -\cot x = \cot x + \frac{1}{\sin x}$

$$\Rightarrow \quad \frac{2\cos x}{\sin x} + \frac{1}{\sin x} = 0$$

$$\Rightarrow \quad 1 + 2\cos x = 0 \text{ and } \sin x \ne 0 \Rightarrow x = \frac{2\pi}{3}.$$

(B) since $\sin\phi + \sin\theta = \frac{1}{2}$ ...(i)

and $\cos\theta + \cos\phi = 2$ ...(ii)

(ii) is true only if $\theta = \phi = 0$ or 2π but $\theta = \phi = 0$ or 2π do not satisfy (i)

Hence given system of equation has no solution.

(C) $\sin^2\alpha + \sin\left(\frac{\pi}{3} - \alpha\right) \cdot \sin\left(\frac{\pi}{3} + \alpha\right)$

$$= \sin^2\alpha + \sin^2\frac{\pi}{3} - \sin^2\alpha = \frac{3}{4}.$$

(D) $\tan\theta = 3\tan\phi$

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{2\tan\phi}{1 + 3\tan\phi}$$

$$= \frac{2}{\cot\phi + 3\tan\phi} \cdot \text{Max if } \tan\phi > 0$$

$$\frac{\cos\phi + 3\tan\phi}{2} \geq \sqrt{3} \quad \text{(Using AM} \geq \text{GM)}$$

$$\Rightarrow \quad (\cot\phi + 3\tan\phi)^2 \geq 12 \Rightarrow \left[\frac{2}{\tan(\theta - \phi)}\right]^2 \geq 12$$

$$\Rightarrow \quad \tan^2(\theta - \phi) \leq \frac{1}{3}.$$

102. (A) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)$

$$= 1 + \left(\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right)\right) - \left(\sin\frac{\pi}{2} - \sin\frac{C}{2}\right)$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) - 2\cos\left(\frac{\pi + C}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

$$= 1 + 2\sin\left(\frac{\pi - C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi + C}{4}\right)\right\}$$

$(\because A + B + C = \pi)$

$$= 1 + 2\sin\left(\frac{\pi - C}{4}\right)\left\{2\sin\left(\frac{\pi + C + A - B}{8}\right)\sin\left(\frac{\pi + C + B - A}{8}\right)\right\}$$

$$= 1 + 4\sin\left(\frac{\pi - C}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - A}{4}\right)$$

$$= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi}{2} - \frac{\pi - A}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

(B) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right)$

$$= -1 + \left(\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right)\right) + \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{C}{2}\right)\right)$$

$$= -1 + 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) + 2\cos\left(\frac{\pi + C}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

$$= -1 + 2\sin\left(\frac{\pi - C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) + \cos\left(\frac{\pi + C}{4}\right)\right\}$$

$(\because A + B + C = \pi)$

$$= -1 + 2\sin\left(\frac{\pi - C}{4}\right)\left\{2\cos\left(\frac{\pi + C + A - B}{8}\right)\cos\left(\frac{\pi + C + B - A}{8}\right)\right\}$$

$$= -1 + 4\sin\left(\frac{\pi - C}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\cos\left(\frac{\pi - A}{4}\right)$$

$$= -1 + 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

$$= -1 + 4\sin\left(\frac{\pi}{2} - \frac{\pi - A}{4}\right)\sin\left(\frac{\pi}{2} - \frac{\pi - B}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi - C}{4}\right)$$

$$= -1 + 4\sin\left(\frac{\pi + A}{4}\right)\sin\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi + C}{4}\right)$$

(C) $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) - \cos\left(\frac{C}{2}\right)$

$$= \left(\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right)\right) - \left(\cos\left(\frac{C}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)$$

$$= 2\cos\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi + C}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$$

$$= 2\cos\left(\frac{\pi - C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi + C}{4}\right)\right\}$$

$(\because A + B + C = \pi)$

$$= 2\cos\left(\frac{\pi - C}{4}\right)\left\{2\sin\left(\frac{\pi + C + A - B}{8}\right)\sin\left(\frac{\pi + C + B - A}{8}\right)\right\}$$

$$= 4\cos\left(\frac{\pi - C}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - A}{4}\right)$$

$$= 4\cos\left(\frac{\pi - C}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi - B}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi - A}{4}\right)$$

$$= 4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$$

103. $\frac{1}{1 + \tan^2\frac{A}{2}} + \frac{1}{1 + \tan^2\frac{B}{2}} + \frac{1}{1 + \tan^2\frac{C}{2}}$

$$= k\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$\Rightarrow \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$$

$$= 2\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right] \quad \text{[by using identity]}$$

104. $\frac{\sin\alpha}{\sin\beta} = \frac{\cos\gamma}{\cos\delta}$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\sin\beta} = \frac{\cos\gamma - \cos\delta}{\cos\delta} \quad \text{(using dividendo)}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)}{\sin\beta} = \frac{2\sin\left(\frac{\gamma + \delta}{2}\right)\sin\left(\frac{\delta - \gamma}{2}\right)}{\cos\delta}$$

105. Let $\frac{\pi}{20} = \theta \Rightarrow 10\theta = \frac{\pi}{2}$

$\Rightarrow \quad 2\theta = 18°$ or $\theta = 9°$

Now, $\tan\theta - \tan 3\theta + \tan 5\theta - \tan 7\theta + \tan 9\theta$

$\tan\theta - \tan 3\theta + \tan 5\theta - \cot 3\theta + \cot\theta$

$(\tan\theta + \cot\theta) - (\tan 3\theta + \cot 3\theta) + \tan 45°$

[using $\tan 5\theta = \tan 45°$]

$$E = \frac{2}{\sin 2\theta} - \frac{2}{\sin 6\theta} + 1$$

$$E = 2\left(\frac{1}{\sin 2\theta} - \frac{1}{\cos 4\theta}\right) + 1$$

$$E = 2\left(\frac{1}{\sin 18^\circ} - \frac{1}{\cos 36^\circ}\right) + 1$$

$$E = 2\left(\frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}\right) + 1 = 5$$

Aliter $$E = 1 + 2\left(\frac{\sin 6\theta - \sin 2\theta}{\sin 2\theta \cdot \sin 6\theta}\right)$$

$$= 1 + 2\left(\frac{2\cos 4\theta \cdot \sin 2\theta}{\sin 2\theta \cdot \cos 4\theta}\right)$$

$$= 1 + 4 = 5$$

106. $$x = \frac{\cos 1^\circ + \cos 2^\circ + \ldots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \ldots + \sin 44^\circ}$$

$$= \frac{\dfrac{\sin 22^\circ}{\sin(1/2)^\circ} \cdot \cos 22.5^\circ}{\dfrac{\sin 22^\circ}{\sin(1/2)^\circ} \cdot \sin 22.5^\circ} = \cot 22.5^\circ$$

[using the formula of sum of cos series]

$$S = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\frac{(n+1)\theta}{2},$$

for sine series, $S = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\dfrac{(n+1)\theta}{2}$

$$\cot\left(\frac{\pi}{8}\right) = \sqrt{2} + 1 = 2.414\ldots$$

$\therefore$ $x = 2.414\ldots$

Greatest integer = 2.

107. LHS $= \tan 15^\circ \cdot \tan(30^\circ - 5^\circ) \cdot \tan(30^\circ + 5^\circ)$

Let $t = \tan 30^\circ$ and $m = \tan 5^\circ$

$$\therefore \text{LHS} = \tan 15^\circ \cdot \frac{t-m}{1+tm} \cdot \frac{t+m}{1-tm} = \tan(3(5^\circ)) \cdot \frac{t^2 - m^2}{1 - t^2m^2}$$

$$= \frac{3m - m^3}{1 - 3m^2} \frac{1 - 3m^2}{3 - m^2}$$

$$= \frac{m(3 - m^2)}{(1 - 3m^2)} \cdot \frac{(1 - 3m^2)}{3 - m^2} = m = \tan 5^\circ$$

Hence, $\tan\theta = \tan 5^\circ$

$\Rightarrow$ $\theta = 5^\circ$.

108. We have, $\dfrac{1}{\sin 10^\circ} + \dfrac{1}{\sin 50^\circ} - \dfrac{1}{\sin 70^\circ}$

$$= \frac{1}{\cos 80^\circ} + \frac{1}{\cos 40^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\cos 40^\circ \cos 20^\circ + \cos 80^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ}{\cos 20^\circ . \cos 40^\circ . \cos 80^\circ}$$

$$= 8\,[\cos 20^\circ (\cos 40^\circ + \cos 80^\circ) - \cos 40^\circ \cos 80^\circ]$$

$$= 8[2\cos 20^\circ \cos 60^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ]$$

$$= 4[2\cos^2 20^\circ - 2\cos 40^\circ \cos 80^\circ]$$

$$= 4[1 + \cos 40^\circ - (\cos 120^\circ + \cos 40^\circ)]$$

$$= 4 \cdot \frac{3}{2} = 6$$

109. We have, $\cos 5\alpha = \cos^5\alpha$

$\cos 5\alpha = \cos(3\alpha + 2\alpha) = \cos 3\alpha \cos 2\alpha - \sin 3\alpha \sin 2\alpha$

$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) -$
$(3\sin\alpha - 4\sin^3\alpha) 2\sin\alpha \cos\alpha$

$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (1\ \cos^2\alpha)(3 -$
$4 + 4\cos^2\alpha) 2\cos\alpha$

$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (2\cos^2\alpha -$
$2\cos^3\alpha)(4\cos^2\alpha - 1)$

$= 8\cos^5\alpha - 4\cos^3\alpha - 6\cos^3\alpha + 3\cos\alpha -$
$[8\cos^3\alpha - 2\cos\alpha - 8\cos^5\alpha + 2\cos^3\alpha]$

$= 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

$\therefore$ $16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha = \cos^5\alpha$

$15\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha = 0$

$5\cos\alpha\,[3\cos^4\alpha - 4\cos^2\alpha + 1] = 0$

Also $\cos\alpha = 0$

$3\cos^4\alpha - 3\cos^2\alpha - \cos^2\alpha + 1 = 0$

$3\cos^2\alpha(\cos^2\alpha - 1) - (\cos^2\alpha - 1) = 0$

$(3\cos^2\alpha - 1)(1 - \cos^2\alpha) = 0$

$\cos^2\alpha = 1$

$\cos^2\alpha = \dfrac{1}{3}$

$\Rightarrow$ $\sec^2\alpha = 3;\ \text{cosec}^2\alpha = \dfrac{3}{2};\ \cot^2\alpha = \dfrac{1}{2}$

$\therefore$ $(\sec^2\alpha + \text{cosec}^2\alpha + \cot^2\alpha) = 3 + \dfrac{3}{2} + \dfrac{1}{2} = 5$

110. We have,

$$\tan^2\frac{\pi}{16} + \tan^2\frac{2\pi}{16} + \tan^2\frac{3\pi}{16} + \ldots + \tan^2\frac{7\pi}{16}$$

$$= \left(\tan^2\frac{\pi}{16} + \cot^2\frac{\pi}{16} + 2\right) + \left(\tan^2\frac{2\pi}{16} + \cot^2\frac{2\pi}{16} + 2\right) + \left(\tan^2\frac{3\pi}{16} + \cot^2\frac{3\pi}{16} + 2\right) + \tan^2\frac{4\pi}{16} - 6$$

$\left[\text{If } A + B = \dfrac{\pi}{2}, \text{ then } \tan B = \tan\left(\dfrac{\pi}{2} - A\right) = \cot A \quad \text{So, } \tan^2\dfrac{7\pi}{16} = \tan^2\left(\dfrac{8\pi}{16} - \dfrac{\pi}{16}\right) = \cot^2\dfrac{\pi}{16} \text{ etc.}\right]$

$$= \left(\tan\frac{\pi}{16} + \cot\frac{\pi}{16}\right)^2 + \left(\tan\frac{2\pi}{16} + \cot\frac{2\pi}{16}\right)^2 + \left(\tan\frac{3\pi}{16} + \cot\frac{3\pi}{16}\right)^2 - 5$$

$$= \frac{4}{\sin^2\frac{\pi}{8}} + \frac{4}{\sin^2\frac{3\pi}{8}} + \left(\frac{4}{\sin^2\frac{\pi}{4}} - 5\right)$$

$$= \frac{4\left(\sin^2\frac{3\pi}{8} + \sin^2\frac{\pi}{8}\right)}{\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}} + 3 = \frac{16}{\sin^2\frac{\pi}{4}} + 3 = 32 + 3 = 35$$

111. We have, $\dfrac{4 + \sec 20^\circ}{\text{cosec } 20^\circ}$

$$= \frac{\sin 20^\circ}{\cos 20^\circ}(4\cos 20^\circ + 1)$$
$$= \frac{2\sin 40^\circ + \sin 20^\circ}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + (\sin 40^\circ + \sin 20^\circ)}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + 2\sin 30^\circ \cos 10^\circ}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + \sin 80^\circ}{\cos 20^\circ}$$
$$= \frac{2\sin 60^\circ \cos 20^\circ}{\cos 20^\circ} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Hence, square of the value of expression = 3

112. $A + B + C = \pi$ ...(i)

$$\frac{\sin A}{3} = \frac{\cos B}{3} = \frac{\tan C}{2} \quad \text{...(ii)}$$

$\Rightarrow \quad \sin A = \cos B \Rightarrow A + B = \dfrac{\pi}{2}$ (rejected)

Or $\quad A - B = \dfrac{\pi}{2}$ ...(iii)

$\Rightarrow \quad 2A + C = \dfrac{3\pi}{2}$ [from Eqs. (i) and (ii)]

Now $\quad \dfrac{\sin A}{\tan C} = \dfrac{3}{2}$ [from Eq. (ii)]

$\Rightarrow \quad \dfrac{\sin A}{\tan\left(\dfrac{3\pi}{2} - 2A\right)} = \dfrac{3}{2}$ [from Eq. (iii)]

$\Rightarrow \quad 2\sin A = 3\cot 2A$

$\Rightarrow \quad 2\sin A = \dfrac{3\cdot(2\cos^2 A - 1)}{2\sin A \cos A}$

$\Rightarrow \quad 4\cos A(1 - \cos^2 A) = 3(2\cos^2 A - 1)$

$\Rightarrow \quad 4\cos^3 A + 6\cos^2 A - 4\cos A - 3 = 0$

Put $\cos A = -\dfrac{1}{2}$

$\Rightarrow \quad (2\cos A + 1)(2\cos^2 A + 2\cos A - 3) = 0$

$\Rightarrow \quad \cos A = -\dfrac{1}{2},$

$$\cos A = \frac{-2 \pm \sqrt{4 + 24}}{4} = -1 \pm \sqrt{7} \text{ (rejected)}$$

$\Rightarrow \quad A = \dfrac{2\pi}{3}, B = \dfrac{\pi}{6}, C = \dfrac{\pi}{6}$

$\therefore \quad \dfrac{\sin A}{\cos 2A} + \dfrac{\cos B}{\cot 2B} + \dfrac{\tan C}{\cot 2C} = \dfrac{1}{2} + \dfrac{1}{2} + 1 = 2$

113. $f(\theta) = \dfrac{1}{1 + g(\theta)}$ ...(i)

Given, $2f(\alpha) - g(\beta) = 1$

$2f(\alpha) = 1 + g(\beta) = \dfrac{1}{f(\beta)}$ [from Eq. (i)]

$2f(\alpha)f(\beta) = 1$ ...(ii)

Now, $\quad 2f(\beta) - g(\alpha) = 2f(\beta) + 1 - (1 + g(\alpha))$

$$= 2f(\beta) + 1 - \frac{1}{f(\alpha)}$$
$$= \frac{2f(\alpha)f(\beta) - 1}{f(\alpha)} + 1 = 1 \quad \text{[from Eq. (ii)]}$$

114. As we know that $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

$$\log_{\left|\frac{1 - \sin x}{\cos x}\right|}\left|\frac{1 + \sin x}{\cos x}\right| = -1$$

Now, series is

$$\text{Let } S = 1 - \frac{x}{2} - \frac{x^2}{4} - \ldots = 1 - \frac{\frac{x}{2}}{1 - \frac{x}{2}}$$
$$= 1 - \frac{-x}{2 - x} = \frac{2 - x - x}{2 - x} = \frac{2(1 - x)}{(2 - x)} = \frac{k(1 - x)}{(2 - x)}$$

Thus, $k = 2$

115. From the second relation $9x\sin^3\theta = 5y\cos^3\theta$.

$\Rightarrow \quad \dfrac{\cos^3\theta}{9x} = \dfrac{\sin^3\theta}{5y} = k^3$ (say)

$\Rightarrow \quad \cos\theta = k(9x)^{\frac{1}{3}}$ and $\sin\theta = k(5y)^{\frac{1}{3}}$

Squaring and adding, we get

$$1 = \cos^2\theta + \sin^2\theta = k^2\left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]$$

and $\quad \dfrac{9x}{k(9x)^{\frac{1}{3}}} + \dfrac{5y}{k(5y)^{\frac{1}{3}}} = 56$ (From Ist relation)

$\Rightarrow \quad (9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}} = 56k$

$\Rightarrow \quad \left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]^2 = (56)^2k^2 = \dfrac{(56)^2}{(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}}$

$\Rightarrow \quad \left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]^3 = (56)^2 = 3136.$

116. $A > \dfrac{\pi}{2} \Rightarrow B + C < \dfrac{\pi}{2}$

$\Rightarrow \quad \tan(B + C) > 0 \Rightarrow \dfrac{\tan B + \tan C}{1 - \tan B \tan C} > 0$

$\Rightarrow \quad \tan B \tan C < 1$ as $\tan B > 0, \tan C > 0$

$\Rightarrow \quad [x] = 2635 - 1 = 2634$

117. $\because \cot\left(7\dfrac{1^\circ}{2}\right) = \dfrac{1 + \cos 15^\circ}{\sin 15^\circ}$

$$= \frac{1 + \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$
$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{2}$$

$$= \frac{2\sqrt{6} + 2\sqrt{2} + 3 + \sqrt{3} + \sqrt{3} + 1}{2}$$
$$= \sqrt{6} + \sqrt{2} + 2 + \sqrt{3}$$
$$= \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$$

and $4\cos 36° = 4\left(\frac{\sqrt{5}+1}{4}\right) = \sqrt{5} + 1 = \sqrt{5} + \sqrt{1}$

Hence, $4\cos 36° + \cot\left(7\frac{1°}{2}\right)$

$$= \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$$

$\therefore \quad n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5$ and $n_6 = 6$

$\therefore \quad \sum_{i=1}^{6} n_i^2 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^3 + n_6^2$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$
$$= 91$$

118. $\because \prod_{r=1}^{4} \sin(rA) = \sin A \sin 2A \sin 3A \sin 4A$

$$= \sin A \cdot 2\sin A\cos A \cdot (3\sin A - 4\sin^3 A \cdot 2\sin 2A\cos 2A)$$
$$= 2\sin^2 A\cos A \cdot \sin A(3 - 4\sin^2 A)$$
$$\cdot\, 4\sin A\cos A \cdot (1 - 2\sin^2 A)$$
$$= 8x^2(1-x)(3-4x)(1-2x)$$
$$= 24x^2 - 104x^3 + 144x^4 - 64x^5$$

On comparing, we get $a = 24, b = -104, c = 144, d = -64$

$10a - 7b + 15c - 5d$

$$= 10 \times 24 - 7 \times -104 + 15 \times 144 - 5 \times -64$$
$$= 240 + 728 + 2160 + 320 = 3448$$

119. Let $x + 5 = 14\cos\theta$ and $y - 12 = 14\sin\theta$

$\therefore x^2 + y^2 = (14\cos\theta - 5)^2 + (14\sin\theta + 12)^2$

$$= 196 + 25 + 144 + 28(12\sin\theta - 5\cos\theta)$$
$$= 365 + 28(12\sin\theta - 5\cos\theta)$$

$\therefore \quad \sqrt{x^2 + y^2}\Big|_{\min} = \sqrt{365 - 28 \times 13} = \sqrt{365 - 364} = 1$

120. $\because 12° \times 5 = 60°$

Let $\quad 12° = \theta$

$\therefore \quad 5\theta = 60°$

$\Rightarrow \quad 3\theta + 2\theta = 60°$

$\therefore \quad \cos(3\theta + 2\theta) = \cos 60°$

$\Rightarrow \quad \cos 3\theta\cos 2\theta - \sin 3\theta\sin 2\theta = \frac{1}{2}$

$\Rightarrow (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)$

$$2\sin\theta\cos\theta = \frac{1}{2}$$

Let $\quad \cos\theta = x$

$\therefore \quad (4x^3 - 3x)(2x^2 - 1) - 2x(3 - 4(1 - x^2))(1 - x^2) = \frac{1}{2}$

$\Rightarrow \quad (8x^5 - 10x^3 + 3x) - (2x - 2x^3)(4x^2 - 1) = \frac{1}{2}$

$\Rightarrow (16x^5 - 20x^3 + 6x) - (4x - 4x^3)(4x^2 - 1) - 1 = 0$

$\Rightarrow \quad 32x^5 - 40x^3 + 10x - 1 = 0$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)(32x^4 + 16x^3 - 32x^2 - 16x + 2) = 0$

but $x \neq \frac{1}{2}$,

$\therefore \quad 16x^4 + 8x^3 - 16x^2 - 8x + 1 = 0$

$\therefore$ Degree is 4.

121. From conditional identities, we have

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$$
$$= \frac{4\sin A\sin B\sin C}{4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}$$
$$= 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \Rightarrow k = 8$$

and $3k^3 + 2k^2 + k + 1 = 1536 + 128 + 8 + 1 = 1673$

122. $x = \cot\frac{11\pi}{8} = \cot\left(\pi + \frac{3\pi}{8}\right) = \cot\frac{3\pi}{8} = \sqrt{2} - 1$

$\Rightarrow \quad (x+1)^2 = 2$

$\therefore \quad x^2 + 2x - 1 = 0$

Now, $\quad f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$

$$= x^2(x^2 + 2x - 1) + 2x^3 + 3x^2 - 4x + 7$$
$$= 0 + 2x^3 + 3x^2 - 4x + 7$$
$$= 2x(x^2 + 2x - 1) - x^2 - 2x + 7 = -x^2 - 2x + 7$$
$$= -(x^2 + 2x - 1) + 6 = 0 + 6 = 6$$

123. $\frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}} = \frac{\sqrt{a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}}$

Now, $\sqrt{b} + \sqrt{c} - \sqrt{a} = \frac{(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{b} + \sqrt{c} + \sqrt{a})}{(\sqrt{b} + \sqrt{c} + \sqrt{a})}$

$$= \frac{(\sqrt{b} + \sqrt{c})^2 - a}{(\sqrt{b} + \sqrt{c} + \sqrt{a})} = \frac{(b + c - a) + 2\sqrt{bc}}{(\sqrt{b} + \sqrt{c} + \sqrt{a})} > 0$$

Hence, $\quad \sqrt{b} + \sqrt{c} - \sqrt{a} > 0$

Now, let $\sqrt{b} + \sqrt{c} - \sqrt{a} = x, \sqrt{c} + \sqrt{a} - \sqrt{b} = y$

and $\quad \sqrt{a} + \sqrt{b} - \sqrt{c} = z$

$\therefore \quad \frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} = \frac{y + z}{2x}$

$\Rightarrow \sum \frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}}$

$$= \frac{1}{2}\left\{\frac{y}{x} + \frac{z}{x}\right\} + \frac{1}{2}\left\{\frac{z}{y} + \frac{x}{y}\right\} + \frac{1}{2}\left\{\frac{x}{z} + \frac{y}{z}\right\}$$
$$= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{2}\left(\frac{y}{z} + \frac{z}{y}\right) + \frac{1}{2}\left(\frac{z}{x} + \frac{x}{z}\right)$$
$$\geq 1 + 1 + 1 \quad (\because \text{AM} \geq \text{GM})$$

$2020\sum \frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}} \leq 6060$

$\therefore$ Minimum value is 6060.

124. We have, $\sin\theta(1+\sin^2\theta)=1-\sin^2\theta$

$\Rightarrow \quad \sin\theta(2-\cos^2\theta)=\cos^2\theta$

Squaring both sides, we get

$\sin^2\theta(2-\cos^2\theta)^2=\cos^4\theta$

$\Rightarrow \quad (1-\cos^2\theta)(4-4\cos^2\theta+\cos^4\theta)=\cos^4\theta$

$\Rightarrow \quad -\cos^6\theta+5\cos^4\theta-8\cos^2\theta+4=\cos^4\theta$

$\therefore \quad \cos^6\theta-4\cos^4\theta+8\cos^2\theta=4$

125. $16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)$

$$=16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)\times\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)$$

$$=16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta+\cos\frac{\pi}{8}\right)\times\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta+\cos\frac{3\pi}{8}\right)$$

$$=16\left(\cos^2\theta-\cos^2\frac{\pi}{8}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{8}\right)$$

$$=16\left(\cos^2\theta-\cos^2\frac{\pi}{8}\right)\left(\cos^2\theta-\sin^2\frac{\pi}{8}\right)$$

$$=16\left(\cos^4\theta-\cos^2\theta+\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$$

$$=16\left(\cos^4\theta-\cos^2\theta+\frac{1}{8}\right)$$

$$=16\left(-\cos^2\theta\sin^2\theta+\frac{1}{8}\right)=16\left(\frac{-\sin^2 2\theta}{4}+\frac{1}{8}\right)$$

$$=16\left(\frac{1-2\sin^2 2\theta}{8}\right)=\frac{16\cos 4\theta}{8}=2\cos 4\theta$$

$\therefore \quad \lambda=2$

126. $2k\cos\cos 40^\circ=\frac{1}{\sin 20^\circ}+\frac{1}{\sqrt{3}\cos 20^\circ}$

$$=\frac{\sqrt{3}\cos 20^\circ+\sin 20^\circ}{\sqrt{3}\sin 20^\circ\cos 20^\circ}$$

$$=\frac{\frac{\sqrt{3}}{2}\cos 20^\circ+\frac{1}{2}\sin 20^\circ}{\frac{\sqrt{3}}{4}\sin 40^\circ}$$

$$=\frac{\sin 60^\circ\cos 20^\circ+\cos 60^\circ\sin 20^\circ}{\left(\frac{\sqrt{3}}{4}\right)\sin 40^\circ}$$

$$=\left(\frac{4}{\sqrt{3}}\right)2\cos 40^\circ$$

$\Rightarrow \quad 2k^2=16$

so $18k^4+162k^2+369=1745$

127. $\tan 82\frac{1}{2}^\circ=\cot 7\frac{1}{2}^\circ=\frac{\cos 7\frac{1}{2}^\circ}{\sin 7\frac{1}{2}^\circ}$

On multiplying numerator and denominator by $2\cos 7\frac{1}{2}^\circ$, we get

$$\tan 82\frac{1}{2}^\circ=\frac{2\cos^2 7\frac{1}{2}^\circ}{2\sin 7\frac{1}{2}^\circ\cos 7\frac{1}{2}^\circ}=\frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$=\frac{1+\cos(45^\circ-30^\circ)}{\sin(45^\circ-30^\circ)}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{2}(\sqrt{3}+1)+(\sqrt{3}+1)^2}{2}=\frac{2\sqrt{2}(\sqrt{3}+1)+(4+2\sqrt{3})}{2}$$

$$=\sqrt{2}(\sqrt{3}+1)+(2+\sqrt{3})=\sqrt{6}+\sqrt{2}+\sqrt{4}+\sqrt{3}$$

$$=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$

128. LHS $=\frac{2-m(\sin 2\alpha+\sin 2\beta)}{1-m(\sin 2\alpha+\sin 2\beta)+m^2\sin 2\alpha\sin 2\beta}$

$$=\frac{2-2m\sin(\alpha+\beta)\cos(\alpha-\beta)}{1-2m\sin(\alpha+\beta).\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$$

$$=\frac{2\{1-\cos^2(\alpha-\beta)\}}{1-2m\sin(\alpha+\beta).\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$$

[using $m\sin(\alpha+\beta)=\cos(\alpha-\beta)$]

$$=\frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2[\sin(\alpha+\beta)+\sin(\alpha-\beta)][\sin(\alpha+\beta)-\sin(\alpha-\beta)]}$$

$$=\frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2\sin^2(\alpha+\beta)-m^2\sin^2(\alpha-\beta)}$$

$$=\frac{2\sin^2(\alpha-\beta)}{1-\cos^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)}$$

$$=\frac{2\sin^2(\alpha-\beta)}{\sin^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)}=\frac{2}{1-m^2}$$

129. Given $\tan\frac{1}{4}(\beta+\gamma-2\alpha).\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma)=1$,

where $\alpha+\beta+\gamma=\pi$

$\Rightarrow \quad \tan\left(\frac{\pi-2\alpha}{4}\right)\tan\left(\frac{\pi-2\beta}{4}\right)\tan\left(\frac{\pi-2\gamma}{4}\right)=1$

$\Rightarrow \quad \left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)\left(1-\tan\frac{\gamma}{2}\right)=\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\gamma}{2}\right)$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=0$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2} \quad$ ...(i)

Also, $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$ ...(ii)

On squaring Eq. (i) and using Eq. (ii); $\left\{\because \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\frac{\pi}{2}\right\}$

$$\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}+2=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \quad ...(iii)$$

The equation to be prove is

$$1+\cos\alpha+\cos\beta+\cos\gamma=0$$

$$\Rightarrow \quad 1+\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}+\frac{1-\tan^2\frac{\gamma}{2}}{1+\tan^2\frac{\gamma}{2}}=0$$

$$\Rightarrow \quad \frac{2}{1+\tan^2\frac{\alpha}{2}}+\frac{2\left(1-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\gamma}{2}\right)}=0$$

$$\Rightarrow \left(1+\tan^2\frac{\alpha}{2}\right)\left(1-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)+\left(1+\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\gamma}{2}\right)=0$$

$$\Rightarrow \left(1+\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)$$
$$+\left(1+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}+\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)=0$$

$$\Rightarrow \ 2+\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \quad ...(iv)$$

From Eq. (iii) and (iv)

$$1+\cos\alpha+\cos\beta+\cos\gamma=0$$

130. We have, $\sin^4 x+\cos^4 x\le\sin^2 x+\cos^2 x$,

as $\quad |\sin x|\le 1$ and $|\cos x|\le 1$

$\Rightarrow \quad a\le 1$ ...(i)

Next, $\sin^4 x+\cos^4 x=a$

$\Rightarrow \quad (\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x=a$

$\Rightarrow \quad 1-\frac{1}{2}\sin^2 2x=a$

$\Rightarrow \quad \frac{1}{2}\sin^2 2x=1-a$

$\Rightarrow \quad 1-a\le\frac{1}{2}$ $\quad\left[\because \frac{1}{2}\sin^2 x\le\frac{1}{2}\right]$

$\Rightarrow \quad a\ge\frac{1}{2}$ ...(ii)

From Eqs. (i) and (ii),

$$\frac{1}{2}\le a\le 1$$

131. Let $a\sec\theta-b\tan\theta=x$

So, $\quad a^2\sec^2\theta=(x+b\tan\theta)^2$

$\Rightarrow \quad a^2(1+\tan^2\theta)=x^2+2bx\tan\theta+b^2\tan^2\theta$

$\Rightarrow \quad \tan^2\theta(a^2-b^2)-2bx\tan\theta+(a^2-x^2)=0$

$\Rightarrow \quad \left(\tan\theta-\frac{bx}{a^2-b^2}\right)^2=\frac{a^2(x^2+b^2-a^2)}{(a^2-b^2)^2}$

Thus, $\quad x^2+(b^2-a^2)\ge 0$

$\Rightarrow \quad x^2\ge a^2-b^2$

Thus, the minimum value of x is $\sqrt{a^2-b^2}$, which is attained at $\theta=\sin^{-1}\left(\frac{b}{a}\right)$.

132. We can write

$(b\tan\gamma-c\tan\beta)^2+(c\tan\alpha-a\tan\gamma)^2+(a\tan\beta-b\tan\alpha)^2$

$=(a^2+b^2+c^2)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-(a\tan\alpha+b\tan\beta+c\tan\gamma)^2$

The minimum value of the LHS being zero, that of

$$(a^2+b^2+c^2)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-k^2\ge 0$$

$$\Rightarrow \quad \tan^2\alpha+\tan^2\beta+\tan^2\gamma\ge\frac{k^2}{a^2+b^2+c^2}$$

Hence, minimum value of $\tan^2\alpha+\tan^2\beta+\tan\gamma$ is $\left(\frac{k^2}{a^2+b^2+c^2}\right)$.

133. Here, $\frac{x}{y}=\frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$. By componendo and dividendo

$$\frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$\therefore \quad \frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\cdot\sin(\alpha-\beta)$$

$$\frac{x+y}{x-y}\cdot\sin^2(\alpha-\beta)=\frac{1}{2}\{\cos 2(\theta+\beta)-\cos 2(\theta+\alpha)\} \quad ...(i)$$

Similarly,

$$\frac{y+z}{y-z}\cdot\sin^2(\beta-\gamma)=\frac{1}{2}\{\cos 2(\theta+\gamma)-\cos 2(\theta+\beta)\} \quad ...(ii)$$

and $\quad \frac{z+x}{z-x}\cdot\sin^2(\gamma-\alpha)=\frac{1}{2}\{\cos 2(\theta+\alpha)-\cos 2(\theta+\gamma)\}$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$\Sigma\frac{x+y}{x-y}\cdot\sin^2(\alpha-\beta)=0$$

134. $f(x)$ may be written as $f(x)=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cos(a_k+x)$

$$=\sum_{k=1}^{n}\frac{1}{2^{k-1}}(\cos a_k\cos x-\sin a_k\sin x)$$

$$=\left(\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\cos a_k\right)\cdot\cos x\left(-\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\sin a_k\right)\cdot\sin x$$

where, $A=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cos a_k$, $B=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\sin a_k$. Now, A and B both cannot be zero, for if they were then $f(x)$ would vanish identically.

Now,

$$f(x_1)=A\cos x_1-B\sin x_1=0$$
$$f(x_2)=A\cos x_2-B\sin x_2=0$$

$\Rightarrow \quad \tan x_1=\frac{A}{B}$ and $\tan x_2=\frac{A}{B}$

$\Rightarrow \quad \tan x_1=\tan x_2 \Rightarrow \ x_2-x_1=m\pi.$

135. $x = \tan(n\theta + \alpha) - \tan(n\theta + \beta)$

$$= \frac{\sin(n\theta + \alpha)}{\cos(n\theta + \alpha)} - \frac{\sin(n\theta + \beta)}{\cos(n\theta + \beta)}$$

$$= \frac{\sin(n\theta + \alpha - n\theta - \beta)}{\cos(n\theta + \alpha)\cos(n\theta + \beta)} = \frac{2\sin(\alpha - \beta)}{\cos(2n\theta + \alpha + \beta) + \cos(\alpha - \beta)}$$

$$\Rightarrow \cos(2n\theta + \alpha + \beta) + \cos(\alpha - \beta) = \frac{2\sin(\alpha - \beta)}{x} \quad \text{...(i)}$$

Again $y = \cot(n\theta + \alpha) - \cot(n\theta + \beta)$

$$= \frac{\cos(n\theta + \alpha)}{\sin(n\theta + \alpha)} - \frac{\cos(n\theta + \beta)}{\sin(n\theta + \beta)} = \frac{\sin(n\theta + \beta - n\theta - \alpha)}{\sin(n\theta + \alpha)\sin(n\theta + \beta)}$$

$$\Rightarrow \quad y = \frac{\sin(\beta - \alpha)}{\cos(\alpha - \beta) - \cos(2n\theta + \alpha + \beta)}$$

$$\Rightarrow \cos(\alpha - \beta) - \cos(2n\theta + \alpha + \beta) = \frac{2\sin(\beta - \alpha)}{y} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2\cos(\alpha - \beta) = \frac{2\sin(\alpha - \beta)}{x} + \frac{2\sin(\beta - \alpha)}{y}$$

$$\Rightarrow \quad \cot(\alpha - \beta) = \frac{1}{x} - \frac{1}{y}$$

136. $\{\sin(\alpha - \beta) + \cos(\alpha + 2\beta)\cdot\sin\beta\}^2 = 4\cos\alpha\cdot\sin\beta\cdot\sin(\alpha + \beta)$

$$\Rightarrow \{\sin\alpha\cos\beta - \sin\beta\cos\alpha + (\cos\alpha\cos 2\beta - \sin\alpha\sin 2\beta)\sin\beta\}^2$$
$$= 4\cos\alpha\sin\beta\sin(\alpha + \beta)$$

$$\Rightarrow \{\tan\alpha - \tan\beta + \cos 2\beta\cdot\tan\beta - \sin 2\beta\cdot\tan\alpha\tan\beta\}^2$$
$$= 4\tan\beta(\tan\alpha + \tan\beta) \quad \{\because \text{dividing by } \cos^2\alpha.\cos^2\beta\}$$

$\Rightarrow$

$$\{\tan\alpha\cdot\cos 2\beta - \tan\beta + \cos 2\beta\cdot\tan\beta\}^2 = 4\tan\beta\{\tan\alpha + \tan\beta\}$$

$$\Rightarrow \{(\tan\alpha + \tan\beta)\cdot\cos 2\beta - \tan\beta\}^2 = 4\tan\beta(\tan\alpha + \tan\beta) \text{ ...(i)}$$

If $\quad (\tan\alpha + \tan\beta) = \dfrac{\tan\beta}{x}$ ...(ii)

Eq. (i) becomes;

$$\left\{\frac{\tan\beta}{x}\cdot\cos 2\beta - \tan\beta\right\}^2 = 4\tan\beta\cdot\frac{\tan\beta}{x}$$

$$\Rightarrow \quad (\cos 2\beta - x)^2 = 4x$$

$$\Rightarrow \quad \cos^2 2\beta + x^2 - 2x\cos 2\beta = 4x$$

$$\Rightarrow \quad x^2 - 2x(\cos 2\beta + 2) + \cos^2 2\beta = 0$$

$$\Rightarrow \quad x = (\cos 2\beta + 2) \pm 2\sqrt{1 + \cos 2\beta}$$

$$\Rightarrow \quad x = \cos 2\beta + 2 \pm 2\sqrt{2\cos^2\beta}$$

$$\Rightarrow \quad x = 2\cos^2\beta - 1 + 2 \pm 2\sqrt{2}\cos\beta$$
$$= (\sqrt{2}\cos\beta \pm 1)^2$$

$$\Rightarrow \quad \tan\alpha + \tan\beta = \frac{\tan\beta}{x} = \frac{\tan\beta}{(\sqrt{2}\cos\beta - 1)^2} \quad [\text{since, } x < 1]$$

137. Let $\quad \Delta = \begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \cos^3 A & \cos^3 B & \cos^3 C \end{vmatrix}$

$$= \cos A\cos B\cos C\begin{vmatrix} \tan A & \tan B & \tan C \\ 1 & 1 & 1 \\ \cos^2 A & \cos^2 B & \cos^2 C \end{vmatrix}$$

$$= \cos A\cos B\cos C$$
$$\begin{vmatrix} \tan A & \tan B - \tan A & \tan C - \tan A \\ 1 & 0 & 0 \\ \cos^2 A & \cos^2 B - \cos^2 A & \cos^2 A - \cos^2 A \end{vmatrix}$$

$\left[\text{since, } \tan B - \tan A = -\dfrac{\sin(A - B)}{\cos A\cos B},\right.$

$\left.\cos^2 B - \cos^2 A = \sin(A - B)\sin(A + B)\right]$

$$\therefore \quad \Delta = -\cos A\cos B\cos C$$
$$\begin{vmatrix} -\dfrac{\sin(A - B)}{\cos A\cos B} & -\dfrac{\sin(A - C)}{\cos A\cos C} \\ \sin(A - B)\sin(A + B) & \sin(A - C)\sin(A + C) \end{vmatrix}$$

$$= \cos A\cos B\cos C\cdot$$
$$\frac{\sin(A - B).\sin(A - C)}{\cos A\cos B\cos C}\begin{vmatrix} \cos C & \cos B \\ \sin(A + B) & \sin(A + C) \end{vmatrix}$$

$$= -\sin(B - C)\sin(C - A)\sin(A - B) = 0$$

If $B = C$ or $C = A$ or $A = B$

Hence, ΔABC is an isosceles.

138. Here, $\quad \dfrac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} = \dfrac{\sqrt{a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}}$

Now, $\quad \sqrt{b} + \sqrt{c} - \sqrt{a} = \dfrac{(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{b} + \sqrt{c} + \sqrt{a})}{(\sqrt{b} + \sqrt{c} + \sqrt{a})}$

$$= \frac{b + c - a + 2\sqrt{bc}}{\sqrt{b} + \sqrt{c} + \sqrt{a}} > 0$$

Hence, $\sqrt{b} + \sqrt{c} - \sqrt{a} = 0$

Let $\sqrt{b} + \sqrt{c} - \sqrt{a} = x$, $\sqrt{c} + \sqrt{a} - \sqrt{b} = y$, $\sqrt{a} + \sqrt{b} - \sqrt{c} = z$

$$\therefore \quad \frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} = \frac{y + z}{2x}$$

$$\Rightarrow \Sigma = \frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}}$$
$$= \frac{1}{2}\left\{\frac{y}{x} + \frac{z}{x}\right\} + \frac{1}{2}\left\{\frac{y}{z} + \frac{z}{y}\right\} + \frac{1}{2}\left\{\frac{z}{x} + \frac{x}{y}\right\}$$

which is greater than or equal to 3, as each term $\left(\dfrac{1}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right)\text{etc.}\right)$ is greater than or equal to 1.

(using AM $\geq$ GM)

Now, equality hold if and only if,

$$\frac{x}{y} = \frac{y}{x}, \frac{y}{z} = \frac{z}{y}$$

and $\quad \dfrac{z}{x} = \dfrac{x}{y} \quad$ i.e. $x = y = z$

$\Rightarrow a = b = c$ i.e. triangle is equilateral.

139. $2(\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)) + 3 = 0$

$$\Rightarrow \quad 2(\cos(\alpha + \theta - (\beta + \theta)) + \cos(\beta + \theta - (\gamma + \theta))$$
$$+ \cos(\gamma + \theta - (\alpha + \theta))) + 3 = 0$$

$$\Rightarrow \quad 2(\cos(\alpha + \theta).\cos(\beta + \theta) + \sin(\alpha + \theta).\sin(\beta + \theta) + \ldots + \ldots))$$
$$+ \{(\sin^2(\alpha + \theta) + \cos^2(\alpha + \theta)) + (\sin^2(\beta + \theta) + \cos^2(\beta + \theta))$$
$$+ (\sin^2(\gamma + \theta) + \cos^2(\gamma + \theta))\} = 0$$

$$\Rightarrow \quad (\sin(\gamma+\theta)+\sin(\beta+\theta)+\sin(\alpha+\theta))^2+(\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta))^2=0$$

which is only possible if;

$$\sin(\alpha+\theta)+\cos(\beta+\theta)+\sin(\gamma+\theta)=0 \quad \text{...(i)}$$

$$\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta)=0 \quad \text{...(ii)}$$

From Eq. (ii), we get

$$d(\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta))=0$$

$$\Rightarrow \sin(\alpha+\theta)\cdot d\alpha+\sin(\beta+\theta)\cdot d\beta+\sin(\gamma+\theta)\cdot d\gamma=0$$

$$\Rightarrow \frac{d\alpha}{\sin(\beta+\theta)\cdot\sin(\gamma+\theta)}+\frac{d\beta}{\sin(\alpha+\theta)\cdot\sin(\gamma+\theta)}+\frac{d\gamma}{\sin(\alpha+\theta)\cdot\sin(\beta+\theta)}=0$$

140. The quadratic equation,

$$4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right)=0 \text{ have real roots}$$

$$\Rightarrow \text{Discriminant}=4-4\cdot 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\geq 0$$

$$\Rightarrow 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\leq 1$$

$$\left[\text{but } 4^{\sec^2\alpha}\geq 4, \beta^2-\beta+\frac{1}{2}=\left(\beta+\frac{1}{2}\right)^2+\frac{1}{4}\geq\frac{1}{4}\right]$$

i.e. the equation will be satisfied only when $4^{\sec^2\alpha}=4$ and

$$\beta^2-\beta+\frac{1}{2}=\frac{1}{4}$$

$$\Rightarrow \sec^2\alpha=1 \text{ and } \left(\beta-\frac{1}{2}\right)^2=0$$

$$\Rightarrow \cos^2\alpha=1 \text{ and } \beta=\frac{1}{2}$$

$$\Rightarrow \alpha=n\pi \text{ and } \beta=\frac{1}{2}$$

$$\cos\alpha+\cos^{-1}\beta=\cos n\pi+\cos^{-1}\left(\frac{1}{2}\right)$$

$$=1+\frac{\pi}{3}, \text{ when } n \text{ is an even integer.}$$

$$=-1+\frac{\pi}{3}, \text{ when } n \text{ is an odd integer.}$$

i.e. values of $\cos\alpha+\cos^{-1}\beta$ is $\frac{\pi}{3}-1, \frac{\pi}{3}+1$.

141. $f(\theta)=1-(a\cos\theta+b\sin\theta)-(A\cos 2\theta+B\sin 2\theta)$

$$\Rightarrow \quad f(\theta)=1-\sqrt{a^2+b^2}\cos(\theta-\alpha)-\sqrt{A^2+B^2}\cos(2\theta-\beta)$$

Now, $f\left(\alpha+\frac{\pi}{4}\right)=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\cos\left(\frac{\pi}{2}+2\alpha-\beta\right)$

$$=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}+\sqrt{A^2+B^2}\sin(2\alpha-\beta) \quad \text{...(i)}$$

and $f\left(\alpha-\frac{\pi}{4}\right)=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\cos\left(2\alpha-\beta-\frac{\pi}{2}\right)$

$$=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\sin(2\alpha-\beta) \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii),

$$f\left(\alpha+\frac{\pi}{4}\right)+f\left(\alpha-\frac{\pi}{4}\right)=2-2\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\geq 0$$

$$\Rightarrow \quad \sqrt{a^2+b^2}\leq\sqrt{2}$$

$$\Rightarrow \quad a^2+b^2\leq 2$$

Similarly putting $\theta=\beta$ and $\beta+\pi$. We have,

$$f(\beta)+f(\beta+\pi)=2-2\sqrt{A^2+B^2}\geq 0$$

$$\Rightarrow \quad \sqrt{A^2+B^2}\leq 1 \Rightarrow A^2+B^2\leq 1$$

142. Clearly θ_1, θ_0 are roots of ; $\frac{\cos\theta}{\cos\theta_2}+\frac{\sin\theta}{\sin\theta_2}=1$

$$\Rightarrow \frac{\cos\theta}{\cos\theta_2}=1-\frac{\sin\theta}{\sin\theta_2}\Rightarrow\frac{\cos^2\theta}{\cos^2\theta_2}=1+\frac{\sin^2\theta}{\sin^2\theta_2}-\frac{2\sin\theta}{\sin\theta_2}$$

$$\Rightarrow \sin^2\theta\left(\frac{1}{\sin^2\theta_2}+\frac{1}{\cos^2\theta_2}\right)-\frac{2\sin\theta}{\sin\theta_2}+\left(1-\frac{1}{\cos^2\theta_2}\right)=0$$

The roots of the equation are θ_0 and θ_1.

$$\Rightarrow \quad \sin\theta_0\cdot\sin\theta_1=\frac{1-\frac{1}{\cos^2\theta_2}}{\frac{1}{\sin^2\theta_2}+\frac{1}{\cos^2\theta_2}}$$

$$=(\cos^2\theta_2-1)\cdot\sin^2\theta_2=-\sin^4\theta_2$$

$$\Rightarrow \quad \frac{\sin\theta_0\cdot\sin\theta_1}{\sin^2\theta_2}=-\sin^2\theta_2 \quad \text{...(i)}$$

Similarly, taking a quadratic in $\cos\theta$, we get

$$\Rightarrow \quad \frac{\cos\theta_0\cdot\cos\theta_1}{\sin^2\theta_2}=-\cos^2\theta_2 \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\frac{\sin\theta_0\sin\theta_1}{\sin^2\theta_2}+\frac{\cos\theta_0\cos\theta_1}{\cos^2\theta_2}=-1$$

143. Let the given expression be E, then E can be written as,

$$E=\sum_{k=1}^{n-1}{}^nC_k \cos kx\cdot\cos(n+k)x+\sum_{k=1}^{n-1}{}^nC_k\sin(n-k)x.\sin(2n-k)x$$

or $$E=\sum_{k=1}^{n-1}{}^nC_k\cos kx \cos(n+k)x+\sum_{k=1}^{n-1}{}^nC_k\sin(k)x\cdot\sin(n+k)x$$

[replacing k by $(n-k)$ in the second]

Sum and using ${}^nC_k={}^nC_{n-k}$

$$E=\sum_{k=1}^{n-1}{}^nC_k(\cos kx\cos(n+k)x+\sin kx\cdot\sin(n+k)x)$$

$$=\sum_{k=1}^{n-1}{}^nC_k\cos nx$$

$$=\cos nx\{({}^nC_0+{}^nC_1+\dots+{}^nC_n)-{}^nC_0-{}^nC_n\}$$

$$=\cos nx\{2^n-2\}$$

$$\therefore \quad E=(2^n-2)\cos nx$$

144. It is evident from the inequality that,

$$|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}| \le \sqrt{2}\ \forall\ x \in [0, 2\pi]$$

as $|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}| \le \sqrt{1+\sin 2x} \le \sqrt{2}$

Now,

$2\cos x \le |\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}|$ holds for all x for which $\cos x \le 0$.

$$\Rightarrow \qquad x \le \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \qquad \text{...(i)}$$

Now, if $\cos x > 0$

Then, $\quad 4\cos^2 x \le 1+\sin 2x + 1 - \sin 2x - \sqrt{1-\sin^2 2x}$

$\Rightarrow \quad 2 + 2\cos 2x \le 2 - 2|\cos x|$

$\Rightarrow \quad |\cos 2x| \le -\cos 2x$

$\Rightarrow \quad \cos 2x \le 0$

$$\Rightarrow \qquad x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right] \qquad \text{...(ii)}$$

Hence, from Eqs. (i) and (ii)

$$x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$$

145. The given equation can be rewritten as, $x^2 - 3 = 3\left[\sin\left(x - \frac{\pi}{6}\right)\right]$

Here, right hand side can take only the values −3, 0, 3.

Case I When $x^2 - 3 = -3 \Rightarrow x = 0$

At $x = 0, \left[\sin\left(x - \frac{\pi}{6}\right)\right] = -1$, so $x = 0$ is a solution.

Case II When $x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

Now at $x = \sqrt{3}, \left[\sin\left(x - \frac{\pi}{6}\right)\right] = 0 \Rightarrow x = \sqrt{3}$

But at $x = -\sqrt{3}, \left[\sin\left(x - \frac{\pi}{6}\right)\right] = -1$, hence $x = -\sqrt{3}$ is not a solution.

Case III When $x^2 - 3 = 3 \Rightarrow x = \pm\sqrt{6}$

But $\left[\sin\left(\pm\sqrt{6} - \frac{\pi}{6}\right)\right] \ne 1 \Rightarrow x = \pm\sqrt{6}$ is not a solution.

Hence, the given equation has only two solutions $x = 0$ and $\sqrt{3}$.

146. $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos(rB - (n-r)A)$

$= \text{real part of } \sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB - (n-r)A\}}$

Now, $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$

$= \sum_{r=0}^{n} {}^nC_r (ae^{iB})^r (be^{-iA})^{n-r} = (ae^{iB} + be^{-iA})^n$

$= (a\cos B + ia\sin B + b\cos A - bi\sin A)^n$

$= \{(a\cos B + b\cos A) + i(a\sin B - b\sin A)\}^n$

$= \{C + i\cdot 0\}^n = C^n$

147. Let $z^5 + 1 = 0 \Rightarrow z^5 = -1 = (\cos(2r+1) + i\sin(2r+1)\pi)$

$\Rightarrow \quad z = e^{i\left(\frac{2r+1}{5}\right)\pi}, r = 0, 1, 2, 3, 4$

$\Rightarrow$ Roots of $z^k + 1 = 0$ are $e^{i\pi/5}, e^{i3\pi/5}, e^{i\pi}, e^{i7\pi/5}, e^{i9\pi/5}$. Clearly $e^{i7\pi/5}, e^{i9\pi/5}$ and $e^{i3\pi/5}, e^{i7\pi/5}$ are pairwise conjugate.

$\Rightarrow z^5 + 1 = (z - e^{i\pi})(z - e^{i\pi/5})(z - e^{-i\pi/5})(z - e^{-i3\pi/5})(z - e^{i7\pi/5})$

$$\Rightarrow z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right) \text{...(i)}$$

It is required factorisation of $z^5 + 1$.

$$\text{Now,} \qquad \frac{z^5+1}{z+1} = 1 - z + z^2 - z^2 + z^4 \qquad \text{...(ii)}$$

$\Rightarrow \quad 1 - z + z^2 - z^3 + z^4 =$

$$\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$

[using Eqs. (i) and (ii)}

On dividing both side by z^2

$$\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 1$$

$$= \left(z + \frac{1}{z} - 2\cos\frac{\pi}{5}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{5}\right)$$

Let $z = e^{i\theta}$

$$\Rightarrow \qquad z^2 + \frac{1}{z^2} = 2\cos 2\theta, z + \frac{1}{z} = 2\cos\theta$$

$$\Rightarrow 2\cos 2\theta - 2\cos\theta + 1 = 4\left(\cos\theta - \cos\frac{\pi}{5}\right)\left(\cos\theta - \cos\frac{3\pi}{5}\right)$$

Putting $\theta = 0$, we get

$$\frac{1}{4} = \left(1 - \cos\frac{\pi}{5}\right)\left(1 - \cos\frac{3\pi}{5}\right)$$

$$\Rightarrow \qquad \frac{1}{4} = 2\sin^2\frac{\pi}{10}\cdot 2\sin^2\frac{3\pi}{10}$$

$$\Rightarrow \qquad \sin\frac{\pi}{10}\cdot\sin\frac{3\pi}{10} = \frac{1}{4}$$

$$\Rightarrow \qquad 4\sin\frac{\pi}{10}\cdot\cos\left(\frac{\pi}{2} - \frac{3\pi}{10}\right) = 1$$

$$\Rightarrow \qquad 4\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{5} = 1$$

148. Let $\theta = \frac{(2n+1)\pi}{7}$, where $n = 0, 1, 2, 3, 4, 5, 6$.

Then, $4\theta = (2n+1)\pi - 3\theta$

$\Rightarrow \quad \cos 4\theta = -\cos 3\theta$

$\Rightarrow \quad 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow \quad 2(2\cos^2\theta - 1)^2 - 1 = -4\cos^3\theta + 3\cos\theta$

$\Rightarrow \quad 2(2x^2 - 1)^2 - 1 = -4x^3 + 3x \qquad$ [put $x = \cos\theta$]

$\Rightarrow \quad 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

$\Rightarrow \quad (x+1)(8x^3 - 4x^2 - 4x + 1) = 0$

The roots of this equation are,

$$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}, \cos\frac{7\pi}{7}, \cos\frac{9\pi}{7}, \cos\frac{11\pi}{7}, \cos\frac{13\pi}{7}$$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$

$$\text{are} \qquad \cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7} \qquad \text{...(i)}$$

Put $x = \frac{1}{y}$ in Eq. (i) (i.e. $y = \sec\theta$), then

$\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}$ are the roots of the equation.

$$\frac{8}{y^3} - \frac{4}{y^2} - \frac{4}{y} + 1 = 0$$

$$\Rightarrow \quad y^3 - 4y^2 - 4y + 8 = 0$$

$$\Rightarrow \quad \sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$$

Again putting $\frac{1}{x^2} = y$ in Eq. (i)

(i.e. $y = \sec^2\theta$)

$$\frac{8}{y^{3/2}} - \frac{4}{y} - \frac{4}{y^{1/2}} + 1 = 0$$

$$\Rightarrow \quad 8 - 4y^{1/2} - 4y + y^{3/2} = 0$$

$$\Rightarrow \quad y^{1/2}(y-4) = 4(y-2)$$

$$\Rightarrow \quad y(y-4)^2 = 16(y-2)^2$$

$$\Rightarrow \quad y^3 - 24y^2 + 80y - 64 = 0 \quad \text{...(ii)}$$

Hence, the roots are

$$\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$$

Now, putting $y = 1 + z$, (i.e. $z = \tan^2\theta$)

We have,

$$(1+z)^3 - 24(1+z)^2 + 80(1+z) - 64 = 0$$

$$\Rightarrow \quad z^3 - 21z^2 + 35z - 7 = 0$$

whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$.

149. We have, $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$

or $4(\cos\beta - \cos\alpha) + 2\cos\alpha\cos\beta = 2$

$$\Rightarrow \quad 1 - \cos\alpha + \cos\beta - \cos\alpha\cos\beta = 3 + 3\cos\alpha - 3\cos\beta - 3\cos\alpha\cos\beta$$

$$\Rightarrow \quad (1-\cos\alpha)(1+\cos\beta) = 3(1+\cos\alpha)(1-\cos\beta)$$

$$\Rightarrow \quad \frac{(1-\cos\alpha)}{(1+\cos\alpha)} = \frac{3(1-\cos\beta)}{1+\cos\beta}$$

$$\Rightarrow \quad \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$$

$$\therefore \quad \tan\frac{\alpha}{2} \pm \sqrt{3}\tan\frac{\beta}{2} = 0$$

150. Here, $x^2 - 2x\sec\theta + 1 = 0$ has roots α_1 and β_1.

$$\therefore \quad \alpha_1, \beta_1 = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2\times 1} = \frac{2\sec\theta \pm 2|\tan\theta|}{2}$$

Since, $\theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)$,

i.e. $\theta \in$ IV quadrant $= \frac{2\sec\theta \mp 2\tan\theta}{2}$

$\therefore \quad \alpha_1 = \sec\theta - \tan\theta$ and $\beta_1 = \sec\theta + \tan\theta$ [as $\alpha_1 > \beta_1$]

and $x^2 + 2x\tan\theta - 1 = 0$ has roots α_2 and β_2.

i.e. $$\alpha_2, \beta_2 = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$$

$\therefore \quad \alpha_2 = -\tan\theta + \sec\theta$

and $\quad \beta_2 = -\tan\theta - \sec\theta$ [as $\alpha_2 > \beta_2$]

Thus, $\quad \alpha_1 + \beta_2 = -2\tan\theta$

151. Here, $$\sum_{k=1}^{13} \frac{1}{\sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\}\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

Converting into differences, by multiplying and dividing by $\sin\left[\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\}\right]$, i.e. $\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sum_{k=1}^{13} \frac{\sin\left[\left(\frac{\pi}{4} + k\frac{\pi}{6}\right) - \left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\right]}{\sin\frac{\pi}{6}\left\{\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\sin\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)\right\}}$$

$$= 2\sum_{k=1}^{13} \frac{\left[\begin{array}{l}\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\cos\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\} \\ -\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\cos\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\end{array}\right]}{\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\sin\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)}$$

$$= 2\sum_{k=1}^{13}\left[\cot\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\} - \cot\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)\right]$$

$$= 2\left[\left\{\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right\} + \left\{\cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{2\pi}{6}\right)\right\} + \dots + \left\{\cot\left(\frac{\pi}{4} + 12\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + 13\frac{\pi}{6}\right)\right\}\right]$$

$$= 2\left\{\cot\frac{\pi}{4} - \cot\left(\frac{\pi}{4} + 13\frac{\pi}{6}\right)\right\}$$

$$= 2\left[1 - \cot\left(\frac{29\pi}{12}\right)\right] = 2\left[1 - \cot\left(2\pi + \frac{5\pi}{12}\right)\right]$$

$$= 2\left[1 - \cot\frac{5\pi}{12}\right] \quad \left[\because \cot\frac{5\pi}{12} = (2 - \sqrt{3})\right]$$

$$= 2(1 - 2 + \sqrt{3})$$

$$= 2(\sqrt{3} - 1)$$

152. $$f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta} \quad \text{...(i)}$$

At $\quad \cos 4\theta = \frac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \frac{1}{3}$

$$\Rightarrow \quad \cos^2 2\theta = \frac{2}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\frac{2}{3}} \quad \text{...(ii)}$$

$$\therefore \quad f(\cos 4\theta) = \frac{2\cdot\cos^2\theta}{2\cos^2\theta - 1} = \frac{1 + \cos 2\theta}{\cos 2\theta}$$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}} \quad \text{[from Eq. (ii)]}$$

153. Given equations can be written as

$$x\sin 3\theta - \frac{\cos 3\theta}{y} - \frac{\cos 3\theta}{z} = 0 \quad \text{...(i)}$$

$$x\sin 3\theta - \frac{2\cos 3\theta}{y} - \frac{2\sin 3\theta}{z} = 0 \quad \text{...(ii)}$$

and $x\sin 3\theta - \frac{2}{y}\cos 3\theta - \frac{1}{z}(\cos 3\theta + \sin 3\theta) = 0$ ...(iii)

Eqs. (ii) and (iii), implies

$$2\sin 3\theta = \cos 3\theta + \sin 3\theta \Rightarrow \sin 3\theta = \cos 3\theta$$

$\therefore \quad \tan 3\theta = 1$

$$\Rightarrow \quad 3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \text{ or } \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$$

154. For $0 < \theta < \frac{\pi}{2}$

$$\sum_{m=1}^{6} \text{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\text{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\sin\left[\theta + \frac{m\pi}{4} - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\frac{\pi}{4}\left\{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)\right\}} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)}{1/\sqrt{2}} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6}\left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right] = 4$$

$$\Rightarrow \cot(\theta) - \cot\left(\theta + \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) - \cot\left(\theta + \frac{2\pi}{4}\right)$$

$$+ \ldots + \cot\left(\theta + \frac{5\pi}{4}\right) - \cot\left(\theta + \frac{6\pi}{4}\right) = 4$$

$$\Rightarrow \cot\theta - \cot\left(\frac{3\pi}{2} + \theta\right) = 4$$

$$\Rightarrow \cot\theta + \tan\theta = 4$$

$$\Rightarrow \tan^2\theta - 4\tan\theta + 1 = 0$$

$$\Rightarrow (\tan\theta - 2)^2 - 3 = 0$$

$$\Rightarrow (\tan\theta - 2 + \sqrt{3})(\tan\theta - 2 - \sqrt{3}) = 0$$

$$\Rightarrow \tan\theta = 2 - \sqrt{3}$$

or $\tan\theta = 2 + \sqrt{3}$

$$\Rightarrow \theta = \frac{\pi}{12};\ \theta = \frac{5\pi}{12} \quad \left[\because \theta \in \left(0, \frac{\pi}{2}\right)\right]$$

155. $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$

$$\Rightarrow \frac{\sin^4 x}{2} + \frac{(1-\sin^2 x)^2}{3} = \frac{1}{5}$$

$$\Rightarrow \frac{\sin^4 x}{2} + \frac{1 + \sin^4 x - 2\sin^2 x}{3} = \frac{1}{5}$$

$$\Rightarrow 5\sin^4 x - 4\sin^2 x + 2 = \frac{6}{5}$$

$$\Rightarrow 25\sin^4 x - 20\sin^2 x + 4 = 0$$

$$\Rightarrow (5\sin^2 x - 2)^2 = 0$$

$$\Rightarrow \sin^2 x = \frac{2}{5}$$

$$\cos^2 x = \frac{3}{5}, \tan^2 x = \frac{2}{3}$$

$$\therefore \frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$$

156. As when $\theta \in \left(0, \frac{\pi}{4}\right)$, $\tan\theta < \cot\theta$

Since, $\tan\theta < 1$ and $\cot\theta > 1$

$\therefore \quad (\tan\theta)^{\cot\theta} < 1$ and $(\cot\theta)^{\tan\theta} > 1$

$\therefore$ $t_4 > t_1$ which only holds in (b).

Therefore, (b) is the answer.

157. Since, $\cos(\alpha - \beta) = 1$

$\Rightarrow \quad \alpha - \beta = 2n\pi$

But $\quad -2\pi < \alpha - \beta < 2\pi \quad$ [as $\alpha, \beta \in (-\pi, \pi)$]

$\therefore \quad \alpha - \beta = 0$ ...(i)

Given, $\cos(\alpha + \beta) = \frac{1}{e}$

$\Rightarrow \cos 2\alpha = \frac{1}{e} < 1$, which is true for four values of α.

[as $-2\pi < 2\alpha < 2\pi$]

158. Given, $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$

$$\Rightarrow 5\left(\frac{1-\cos 2x}{1+\cos 2x} - \frac{1+\cos 2x}{2}\right) = 2\cos 2x + 9$$

Put $\cos 2x = y$, we have

$$5\left(\frac{1-y}{1+y} - \frac{1+y}{2}\right) = 2y + 9$$

$$\Rightarrow 5(2 - 2y - 1 - y^2 - 2y) = 2(1+y)(2y+9)$$

$$\Rightarrow 5(1 - 4y - y^2) = 2(2y + 9 + 2y^2 + 9y)$$

$$\Rightarrow 5 - 20y - 5y^2 = 22y + 18 + 4y^2$$

$$\Rightarrow 9y^2 + 42y + 13 = 0$$

$$\Rightarrow 9y^2 + 3y + 39y + 13 = 0$$

$$\Rightarrow 3y(3y+1) + 13(3y+1) = 0$$

$$\Rightarrow (3y+1)(3y+13) = 0$$

$$\Rightarrow y = -\frac{1}{3}, -\frac{13}{3}$$

$$\therefore \cos 2x = -\frac{1}{3}, -\frac{13}{3}$$

$$\Rightarrow \cos 2x = -\frac{1}{3} \quad \left[\because \cos 2x \neq -\frac{13}{3}\right]$$

Now, $\cos 4x = 2\cos^2 2x - 1 = 2\left(-\frac{1}{3}\right)^2 - 1$

$$= \frac{2}{9} - 1 = -\frac{7}{9}$$

159. $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$

Now, $f_4(x) - f_6(x)$

$$= \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$$

$$= \frac{1}{4}(1 - 2\sin^2 x \cdot \cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x \cdot \cos^2 x)$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

160. Given expression is

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1 + \sin A \cos A}{\sin A \cos A}$$

$$= 1 + \sec A \operatorname{cosec} A$$

161. **Given** A ΔPQR such that

$$3 \sin P + 4 \cos Q = 6 \quad \text{...(i)}$$

$$4 \sin Q + 3 \cos P = 1 \quad \text{...(ii)}$$

On squaring and adding Eqs. (i) and (ii) both sides we get

$$(3 \sin P + 4 \cos Q)^2 + (4 \sin Q + 3 \cos P)^2 = 36 + 1$$

$$\Rightarrow 9(\sin^2 P + \cos^2 P) + 16(\sin^2 Q + \cos^2 Q) + 2 \times 3 \times 4(\sin P \cos Q + \sin Q \cos P) = 37$$

$$\Rightarrow 24[\sin(P + Q)] = 37 - 25$$

$$\Rightarrow \sin(P + Q) = \frac{1}{2}$$

Since, P and Q are angles of ΔPQR, hence $0° < P, Q < 180°$.

$$\Rightarrow P + Q = 30° \text{ or } 150°$$

$$\Rightarrow R = 150° \text{ or } 30°$$

Hence, two cases arise here.

***Case* I** $R = 150°$

$$R = 150° \Rightarrow P + Q = 30°$$

$$\Rightarrow 0 < P, Q < 30°$$

$$\Rightarrow \sin P < \frac{1}{2}, \cos Q < 1$$

$$\Rightarrow 3 \sin P + 4 \cos Q < \frac{3}{2} + 4$$

$$\Rightarrow 3 \sin P + 4 \cos Q < \frac{11}{2} < 6$$

$\Rightarrow$ $3 \sin P + 4 \cos Q \Rightarrow 6$ is not possible.

***Case* II** $R = 30°$

Hence, $R = 30°$ is the only possibility.

162. $A = \sin^2 x + \cos^4 x$

$$\Rightarrow A = 1 - \cos^2 x + \cos^4 x$$

$$= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4}$$

$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4} \quad \text{...(i)}$$

where, $0 \leq \left(\cos^2 x - \frac{1}{2}\right)^2 \leq \frac{1}{4}$ ...(ii)

$$\therefore \frac{3}{4} \leq A \leq 1$$

163. $\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \alpha + \beta \in$ Ist quadrant

and $\sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \alpha - \beta \in$ Ist quadrant

Now, $2\alpha = (\alpha + \beta) + (\alpha - \beta)$

$$\therefore \tan 2\alpha = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33}$$

164. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + 3 = 0$$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta + \sin^2\gamma + \cos^2\gamma = 0$$

$$\Rightarrow (\sin \alpha + \sin \beta + \sin \gamma)^2 + (\cos \alpha + \cos \beta + \cos \gamma)^2 = 0$$

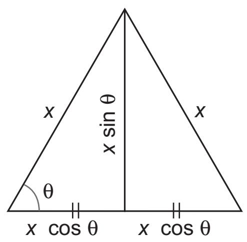
It is possible when,

$$\sin\alpha + \sin\beta + \sin\gamma = 0$$

and $\cos\alpha + \cos\beta + \cos\gamma = 0$

Hence, both statements A and B are true.

165. Area $= \frac{1}{2} \times$ Base $\times$ Altitude



$$= \frac{1}{2} \times (2x \cos\theta) \times (x \sin\theta) = \frac{1}{2}x^2 \sin 2\theta$$

[since, maximum value of $\sin 2\theta$ is 1]

$\therefore$ Maximum area $= \frac{1}{2}x^2$

166. Given, $\cos x + \sin x = \frac{1}{2}$

$$\therefore \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{1}{2}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} = \frac{1}{2}$

$\Rightarrow \quad 2(1 - t^2 + 2t) = 1 + t^2 \Rightarrow 3t^2 - 4t - 1 = 0$

$\Rightarrow \quad t = \dfrac{2 \pm \sqrt{7}}{3}$

As $\quad 0 < x < \pi \Rightarrow 0 < \dfrac{x}{2} < \dfrac{\pi}{2}$

So, $\tan \dfrac{x}{2}$ is positive.

$\therefore \quad t = \tan \dfrac{x}{2} = \dfrac{2 + \sqrt{7}}{3}$

Now, $\quad \tan x = \dfrac{2 \tan \dfrac{x}{2}}{1 - \tan^2 \dfrac{x}{2}} = \dfrac{2t}{1 - t^2}$

$$\Rightarrow \quad \tan x = \frac{2\left(\dfrac{2 + \sqrt{7}}{3}\right)}{1 - \left(\dfrac{2 + \sqrt{7}}{3}\right)^2}$$

$$\Rightarrow \quad \tan x = \frac{-3\,(2 + \sqrt{7})}{1 + 2\sqrt{7}} \times \frac{1 - 2\sqrt{7}}{1 - 2\sqrt{7}}$$

$$\Rightarrow \quad \tan x = -\left(\frac{4 + \sqrt{7}}{3}\right)$$

167. Since, $\tan \dfrac{P}{2}$ and $\tan \dfrac{Q}{2}$ are the roots of equation

$$ax^2 + bx + c = 0$$

$\therefore \quad \tan \dfrac{P}{2} + \tan \dfrac{Q}{2} = -\dfrac{b}{a}$ ...(i)

and $\quad \tan \dfrac{P}{2} \tan \dfrac{Q}{2} = \dfrac{c}{a}$

Also, $\quad \dfrac{P}{2} + \dfrac{Q}{2} + \dfrac{R}{2} = \dfrac{\pi}{2}$ $\quad [\because P + Q + R = \pi]$

$\Rightarrow \quad \dfrac{P + Q}{2} = \dfrac{\pi}{2} - \dfrac{R}{2}$

$\Rightarrow \quad \dfrac{P + Q}{2} = \dfrac{\pi}{4}$ $\quad [\because \angle R = \dfrac{\pi}{2}$ (given)$]$

$\Rightarrow \quad \tan\left(\dfrac{P}{2} + \dfrac{Q}{2}\right) = 1$

$$\Rightarrow \quad \frac{\tan \dfrac{P}{2} + \tan \dfrac{Q}{2}}{1 - \tan \dfrac{P}{2} \tan \dfrac{Q}{2}} = 1$$

$$\Rightarrow \quad \frac{-\dfrac{b}{a}}{1 - \dfrac{c}{a}} = 1$$

$\Rightarrow \quad -\dfrac{b}{a} = 1 - \dfrac{c}{a}$ $\quad$ [from Eq. (i)]

$\Rightarrow \quad -b = a - c$

$\Rightarrow \quad c = a + b$

Alternate Solution

Since, $\quad \angle R = \dfrac{\pi}{2}$

$\Rightarrow \quad \angle P + \angle Q = \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{\angle P}{2} = \dfrac{\pi}{4} - \dfrac{\angle Q}{2}$

$$\therefore \quad \tan \frac{P}{2} = \tan\left(\frac{\pi}{4} - \frac{Q}{2}\right) = \frac{\tan \dfrac{\pi}{4} - \tan \dfrac{Q}{2}}{1 + \tan \dfrac{\pi}{4} \tan \dfrac{Q}{2}}$$

$\Rightarrow \quad \tan \dfrac{P}{2} + \tan \dfrac{P}{2} \tan \dfrac{Q}{2} = 1 - \tan \dfrac{Q}{2}$

$\Rightarrow \quad \tan \dfrac{P}{2} + \tan \dfrac{Q}{2} = 1 - \tan \dfrac{P}{2} \tan \dfrac{Q}{2}$

$\Rightarrow \quad -\dfrac{b}{a} = 1 - \dfrac{c}{a}$

$\Rightarrow \quad -b = a - c$

$\Rightarrow \quad c = a + b$