CHAPTER 2 Complex Numbers

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. If the expression *(1987 - 2 Marks)*

$$\frac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)+i\tan(x)\right]}{\left[1+2\,i\,\sin\left(\frac{x}{2}\right)\right]}$$

is real, then the set of all possible values of x is ....

2. For any two complex numbers z_1, z_2 and any real number a and b. *(1988 - 2 Marks)*
$|az_1-bz_2|^2+|bz_1+az_2|^2=$ ....
3. If a, b, c, are the numbers between 0 and 1 such that the points $z_1=a+i, z_2=1+bi$ and $z_3=0$ form an equilateral triangle, then $a=$ ....and $b=$ .... *(1989 - 2 Marks)*
4. *ABCD* is a rhombus. Its diagonals *AC* and *BD* intersect at the point *M* and satisfy *BD* = 2*AC*. If the points *D* and *M* represent the complex numbers 1 + *i* and 2 - *i* respectively, then A represents the complex number ....or.......... *(1993 - 2 Marks)*
5. Suppose Z_1, Z_2, Z_3 are the vertices of an equilateral triangle inscribed in the circle $|Z|=2$. If $Z_1=1+i\sqrt{3}$ then $Z_2=$ ...., $Z_3=$ .... *(1994 - 2 Marks)*
6. The value of the expression $1\cdot(2-\omega)(2-\omega^2)+2\cdot(3-\omega)(3-\omega^2)+....+(n-1).(n-\omega)(n-\omega^2)$, where ω is an imaginary cube root of unity, is..... *(1996 - 2 Marks)*

B True / False

1. For complex number $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, we write $z_1\cap z_2$, if $x_1\le x_2$ and $y_1\le y_2$. Then for all complex numbers z with $1\cap z$, we have $\frac{1-z}{1+z}\cap\theta$. *(1981 - 2 Marks)*
2. If the complex numbers, Z_1, Z_2 and Z_3 represent the vertices of an equilateral triangle such that $|Z_1|=|Z_2|=|Z_3|$ then $Z_1+Z_2+Z_3=0$. *(1984 - 1 Mark)*
3. If three complex numbers are in A.P. then they lie on a circle in the complex plane. *(1985 - 1 Mark)*
4. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. *(1988 - 1 Mark)*

C MCQs with One Correct Answer

1. If the cube roots of unity are 1, ω, ω^2, then the roots of the equation $(x-1)^3+8=0$ are *(1979)*
(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$
(c) $-1,-1,-1$ (d) None of these
2. The smallest positive integer n for which *(1980)*
$\left(\frac{1+i}{1-i}\right)^n=1$ is
(a) $n=8$ (b) $n=16$
(c) $n=12$ (d) none of these
3. The complex numbers $z=x+iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right|=1$ lie on *(1981 - 2 Marks)*
(a) the x-axis
(b) the straight line $y=5$
(c) a circle passing through the origin
(d) none of these
4. If $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then *(1982 - 2 Marks)*
(a) $Re(z)=0$ (b) $Im(z)=0$
(c) $Re(z)>0, Im(z)>0$ (d) $Re(z)>0, Im(z)<0$
5. The inequality $|z-4|<|z-2|$ represents the region given by *(1982 - 2 Marks)*
(a) $Re(z)\ge 0$ (b) $Re(z)<0$
(c) $Re(z)>0$ (d) none of these
6. If $z=x+iy$ and $\omega=(1-iz)/(z-i)$, then $|\omega|=1$ implies that, in the complex plane, *(1983 - 1 Mark)*
(a) *z* lies on the imaginary axis
(b) *z* lies on the real axis
(c) *z* lies on the unit circle
(d) None of these

7. The points $z_1, z_2, z_3 z_4$ in the complex plane are the vertices of a parallelogram taken in order if and only if *(1983 - 1 Mark)*

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

8. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles *(1985 - 2 Marks)*

(a) have the same area (b) are similar
(c) are congruent (d) none of these

9. If $\omega\ (\neq 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$ then A and B are respectively *(1995S)*

(a) 0, 1 (b) 1, 1 (c) 1, 0 (d) $-1, 1$

10. Let z and ω be two non zero complex numbers such that $|z| = |\omega|$ and $\text{Arg}\, z + \text{Arg}\, \omega = \pi$, then z equals *(1995S)*

(a) ω (b) $-\omega$ (c) $\overline{\omega}$ (d) $-\overline{\omega}$

11. Let z and ω be two complex numbers such that $|z| \leq 1$, $|\omega| \leq 1$ and $|z + i\omega| = |z - i\overline{\omega}| = 2$ then z equals *(1995S)*

(a) 1 or i (b) i or $-i$ (c) 1 or -1 (d) i or -1

12. For positive integers n_1, n_2 the value of the expression

$(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$ is a real number if and only if *(1996 - 1 Marks)*

(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
(c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$

13. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$

is equal to *(1999 - 2 Marks)*

(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$ (c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

14. If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$ *(2000S)*

(a) π (b) $-\pi$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

15. If z_1, z_2 and z_3 are complex numbers such that *(2000S)*

$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is

(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

16. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form *(2001S)*

(a) $4k + 1$ (b) $4k + 2$ (c) $4k + 3$ (d) $4k$

17. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is *(2001S)*

(a) of area zero
(b) right-angled isosceles
(c) equilateral
(d) obtuse-angled isosceles

18. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is *(2002S)*

(a) 0 (b) 2 (c) 7 (d) 17

19. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), then $\text{Re}(\omega)$ is *(2003S)*

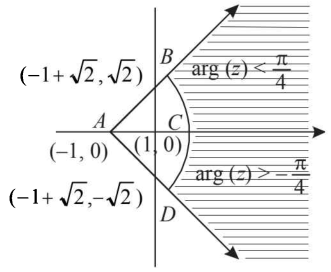
(a) 0 (b) $-\frac{1}{|z+1|^2}$

(c) $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

20. If $\omega\ (\neq 1)$ be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is *(2004S)*

(a) 2 (b) 3 (c) 5 (d) 6

21. The locus of z which lies in shaded region (excluding the boundaries) is best represented by *(2005S)*



(a) $z : |z + 1| > 2$ and $|\arg(z+1)| < \pi/4$
(b) $z : |z - 1| > 2$ and $|\arg(z-1)| < \pi/4$
(c) $z : |z + 1| < 2$ and $|\arg(z+1)| < \pi/2$
(d) $z : |z - 1| < 2$ and $|\arg(z+1)| < \pi/2$

22. a, b, c are integers, not all simultaneously equal and ω is cube root of unity $(\omega \neq 1)$, then minimum value of $|a + b\omega + c\omega^2|$ is *(2005S)*

(a) 0 (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

23. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then the value of the det.

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is *(2002 - 2 Marks)*

(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

24. If $\frac{w - \overline{w}z}{1 - z}$ is purely real where $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$, then the set of the values of z is *(2006 - 3M, –1)*

(a) $\{z : |z| = 1\}$ (b) $\{z : z = \overline{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$

25. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is ***(2007 -3 marks)***

(a) $3e^{i\pi/4} + 4i$ (b) $(3-4i)e^{i\pi/4}$
(c) $(4+3i)e^{i\pi/4}$ (d) $(3+4i)e^{i\pi/4}$

26. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\dfrac{z}{1-z^2}$ lie on

(a) a line not passing through the origin ***(2007 -3 marks)***
(b) $|z| = \sqrt{2}$
(c) the x-axis
(d) the y-axis

27. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\dfrac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by ***(2008)***

(a) $6+7i$ (b) $-7+6i$
(c) $7+6i$ (d) $-6+7i$

28. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is ***(2009)***

(a) $\dfrac{1}{\sin 2°}$ (b) $\dfrac{1}{3\sin 2°}$ (c) $\dfrac{1}{2\sin 2°}$ (d) $\dfrac{1}{4\sin 2°}$

29. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation : $z\bar{z}^3 + \bar{z}z^3 = 350$ is ***(2009)***

(a) 48 (b) 32 (c) 40 (d) 80

30. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then a cannot take the value ***(2012)***

(a) -1 (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{3}{4}$

31. Let complex numbers α and $\dfrac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2 + (y-y_0)^2 = r^2$ and $(x-x_0)^2 + (y-y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ ***(JEE Adv. 2013)***

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sqrt{7}}$ (d) $\dfrac{1}{3}$

D MCQs with One or More than One Correct

1. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies – ***(1985 - 2 Marks)***

(a) $|w_1| = 1$ (b) $|w_2| = 1$
(c) $\text{Re}(w_1\bar{w}_2) = 0$ (d) none of these

2. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\dfrac{z_1 + z_2}{z_1 - z_2}$ may be ***(1986 - 2 Marks)***

(a) zero (b) real and positive
(c) real and negative (d) purely imaginary
(e) none of these.

3. If z_1 and z_2 are two nonzero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then Arg z_1 – Arg z_2 is equal to ***(1987 - 2 Marks)***

(a) $-\pi$ (b) $-\dfrac{\pi}{2}$ (c) 0 (d) $\dfrac{\pi}{2}$
(e) π

4. The value of $\sum_{k=1}^{6}\left(\sin\dfrac{2\pi k}{7} - i\cos\dfrac{2\pi k}{7}\right)$ is ***(1987 - 2 Marks)***

(a) -1 (b) 0 (c) $-i$ (d) i
(e) None

5. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals ***(1998 - 2 Marks)***

(a) 128ω (b) -128ω (c) $128\omega^2$ (d) $-128\omega^2$

6. The value of the sum $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals ***(1998 - 2 Marks)***

(a) i (b) $i-1$ (c) $-i$ (d) 0

7. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then ***(1998 - 2 Marks)***

(a) $x=3, y=2$ (b) $x=1, y=3$
(c) $x=0, y=3$ (d) $x=0, y=0$

8. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a non-zero complex number w, then ***(2010)***

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(b) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$
(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix}$
(d) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

9. Let $w=\frac{\sqrt{3}+i}{2}$ and $P=\{w^n : n=1,2,3,\ldots\}$. Further $H_1=\left\{z\in\mathbb{C} : \operatorname{Re} z>\frac{1}{2}\right\}$ and $H_2=\left\{z\in\mathbb{C} : \operatorname{Re} z<\frac{-1}{2}\right\}$, where c is the set of all complex numbers. If $z_1 \in P\cap H_1, z_2 \in P\cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **(JEE Adv. 2013)**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

10. Let $a, b \in \mathbb{R}$ and $a^2+b^2 \neq 0$.

Suppose $S=\left\{Z\in C : Z=\frac{1}{a+ibt}, +\in\mathbb{R}, t\neq 0\right\}$, where $i=\sqrt{-1}$. If $z = x+iy$ and $z \in S$, then (x, y) lies on **(JEE Adv. 2016)**

(a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a},0\right)$ for $a>0$, $b\neq 0$

(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a},0\right)$ for $a<0$, $b\neq 0$

(c) the x-axis for $a\neq 0$, $b=0$

(d) the y-axis for $a=0$, $b\neq 0$

E Subjective Problems

1. Express $\frac{1}{1-\cos\theta+2i\sin\theta}$ in the form $x+iy$. **(1978)**

2. If $x=a+b, y=a\gamma+b\beta$ and $z=a\beta+b\gamma$ where γ and β are the complex cube roots of unity, show that $xyz=a^3+b^3$. **(1978)**

3. If $x+iy=\sqrt{\frac{a+ib}{c+id}}$, prove that $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$. **(1979)**

4. Find the real values of x and y for which the following equation is satisfied $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ **(1980)**

5. Let the complex number z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2+z_2^2+z_3^2=3z_0^2$. **(1981 - 4 Marks)**

6. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2+z_2^2-z_1z_2=0$. **(1983 - 3 Marks)**

7. If $1, a_1, a_2, \ldots\ldots, a_{n-1}$ are the n roots of unity, then show that $(1-a_1)(1-a_2)(1-a_3)\ldots.(1-a_{n-1})=n$ **(1984 - 2 Marks)**

8. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z+iz$ is $\frac{1}{2}|z|^2$. **(1986 - 2½ Marks)**

9. Let $Z_1=10+6i$ and $Z_2=4+6i$. If Z is any complex number such that the argument of $\frac{(Z-Z_1)}{(Z-Z_2)}$ is $\frac{\pi}{4}$, then prove that $|Z-7-9i|=3\sqrt{2}$. **(1990 - 4 Marks)**

10. If $iz^3+z^2-z+i=0$, then show that $|z|=1$. **(1995 - 5 Marks)**

11. If $|Z|\leq 1, |W|\leq 1$, show that

$$|Z-W|^2 \leq (|Z|-|W|)^2 + (Arg\,Z - Arg\,W)^2$$

(1995 - 5 Marks)

12. Find all non-zero complex numbers Z satisfying $\overline{Z}=iZ^2$. **(1996 - 2 Marks)**

13. Let z_1 and z_2 be roots of the equation $z^2+pz+q=0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB=\alpha\neq 0$ and $OA=OB$, where O is the origin, prove that

$p^2=4q\cos^2\left(\frac{\alpha}{2}\right)$. **(1997 - 5 Marks)**

14. For complex numbers z and w, prove that $|z|^2w-|w|^2z=z-w$ if and only if $z=w$ or $z\overline{w}=1$. **(1999 - 10 Marks)**

15. Let a complex number α, $\alpha\neq 1$, be a root of the equation $z^{p+q}-z^p-z^q+1=0$, where p, q are distinct primes. Show that either $1+\alpha+\alpha^2+\ldots.+\alpha^{p-1}=0$ or $1+\alpha+\alpha^2+\ldots+\alpha^{q-1}=0$, but not both together. **(2002 - 5 Marks)**

16. If z_1 and z_2 are two complex numbers such taht $|z_1|<1<|z_2|$ then prove that $\left|\frac{1-z_1\overline{z}_2}{z_1-z_2}\right|<1$. **(2003 - 2 Marks)**

17. Prove that there exists no complex number z such that $|z|<\frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r|<2$. **(2003 - 2 Marks)**

18. Find the centre and radius of circle given by

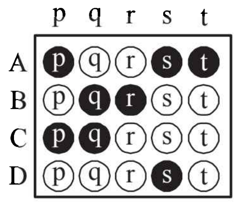
$$\left|\frac{z-\alpha}{z-\beta}\right|=k, k\neq 1$$

where, $z=x+iy$, $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$ **(2004 - 2 Marks)**

19. If one the vertices of the square circumscribing the circle $|z-1|=\sqrt{2}$ is $2+\sqrt{3}\,i$. Find the other vertices of the square. **(2005 - 4 Marks)**

F Match the Following

DIRECTIONS (Q. 1 and 2) : *Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :*
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. $z \neq 0$ is a complex number *(1992 - 2 Marks)*

Column I	**Column II**
(A) $\text{Re}\, z = 0$	(p) $\text{Re}\, z^2 = 0$
(B) $\text{Arg}\, z = \dfrac{\pi}{4}$	(q) $\text{Im}\, z^2 = 0$
	(r) $\text{Re}\, z^2 = \text{Im}\, z^2$

2. Match the statements in **Column I** with those in **Column II**. *(2010)*
[Note : Here z takes values in the complex plane and Im z and Re z denote , respectively, the imaginary part and the real part of z.]

Column I	**Column II**
(A) The set of points z satisfying $\|z - i\|z\|\| = \|z + i\|z\|\|$ is contained in or equal to	(p) an ellipse with eccentricity $\dfrac{4}{5}$
	(q) the set of points z satisfying $\text{Im}\, z = 0$
(B) The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$ is contained in or equal to	(r) the set of points z satisfying $\|\text{Im}\, z\| \leq 1$
(C) If $\|w\| = 2$, then the set of points $z = w - \dfrac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\|\text{Re}\, z\| < 2$
(D) If $\|w\| = 1$, then the set of points $z = w + \dfrac{1}{w}$ is contained in or equal to.	(t) the set of points z satisfying $\|z\| \leq 3$

DIRECTIONS (Q. 3) : *Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

3. Let $z_k = \cos\left(\dfrac{2k\pi}{10}\right) + i\sin\left(\dfrac{2k\pi}{10}\right)$; $k = 1, 2, ..., 9.$ *(JEE Adv. 2014)*

List-I	**List-II**
P. For each z_k there exists as z_j such that $z_k . z_j = 1$	**1.** True
Q. There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 . z = z_k$ has no solution z in the set of complex numbers	**2.** False
R. $\dfrac{\|1 - z_1\|\|1 - z_2\|....\|1 - z_9\|}{10}$ equals	**3.** 1
S. $1 - \sum_{k=1}^{9} \cos\left(\dfrac{2k\pi}{10}\right)$ equals	**4.** 2

	P	Q	R	S
(a)	1	2	4	3
(b)	2	1	3	4
(c)	1	2	3	4
(d)	2	1	4	3

G Comprehension Based Questions

PASSAGE-1

Let A, B, C be three sets of complex numbers as defined below

$$A = \{z : \text{Im } z \geq 1\}$$

$$B = \{z : |z - 2 - i| = 3\}$$

$$C = \{z : \text{Re}((1-i)z) = \sqrt{2}\}$$

1. The number of elements in the set $A \cap B \cap C$ is *(2008)*
   (a) 0 (b) 1 (c) 2 (d) ∞
2. Let z be any point in $A \cap B \cap C$.
   Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between *(2008)*
   (a) 25 and 29 (b) 30 and 34
   (c) 35 and 39 (d) 40 and 44
3. Let z be any point $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between
   (a) –6 and 3 (b) –3 and 6 *(2008)*
   (c) –6 and 6 (d) –3 and 9

PASSAGE-2

Let $S = S_1 \cap S_2 \cap S_3$, where

$$S_1 = \{z \in \mathbb{C} : |z| < 4\},\ S_2 = \left\{z \in \mathbb{C} : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$$

and $S_3 = \{z \in \mathbb{C} : \text{Re } z > 0\}$.

4. Area of $S =$ *(JEE Adv. 2013)*
   (a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$ (c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$
5. $\min_{z \in S} |1 - 3i - z| =$ *(JEE Adv. 2013)*
   (a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$
   (c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

I Integer Value Correct Type

1. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is *(2011)*
2. Let $\omega = e^{\frac{i\pi}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that *(2011)*
   $a + b + c = x$
   $a + b\omega + c\omega^2 = y$
   $a + b\omega^2 + c\omega = z$
   Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is
3. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$ is *(JEE Adv. 2015)*

Section-B JEE Main / AIEEE

1. z and w are two nonzero complex numbers such that $|z| = |w|$ and Arg z + Arg w = π then z equals **[2002]**
   (a) $\overline{\omega}$ (b) $-\overline{\omega}$ (c) ω (d) $-\omega$
2. If $|z - 4| < |z - 2|$, its solution is given by **[2002]**
   (a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$
   (c) $\text{Re}(z) > 3$ (d) $\text{Re}(z) > 2$
3. The locus of the centre of a circle which touches the circle $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 & z_2 are complex numbers) will be **[2002]**
   (a) an ellipse (b) a hyperbola
   (c) a circle (d) none of these
4. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$ and $Arg(z) - Arg(\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to **[2003]**
   (a) $-i$ (b) 1 (c) -1 (d) i
5. Let Z_1 and Z_2 be two roots of the equation $Z^2 + aZ + b = 0$, Z being complex. Further, assume that the origin, Z_1 and Z_2 form an equilateral triangle. Then **[2003]**
   (a) $a^2 = 4b$ (b) $a^2 = b$ (c) $a^2 = 2b$ (d) $a^2 = 3b$
6. If $\left(\frac{1+i}{1-i}\right)^x = 1$ then **[2003]**
   (a) $x = 2n + 1$, where n is any positive integer
   (b) $x = 4n$, where n is any positive integer
   (c) $x = 2n$, where n is any positive integer
   (d) $x = 4n + 1$, where n is any positive integer.
7. Let z and w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg $zw = \pi$. Then arg z equals **[2004]**
   (a) $\frac{5\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{4}$

8. If $z = x - i\,y$ and $z^{\frac{1}{3}} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{q}\right) \Big/ (p^2 + q^2)$ is equal to **[2004]**

(a) –2 (b) –1 (c) 2 (d) 1

9. If $|z^2 - 1| = |z|^2 + 1$, then z lies on **[2004]**

(a) an ellipse (b) the imaginary axis
(c) a circle (d) the real axis

10. If the cube roots of unity are **1**, ω, ω^2 then the roots of the equation $(x-1)^3 + 8 = 0$, are **[2005]**

(a) $-1, -1+2\omega, -1-2\omega^2$
(b) $-1, -1, -1$
(c) $-1, 1-2\omega, 1-2\omega^2$
(d) $-1, 1+2\omega, 1+2\omega^2$

11. If z_1 and z_2 are two non- zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg z_1 – arg z_2 is equal to **[2005]**

(a) $\frac{\pi}{2}$ (b) $-\pi$ (c) 0 (d) $\frac{-\pi}{2}$

12. If $\omega = \frac{z}{z - \frac{1}{3}i}$ and $|\omega| = 1$, then z lies on **[2005]**

(a) an ellipse (b) a circle
(c) a straight line (d) a parabola

13. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$ is **[2006]**

(a) i (b) 1 (c) –1 (d) $-i$

14. If $z^2 + z + 1 = 0$, where z is complex number, then the value of

$\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 + ... + \left(z^6+\frac{1}{z^6}\right)^2$ is **[2006]**

(a) 18 (b) 54
(c) 6 (d) 12

15. If $|z + 4| \le 3$, then the maximum value of $|z+1|$ is **[2007]**

(a) 6 (b) 0 (c) 4 (d) 10

16. The conjugate of a complex number is $\frac{1}{i-1}$ then that complex number is **[2008]**

(a) $\frac{-1}{i-1}$ (b) $\frac{1}{i+1}$ (c) $\frac{-1}{i+1}$ (d) $\frac{1}{i-1}$

17. Let R be the real line. Consider the following subsets of the plane $R \times R$:
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
$T = \{(x, y) : x - y \text{ is an integer}\}$,
Which one of the following is true? **[2008]**

(a) Neither S nor T is an equivalence relation on R
(b) Both S and T are equivalence relation on R
(c) S is an equivalence relation on R but T is not
(d) T is an equivalence relation on R but S is not

18. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$ equals **[2010]**

(a) 1 (b) 2 (c) ∞ (d) 0

19. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that : **[2011]**

(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
(c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$

20. If $\omega(\ne 1)$ is a cube root of unity, and $(1+\omega)^7 = A + B\omega$. Then (A, B) equals **[2011]**

(a) (1, 1) (b) (1, 0)
(c) (–1, 1) (d) (0, 1)

21. If $z \ne 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[2012]**

(a) either on the real axis or on a circle passing through the origin.
(b) on a circle with centre at the origin
(c) either on the real axis or on a circle not passing through the origin.
(d) on the imaginary axis.

22. If z is a complex number of unit modulus and argument θ, then arg $\left(\frac{1+z}{1+\bar{z}}\right)$ equals: **[JEE M 2013]**

(a) $-\theta$ (b) $\frac{\pi}{2} - \theta$ (c) θ (d) $\pi - \theta$

23. If z is a complex number such that $|z| \ge 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$: **[JEE M 2014]**

(a) is strictly greater than $\frac{5}{2}$
(b) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$
(c) is equal to $\frac{5}{2}$
(d) lie in the interval (1, 2)

24. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a: **[JEE M 2015]**

(a) circle of radius 2.

(b) circle of radius $\sqrt{2}$.

(c) straight line parallel to x-axis

(d) straight line parallel to y-axis.

25. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is: **[JEE M 2016]**

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (b) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

2 Complex Numbers

Section-A : JEE Advanced/ IIT-JEE

A
1. $2n\pi,\ n\pi+\frac{\pi}{4}$
2. $(a^2+b^2)(|z_1|^2+|z_2|^2)$
3. $2-\sqrt{3},\ 2-\sqrt{3}$
4. $3-\frac{i}{2}$ or $1-\frac{3}{2}i$
5. $-2,\ 1-i\sqrt{3}$
6. $\frac{1}{4}n(n-1)(n^2+3n+4)$

B	**1.** T	**2.** T	**3.** F	**4.** T		
C	**1.** (b)	**2.** (d)	**3.** (a)	**4.** (b)	**5.** (d)	**6.** (b)
	7. (b)	**8.** (b)	**9.** (b)	**10.** (d)	**11.** (c)	
	12. (d)	**13.** (c)	**14.** (a)	**15.** (a)	**16.** (d)	**17.** (c)
	18. (b)	**19.** (a)	**20.** (b)	**21.** (a)	**22.** (b)	**23.** (b)
	24. (d)	**25.** (d)	**26.** (d)	**27.** (d)	**28.** (d)	**29.** (a)
	30. (d)	**31.** (c)				
D	**1.** (a, b, c)	**2.** (a, d)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (b)
	7. (d)	**8.** (a, c, d)	**9.** (c, d)	**10.** (a, c, d)		

E
1. $\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$
4. $x=3, y=-1$
12. $i,\ \frac{\pm\sqrt{3}}{2}-\frac{i}{2}$
18. centre $=\frac{\alpha-k^2\beta}{1-k^2}$, radius $=\frac{k}{|1-k^2|}|\alpha-\beta|$
19. $(1-\sqrt{3})+i,\ -i\sqrt{3},\ (\sqrt{3}+1)-i$

F 1. (A) - q ; (B) - p **2.** (A) - q, r ; (B) - p; (C) - p, s, t; (D) - q, r, s, t **3.** (c)

G 1. (b) **2.** (c) **3.** (d) **4.** (b) **5.** (c)

I 1. 5 **2.** 3 **3.** 4

Section-B : JEE Main/ AIEEE

1. (b)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (d)	**6.** (b)
7. (c)	**8.** (a)	**9.** (b)	**10.** (c)	**11.** (c)	**12.** (c)
13. (d)	**14.** (d)	**15.** (a)	**16.** (c)	**17.** (d)	**18.** (a)
19. (c)	**20.** (a)	**21.** (a)	**22.** (c)	**23.** (b)	**24.** (a)
25. (b)					

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Let $z=\frac{\sin x/2+\cos x/2+i\tan x}{1+2i\sin x/2}$

$$=\frac{(\sin x/2+\cos x/2+i\tan x)(1-2i\sin x/2)}{(1+2i\sin x/2)(1-2i\sin x/2)}$$

$$=\frac{[\sin x/2+\cos x/2+2\sin x/2\tan x)+i(\tan x-2\sin^2 x/2-2\sin x/2\cos x/2)]}{(1+4\sin^2 x/2)}$$

But ATQ, $I_m(z)=0$ (as z is real)

$$\Rightarrow \tan x-2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=0$$

$$\Rightarrow \frac{\sin x}{\cos x}-2\sin^2 x/2-2\sin x/2\cos x/2=0$$

$$\Rightarrow \frac{\sin x}{\cos x}-(1-\cos x)-\sin x=0$$

$\Rightarrow \quad \sin x\left[\frac{1}{\cos x}-1\right]-[1-\cos x]=0$

$\Rightarrow \quad \left(\frac{1-\cos x}{\cos x}\right)\sin x-[1-\cos x]=0$

$\Rightarrow \quad (1-\cos x)\left(\frac{\sin x}{\cos x}-1\right)=0$

$\Rightarrow \quad \cos x=1 \quad \Rightarrow \quad x=2n\pi$ and

$\tan x=1 \quad \Rightarrow \quad x=n\pi+\pi/4$

$\therefore \quad x=2n\pi,\ n\pi+\pi/4$ **Ans.**

2. $|az_1-bz_2|^2+|bz_1+az_2|^2$

$= t^2|z_1|^2+b^2|z_2|^2-2ab\ \mathrm{Re}(z_1\bar{z}_2)+b^2|z_1|^2$

$+a^2|z_2|^2+2ab\ \mathrm{Re}(z_1\bar{z}_2)$

$=(a^2+b^2)(|z_1|^2+|z_2|^2)$

3. **KEY CONCEPT :** $|z_1-z_2|$ = distance between two points represented by z_1 and z_2.

As $z_1=a+i$, $z_2=1+bi$ and $z_3=0$ form an equilateral triangle, therefore $|z_1-z_3|=|z_2-z_3|=|z_1-z_2|$

$\Rightarrow \quad |a+i|=|1+bi|=|(a-1)+i(1-b)|$

$\Rightarrow \quad a^2+1=1+b^2=(a-1)^2+(1-b)^2$

$\Rightarrow \quad a^2=b^2=a^2+b^2-2a-2b+1$

$\Rightarrow \quad a=b$ ....(1)

($\because$ $a,b>0$ $\therefore$ $a\neq -b$) and

$b^2-2a-2b+1=0$

or $\quad a^2-2a-2b+1=0$ ....(2)

$\Rightarrow \quad a^2-2a-2a+1=0 \qquad [\because a=b]$

$\Rightarrow \quad a^2-4a+1=0$

$\Rightarrow \quad a=\frac{4\pm 2\sqrt{3}}{2}=2\pm\sqrt{3}$ But $0<a,b<1$

$\therefore \quad a=2-\sqrt{3}$ also $b=2-\sqrt{3}$

4. If we see the problem as in co-ordinate geometry we have $D\equiv(1,1)$ and $M\equiv(2,-1)$

We know that diagonals of rhombus bisect each other at 90°

$\therefore$ AC passes through M and is $\perp$ to BD

$\therefore$ Eq. of AC in symmetric form can be written as

$$\frac{x-2}{2/\sqrt{5}}=\frac{y+1}{1/\sqrt{5}}=r$$

Now for pt. A, as

$$AM=\frac{1}{2}DM=\frac{1}{2}\sqrt{(2-1)^2+(-1-1)^2}=\sqrt{5}/2$$

Putting $r=\pm\sqrt{5}/2$ *we get*,

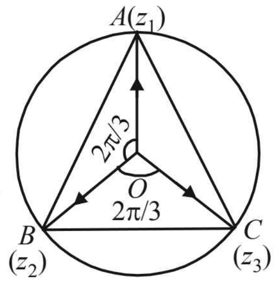
$$\frac{x-2}{2/\sqrt{5}}=\frac{y+1}{1/\sqrt{5}}=\pm\sqrt{5}/2$$

$\Rightarrow \quad x=\pm 1+2,\ y=\pm\frac{1}{2}-1$

$\Rightarrow \quad x=3$ or $1, y=\frac{-1}{2}$ *or* $\frac{-3}{2}$

$\therefore$ Pt. A is $3-i/2$ or $1-(3/2)i$

5. Let z_1, z_2, z_3 be the vertices A, B and C respectively of equilateral ΔABC, inscribed in a circle $|z|=2$, centre $(0,0)$ rasius $=2$



Given $\quad z_1=1+i\sqrt{3}$

$z_2=e^{\frac{2\pi i}{3}}z_1$

$=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)(1+i\sqrt{3})$

$=\frac{-1-3}{2}=-2$

and $z_3=e^{4(\pi/3)i}z_1$

$=\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)(1+i\sqrt{3})$

$=\left(\frac{-1-i\sqrt{3}}{2}\right)(1+i\sqrt{3}) = \frac{-1-2i\sqrt{3}+3}{2}=1-i\sqrt{3}$

6. rth term of the given series,

$=r[(r+1)-\omega](r+1)-\omega^2]$

$=r[(r+1)^2-(\omega+\omega^2)(r+1)+\omega^3]$

$=r[(r+1)^2-(-1)(r+1)+1]$

$=r[(r^2+3r+3]=r^3+3r^2+3r$

Thus, sum of the given series,

$$=\sum_{r=1}^{(n-1)}(\mathrm{r}^3+3r^2+3r)$$

$=\frac{1}{4}(n-1)^2n^2+3.\frac{1}{6}(n-1)(n)(2n-1)+3.\frac{1}{2}(n-1)n$

$=(n-1)(n)\left[\frac{1}{4}(n-1)n+\frac{1}{2}(2n-1)+\frac{3}{2}\right]$

$=\frac{1}{4}(n-1)\,n[n^2-n+4n-2+6]$

$=\frac{1}{4}(n-1)\,n[n^2+3n+4]$

B. True / False

1. Let $z = x + iy$

then $1 \cap z \Rightarrow 1 \le x \,\&\, 0 \le y$ (by def.)

Consider

$$\frac{1-z}{1+z} = \frac{1-(x+iy)}{1+(x+iy)} = \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{iy(1-x+1+x)}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2}$$

$$\frac{1-z}{1+z} \cap 0 \quad \Rightarrow \quad \frac{1-x^2-y^2}{(1+x)^2+y^2} \le 0$$

and $\dfrac{-2y}{(1+x)^2+y^2} \le 0$

$\Rightarrow \ 1-x^2-y^2 \le 0$ and $-2y \le 0$

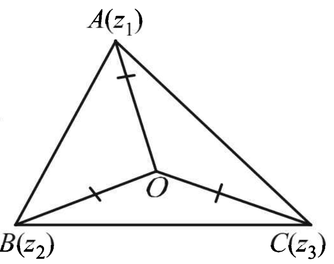
$\Rightarrow \ x^2+y^2 \ge 1$ and $y \ge 0$ which is true as

$x \ge 1 \ \& \ y \ge 0$

$\therefore$ The given statement is true $\forall \ z \in C$.

2. As $|z_1| = |z_2| = |z_3|$

$\therefore$ z_1, z_2, z_3 are equidistant from origin. Hence O is the circumcentre of ΔABC.



But according to question ΔABC is equilateral and we know that in an equilateral Δ circumcentre and centriod coincide.

$\therefore$ Centriod of $\Delta ABC = 0$

$\Rightarrow \ \dfrac{z_1+z_2+z_3}{3} = 0 \Rightarrow z_1+z_2+z_3 = 0$

$\therefore$ Statement is true.

3. If z_1, z_2, z_3 are in A.P. then, $\dfrac{z_1+z_3}{2} = z_2$

$\Rightarrow$ z_2 is mid pt. of line joining z_1 and z_3.

$\Rightarrow$ z_1, z_2, z_3 lie on a st. line

$\therefore$ Given statement is false

4. $\because$ Cube roots of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-\sqrt{3}}{2}$

$\therefore$ Vertices of triangle are

$$A(1,0), B\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), C\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$$

$\Rightarrow AB = BC = CA \quad \therefore \Delta$ is equilateral.

C. MCQs with ONE Correct Answer

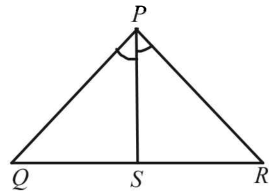
1. **(b)** $(x-1)^3 + 8 = 0$

$\Rightarrow \ (x-1)^3 = -8 = (-2)^3$

$\Rightarrow \ x - 1 = -2$

or -2ω or $-2\omega^2$

$\Rightarrow \ x = -1, 1-2\omega, 1-2\omega^2$



2. **(d)** $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)\,(1+i)} = \dfrac{1-1+2i}{2} = i$

Now $i^n = 1 \Rightarrow$ the smallest positive integral value of n should be 4.

3. **(a)** ATQ $|x+iy-5i| = |x+iy+5i|$

$\Rightarrow \ |x+(y-5)\,i| = |x+(y+5)\,i|$

$\Rightarrow \ x^2+(y-5)^2 = x^2+(y+5)^2$

$\Rightarrow \ x^2+y^2-10y+25 = x^2+y^2+10y+25$

$\Rightarrow \ 20y = 0 \ \Rightarrow y = 0$

$\therefore$ 'a' is the correct alternative.

4. **(b)** $\left(\dfrac{\sqrt{3}}{2}+\dfrac{i}{2}\right) = -i\left(\dfrac{-1}{2}+\dfrac{i\sqrt{3}}{2}\right) = i\omega$

$\dfrac{\sqrt{3}}{2}-\dfrac{i}{2} = i\left(\dfrac{-1}{2}-\dfrac{i\sqrt{3}}{2}\right) = i\omega^2$

$\therefore \quad z = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$

$= i(\omega-\omega^2) = i\,(i\sqrt{3}) = -\sqrt{3}$

$\Rightarrow$ Re$(z) < 0$ and Im$(z) = 0$

$\therefore$ (b) is the correct choice.

5. **(d)** $|z-4| < |z-2|$

$\Rightarrow \ |(x-4)+iy| < |(x-2)+iy|$

$\Rightarrow \ (x-4)^2+y^2 < (x-2)^2+y^2$

$\Rightarrow \ -8x+16 < -4x+4 \ \Rightarrow \ 4x-12 > 0$

$\Rightarrow \ x > 3 \Rightarrow$ Re$(z) > 3$

6. **(b)** $|\omega| = 1 \ \Rightarrow \ \left|\dfrac{1-iz}{z-i}\right| = 1$

$\Rightarrow \ |1-iz| = |z-i|$

$\Rightarrow \ |1-i\,(x+iy)| = |x+iy-i|$

$\Rightarrow \ |(y+1)-ix| = |x+i\,(y-1)|$

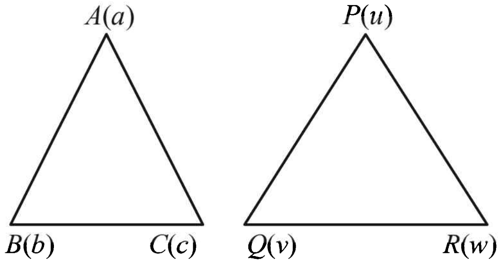
$\Rightarrow \ x^2+(y+1)^2 = x^2+(y-1)^2$

$\Rightarrow \ 4y = 0 \ \Rightarrow \ y = 0 \Rightarrow z$ lies on real axis

7. **(b)** If vertices of a parallelogram are z_1, z_2, z_3, z_4 then as diagonals bisect each other

$\therefore \ \dfrac{z_1+z_3}{2} = \dfrac{z_2+z_4}{2} \quad \Rightarrow z_1+z_3 = z_2+z_4$

8. **(b)** Let ABC be the Δ with vertices a, b, c and PQR be the Δ with vertices u, v, w.
Then $c = (1-r)\,a + rb$



$$\Rightarrow \quad c - a = r(b-a) \quad \Rightarrow \quad \frac{c-a}{b-a} = r \quad ....(1)$$

$$\Rightarrow \quad w = (1-r)\,u + rv \quad \Rightarrow \quad \frac{w-u}{v-u} = r \quad ....(2)$$

From (1) and (2) $\left|\frac{c-a}{b-a}\right| = \left|\frac{w-u}{v-u}\right|$ and

$$\arg\left(\frac{c-a}{b-a}\right) = \arg\left(\frac{w-u}{v-u}\right)$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ} \text{ and } \angle CAB = \angle RPQ$$

$$\Rightarrow \quad \Delta ABC \sim \Delta PQR$$

9. **(b)** $(1+\omega)^7 = A + B\omega$

$$\Rightarrow (-\omega^2)^7 = A + B\omega \qquad (\because\ 1+\omega+\omega^2 = 0)$$

$$\Rightarrow -\omega^{14} = A + B\omega$$

$$\Rightarrow -\omega^2 = A + B\omega \qquad (\because\ \omega^3 = 1)$$

$$\Rightarrow 1+\omega = A + B\omega \Rightarrow A = 1, B = 1$$

10. **(d)** $\because\ |z| = |\omega|$ and $\arg z = \pi - \arg\omega$

Let $\omega = re^{i\theta}$ then $z = re^{i(\pi-\theta)}$

$$\Rightarrow \quad z = re^{i\pi}\,.\,e^{-i\theta}$$

$$= (re^{-i\theta})\,(\cos\pi + i\sin\pi) = \overline{\omega}\,(-1) = -\overline{\omega}$$

11. **(c)** Given that $|z + i\omega| = |z - i\overline{\omega}|$

$\Rightarrow |z - (-i\omega)| = |z - (-\overline{i\omega})|$

$\Rightarrow$ z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(-\overline{i\omega})$, which is real axis, $(-i\omega)$ and $(-\overline{i\omega})$ being mirror images of each other.

$\therefore$ Im$(z) = 0$.

If $z = x$ then $|z| \le 1 \Rightarrow x^2 \le 1 \Rightarrow -1 \le x \le 1$

$\therefore$ (c) is the correct option.

12. **(d)** $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$

$$= (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$$

Using $1 + i = \sqrt{2}\,(\cos\pi/4 + i\sin\pi/4)$

and $\quad 1 - i = \sqrt{2}\,(\cos\pi/4 - i\sin\pi/4)$

We get the given expression as

$$= \left(\sqrt{2}\right)^{n_1}\left[\cos\frac{n_1\pi}{4} + i\sin\frac{n_1\pi}{4}\right]$$

$$+ \left(\sqrt{2}\right)^{n_2}\left[\cos\frac{n_2\pi}{4} + i\sin\frac{n_2\pi}{4}\right]$$

$$+ \left(\sqrt{2}\right)^{n_2}\left[\cos\frac{n_2\pi}{4} - i\sin\frac{n_2\pi}{4}\right]$$

$$= \left(\sqrt{2}\right)^{n_1}\left[2\cos\frac{n_1\pi}{4}\right] + \left(\sqrt{2}\right)^{n_2}\left[2\cos\frac{n_2\pi}{4}\right]$$

= real number irrespective the values of n_1 and n_2

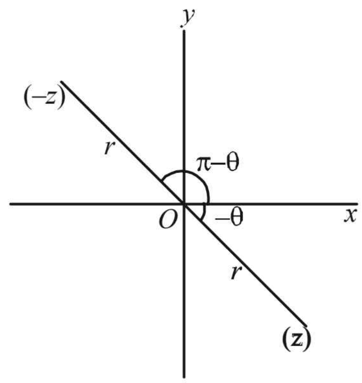
$\therefore$ (d) is the most appropriate answer.

13. **(c)** $E = 4 + 5(\omega)^{334} + 3(\omega)^{365} = 4 + 5\omega + 3\omega^2$

$$= 1 + 2\omega + 3(1+\omega+\omega^2) = 1 + (-1 + i\sqrt{3}) = i\sqrt{3}$$

14. **(a)** $\arg(z) < 0$ (given) $\Rightarrow \arg(z) = -\,\theta$

Now



$$z = r\cos(-\theta) + i\sin(-\theta) = r[\cos(\theta) - i\sin(\theta)]$$

Again $-z = -r\,[\cos(\theta) - i\sin(\theta)]$

$$= r\,[\cos(\pi-\theta) + i\sin(\pi-\theta)]$$

$\therefore \quad \arg(-z) = \pi - \theta;$

Thus $\arg(-z) - \arg(z) = \pi - \theta - (-\theta) = \pi - \theta + 0 = \pi$

15. **(a)** $|z_1| = |z_2| = |z_3| = 1$ (given)

Now, $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\overline{z}_1 = 1$

Similarly $z_2\overline{z}_2 = 1,\ z_3\overline{z}_3 = 1$

Now, $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1 \Rightarrow |\overline{z}_1 + \overline{z}_2 + \overline{z}_3| = 1$

$\Rightarrow \quad |\overline{z_1 + z_2 + z_3}| = 1$ **NOTE THIS STEP**

$\Rightarrow \quad |z_1 + z_2 + z_3| = 1$

16. (d) Let $z = (1)^{1/n} = (\cos 2k\pi + i \sin 2k\pi)^{1/n}$

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0, 1, 2, \ldots, n-1.$$

Let $\quad z_1 = \cos\left(\frac{2k_1\pi}{n}\right) + i\sin\left(\frac{2k_1\pi}{n}\right)$

and $\quad z_2 = \cos\left(\frac{2k_2\pi}{n}\right) + i\sin\frac{2k_2\pi}{n}$

be the two values of z. s.t. they subtend $\angle$ of 90° at origin.

$\therefore \quad \frac{2k_1\pi}{n} - \frac{2k_2\pi}{n} = \pm\frac{\pi}{2} \Rightarrow 4(k_1 - k_2) = \pm n$

As k_1 and k_2 are integers and $k_1 \neq k_2$.

$\therefore \quad n = 4k, k \in I$

17. (c) $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$

$\Rightarrow \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{1 - i\sqrt{3}}{2}\right)$

$\Rightarrow \arg\ (\cos(-\pi/3) + i\sin(-\pi/3))$

$\Rightarrow$ angle between $z_1 - z_3$ and $z_2 - z_3$ is 60°.

and $\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1 - i\sqrt{3}}{2}\right|$

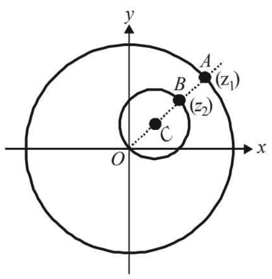
$\Rightarrow \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = 1 \Rightarrow |z_1 - z_3| = |z_2 - z_3|$

NOTE THIS STEP

$\Rightarrow$ The Δ with vertices z_1, z_2 and z_3 is isosceles with vertical $\angle 60°$. Hence rest of the two angles should also be 60° each.

$\Rightarrow$ Req. Δ is an equilateral Δ.

18. (b) $|z_1| = 12 \Rightarrow z_1$ lies on a circle with centre (0, 0) and radius 12 units, and $|z_2 - 3 - 4i| = 5 \Rightarrow z_2$ lies on a circle with centre (3, 4) and radius 5 units.



From fig. it is clear that $|z_1 - z_2|$ i.e., distance between z_1 and z_2 will be min when they lie at A and B resp. i.e., O, C, B, A are collinear as shown. Then $z_1 - z_2 = AB = OA - OB = 12 - 2(5) = 2$. As above is the min, value, we must have $|z_1 - z_2| \geq 2$.

19. (a) Given that $|z| = 1$ and $\omega = \frac{z-1}{z+1} (z \neq -1)$

Now we know that $z\bar{z} = |z|^2$

$\Rightarrow \quad z\bar{z} = 1 \qquad$ (for $|z| = 1$)

$\therefore \quad \omega = \left(\frac{z-1}{z+1}\right) \times \frac{(\bar{z}+1)}{(\bar{z}+1)} = \frac{z\bar{z} + z - \bar{z} - 1}{z\bar{z} + z + \bar{z} + 1} = \frac{2iy}{2 + 2x}$

[$\because z\bar{z} = 1$ and taking $z = x + iy$ so that

$z + \bar{z} = 2x$ and $z - \bar{z} = 2iy$]

$\Rightarrow \quad \text{Re}(\omega) = 0$

20. (b) $(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow (-\omega)^n = (1 + \omega)^n = (-\omega^2)^n \Rightarrow \omega^n = 1 \Rightarrow n = 3$

21. (a) Here we observe that.

$AB = AC = AD = 2$

$\therefore$ BCD is an arc of a circle with centre at A and radius 2. Shaded region is outer (exterior) part of this sector $ABCDA$.

$\therefore$ For any pt. z on arc BCD we should have

$|z - (-1)| = 2$

and for shaded region, $|z + 1| > 2 \qquad$ ....(i)

For shaded region we also have

$-\pi/4 < \arg(z + 1) < \pi/4$

or $\qquad |\arg(z + 1)| < \pi/4 \qquad$ ...(ii)

Combining (i) and (ii), (a) is the correct option.

22. (b) Given that a, b, c are integers not all equal, ω is cube root of unity $\neq 1$, then

$|a + b\omega + c\omega^2|$

$= \left|a + b\left(\frac{-1 + i\sqrt{3}}{2}\right) + c\left(\frac{-1 - i\sqrt{3}}{2}\right)\right|$

$= \left|\left(\frac{2a - b - c}{2}\right) + i\left(\frac{b\sqrt{3} - c\sqrt{3}}{2}\right)\right|$

$= \frac{1}{2}\sqrt{(2a - b - c)^2 + 3(b - c)^2}$

$= \sqrt{\frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]}$

R.H.S. will be min. when $a = b = c$, but we cannot take $a = b = c$ as per question.

$\therefore$ The min value is obtained when any two are zero and third is a minimum magnitude integer i.e. 1.
Thus $b = c = 0, a = 1$ gives us the minimum value 1.

23. (b) Operating $R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega4 \end{vmatrix} = 3[-\omega-1-\omega] = 3(\omega^2 - \omega)$$

24. (d) $\because \dfrac{w - wz}{1 - z}$ is purely real

$$\therefore \overline{\left(\frac{w-\overline{w}z}{1-z}\right)} = \left(\frac{w-\overline{w}z}{1-z}\right) \Rightarrow \frac{\overline{w}-w\overline{z}}{1-\overline{z}} = \frac{w-\overline{w}z}{1-z}$$

$$\Rightarrow \overline{w} - \overline{w}z - w\overline{z} + wz\overline{z} = w - w\overline{z} - \overline{w}z + \overline{w}z\overline{z}$$

$$\Rightarrow w - \overline{w} = (w - \overline{w})|z|^2$$

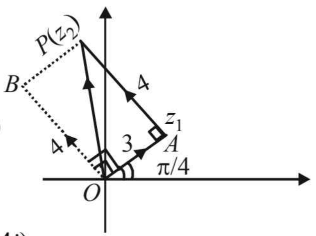
$$\Rightarrow |z|^2 = 1 \quad (\because w = \alpha + i\beta \text{ and } \beta \neq 0)$$

$\Rightarrow |z| = 1$ also given $z \neq 1$

$\therefore$ The required set is $\{z : |z| = 1, z \neq 1\}$

$= 3\omega(\omega - 1)$

25. (d) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$



$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP} = 3e^{i\pi/4} + 4e^{i(\pi/2+\pi/4)}$

$= 3e^{i\pi/4} + 4e^{i\pi/2}.e^{i\pi/4}$

$= 3e^{i\pi/4} + 4ie^{i\pi/4} = e^{i\pi/4}(3 + 4i)$.

26. (d) Given $|z| = 1$ and $z \neq \pm 1$

To find locus of $\omega = \dfrac{z}{1 - z^2}$

We have $\omega = \dfrac{z}{1-z^2} = \dfrac{z}{z\overline{z} - z^2}$

$[\because |z| = 1 \Rightarrow |z|^2 = z\overline{z} = 1]$

$= \dfrac{1}{\overline{z} - z}$ = purely imaginary number

$\therefore$ ω must lie on y – axis.

27. (d) The initial position of point is $Z_0 = 1 + 2i$

$\therefore Z_1 = (1 + 5) + (2 + 3)i = 6 + 5i$

Now Z_1 is moved through a distance of $\sqrt{2}$ units in the direction $\hat{i} + \hat{j}$. (i.e. by $1 + i$)

$\therefore$ It becomes $Z_1' = Z_1 + (1 + i) = 7 + 6i$

Now OZ_1' is rotated through an angle $\dfrac{\pi}{2}$ in anticlockwise direction, therefore $Z_2 = iZ_1' = -6 + 7i$

28. (d) $z = \cos\theta + i\sin\theta$

$\Rightarrow z^{2m-1} = (\cos\theta + i\sin\theta)^{2m-1}$

$= \cos(2m-1)\theta + i\sin(2m-1)\theta$

[using De Moivre's theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$]

$\therefore \quad \text{Im}(z^{2m-1}) = \sin(2m-1)\theta$

$$\therefore \quad \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \sum_{m=1}^{15} \sin(2m-1)\theta$$

$= \sin\theta + \sin 3\theta + \sin 5\theta + ...+$ upto 15 terms

$$= \frac{\sin\left[15\left(\frac{2\theta}{2}\right)\right].\sin[\theta + 14 \times \theta]}{\sin\theta}$$

[Using $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ...n$ terms $= \dfrac{\sin(n\beta/2).\sin[\alpha + (n-1)\beta/2]}{\sin(\beta/2)}$]

$$= \frac{\sin 15\theta.\sin 15\theta}{\sin\theta} = \frac{\sin 30°.\sin 30°}{\sin 2°} = \frac{1}{4\sin 2°}$$

29. (a) Given $z = x + iy$ where x and y are integer

Also $z\overline{z}^3 + \overline{z}z^3 = 350 \Rightarrow |z|^2(\overline{z}^2 + z^2) = 350$

$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 175$

$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 25 \times 7$ ...(i)

or $(x^2+y^2)(x^2-y^2) = 35 \times 5$ ...(ii)

$\because$ x and y are integers,

$\therefore x^2 + y^2 = 25$ and $x^2 - y^2 = 7$ [From eq (i)]

$\Rightarrow x^2 = 16$ and $y^2 = 9$

$\Rightarrow x = \pm 4$ and $y = \pm 3$

$\therefore$ Vertices of rectangle are $(4, 3), (4, -3), (-4, -3), (-4, 3)$.

So, area of rectangle $= 8 \times 6 = 48$ sq. units

Now from eq. (ii)

or $x^2 + y^2 = 35$ and $x^2 - y^2 = 5$

$\Rightarrow x^2 = 20$, which is not possible for any integral value of x

30. (d) $\because$ $Im(z) \neq 0 \Rightarrow z$ is non real

and equation $z^2 + z + (1 - a) = 0$ will have non real roots, if $D < 0$

$\Rightarrow 1 - 4(1 - a) < 0 \Rightarrow 4a < 3 \Rightarrow a < \dfrac{3}{4}$

$\therefore$ a can not take the value $\dfrac{3}{4}$

31. **(c)** As α lies on the circle $(x-x_0)^2+(y-y_0)^2=r^2$

$\therefore\ |\alpha-z_0|^2=r^2$

$\Rightarrow\ (\alpha-z_0)(\bar{\alpha}-\bar{z}_0)=r^2$

$\Rightarrow\ \alpha\bar{\alpha}-\alpha\bar{z}_0-\bar{\alpha}z_0+z_0\bar{z}_0=r^2$

$\Rightarrow\ |\alpha|^2+|z_0|^2-\alpha\bar{z}_0-\bar{\alpha}z_0=r^2$ (i)

Also $\frac{1}{\alpha}$ lies on the circle $(x-x_0)^2+(y-y_0)^2=4r^2$

$\therefore\ \left|\frac{1}{\alpha}-z_0\right|^2=4r^2 \Rightarrow \left(\frac{1}{\alpha}-z_0\right)\left(\frac{1}{\bar{\alpha}}-\bar{z}_0\right)=4r^2$

$\Rightarrow\ \frac{1}{\alpha\bar{\alpha}}-\frac{z_0}{\bar{\alpha}}-\frac{\bar{z}_0}{\alpha}+z_0\bar{z}_0=4r^2$

$\Rightarrow\ \frac{1}{|\alpha|^2}-\frac{z_0\alpha}{|\alpha|^2}-\frac{\bar{z}_0\alpha}{|\alpha|^2}+|z_0|^2=4r^2$

$\Rightarrow\ 1+|\alpha|^2|z_0|^2-z_0\bar{\alpha}-\bar{z}_0\alpha=4r^2|\alpha|^2$ (ii)

Subtracting eq^n (i) from (ii) we get

$1-|\alpha|^2+|z_0|^2(|\alpha|^2-1)=r^2(4|\alpha|^2-1)$

or $(|\alpha|^2-1)(|z_0|^2-1)=r^2(4|\alpha|^2-1)$

Using $|z_0|^2=\frac{r^2+2}{2}$ we get

$(|\alpha|^2-1)\frac{r^2}{2}=r^2(4|\alpha|^2-1)$

$\Rightarrow\ |\alpha|^2-1=8|\alpha|^2-2 \Rightarrow |\alpha|=\frac{1}{\sqrt{7}}$

D. MCQs with ONE or MORE THAN ONE Correct

1. **(a, b, c)** $z_1=a+ib$ and $z_2=c+id.$

ATQ $|zi|^2=|z_2|^2=1$

$\Rightarrow\ a^2+b^2=1$ and $c^2+d^2=1.$ ....(1)

Also Re $(z_1\overline{z_2})=0 \Rightarrow ac+bd=0$

$\Rightarrow\ \frac{a}{b}=\frac{-d}{c}=\alpha\ (say)$ ....(2)

From (1) and (2), we get

$b^2\alpha^2+b^2=c^2\alpha^2+c^2 \Rightarrow b^2=c^2;$

Similarly $a^2=d^2$

$\therefore\ |\omega_1|=\sqrt{a^2+c^2}=\sqrt{c^2+d^2}=1$

and $|\omega_2|=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1$

Also Re $(\omega_1\bar{\omega}_2)=ab+cd=(b\alpha)b+c(-c\alpha)$

$=\alpha(b^2-c^2)=0$

2. **(a, d)** Let $z_1=a+ib,\ a>0$ and $b\in R;\ z_2=c+id,$ $d<0, c\in R$

then $|z_1|=|z_2| \Rightarrow a_2+b_2=c_2+d_2$

$\Rightarrow a^2-c^2=d^2-b^2$ ....(1)

Now, $\frac{z_1+z_2}{z_1-z_2}=\frac{(a+c)+i(b+d)}{(a-c)+i(b-d)}$

$=\frac{[(a^2-c^2)+(b^2-d^2)]+i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$

$=\frac{i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$ [Using (1)]

= purely imaginary number or zero in case $a+c=b+d=0.$

3. **(c)** Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$

and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$

where $r_1=|z_1|,\ r_2=|z_2|,\ \theta_1=\arg(z_1),\ \theta_2=\arg(z_2)$

$\therefore\ z_1+z_2=r_1(\cos\theta_1+i\sin\theta_1)+r_2(\cos\theta_2+i\sin\theta_2)$

$=(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)$

$=r_1^2\cos^2\theta_1+r_2^2\cos^2\theta_2+2r_1r_2\cos\theta_1\cos\theta_2$
$+r_1^2\sin^2\theta_1+r_2^2\sin^2\theta_2+2r_1r_2\sin\theta_1\sin\theta_2$

$=r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)$

and $|z_1|+|z_2|=r_1+r_2$

Since $|z_1+z_2|=|z_1|+|z_2|$ (given)

$\Rightarrow\ |z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow\ r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)=r_1^2+r_2^2+2r_1r_2$

$\Rightarrow\ \cos(\theta_1-\theta_2)=1 \Rightarrow \theta_1-\theta_2=0$

$\Rightarrow$ Arg $(z_1)=\arg(z_2)$

4. **(d)** Let $z=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$

Then by DeMoivre's theorem, we have

$z^k=\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}$

Now, $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$

$=\sum_{k=1}^{6}(-i)\left(\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}\right)$

$=(-i)\sum_{k=1}^{6}z^k=-i\,z\frac{(1-z^6)}{1-z}=-i\left(\frac{z-z^7}{1-z}\right)$

$= (-i)\left(\frac{z-1}{1-z}\right) =$ [Using $z^7 = \cos 2\pi + i \sin 2\pi = 1$]

$= i\left(\frac{1-z}{1-z}\right) = i$

5. (d) We have $(1+\omega+\omega^2)^7 = (-\omega^2 - \omega^2)^7$

$(-2)^7 (\omega^2)^7 = -128\omega^{14} = -128\omega^2$

6. (b) $\sum_{i=1}^{13}(i^n + i^{n+1}) = \sum_{i=1}^{13} i^n (1+i) = (1+i)\sum_{i=1}^{13} i^n$

This forms a G.P.

Sum of G.P. $= i\,(1+i)\,\frac{(1-i^{13})}{1-i} = i - 1 \; as \; i^{13} = i$

7. (d) Taking $-3i$ common from C_2, we get

$$-3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0 \quad (\because C_2 \equiv C_3)$$

$\Rightarrow x = 0, \; y = 0$

8. (a,c,d) Given that $z = (1-t)\, z_1 + t\, z_2$ where $0 < t < 1$

$\Rightarrow z = \frac{(1-t)z_1 + tz_2}{(1-t)+t}$

$\Rightarrow$ z divides the join of z_1 and z_2 internally in the ratio $t : (1-t)$.

$\therefore z_1, z$ and z_2 are collinear $\underset{z_1}{\bullet}\overset{t}{-}\underset{z}{\bullet}\overset{1-t}{---}\underset{z_2}{\bullet}$

$\Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$

Also $z = (1-t)z_1 + t\, z_2$

$\Rightarrow \frac{z - z_1}{z_2 - z_1} = t =$ purely real number

$\therefore \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \Rightarrow \arg(z - z_1) = \arg(z_2 - z_1)$

Also $\frac{z - z_1}{z_2 - z_1} = t \Rightarrow \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} = t$

$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

$\Rightarrow (z - z_1)(\bar{z}_2 - \bar{z}_1) = (\bar{z} - \bar{z}_1)(z_2 - z_1)$

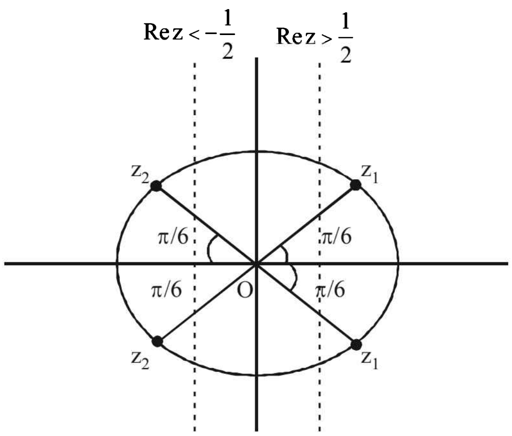
$$\Rightarrow \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$$

9. (c, d) $w = \frac{\sqrt{3}+i}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

and $w^n = \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}$

$\therefore$ P contains all those points which lie on unit circle and have arguments $\frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}$ and so on.

As $z_1 \in P \cap H_1$ and $z_2 \in P \cap H_2$, therefore z_1 and z_2 can have possible positions as shown in the figure.



$\therefore \angle z_1 O z_2$ can be $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

10. (a, c, d) $z = \frac{1}{a + ibt} = x + iy$

$\Rightarrow x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

$\Rightarrow x = \frac{a}{a^2 + b^2t^2}, \; y = \frac{-bt}{a^2 + b^2t^2}$

$\Rightarrow x^2 + y^2 = \frac{1}{a^2 + b^2t^2} = \frac{x}{a}$

$\Rightarrow x^2 + y^2 - \frac{x}{a} = 0$

$\therefore$ Locus of z is a circle with centre $\left(\frac{1}{2a}, 0\right)$ and radius $= \frac{1}{2|a|}$ irrespective of 'a' +ve or –ve

Also for $b = 0, a \neq 0$, we get, $y = 0$

$\therefore$ locus is x-axis

and for $a = 0, b \neq 0$ we get $x = 0$

$\Rightarrow$ locus is y-axis.

$\therefore$ a, c, d are the correct options.

E. Subjective Problems

1. $\dfrac{1}{1-\cos\theta+2i\sin\theta}$

$= \dfrac{1}{2\sin^2\theta/2+4i\sin\theta/2\cos\theta/2} = \dfrac{1}{2\sin\theta/2}$

$\left[\dfrac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$

$= \dfrac{1}{2\sin\theta/2}\left[\dfrac{\sin\theta/2-2i\cos\theta/2}{(\sin^2\theta/2+4\cos^2\theta/2}\right]$

$= \dfrac{1}{2\sin\theta/2}\left[\dfrac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$

$= \dfrac{2}{2\sin\theta/2}\left[\dfrac{2\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$

$= \left(\dfrac{1}{5+3\cos\theta}\right)+\left(\dfrac{-2\cot\theta/2}{5+3\cos\theta}\right)i$

which is of the form $X + iY$.

2. As β and γ are the complex cube roots of unity therefore, let $\beta = \omega$ and $\gamma = \omega^2$ so that $\omega + \omega^2 + 1 = 0$ and $\omega^3 = 1$.
Then $xyz = (a + b)(a\omega^2 + b\omega)(a\omega + b\omega^2)$
$= (a + b)(a^2\omega^3 + ab\omega^4 + ab\omega^2 + b^2\omega^3)$
$= (a + b)(a^2 + ab\omega + ab\omega^2 + b^2)$ (using $\omega^3 = 1$)
$= (a + b)(a^2 + ab(\omega + \omega^2) + b^2)$
$= (a + b)(a^2 - ab + b^2)$ (using $\omega + \omega^2 = -1$)
$= a^3 + b^3$ Hence proved.

3. Given $x+iy=\sqrt{\dfrac{c+ib}{c+id}}$

$\Rightarrow (x+iy)^2=\dfrac{a+ib}{c+id}$ ....(1)

Taking conjugate on both sides, we get

$(x-iy)^2=\dfrac{a-ib}{c-id}$ ....(2)

Multiply (1) and (2), we get

$(x^2+y^2)^2=\dfrac{a^2+b^2}{c^2+d^2}$

4. $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

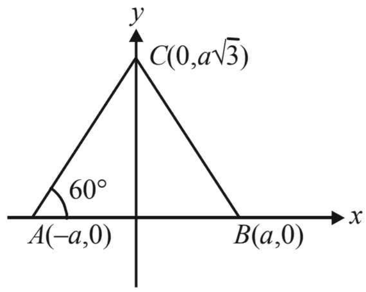
$\Rightarrow (4+2i)x-6i-2+(9-7i)y+3i-1=10i$

$\Rightarrow (4x+9y-3)+(2x-7y-3)i=10i$

$\Rightarrow 4x+9y-3=0$ and $2x-7y-3=10$

On solving these two, we get $x = 3, y = -1$

5.



Let us consider the equilateral Δ with each side of length $2a$ and having two of its vertices on x-axis namely $A(-a,0)$ and $B(a, 0)$, then third vertex C will clearly lie on y-axis s.t.

$OC = 2a\sin 60° = a\sqrt{3}$ $\therefore$ C has the co-ordinates $(0, a\sqrt{3})$.

Now in the form of complex numbers if A, B and C are represented by z_1, z_2, z_3 then $z_1=-a$; $z_2=a$; $z_3=a\sqrt{3}\,i$

As in an equilateral Δ, centriod and circumcentre coincide, we get

Circumcentre, $z_0=\dfrac{z_1+z_2+z_3}{3}$

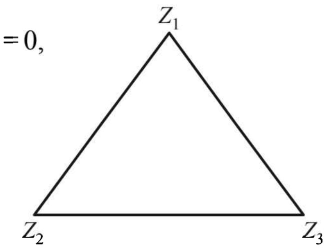
$\Rightarrow z_0=\dfrac{-a+a+a\sqrt{3}\,i}{3}=\dfrac{ia}{\sqrt{3}}$

Now, $z_1^2+z_2^2+z_3^2=a^2+a^2-3a^2=-a^2$

and $3z_0^2=(ia)^2=-a^2$ $\therefore$ Clearly $3z_0^2=z_1^2+z_2^2+z_3^2$

6. We know that if z_1, z_2, z_3 are vertices of an equilateral Δ then

$\dfrac{z_1-z_2}{z_3-z_2}=\dfrac{z_3-z_1}{z_2-z_1}$ Here $z_3 = 0$,



We get $\dfrac{z_1-z_2}{-z_2}=\dfrac{-z_1}{z_2-z_1}$

$\Rightarrow -(z_1-z_2)^2=z_1z_2$

$\Rightarrow -z_1^2-z_2^2+2z_1z_2=z_1z_2 \Rightarrow z_1^2+z_2^2-z_1z_2=0.$

7. $1, a_1, a_2, \ldots, a_{n-1}$ are the n roots of unity. Clearly above n values are roots of eq. $x^n - 1 = 0$
Therefore we must have (by factor theorem)
$x^n - 1 = (x - 1)(x - a_1)(x - a_2)\ldots(x - a_{n-1})$ ....(1)

$\Rightarrow \dfrac{x^n-1}{x-1}=(x-a_1)(x-a_2)\ldots(x-a_{n-1})$ ....(2)

Differentiating both sides of eq. (1), we get
$nx^{n-1} = (x - a_1)(x - a_2)\ldots(x - a_{n-1}) + (x - 1)(x - a_2)$
$\ldots(x - a_{n-1}) + \ldots + (x - 1)(x - a_1)\ldots(x - a_{n-2})$
For $x = 1$, we get $n = (1 - a_1)(1 - a_2)\ldots(1 - a_{n-1})$
[All the terms except first contain $(x - 1)$ and hence become zero for $x = 1$] Proved.

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8. Let $A = z = x + iy$, $B = iz = -y + ix$,
$C = z + iz = (x-y) + i(x+y)$

Now, area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix}$

Operating $R_2 - R_1, R_3 - R_1$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y-x & x-y & 0 \\ -y & x & 0 \end{vmatrix}$$

$$= \frac{1}{2}|x(-y-x) + y(x-y)|$$

$$= \frac{1}{2}|-xy - x^2 + xy - y^2| = \frac{1}{2}|-x^2 - y^2|$$

$$= \frac{1}{2}|x^2 + y^2| = \frac{1}{2}|z|^2 \text{ Hence Proved.}$$

9. We are given that $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$

Also $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$

$\Rightarrow \arg(z - z_1) - \arg(z - z_2) = \frac{\pi}{4}$ **NOTE THIS STEP**

$\Rightarrow \arg((x+iy) - (10+6i)) - \arg((x+iy) - (4+6i)) = \frac{\pi}{4}$

$\Rightarrow \arg[(x-10) + i(y-6)] - \arg[(x-4) + i(y-6)] = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{y-6}{x-10}\right) - \tan^{-1}\left(\frac{y-6}{x-4}\right) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{\frac{y-6}{x-10} - \frac{y-6}{x-4}}{1 + \frac{(y-6)^2}{(x-4)(x-10)}}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{(x-4)(y-6) - (x-10)(y-6)}{(x-4)(x-10) + (y-6)^2} = \tan\frac{\pi}{4}$

$\Rightarrow (x - 4 - x + 10)(y-6) = (x-4)(x-10) + (y-6)^2$

$\Rightarrow 6y - 36 = x^2 + y^2 - 14x - 12y + 40 + 36$

$\Rightarrow x^2 + y^2 - 14x - 18y + 112 = 0$

$\Rightarrow (x^2 - 14x + 49) + (y^2 - 18y + 81) = 18$

$\Rightarrow (x-7)^2 + (y-9)^2 = (3\sqrt{2})^2$

$\Rightarrow |(x+iy) - (7+9i)| = 3\sqrt{2}$

$\Rightarrow |z - (7+9i)| = 3\sqrt{2}$. Hence Proved.

10. Dividing through out by i and knowing that $\frac{1}{i} = -i$, we get

$z^3 - iz^2 + iz + 1 = 0$

or $z^2(z - i) + i(z - i) = 0$ as $1 = -i^2$

or $(z - i)(z^2 + i) = 0$ $\therefore z = i$ or $z^2 = -i$

$\therefore$ $|z| = |i| = 1$ or $|z^2| = |z|^2 = |-i| = 1 \Rightarrow |z| = 1$

Hence in either case $|z| = 1$

11. Let $Z = r_1(\cos\theta_1 + i\sin\theta_1)$

and $W = r_2(\cos\theta_2 + i\sin\theta_2)$

We have $|Z| = r_1$, $|W| = r_2$, Arg $Z = \theta_1$ and Arg $W = \theta_2$

Since $|Z| \le 1, |W| \le 1$, it follows that $r_1 \le$ *and* $r_2 \le 1$

We have $Z - W = (r_1\cos\theta_1 - r_2\cos\theta_2)$
$+ i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$|Z - W|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1$
$+ r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2$

$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$
$- 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2[1 - \cos(\theta_1 - \theta_2)]$

$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$= |r_1 - r_2|^2 + 4r_1r_2\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|^2$

$\le |r_1 - r_2|^2 + 4\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|$ $[\because r_1, r_2 \le 1]$

But $|\sin\theta| \le |\theta| \;\forall\; \theta \in R$ **NOTE THIS STEP**

Therefore,

$|Z - W|^2 \le |r_1 - r_2|^2 + 4\left|\frac{\theta_1 - 0_2}{2}\right|^2 \le |r_1 - r_2|^2 + |\theta_1 - \theta_2|^2$

Thus $|Z - W|^2 \le (|Z| - |W|)^2 + (\text{Arg } Z - \text{Arg } W)^2$

12. Let $z = x + iy$ then $\bar{z} = iz^2$

$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$

$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow\ x(1+2y)=0\ ;\ x^2-y^2+y=0$

$\Rightarrow\ x=0$ or $y=-\dfrac{1}{2}\ \Rightarrow\ x=0,\ y=0,\ 1$

or $\quad y=-\dfrac{1}{2},\ x=\pm\dfrac{\sqrt{3}}{2}$

For non zero complex number z

$$x=0, y=1;\ x=\frac{\sqrt{3}}{2},\ y=-\frac{1}{2};\ x=\frac{-\sqrt{3}}{2},\ y=-\frac{1}{2}$$

$\therefore\quad z=i,\ \dfrac{\sqrt{3}}{2}-\dfrac{i}{2},\ -\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}$

13. $z^2+pz+q=0$

$z_1+z_2=-p,\ z_1z_2=q$

By rotation through α in anticlockwise direction

$z_2=z_1e^{i\alpha}$ ....(1)

$$\frac{z_2}{z_1}=\frac{e^{i\alpha}}{1}=\frac{\cos\alpha+i\sin\alpha}{1}$$

Add 1 in both sides to get $z_1+z_2=-p$

$$\therefore\ \frac{z_1+z_2}{z_1}=\frac{1+\cos\alpha+i\sin\alpha}{1}=2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]$$

or $\quad \dfrac{(z_2+z_1)}{z_1}=2\cos\dfrac{\alpha}{2}e^{i\alpha/2}$

On squaring $(z_2+z_1)^2=4\cos^2(\alpha/2)z_1^2.e^{i\alpha}$

$$=4\cos^2\frac{\alpha}{2}z_1^2.\frac{z_2}{z_1}=4\cos^2\frac{\alpha}{2}z_1z_2$$

or $\quad p^2=4q\cos^2\dfrac{\alpha}{2}$

14. Given that z and w are two complex numbers.

To prove $|z|^2w-|w|^2z=z-w \Leftrightarrow z=w$ or $z\overline{w}=1$

First let us consider

$|z|^2w-|w|^2z=z-w$ ....(1)

$\Rightarrow\ z(1+|w|^2=w(1+|z|^2)$

$$\Rightarrow\ \frac{z}{w}=\frac{1+|z|^2}{1+|w|^2}=\text{a real number}$$

$$\Rightarrow\ \overline{\left(\frac{z}{w}\right)}=\frac{z}{w}\Rightarrow\frac{\overline{z}}{\overline{w}}=\frac{z}{w}$$

$\Rightarrow\ \overline{z}w=z\overline{w}$ ....(2)

Again from equation (1),

$z\overline{z}w-w\overline{w}z=z-w$

$z(\overline{z}w-1)-w(\overline{w}z-1)=0$

$z(z\overline{w}-1)-w(z\overline{w}-1)=0$ (Using equation (2))

$\Rightarrow\ (z\overline{w}-1)(z-w)=0\ \Rightarrow\ z\overline{w}=1$ *or* $z=w$

Conversely if $z=w$ then

L.H.S. of $(1)=|w|^2w-|w|^2w=0$

R.H.S. of $(1)=w-w=0$

$\therefore$ (1) holds

Also if $z\overline{w}=1$ then

L.H.S. of $(1)=z\overline{z}w-w\overline{w}z$

$=z\overline{z}w-w\overline{w}z=z-w=$ R.H.S. Hence proved.

15. The given equation can be written as

$(z^p-1)(z^q-1)=0$

$\therefore\quad z=(1)^{1/p}$ or $(1)^{1/q}$ ....(1)

where p and q are distinct prime numbers.

Hence both the equations will have distinct roots and as $z\neq 1$, both will not be simultaneously zero for any value of z given by equations in (1)

NOTE THIS STEP

Also $1+\alpha+\alpha^2+...+\alpha^{p-1}=\dfrac{1-\alpha^p}{1-\alpha}=0\ (\alpha\neq 1)$

or $\quad 1+\alpha+\alpha^2+...+\alpha^p=\dfrac{1-\alpha^q}{1-\alpha}=0\ (\alpha\neq 1)$

Because of (1) either $\alpha^p=1$ and if $\alpha^q=1$ but not both simultaneously as p and q are distinct primes.

16. Given that $|z_1|<1<|z_2|$

Then $\left|\dfrac{1-z_1\overline{z}_2}{z_1-z_2}\right|<1$ is true

if $|1-z_1\overline{z}_2|<|z_1-z_2|$ is true

if $|1-z_1\overline{z}_2|^2<|z_1-z_2|^2$ is true

if $(1-z_1\overline{z}_2)\overline{(1-z_1\overline{z}_2)}<(z_1-z_2)\overline{(z_1-z_2)}$ is true

if $(1-z_1\overline{z}_2)(1-\overline{z}_1z_2)<(z_1-z_2)(\overline{z}_1-\overline{z}_2)$

if $1-z_1\overline{z}_2-\overline{z}_1z_2+z_1\overline{z}_1z_2\overline{z}_2<z_1\overline{z}_1-z_1\overline{z}_2$

$-\overline{z}_1z_2+z_2\overline{z}_2$ is true

if $1+|z_1|^2|z_2|^2<|z_1|^2+|z_2|^2$ is true

if $(1-|z_1|^2)(1-|z_2|^2)<0$ is true.

which is obviously true

as $|z_1|<1<|z_2|$

$\Rightarrow\ |z_1|^2<1<|z_2|^2$

$\Rightarrow\ |1-|z_1|^2>0$ and $(1-|z_2|^2)<0$ Hence proved.

17. Let us consider, $\sum_{r=1}^{n}a_rz^r=1$ where $|a_r|<2$

$\Rightarrow\ a_1z+a_2z^2+a_3z^3+...+a_nz^n=1$

$\Rightarrow\ |a_1z+a_2z^2+a_3z^3+...+a_nz^n|=1$ ....(1)

But we know that $|z_1+z_2|\leq|z_1|+|z_2|$

$\therefore$ Using its generalised form, we get

$|a_1z+a_2z^2+a_3z^3+...+a_nz^n|$

$\leq|a_1z|+|a_2z^2|+...+|a_nz^n|$

$\Rightarrow\ 1 \le |a_1||z| + |a_2|\ |z^2| + |a_3||z^3| + ... + |a_n||z^n|$ (Using eqⁿ (1))

But given that $|a_r| < 2\ \forall\ r = 1(1)^n$

$\therefore\ 1 < 2\,[\,|z| + |z|^2 + |z|^3 + ... + |z|^n]$ [Using $|z^n| = |z|^n$]

$\Rightarrow\ 1 < 2\left[\dfrac{|z|(1-|z|^n)}{1-|z|}\right] \Rightarrow 2\left[\dfrac{|z| - |z|^{n+1}}{1-|z|}\right] > 1$

$\Rightarrow\ 2[\,|z| - |z|^{n+1}] > 1 - |z|$ $(\because 1 - |z| > 0 \text{ as } |z| < 1/3)$

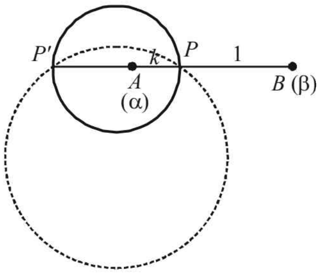
$\Rightarrow\ [|z| - |z|^{n+1}] > \dfrac{1}{2} - \dfrac{1}{2}|z| \Rightarrow \dfrac{3}{2}|z| > \dfrac{1}{2} + |z|^{n+1}$

$\Rightarrow\ |z| > \dfrac{1}{3} + \dfrac{2}{3}|z|^{n+1} \Rightarrow |z| > \dfrac{1}{3}$

which is a contradiction as given that $|z| < \dfrac{1}{3}$

$\therefore$ There exist no such complex number.

18. We are given that



$\left|\dfrac{z-\alpha}{z-\beta}\right| = k \Rightarrow |z-\alpha| = k|z-\beta|$

Let pt. A represents complex number α and B that of β, and P represents z. then $|z-\alpha| = k|z-\beta|$

$\Rightarrow$ z is the complex number whose distance from A is k times its distance from B.

i.e. $PA = k\ PB$

$\Rightarrow$ P divides AB in the ratio k : 1 internally or externally (at P').

Then $P\left(\dfrac{k\beta+\alpha}{k+1}\right)$ and $P'\left(\dfrac{k\beta-\alpha}{k-1}\right)$

Now through PP' there can pass a number of circles, but with given data we can find radius and centre of that circle for which PP' is diameter.

And hence then centre = mid. point of PP'

$$= \left(\dfrac{\dfrac{k\beta+\alpha}{k+1} + \dfrac{k\beta-\alpha}{k-1}}{2}\right) = \dfrac{k^2\beta + k\alpha - k\beta - \alpha + k^2\beta - k\alpha + k\beta - \alpha}{2(k^2-1)}$$

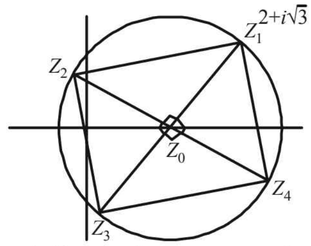
$$= \dfrac{k^2\beta - \alpha}{k^2-1} = \dfrac{\alpha - k^2\beta}{1-k^2}$$

Also radius

$$= \dfrac{1}{2}|PP'| = \dfrac{1}{2}\left|\dfrac{k\beta+\alpha}{k+1} - \dfrac{k\beta-\alpha}{k-1}\right|$$

$$= \dfrac{1}{2}\left|\dfrac{k^2\beta + k\alpha - k\beta - \alpha - k^2\beta + k\alpha - k\beta + \alpha}{k^2-1}\right| = \dfrac{k|\alpha-\beta|}{|1-k^2|}$$

19. The given circle is $|z-1| = \sqrt{2}$ where $z_0=1$ is the centre and $\sqrt{2}$ is radius of circle. z_1 is one of the vertex of square inscribed in the given circle.



Clearly z_2 can be obtained by rotating z_1 by an $\angle 90°$ in anticlockwise sense, about centre z_0

Thus, $z_2 - z_0 = (z_1 - z_0)\,e^{i\pi/2}$

or $z_2 - 1 = (2 + i\sqrt{3} - 1)i \Rightarrow z_2 = i - \sqrt{3} + 1$

$z_2 = (1 - \sqrt{3}) + i$

Again rotating z_2 by 90° about z_0 we get

$z_3 - z_0 = (z_2 - z_0)\,i$

$\Rightarrow\ z_3 - 1 = [(1-\sqrt{3}) + i - 1]\,i = -\sqrt{3}i - 1 \Rightarrow z_3 = -i\sqrt{3}$

and similarly $1 = (-i\sqrt{3} - 1)\,i = \sqrt{3} - i$

$\Rightarrow\ z_4 = (\sqrt{3} + 1) - i$

Thus the remaining vertices are

$(1-\sqrt{3}) + i, -i\sqrt{3}, (\sqrt{3}+1) - i$

F. Match the Following

1. $z \ne 0$ Let $z = a + ib$

Re (z) = 0 $\Rightarrow z = ib \Rightarrow z^2 = -b^2$

$\therefore\ Im\,(z)^2 = 0$

$\therefore$ (A) corresponds to (q)

Arg $z = \dfrac{\pi}{4} \Rightarrow a = b \Rightarrow z = a + ia$

$z^2 = a^2 - a^2 + 2ia^2$; $z^2 = 2ia^2 \Rightarrow \text{Re}\,(z)^2 = 0$

$\therefore$ (B) corresponds to (p).

2. (A) → (q, r) $\big|z - i|z|\big| = \big|z + i|z|\big|$

$\Rightarrow z$ is equidistant from two points $(0, |z|)$ and $(0, -|z|)$ which lie on imaginary axis.

$\therefore z$ must lie on real axis $\Rightarrow \text{Im}(z) = 0$ also $|I_m(z)| \le 1$

(B) → p

Sum of distances of z from two fined points (–4, 0) and (4, 0) is 10 which is greater than 8.

$\therefore z$ traces an ellipse with $2a = 10$ and $2ae = 8$

$\Rightarrow e = 4/5$

(C) $\to$ (p, s, t)

Let $\omega = 2(\cos\theta + i\sin\theta)$

then $z = \omega - \frac{1}{\omega} = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$

$\Rightarrow x + iy = \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta$

Here $|z| = \sqrt{\frac{9+25}{4}} = \sqrt{\frac{34}{4}} \le 3$ and $|R_e(z)| \le 2$

Also $x = \frac{3}{2}\cos\theta, y = \frac{5}{2}\sin\theta \Rightarrow \frac{4x^2}{9} + \frac{4y^2}{25} = 1$

Which is an ellipse with $e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

(D) $\to$ (q, r, s, t)

Let $\omega = \cos\theta + i\sin\theta$ then $z = 2\cos\theta \Rightarrow \text{Im} z = 0$

Also $|z| \le 3$ and $|\text{Im}(z)| \le 1, |R_e(z)| \le 2$

3. **(c)** (P) $\to$ (1): $z_k = \cos\frac{2k\pi}{10} + i\sin\frac{2k\pi}{10}$, $k = 1$ to 9

$\therefore z_k = e^{i\frac{2k\pi}{10}}$

Now $z_k . z_j = 1 \Rightarrow z_j = \frac{1}{z_k} = e^{-i\frac{2k\pi}{10}} = \overline{z_k}$

We know if z_k is 10th root of unity so will be $\overline{z}_k$.

$\therefore$ For every z_k, there exist $z_i = \overline{z}_k$

Such that $z_k . z_j = z_k . \overline{z_k} = 1$

Hence the statement is true.

(Q) $\to$ (2) $z_1 = z_k \Rightarrow z = \frac{z_k}{z_1}$ for $z_1 \ne 0$

$\therefore$ We can always find a solution to $z_1 . z = z_k$

Hence the statement is false.

(R) $\to$ (3) : We know $z^{10} - 1 = (z-1)(z-z_1)....(z-z_9)$

$\Rightarrow (z-z_1)(z-z_2).....(z-z_9) = \frac{z^{10}-1}{z-1}$

$= 1 + z + z^2 + ... z^9$

For z = 1 we get

$(1-z_1)(1-z_2).....(1-z_9) = 10$

$\therefore \frac{|1-z_1||1-z_2|.....|1-z_9|}{10} = 1$

(S) $\to$ (4) : $1, Z_1, Z_2, ... Z_9$ are 10th roots of unity.

$\therefore Z^{10} - 1 = 0$

From equation $1 + Z_1 + Z_2 + ... + Z_9 = 0$

$\text{Re}(1) + \text{Re}(Z_1) + \text{Re}(Z_2) + ... + \text{Re}(Z_9) = 0$

$\Rightarrow \text{Re}(Z_1) + \text{Re}(Z_2) + ... \text{Re}(Z_9) = -1$

$\Rightarrow \sum_{K=1}^{9} \cos\frac{2k\pi}{10} = -1 \Rightarrow 1 - \sum_{K=1}^{9} \cos\frac{2k\pi}{10} = 2$

Hence (c) is the correct option.

G. Comprehension Based Questions

For (Q. 1 - 3)

We have $A = \{z : \text{Im}(z) \ge 1\} = \{(x, y) : y \ge 1\}$

Clearly A is the set of all points lying on or above the line $y = 1$ in cartesian plane.

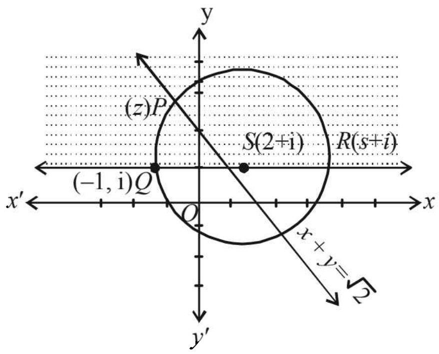
$B = \{z : |z - 2 - i| = 3\} = \{(x, y) : (x-2)^2 + (y-1)^2 = 9\}$

$\Rightarrow B$ is the set of all points lying on the boundary of the circle with centre (2, 1) and radius 3.

$C = \{z : \text{Re}[(1-i)z] = \sqrt{2}\} = \{(x, y) : x + y = \sqrt{2}\}$

$\Rightarrow$ C is the set of all points lying on the straight line represented by $x + y = \sqrt{2}$.

Graphically, the three sets are represented as shown below :



1. **(b)** From graph $A \cap B \cap C$ consists of only one point P [the common point of the region $y \ge 1$, $(x-2)^2 + (y-1)^2 = 9$ and $x + y = \sqrt{2}$] $\therefore n(A \cap B \cap C) = 1$

2. **(c)** As z is a point of $A \cap B \cap C \Rightarrow z$ represents the point P

$\therefore |z+1-i|^2 + |z-5-i|^2 \Rightarrow |z-(-1+i)|^2 + |z-(5+i)|^2$

$\Rightarrow PQ^2 + PR^2 = QR^2 = 6^2 = 36$

which lies between 35 and 39

$\therefore$ (c) is correct option.

3. **(d)** Given that $|w - 2 - i| < 3$

$\Rightarrow$ Distance between w and $2 + i$ i.e. S is smaller than 3.

$\Rightarrow w$ is a point lying inside the circle with centre S and radius 3.

$\Rightarrow$ Distance between z (i.e. the point P) and w should be smaller than 6 (the diameter of the circle)

i.e. $|z-w|<6$

But we know that $||z|-|w|| < |z-w|$

$\Rightarrow ||z|-|w|| < 6 \Rightarrow -6 < |z|-|w| < 6$

$-3 < |z|-|w|+3 < 9$

For (Q. 4 & 5)

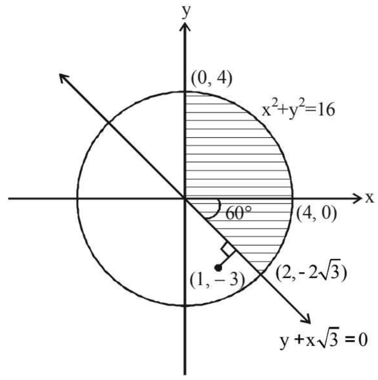
$S_1 : x^2+y^2<16$

$S_2 : \text{Im}\left[\dfrac{(x-1)+i(y+\sqrt{3})}{1-i\sqrt{3}}\right]>0$

$\Rightarrow \sqrt{3}(x-1)+(y+\sqrt{3})>0 \Rightarrow y+\sqrt{3}x>0$

$S_3 : x>0$

Then S : $S_1 \cap S_2 \cap S_3$ is as shown in the figure given below.



4. **(b)** Area of shaded region

$$=\frac{\pi}{4}\times 4^2+\frac{\pi\times 4^2\times 60^\circ}{360^\circ}=4\pi+\frac{8\pi}{3}=\frac{20\pi}{3}$$

5. **(c)** $\min\limits_{z\in s}|1-3i-z|$ = min distance between z and (1, –3)

Clearly (from figure) minimum distance between $z \in s$ and (1, –3) from line $y+x\sqrt{3}=0$ i.e. $\left|\dfrac{\sqrt{3}-3}{\sqrt{3+1}}\right|=\dfrac{3-\sqrt{3}}{2}$

I. Integer Value Correct Type

1. **(5)**

Given $|z-3-2i|\le 2$

which represents a circular region with centre (3, 2) and radius 2.

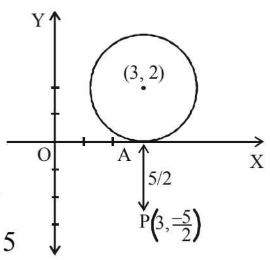
Now $|2z-6+5i| = 2\left|z-\left(3-\dfrac{5}{2}i\right)\right|$

= 2 × distance of z from P

(where Z lies in or on the circle)

Also min distance of z from $P=\dfrac{5}{2}$

$\therefore$ Minimum value of $|2z-6+5i|=5$



2. **(3)**

The expression may not attain integral value for all a, b, c.

If we consider a = b = c then

$x=3a, y=a(1+\omega+\omega^2)=a(1+i\sqrt{3})$

$Z=a(1+\omega^2+\omega)=a(1+i\sqrt{3})$

$\Rightarrow |x|^2+|y|^2+|z|^2=9|a|^2+4|a|^2+4|a|^2=17|a|^2$

$\Rightarrow \dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}=\dfrac{17}{3}$ (which is not an integer)

Note : However if $\omega=e^{i(2\frac{\pi}{3})}$, then the value of expression can be evaluated as follows

$$\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}=\frac{x\bar{x}+y\bar{y}+z\bar{z}}{|a|^2+|b|^2+|c|^2}$$

$$=\frac{(a+b+c)(\bar{a}+\bar{b}+\bar{c})+(a+b\omega+c\omega^2)(\bar{a}+\bar{b}\omega^2+\bar{c}\omega)+(a+b\omega^2+c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$$=\frac{3|a|^2+3|b|^2+3|c|^2+(a\bar{b}+\bar{a}b+b\bar{c}+\bar{b}c+a\bar{c}+\bar{a}c)(1+\omega+\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$=3 \qquad (\because 1+\omega+\omega^2=0)$

3. **(4)** $\alpha_k=\cos\dfrac{k\pi}{7}+i\sin\dfrac{k\pi}{7}=e^{\frac{i\pi k}{7}}$

$$\alpha_{k+1}-\alpha_k=e^{\frac{i\pi(k+1)}{7}}-e^{\frac{i\pi k}{7}}=e^{\frac{i\pi k}{7}}(e^{i\pi/7}-1)$$

$$|\alpha_{k+1}-\alpha_k|=|e^{i\pi/7}-1|$$

$$\Rightarrow \sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|=12|e^{i\pi/7}-1|$$

Similarly $\displaystyle\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|=3|e^{i\pi/7}-1|$

$$\therefore \frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}=4$$

Section-B JEE Main/ AIEEE

1. **(b)** Let $|z|=|\omega|=r$

$\therefore z=re^{i\theta},\ \omega=re^{i\phi}$ where $\theta+\phi=\pi$.

$\therefore z=re^{i(\pi-\phi)}=re^{i\pi}.e^{-i\phi}=-re^{-i\phi}=-\bar{\omega}.[\because \bar{\omega}=re^{-i\phi}]$

2. **(c)** Given $|z-4|<|z-2|$ Let $z=x+iy$

$\Rightarrow |(x-4)+iy)|<|(x-2)+iy|$

$\Rightarrow (x-4)^2+y^2<(x-2)^2+y^2$

$\Rightarrow x^2-8x+16<x^2-4x+4 \Rightarrow 12<4x$

$\Rightarrow x>3 \Rightarrow \text{Re}(z)>3$

3. **(b)** Let the circle be $|z-z_0|=r$. Then according to given conditions $|z_0-z_1|=r+a$ and $|z_0-z_2|=r+b$. Eliminating r, we get $|z_0-z_1|-|z_0-z_2|=a-b$.

$\therefore$ Locus of centre z_0 is $|z-z_1|-|z-z_2|=a-b$, which represents a hyperbola.

4. **(a)** $|\bar{z}\omega|=|\bar{z}||\omega|=|z||\omega|=|z\omega|=1$

$\text{Arg}(\bar{z}\omega)=\arg(\bar{z})+\arg(\omega)=-\arg(z)+\arg\omega$

$=-\dfrac{\pi}{2}\quad \therefore \bar{z}\omega=-1$

5. **(d)** $z^2+az+b=0$; $z_1+z_2=-a\ \&\ z_1z_2=b$

$0, z_1, z_2$ form an equilateral Δ

$\therefore 0^2+z_1^2+z_2^2=0.z_1+z_1.z_2+z_2.0$

(for an equilateral triangle,

$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$)

$\Rightarrow z_1^2+z_2^2=z_1z_2 \Rightarrow (z_1+z_2)^2=3z_1z_2\quad \therefore a^2=3b$

6. **(b)** $\left(\dfrac{1+i}{1-i}\right)^x=1 \Rightarrow \left[\dfrac{(1+i)^2}{1-i^2}\right]^x=1$

$\left(\dfrac{1+i^2+2i}{1+1}\right)^x=1\Rightarrow (i)^x=1;\ \therefore x=4n;\ n\in I^+$

7. **(c)** $\arg zw=\pi\Rightarrow \arg z+\arg w=\pi$...(1)

$\bar{z}+i\bar{w}=0\Rightarrow \bar{z}=-i\bar{w}$

$\therefore z=iw\Rightarrow \arg z=\dfrac{\pi}{2}+\arg w$

$\Rightarrow \arg z=\dfrac{\pi}{2}+\pi-\arg z$ (from(1)) $\therefore \arg z=\dfrac{3\pi}{4}$

8. **(a)** $z^{\frac{1}{3}}=p+iq \Rightarrow z=p^3+(iq)^3+3p(iq)(p+iq)$

$\Rightarrow x-iy=p^3-3pq^2+i(3p^2q-q^3)$

$\therefore x=p^3-3pq^2\Rightarrow \dfrac{x}{p}=p^2-3q^2$

$y=q^3-3p^2q\Rightarrow \dfrac{y}{q}=q^2-3p^2$

$\therefore \dfrac{x}{p}+\dfrac{y}{q}=-2p^2-2q^2\ \therefore \left(\dfrac{x}{p}+\dfrac{y}{q}\right)\Big/(p^2+q^2)=-2$

9. **(b)** $|z^2-1|=|z|^2+1\Rightarrow |z^2-1|^2=(z\bar{z}+1)^2$

$\Rightarrow (z^2-1)(\bar{z}^2-1)=(z\bar{z}+1)^2$

$\Rightarrow z^2\bar{z}^2-z^2-\bar{z}^2+1=z^2\bar{z}^2+2z\bar{z}+1$

$\Rightarrow z^2+2z\bar{z}+\bar{z}^2=0\Rightarrow (z+\bar{z})^2=0\Rightarrow z=-\bar{z}$

$\Rightarrow z$ is purely imaginary

10. **(c)** $(x-1)^3+8=0 \Rightarrow (x-1)=(-2)\,(1)^{1/3}$

$\Rightarrow x-1=-2$ or -2ω or $-2\omega^2$

or $x=-1$ or $1-2\omega$ or $1-2\omega^2$.

11. **(c)** $|z_1+z_2|=|z_1|+|z_2| \Rightarrow z_1$ and z_2 are collinear and are to the same side of origin; hence $\arg z_1-\arg z_2=0$.

12. **(c)** As given $w=\dfrac{z}{z-\frac{1}{3}i}\Rightarrow |w|=\dfrac{|z|}{|z-\frac{1}{3}i|}=1$

$\Rightarrow |z|=\left|z-\dfrac{1}{3}i\right|$

$\Rightarrow$ distance of z from origin and point $\left(0,\dfrac{1}{3}\right)$ is same hence z lies on bisector of the line joining points (0, 0) and (0, 1/3).

Hence z lies on a straight line.

13. **(d)**

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)=i\sum_{k=1}^{10}\left(\cos\frac{2k\pi}{11}-i\sin\frac{2k\pi}{11}\right)$$

$$=i\sum_{k=1}^{10}e^{-\frac{2k\pi}{11}i}=i\left\{\sum_{k=0}^{10}e^{-\frac{2k\pi}{11}i}-1\right\}$$

$$=i\left[1+e^{-\frac{2\pi}{11}i}+e^{-\frac{4\pi}{11}i}+....11\text{ terms}\right]-i$$

$$=i\left[\frac{1-\left(e^{-\frac{2\pi}{11}}\right)^{11}}{1-e^{-\frac{2\pi}{11}i}}\right]-i=i\left[\frac{1-e^{-2\pi i}}{1-e^{-\frac{2\pi}{11}i}}\right]-i$$

$=i\times 0-i\qquad [\because e^{-2\pi i}=1]=-i$

14. **(d)** $z^2+z+1=0 \Rightarrow z=\omega$ or ω^2

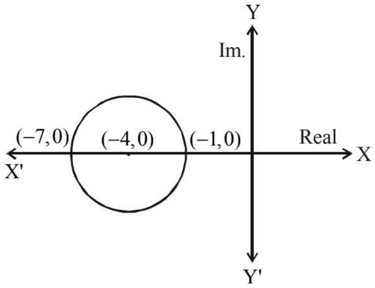
So, $z+\dfrac{1}{z}=\omega+\omega^2=-1$

$z^2+\dfrac{1}{z^2}=\omega^2+\omega=-1,\ z^3+\dfrac{1}{z^3}=\omega^3+\omega^3=2$

$z^4+\frac{1}{z^4}=-1,\ z^5+\frac{1}{z^5}=-1$ and $z^6+\frac{1}{z^6}=2$

$\therefore$ The given sum $=1+1+4+1+1+4=12$

15. (a) z lies on or inside the circle with centre $(-4, 0)$ and radius 3 units.



From the Argand diagram maximum value of $|z+1|$ is 6

16. (c) $\left(\frac{1}{i-1}\right)=\frac{1}{-i-1}=\frac{-1}{i+1}$

17. (d) Given $S=\{(x,y): y=x+1 \text{ and } 0<x<2\}$

$\because x \neq x+1$ for any $x \in (0,2) \Rightarrow (x,x) \notin S$

$\therefore S$ is not reflexive.

Hence S in not an equivalence relation.

Also $T=\{x,y): x-y$ is an integer$\}$

$\because x-x=0$ is an integer $\forall x \in R$

$\therefore T$ is reflexive.

If $x-y$ is an integer then $y-x$ is also an integer

$\therefore T$ is symmetric

If $x-y$ is an integer and $y-z$ is an integer then $(x-y)+(y-z)=x-z$ is also an integer.

$\therefore T$ is transitive

Hence T is an equivalence relation.

18. (a) Let $z=x+iy$

$|z-1|=|z+1|(x-1)^2+y^2=(x+1)^2+y^2$

$\Rightarrow \text{Re}\, z=0 \quad \Rightarrow x=0$

$|z-1|=|z-i|(x-1)^2+y^2=x^2+(y-1)^2$

$\Rightarrow x=y$

$|z+1|=|z-i|(x+1)^2+y^2=x^2+(y-1)^2$

Only (0, 0) will satisfy all conditions.

$\Rightarrow$ Number of complex number $z=1$

19. (c) $\because$ Real part of roots is 1

Let roots are $1+pi,\ 1+q$

$\therefore$ sum of roots $=1+pi+1+qi=-\alpha$ which is real

$\Rightarrow q=-p$ or root are

$1+pi$ and $1-pi$ product of roots $=1+p^2=\beta \in (1,\infty)$

$p \neq 0$ as roots are distinct.

20. (a) $(1+\omega)^7=A+B\omega; \qquad (-\omega^2)^7=A+B\omega$

$-\omega^2=A+B\omega; \qquad 1+\omega=A+B\omega$

$\Rightarrow A=1, B=1.$

21. (a) Let $z=x+iy \ \therefore\ z^2=x^2-y^2+2ixy$

Now $\frac{z^2}{z-1}$ is real $\Rightarrow \text{Im}\left(\frac{z^2}{z-1}\right)=0$

$\Rightarrow \text{Im}\left(\frac{x^2-y^2+2ixy}{(x-1)+iy}\right)=0$

$\Rightarrow \text{Im}\,[(x^2-y^2+2ixy)(x-1)-iy)]=0$

$\Rightarrow 2xy(x-1)-y(x^2-y^2)=0$

$\Rightarrow y(x^2+y^2-2x)=0 \Rightarrow y=0;\ x^2+y^2-2x=0$

$\therefore z$ lies either on real axis or on a circle through origin.

22. (c) Given $|z|=1, \arg z=\theta$

As we know, $\bar{z}=\frac{1}{z}$

$\therefore \quad \arg\left(\frac{1+z}{1+\bar{z}}\right)=\arg\left(\frac{1+z}{1+\frac{1}{z}}\right)=\arg(z)=\theta.$

23. (b) We know minimum value of $|Z_1+Z_2|$ is $||Z_1|-|Z_2||$

Thus minimum value of $\left|Z+\frac{1}{2}\right|$ is $\left||Z|-\frac{1}{2}\right|$

$\leq\left|Z+\frac{1}{2}\right|\leq|Z|+\frac{1}{2}$

Since, $|Z|\geq 2$ therefore $2-\frac{1}{2}<\left|Z+\frac{1}{2}\right|<2+\frac{1}{2}$

$\Rightarrow \quad \frac{3}{2}<\left|Z+\frac{1}{2}\right|<\frac{5}{2}$

24. (a) $\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1 \Rightarrow |z_1-2z_2|^2=|2-z_1\bar{z}_2|^2$

$\Rightarrow (z_1-2z_2)\overline{(z_1-2z_2)}=(2-z_1\bar{z}_2)\overline{(2-z_1\bar{z}_2)}$

$\Rightarrow (z_1-2z_2)(\bar{z}_1-2\bar{z}_2)=(2-z_1\bar{z}_2)(2-\bar{z}_1z_2)$

$\Rightarrow \quad (z_1\bar{z}_1)-2z_1\bar{z}_2-2\bar{z}_1z_2+4z_2\bar{z}_2$

$=4-2\bar{z}_1z_2-2z_1\bar{z}_2+z_1\bar{z}_1z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2+4|z_2|^2=4+|z_1|^2|z_2|^2$

$\Rightarrow \quad |z_1|^2+4|z_2|^2-4-|z_1|^2|z_2|^2=0$

$\left(|z_1|^2-4\right)\left(1-|z_2|^2\right)=0$

$\because \quad |z_2|\neq 1 \ \therefore\ |z_1|^2=4 \Rightarrow |z_1|=2$

$\Rightarrow$ Point z_1 lies on circle of radius 2.

25. (b) Rationalizing the given expression

$\frac{(2+3i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta}$

For the given expression to be purely imaginary, real part of the above expression should be equal to zero.

$\Rightarrow \frac{2-6\sin^2\theta}{1+4\sin^2\theta}=0 \quad \Rightarrow \sin^2\theta=\frac{1}{3} \Rightarrow \sin\theta=\pm\frac{1}{\sqrt{3}}$