Trigonometric Identities and Equations

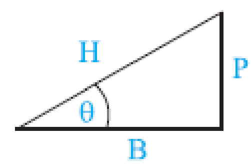
TRIGONOMETRIC IDENTITIES AND EQUATIONS Section - 1

1. INTRODUCTION

1.1 Trigonometric Ratios for acute angles :

For an acute angle θ, *Trignometric Ratios* (T-ratios) can be defined using a right angled triangle with angles θ, 90° – θ, 90°.

$\sin\theta = \frac{P}{H}$ $\cos\theta = \frac{B}{H}$ $\tan\theta = \frac{P}{B}$



cosecθ, secθ, cotθ are reciprocals of sinθ, cosθ and tanθ respectively.

Some standard identities for above trignometric ratios are :

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

For acute angles, all T-ratios are positive.

1.2. Angle :

An angle is the amount of rotation of a revolving line with respect to a fixed line:

There are three system of measuring an angle :

(i) Sexagecimal system or English system

(ii) Centesimal or french system

(iii) Circular System

(i) Sexagecimal System :

In this system a right angle is divided into 90 equal parts, called degrees. The symbol 1° is used to denote one degree. Thus, one degree is one-ninetieth part of right angle. Each degree is divided into 60 equal parts, called minutes and one minute is divided into 60 equal parts, called seconds. The symbol 1′ and 1′′ are used to denote one minute and one second, respectively.

Thus, 1 right angle = 90 degrees (= 90°)

1° = 60 minutes (60′)

1′ = 60 second (60′′)

(ii) Centesimal System :

In this system a right angle is divided into 100 equal parts, called grades; each grade is subdivided into 100 minutes each minute is divided into 100 seconds.

The symbol 1^g, $1'$ and $1''$ are used to denote a grade, a minute and a second respectively.

Thus, 1 right angle = 100 grades (= 100^g)

1 grade = 100 minutes (= $100'$)

1 minute = 100 seconds (= $100''$)

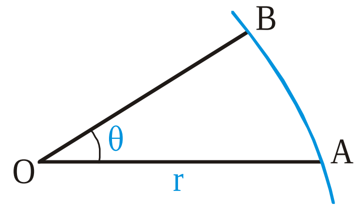
(iii) Circular System :

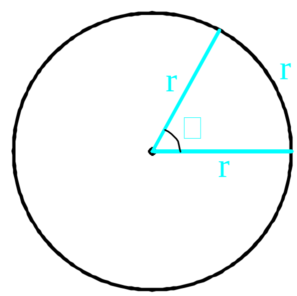
In this system the unit of measurement is radian as defined below

Radians : One radian, written as 1^c, is the measure of an angle subtended at the centre of circle by an arc of length equal to radius of the circle.

Consider an arc *AB* of a circle of radius *r* subtending an angle θ at its centre. The ratio between length of arc *AB* and the radius of circle represents the measure of angle θ in the radians. i.e.

$$\theta \text{ (in radians)} = \frac{\text{arc } AB}{r}$$



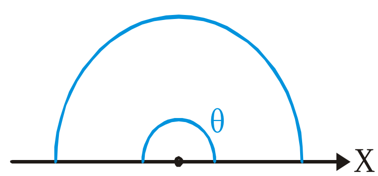


One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of circle.

Relation between degree and radians :

Let θ be the angle subtended at centre by a semicircle = 180°.
From figure, θ = 180°.

In radians: $\theta = \dfrac{\text{arc}}{\text{radius}} = \dfrac{\pi r}{r} = \pi$



$\Rightarrow$ 180° degrees = π radian

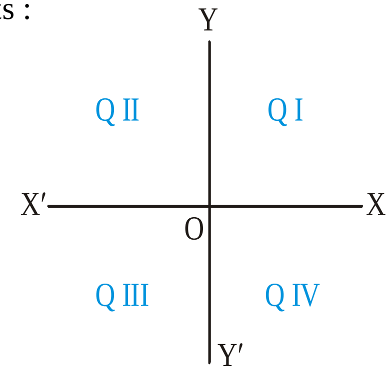
Radian is an important unit for measuring angles. So the following conversions must be remembered

Degree	0	30	45	60	90	120	135	150	180	270
Radian	0	π/6	π/4	π/3	π/2	2π/3	3π/4	5π/6	π	3π/2

Note : θ radian is written as θ^c or can be written simply as θ. When the unit of angle is not mentioned. it must be taken as radians.

Measuring of angle in various quadrants :

The perpendicular lines XOX' and YOY' divide the plane in four parts :



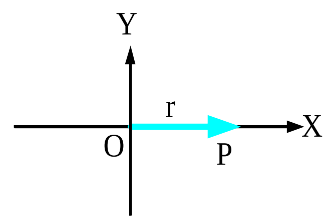
Q I : First Quadrant (all points have + X and + Y)

Q II : Second Quadrant (all points have – X and + Y)

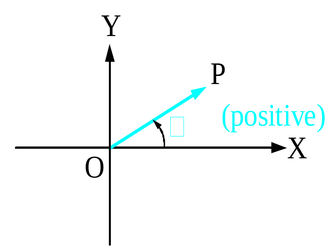
Q III: Third Quadrant (all points have – X and – Y)

Q IV: Fourth Quadrant (all points have + X and – Y)

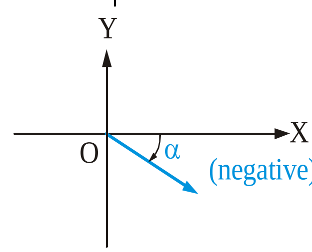
- The angles in trigonometry can be positive or negative and can have any magnitude. Every angle is represented by one position of a revolving ray OP of length r. The starting position for ray OP is taken along $+X$ axis.



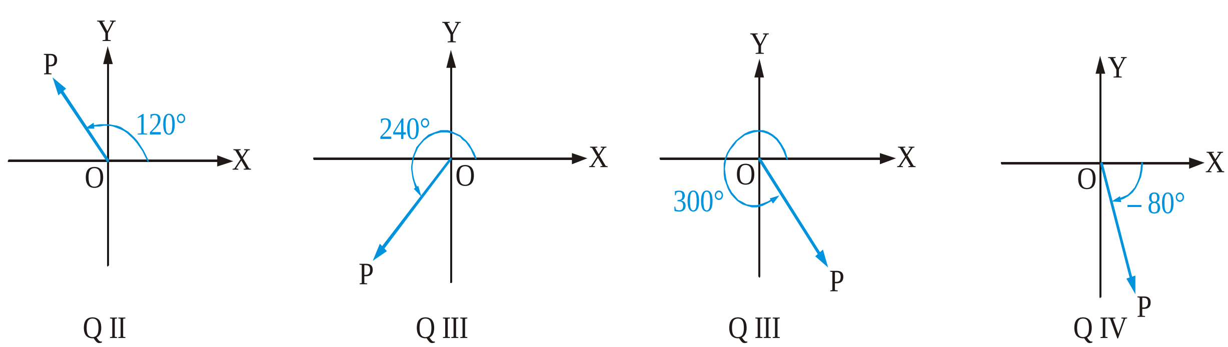
- If an angle α (alpha) is positive, OP rotates through angle α in anticlockwise direction.



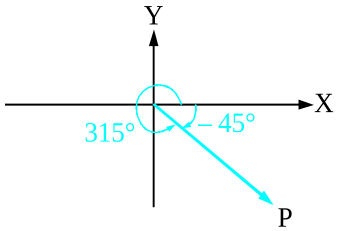
- If an angle α is negative, OP rotates through angle α in clockwise direction.

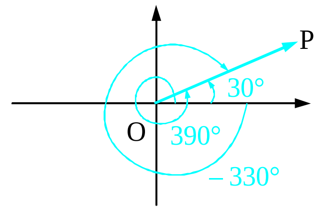


- An angle can lie in any of four quadrants according to the position of revolving ray for the angle.



- Two or more angles may correspond to same position of revolving ray OP

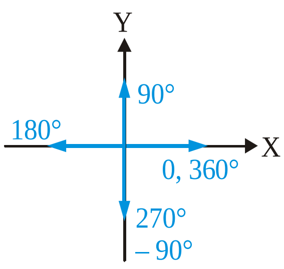


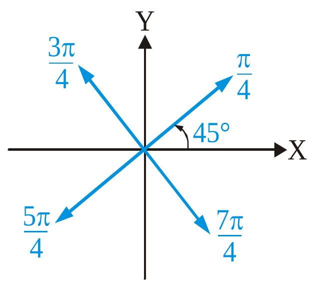


This position of OP represents –45° and +315°

This position of OP represents +30°, +390° and –330°

- The position of revolving ray for angles 0°, 30°, 45°, 60°, 90°, 120°, 135°, 180°, 270°, 360° must be remembered.





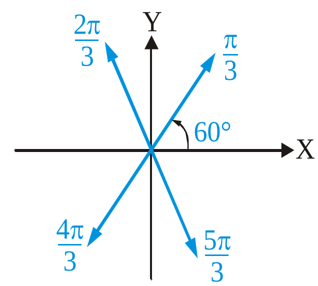


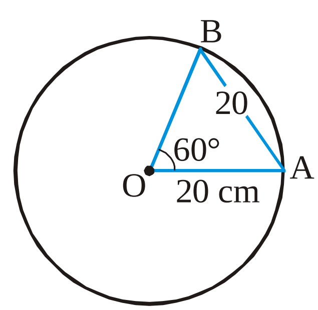
Illustration - 1 *In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.*

(A) $\dfrac{10\pi}{3}$ **(B)** $\dfrac{20\pi}{3}$ **(C)** $\dfrac{5\pi}{3}$ **(D)** $\dfrac{30\pi}{3}$

SOLUTION : (B)

Let arc $AB = S$. It is given that $OA = 20$ cm and chord $AB = 20$ cm.

Therefore, ΔOAB is an equilateral triangle.



$$\Rightarrow \qquad \angle AOB = 60° = \left(60 \times \frac{\pi}{180}\right) = \left(\frac{\pi}{3}\right)^c \qquad \text{(in radians)}$$

We know that $\theta = \dfrac{\text{arc}}{\text{radius}} \Rightarrow \dfrac{\pi}{3} = \dfrac{S}{20} \Rightarrow S = \dfrac{20\pi}{3}$ cm

Illustration - 2 *Find the radian measure corresponding to –37° 30′.*

(A) $\dfrac{-5\pi}{24}$ **(B)** $\dfrac{\pi}{24}$ **(C)** $\dfrac{5\pi}{24}$ **(D)** $\dfrac{-7\pi}{24}$

SOLUTION :(A)

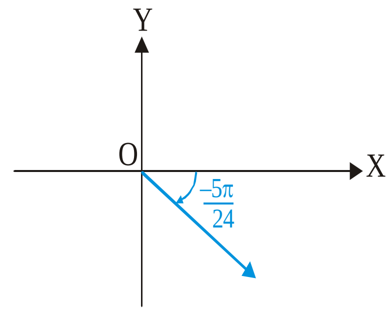
In such kind of problems first of all we convert minute into degree and then degree into radians

Therefore,

$60' = 1°$

$\Rightarrow \quad 30' = \left(\frac{1}{2}\right)^\circ$

$\Rightarrow \quad -37° \ 30' = -37\frac{1°}{2} = -\frac{75°}{2}$



Now, $360° = 2\pi$ radians

$\Rightarrow \quad \frac{-75°}{2} = -\frac{2\pi}{360} \times \frac{75}{2}$ radians $= \frac{-5\pi}{24}$ radians [using $1° = \frac{\pi}{180}$ radians]

This (–) sign indicates that measure of angle is the clockwise direction.

Illustration - 3 *If the angles of a triangle are in the ratio* 3 : 4 : 5, *find he smallest angle in degrees and the greatest angle in radians.*

(A) $50^o, \frac{\pi}{12}$ **(B)** $70^o, \frac{\pi}{12}$ **(C)** $75^o, \frac{5\pi}{12}$ **(D)** $85^o, \frac{\pi}{12}$

SOLUTION : (C)

Let the three angles be $3x$, $4x$ and $5x$ degrees,

In triangle,

$3x + 4x + 5x = 180$

$\Rightarrow \quad 12x = 180 \quad \Rightarrow \quad x = 15$

$\Rightarrow$ The smallest angle = $3x$ degrees

$= 3 \times 15$ degree $= 45°$

and the greatest angle = $5x$ degree

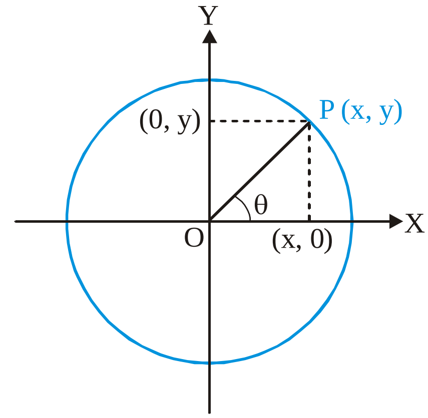
$= 5 \times 15$ degree $= 75°$

$= \left(75 \times \frac{\pi}{180}\right)$ radians $= \frac{5\pi}{12}$ radians

TRIGNOMETRIC FUNCTIONS OF AN ANGLE — Section - 2

2.1 The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle θ, 0° < θ < 90° are defined as the ratios of two sides of a right-angled triangle with θ as one of acute angle. However, we can also define these trignometric ratios through a unit circle.

Draw a unit circle (radius = 1 unit) and take any two diameters at right angle as *X* and *Y* axes. Taking *OX* as the initial line, let $\overrightarrow{OP}$ be the radius vector corresponding to an angle θ, where *P* lies on the unit circle. Let (*x*, *y*) be the coordinates of *P*.



Then by definition in Section 1.1.

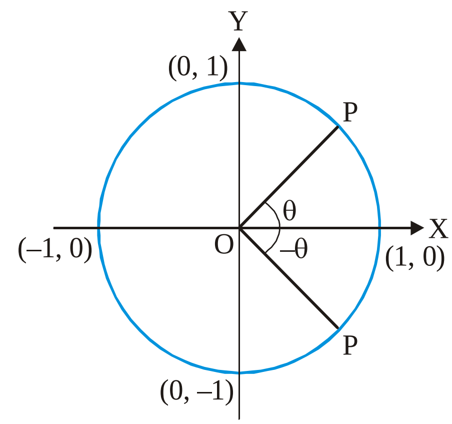
$$\sin\theta = \frac{p}{h} = \frac{y}{1}, \qquad \text{the } y\text{-coordinate of } P$$

$$\cos\theta = \frac{b}{h} = \frac{x}{1}, \qquad \text{the } x\text{-coordinate of } P$$

$$\tan\theta = \frac{p}{b} = \frac{y}{x}, \qquad x \neq 0$$

$$\sec\theta = \frac{h}{b} = \frac{1}{x}, \qquad x \neq 0,$$

$$\text{cosec}\theta = \frac{h}{p} = \frac{1}{y}, \qquad y \neq 0 \quad \text{and} \quad \cot\theta = \frac{b}{p} = \frac{x}{y}.$$

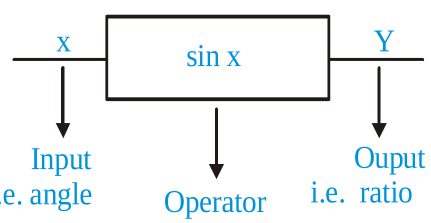


Angle measured anticlockwise from the initial line *OX* is positive and angles measured clockwise are considered to be negative.

As we can associate a unique radius vector $\overrightarrow{OP}$ for unique *P* with each angle θ, we can say ratios of '*x*' and '*y*' are functions of θ. From this, we have an idea that we can study the Trignometric ratios as trigonometric 'function' this holds for all angles.

Therefore, we can study y or *f* (*x*) = sin*x* as trignometric function.

i.e.,



i.e., for every angle θ there exists a unique corresponding ratio *i.e.* output.

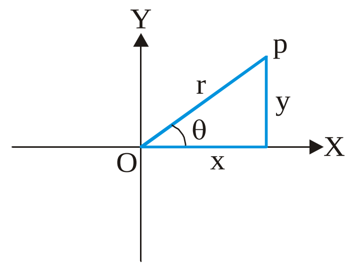
Similarly, we can study other trignometric ratios as trignometric functions in the same manner.

2.2 Signs of Trigonometrical Functions

We have introduced six trigonometric functions. Signs of these ratios depend upon the quadrant in which the terminal side of the angle lies. We always take the length of $\overrightarrow{OP}$ vector is denoted by r which is always positive.

Thus, $\sin\theta = \dfrac{y}{r}$ has the sign of y and $\cos\theta = \dfrac{x}{r}$ has the sign of x.

The sign of tanθ depends on the signs of x and y and similarly the signs of other trignometric functions can be observed by the signs of x and /or y.



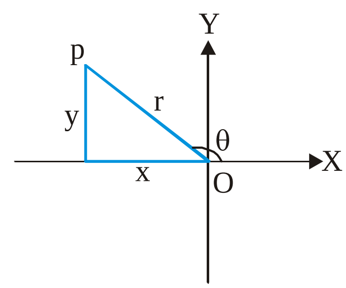
In First quadrant, we have

$x > 0,\ y > 0$

$\therefore \quad \sin\theta = \dfrac{y}{r} > 0 \qquad \cos\theta = \dfrac{x}{r} > 0, \qquad \tan\theta = \dfrac{y}{x} > 0$

$\text{cosec}\theta = \dfrac{r}{y} > 0, \qquad \sec\theta = \dfrac{r}{x} > 0 \quad \text{and} \quad \cot\theta = \dfrac{x}{y} > 0$

Thus, in the first quadrant all trigonometric functions are positive.



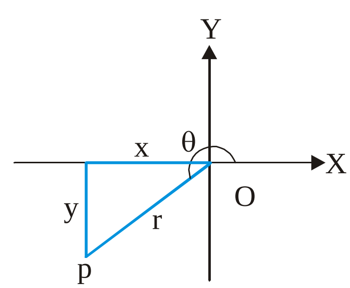
In second quadrant, we have

$x < 0,\ y > 0$

$\therefore \quad \sin\theta = \dfrac{y}{r} > 0 \qquad \cos\theta = \dfrac{x}{r} < 0, \qquad \tan\theta = \dfrac{y}{x} < 0$

$\text{cosec}\theta = \dfrac{r}{y} > 0, \qquad \sec\theta = \dfrac{r}{x} < 0 \quad \text{and} \quad \cot\theta = \dfrac{x}{y} < 0$

Thus, in the second quadrant all trigonometric function are negative other than sine and cosecent.



In third quadrant, we have

$x < 0,\ y < 0$

$\therefore \quad \sin\theta = \dfrac{y}{r} < 0 \qquad \cos\theta = \dfrac{x}{r} < 0, \qquad \tan\theta = \dfrac{y}{x} > 0$

$\text{cosec}\theta = \dfrac{r}{y} < 0 \qquad \sec\theta = \dfrac{r}{x} < 0 \quad \text{and} \quad \cot\theta = \dfrac{x}{y} > 0$

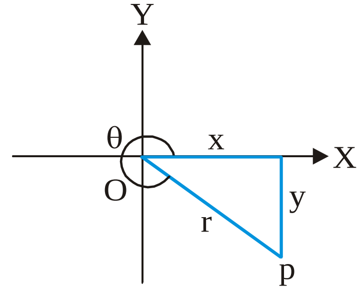
Thus, in the third quadrant all trigonometric function are negative other than tangent and cotangent.

In fourth quadrant, we have

$x > 0,\ y < 0$

$\therefore \quad \sin\theta = \frac{y}{r} < 0 \qquad \cos\theta = \frac{x}{r} > 0, \qquad \tan\theta = \frac{y}{x} < 0$

$\operatorname{cosec}\theta = \frac{r}{y} < 0 \qquad \sec\theta = \frac{r}{x} > 0 \quad \text{and} \quad \cot\theta = \frac{x}{y} < 0.$

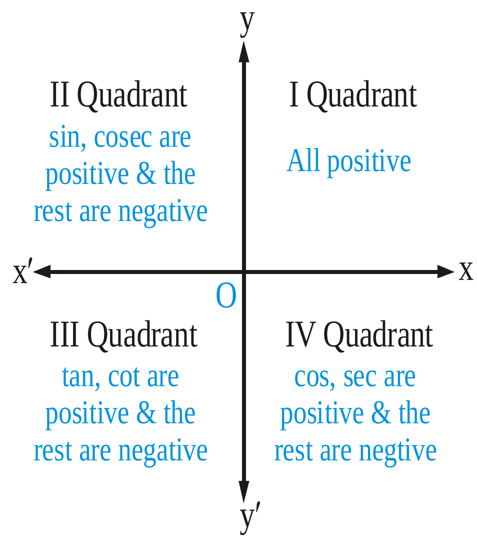


Thus, in the fourth quadrant all trigonometric functions are negative other than cosine and secant.

It follows from the above discussion that the signs of the trigonometric ratios in different quadrants are as follows:

2.3 Sign of T-ratios in four Quadrants:

- If revolving ray lies in Q-I, x and y are positive, hence all T-ratios are positive.
- If revolving ray lies in Q-II, x is negative and y is positive, hence only sinθ and cosecθ are positive.
- If revolving ray lies in Q-III, x is negative and y is negative, hence only tanθ and cotθ positive.
- If revolving ray lies in Q-IV, x is positive and y is negative, hence only cosθ and secθ are positive.



2.4. Graph and Properties of Trignometric Functions

1. y = sinθ or sin$x$

As we have explained all trignometric ratios are functions (i.e., relations between angle and ratio) which implies that each trignometric function must possess pictorial representation i.e. graph.

Now, for graph of trignometric y = sinx, we have to observe nature of y = sinx in different quadrants.

In Ist quadrant :

As x varies from 0 to $\frac{\pi}{2}$ then corresponding ratio of $y = \sin x$ is positive and increases from 0 to 1.

In 2nd quadrant :

As x varies from $\frac{\pi}{2}$ to π then corresponding ratio of $y = \sin x$ is positive and decreases from 1 to 0.

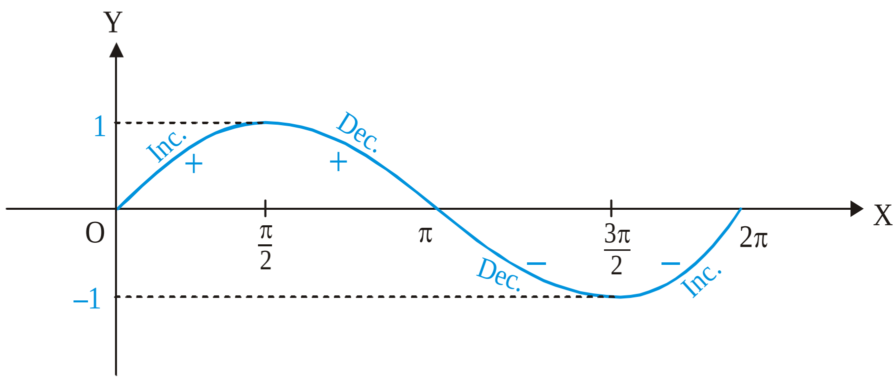
In 3rd quadrant :

As x varies from π to $\frac{3\pi}{2}$ then corresponding ratio of $y = \sin x$ is negative and decreases from 0 to –1.

In 4th quadrant :

As θ varies from $\frac{3\pi}{2}$ to 2π then corresponding ratio of $y = \sin x$ is negative and increases from –1 to 0.

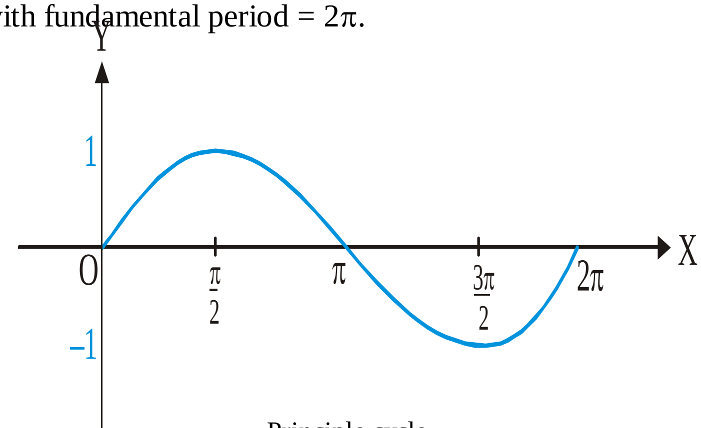
Now, from the above discussion, we have the graph of $y = \sin x$ in 0 to 2π is



If angle is increased further from 2π then we observe that its ratio i.e. output starts repeating.

Which implies $y = \sin x$ is periodic function with fundamental period = 2π.

Properties of $y = \sin x$:



Principle cycle

(i) Domain of $y = \sin x$ is $x \in R$

(ii) Range of $y = \sin x$ is $y \in [-1, 1]$

(iii) It is periodic function with fundamental period of 2π.

(iv) variation of $y = \sin x$ is

$y \in [-1, 1] \Rightarrow -1 \leq \sin x \leq 1$

$\Rightarrow \quad y_{max} = 1$ and $y_{min} = -1$

(v) Variation of $y = A \sin(mx)$

$y \in [-A, A] \Rightarrow -A \leq A \sin(mx) \leq A$

$\Rightarrow \quad A \sin(mx)$ can never be greater than A or less than $-A$

$\Rightarrow \quad y_{max} = A$ and $y_{min} = -A$

(vi) Period of A sin mx is $T = \frac{2\pi}{m}$

2. $y = \cos\theta$ or $\cos x$

In 1st quadrant :

As x varies from 0 to $\frac{\pi}{2}$ then corresponding ratio of $y = \cos x$ is positive and decreases from 1 to 0.

In 2nd quadrant :

As x varies from $\frac{\pi}{2}$ to π then corresponding ratio of $y = \cos x$ is negative and decreases from 0 to –1.

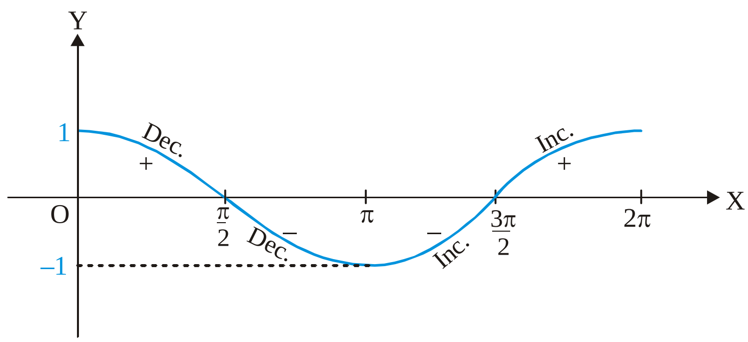
In 3rd quadrant :

As x varies from π to $\frac{3\pi}{2}$ then corresponding ratio of $y = \cos x$ is negative and increases from –1 to 0.

In 4th quadrant :

As x varies from $\frac{3\pi}{2}$ to 2π then corresponding ratio of $y = \cos x$ is positive and increases from 0 to 1.

Now, from the above discussion, we have the graph of $y = \cos x$ in 0 to 2π is



If angle is increases further from 2π then we observe that its ratio i.e. output starts repeating.

Which implies $y = \cos x$ is periodic function with fundamental period = 2π.

Properties of $y = \cos x$:

(i) Domain of $y = \cos x$ is $x \in R$

(ii) Range of $y = \cos x$ is $y \in [-1, 1]$

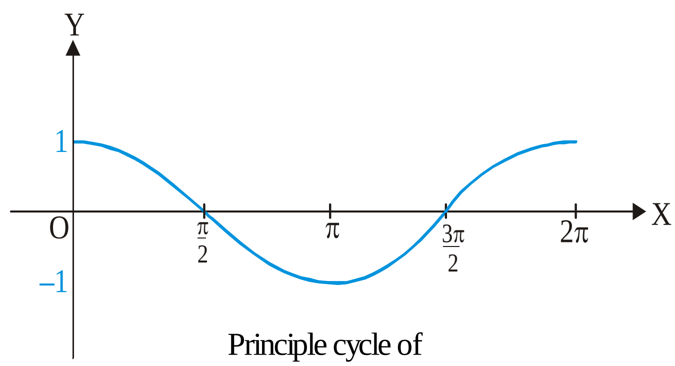
(iii) It is periodic function with fundamental period of 2π.

(iv) variation of $y = \cos x$

$$y \in [-1, 1]$$

$$\Rightarrow \quad -1 \le \cos x \le 1$$

$$\Rightarrow \quad y_{max} = 1 \text{ and } y_{min} = -1$$



Principle cycle of

(v) Variation of $y = A \cos(mx)$

is $y \in [-A, A] \quad \Rightarrow \quad -A \leq A \cos(mx) \leq A$

$\Rightarrow$ $A \cos(mx)$ can never be greater than A or less than $-A$

$\Rightarrow$ $y_{max} = A$ and $y_{min} = -A$

(vi) Period of A cos mx is $T = \dfrac{2\pi}{m}$

3. $y = \tan\theta$ or $\tan x$

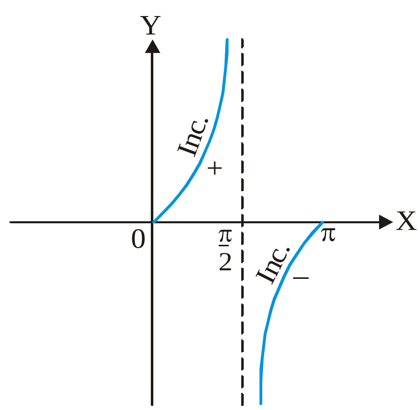
In Ist quadrant :

As x varies from 0 to $\frac{\pi}{2}$ then corresponding ratio of $y = \tan x$ is positive and increases from 0 to ∞ and at $x = \frac{\pi}{2}$ $y = \tan x$ is not defined.

In 2nd quadrant :

As x varies from $\frac{\pi}{2}$ to π then corresponding ratio of $y = \tan x$ is negative and increases from $-\infty$ to 0.

Now, from the above discussion, we have the graph of $y = \tan x$ in 0 to π is



If angle is increased further from π then we observe that its ratio i.e. output starts repeating. Which implies $y = \tan x$ is periodic function with fundamental period = π.

Properties of $y = \tan x$:

(i) Domain of $y = \tan x$ is $x \in R - (2n + 1)\frac{\pi}{2}$

(ii) Range of $y = \tan x$ is $y \in (-\infty, \infty)$

(iii) It is periodic function with fundamental period of π.

(iv) variation of $y = \tan x$

$y \in (-\infty, \infty)$

$\Rightarrow \quad -\infty < \tan(mx) < \infty$

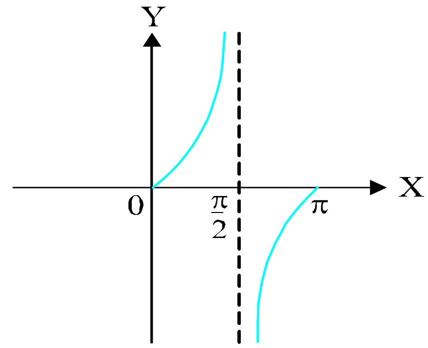
$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$

(v) Variation of $y = A\tan(mx)$ is $y \in (-\infty, \infty)$

$\Rightarrow \quad -\infty < A\tan(mx) < \infty$

$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$

(vi) Period of A tan (mx) is $T = \pi/m$



y = tan x

4. *y* = cot (x)

Properties of $y = \cot(x)$

(i) Domain of the $y = \cot(x)$ is $x \in R - (n\pi)$

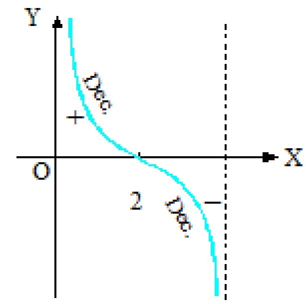
(ii) Range of the $y = \cot(x)$ is $y \in (-\infty, \infty)$

(iii) It is a periodic function with period of π.

(iv) Variation of $y = \tan x$

$y \in (-\infty, \infty)$

$\Rightarrow \quad -\infty < \cot(mx) < \infty \quad \Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$



Principal cycle of A *cot x*

(v) Variation of $y = A\cot(mx)$

as $\quad y \in (-\infty, \infty) \quad \Rightarrow \quad -\infty < A\cot(mx) < \infty \quad \Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$

(vi) Period of A cot (mx) is $T = \pi/m$

5. *y* = cosec (x)

Properties of $y = \text{cosec}(x)$

(i) Domain of the $y = \text{cosec}(x)$ is $x \in R - (n\pi)$

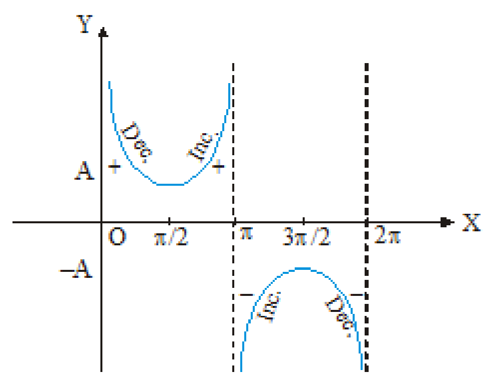
(ii) Range of the $y = \text{cosec}(x)$ is $y \in (-\infty, -1] \cup [1, \infty)$

(iii) It is periodic function with period of 2π

(iv) variation of $y = \text{cosec}\ x$

$y \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$



Principal cycle of A *cosec x*

(v) Variation of $y = A\,\text{cosec}(mx)$ as $y \in (-\infty, -A] \cup [A, \infty)$

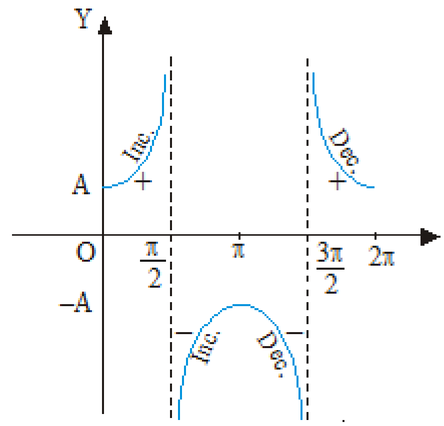
$\Rightarrow \quad A\,\text{cosec}(mx)$ can be greater than A or less than $-A$

$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$

(vi) Period of A cosec (mx) is T = 2π/m

6. *y* = sec (x)

Properties of $y = \sec(x)$



Principal cycle of $A\sec x$

(i) Domain of the $y = \sec(x)$ is $x \in R - (2n+1)\dfrac{\pi}{2}$

(ii) Range of the $y = \sec(x)$ is $y \in (-\infty, -1] \cup [1, \infty)$

(iii) It is periodic function with period of 2π

(iv) variation of $y = \sec x$

$y \in (-\infty, -1] \cup [1, \infty) \quad \Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$

(iv) Variation of $y = A\sec(mx)$ as $y \in (-\infty, -A] \cup [A, \infty)$

$\Rightarrow$ $A\sec(mx)$ can be greater than A or less than $-A$

$\Rightarrow$ $y_{max} = \infty$ and $y_{min} = -\infty$

(vi) Period of A sec (mx) is T = 2π/m

2.5. Trigonometric Ratios of some Standard Acute Angles

The values or trigonometric ratios of standard acute angles are put in the following tabular form for ready reference.

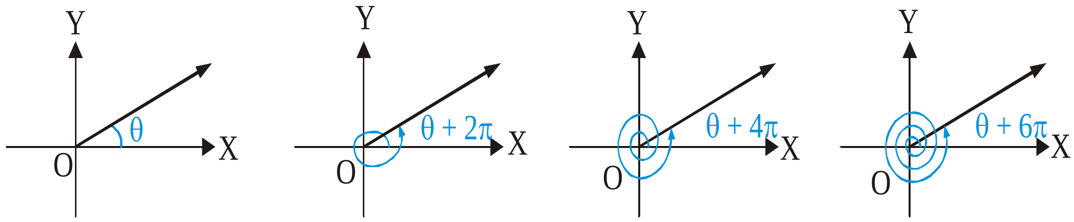
Trigonometric Ratios of Standard Angles

Degree	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**
Radian	0	π/6	π/4	π/3	π/2	2π/3	3π/4	5π/6	π	3π/2
Sinθ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1
cosθ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1/2	$-1/\sqrt{2}$	$-\sqrt{3}/2$	–1	0
tanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	–1	$-1/\sqrt{3}$	0	∞

2.6. Ratios of Allied Angles

T-ratios for $2n\pi + \theta$:

Revolving ray assumes the same position for θ, 2π + θ, 4π + θ, 6π + θ.



***Note*:** That T-ratios will be same for θ, $2\pi + \theta$, $4\pi + \theta$, $6\pi + \theta$, or one can say same for θ and $\theta + 2n\pi$, $n \in I$

$\Rightarrow$ $\sin(2n\pi + \theta) = \sin\theta$

$\cos(2n\pi + \theta) = \cos\theta$

$\tan(2n\pi + \theta) = \tan\theta$

Hence adding and subtracting a multiple of 2π in an angle does not change the value of T-ratio.

T-Ratios for $\frac{\pi}{2} - \theta$:

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\quad$ $\operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ $\quad$ $\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ $\quad$ $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

T-Ratios for $\frac{\pi}{2} + \theta$:

$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos\theta$ $\quad$ $\operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta$

$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ $\quad$ $\sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec}\theta$

$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$ $\quad$ $\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$

T-Ratios for $\pi - \theta$:

$\sin(\pi - \theta) = \sin\theta$ $\quad$ $\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$ $\quad$ $\cot(\pi - \theta) = -\cot\theta$

$\sec(\pi - \theta) = -\sec\theta$ $\quad$ $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta$

T-Ratios for $\pi + \theta$:

$\sin(\pi + \theta) = -\sin\theta$ $\quad$ $\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta$ $\quad$ $\cot(\pi + \theta) = \cot\theta$

$\sec(\pi + \theta) = -\sec\theta$ $\quad$ $\operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}\theta$

T-Ratios for $\frac{3\pi}{2} - \theta$:

$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$ $\quad$ $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta$

$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$ $\quad$ $\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$

T-Ratios for $\frac{3\pi}{2} + \theta$:

$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$ $\quad$ $\cos\left(\frac{3\pi}{2} + \theta\right) = +\sin\theta$

$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$ $\quad$ $\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta$

$\sec\left(\frac{3\pi}{2}-\theta\right)=-\text{cosec}\,\theta$ $\text{cosec}\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$ $\sec\left(\frac{3\pi}{2}+\theta\right)=\text{cosec}\,\theta$ $\text{cosec}\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$

T-Ratios for $2\pi - \theta$:

$\sin(2\pi-\theta) = -\sin\theta$ $\cos(2\pi-\theta) = \cos\theta$

$\tan(2\pi-\theta) = -\tan\theta$ $\cot(2\pi-\theta) = -\cot\theta$

$\sec(2\pi-\theta) = \sec\theta$ $\text{cosec}(2\pi-\theta) = -\text{cosec}\,\theta$

T-Ratios for $2\pi + \theta$:

$\sin(2\pi+\theta) = \sin\theta$ $\cos(2\pi+\theta) = \cos\theta$

$\tan(2\pi+\theta) = \tan\theta$ $\cot(2\pi+\theta) = \cot\theta$

$\sec(2\pi+\theta) = \sec\theta$ $\text{cosec}(2\pi+\theta) = \text{cosec}\,\theta$

T-Ratios for negative θ $(-\theta)$:

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$

$\text{cosec}(-\theta) = -\text{cosec}\,\theta$ $\sec(-\theta) = \sec\theta$

Illustrating the above concepts :

- $\sin(1050°) = \sin(1080° - 30°) = \sin(6\pi - \pi/6) = \sin(-\pi/6) = -\sin\pi/6 = -1/2$
- $\sin(120°) = \sin 2\pi/3 = \sin(\pi - \pi/3) = \sin\pi/3 = \sqrt{3}/2$
- $\cos(120°) = \cos 2\pi/3 = \cos(\pi - \pi/3) = -\cos\pi/3 = -1/2$
- $\tan(120°) = \tan 2\pi/3 = \tan(\pi - \pi/3) = -\tan\pi/3 = -\sqrt{3}$
- $\sin(135°) = \sin 3\pi/4 = \sin(\pi - \pi/4) = \sin\pi/4 = 1/\sqrt{2}$
- $\sin 9\pi/4 = \sin(2\pi + \pi/4) = \sin\pi/4 = 1/\sqrt{2}$
- $\sin 11\pi/3 = \sin(4\pi - \pi/3) = -\sin\pi/3 = -\sqrt{3}/2$
- $\cos 31\pi/6 = \cos(5\pi + \pi/6) = \cos(\pi + \pi/6) = -\cos\pi/6 = -\sqrt{3}/2$
- $\tan 41\pi/6 = \tan(7\pi - \pi/6) = \tan(\pi - \pi/6) = -\tan\pi/6 = -1/\sqrt{3}$

STANDARD RESULTS IN TRIGONOMETRY — Section - 3

3.1. Trigonometric Ratios for sum and difference of angles :

- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\sin(A-B) = \sin A \cos B - \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$
- $$\left.\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1-\tan A \tan B}\\ \tan(A-B) &= \frac{\tan A - \tan B}{1+\tan A \tan B}\end{aligned}\right] \begin{aligned}&\text{where } A \neq n\pi + \frac{\pi}{2}, B \neq n\pi + \frac{\pi}{2}\\ &\text{and } A \pm B \neq m\pi + \frac{\pi}{2}\end{aligned}$$
- $$\left.\begin{aligned}\cot(A+B) &= \frac{\cot A \cdot \cot B - 1}{\cot A + \cot B}\\ \cot(A-B) &= \frac{\cot A \cdot \cot B + 1}{\cot B - \cot A}\end{aligned}\right] \begin{aligned}&\text{where } A \neq n\pi, B \neq n\pi\\ &\text{and } A \pm B \neq m\pi\end{aligned}$$
- $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$
- $\cot(A+B+C) = \dfrac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}$
- $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

  or,

  $\sin(A+B+C) = \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$
- $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

  or,

  $\cos(A+B+C) = \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$
- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\tan(A_1 + A_2 + \dots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}$, where

  $S_1 = \tan A_1 + \tan A_2 + \dots + \tan A_n$ = the sum of the tangents of the separate angles,

$S_2 = \tan A_1 \tan A_2 + \tan A_2 \tan A_3 + ...$ = the sum of the tangents taken two at a time,

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$ = the sum of the tangents taken three at a time, and so on.

3.2. Trigonometric Ratios of Multiple and Submultiple Angles

(i) $\sin 2A = 2 \sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

(iii) $\cos 2A = 2 \cos^2 A - 1$ **or,** $1 + \cos 2A = 2 \cos^2 A$

(iv) $\cos 2A = 1 - 2 \sin^2 A$ **or,** $1 - \cos 2A = 2 \sin^2 A$

(v) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(vi) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(vii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(ix) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(x) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(xi) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

3.3 Tranformation Formulae

3.3A Expressing Product of Trigonometric Functions as Sum or Difference

(i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(ii) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

(iv) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

The above four formula can be obtained by expanding the right hand side and simplifying.

> ***Note :*** In the fourth formula, there is a change in the pattern. Angle $(A - B)$ comes first and $(A + B)$ later. In the first quadrant, the greater the angle, the less the cosine. Hence cosine of the smaller angle is written first [to get a positive result]

3.3B Expressing Sum or Difference of Two Sines or Two Cosines as a Product

In the formulae derived in the earlier section if we put $A + B = C$ and $A - B = D$, then $A = \dfrac{C + D}{2}$

and $B = \dfrac{C - D}{2}$, these formulae can be rewritten as

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cdot \cos \frac{C - D}{2}$$

$$\sin C - \sin D = 2\sin\frac{C-D}{2}\cdot\cos\frac{C+D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2} \quad \textbf{or} \quad 2\sin\frac{C+D}{2}\cdot\sin\frac{D-C}{2}$$

3.4 General formulae

- $\tan A \pm \tan B = \left.\dfrac{\sin(A \pm B)}{\cos A\cos B}\right]$ where $A, B \neq n\pi + \dfrac{\pi}{2}, n \in Z$
- $\cot A \pm \cot B = \left.\dfrac{\sin(B \pm A)}{\sin A\sin B}\right]$ where $A, B \neq n\pi, n \in Z$
- $1 \pm \tan A\cdot\tan B = \left.\dfrac{\cos(A \mp B)}{\cos A\cos B}\right]$ where $A, B \neq n\pi + \dfrac{\pi}{2}, n \in Z$
- $1 \pm \cot A\cdot\cot B = \pm\left.\dfrac{\cos(A \mp B)}{\sin A\sin B}\right]$ where $A, B \neq n\pi, n \in Z$
- $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta}$ where $\theta \neq n\pi$
- $\dfrac{1+\cos\theta}{\sin\theta} = \cot\dfrac{\theta}{2}$, where $\theta \neq (2n+1)\,n\pi$
- $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\dfrac{\theta}{2}$, where $\theta \neq (2n+1)\,\pi$
- $\dfrac{1+\cos\theta}{1-\cos\theta} = \cot^2\dfrac{\theta}{2}$, where $\theta \neq 2n\,\pi$
- $\tan\left(\dfrac{\pi}{4}+\theta\right) = \dfrac{1+\tan\theta}{1-\tan\theta} = \dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \dfrac{1+\sin 2\theta}{\cos 2\theta}$
- $\tan\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{1-\sin 2\theta}{\cos 2\theta}$

3.5 Values of Trigonometrical Ratios of Some Important Angles and Some Important Results

- $\sin 15° = \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$
- $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$
- $\tan 15° = 2 - \sqrt{3} = \cot 75°$
- $\cot 15° = 2 + \sqrt{3} = \tan 75°$
- $\sin 22\dfrac{1°}{2} = \dfrac{1}{2}\left(\sqrt{2-\sqrt{2}}\right)$
- $\cos 22\dfrac{1°}{2} = \dfrac{1}{2}\left(\sqrt{2+\sqrt{2}}\right)$
- $\tan 22\dfrac{1°}{2} = \sqrt{2} - 1$
- $\cot 22\dfrac{1°}{2} = \sqrt{2} + 1$
- $\sin 18° = \dfrac{\sqrt{5}-1}{4} = \cos 72°$
- $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$
- $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$
- $\cos 36° = \dfrac{\sqrt{5}+1}{4} = \sin 54°$
- $\sin 9° = \dfrac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4} = \cos 81°$
- $\cos 9° = \dfrac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4} = \sin 81°$
- $\cos 36° - \cos 72° = \dfrac{1}{2}$
- $\cos 36° \cos 72° = \dfrac{1}{4}$
- $\sin\theta \sin(60° - \theta) \sin(60° + \theta) = 1/4 \sin 3\theta$
- $\cos\theta \cos(60° - \theta) \cos(60° + \theta) = 1/4 \cos 3\theta$
- $\tan\theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta$

3.6 Expressions of sin A/2 and cos A/2 in terms of sin A

$$\sqrt{1+\sin A} = \left|\sin\frac{A}{2} + \cos\frac{A}{2}\right|$$

Note : We must be careful while determining the square root of trigonometrical function e.g.

$$\sqrt{\sin^2 x} = |\sin x| \text{ not } \sin x$$

Illustration - 4 *Show that :*

(i) $\dfrac{1-\cos\theta}{\sin\theta}=\tan\dfrac{\theta}{2}$

(ii) $\dfrac{\sin\theta}{1+\cos\theta}=\tan\dfrac{\theta}{2}$

(iii) $(\cos\,\theta+\sin\,\theta)^2 = 1 + \sin 2\theta$

(iv) $(\cos\,\theta-\sin\,\theta)^2 = 1 - \sin 2\theta$

(v) $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{1+\tan\theta}{1-\tan\theta}=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

$\tan\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

(vi) $\cot\,\theta-\tan\,\theta = 2\cot 2\theta$

SOLUTION : (Hints)

(i) $\text{LHS}=\dfrac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}=\tan\theta/2$

(ii) $\text{LHS}=\dfrac{2\sin\theta/\cos\theta/2}{2\cos^2\theta/2}=\tan\theta/2$

(iii) & (iv) expand LHS to get answer.

(v) expand using tan (A+B), tan (A-B) and $\tan\dfrac{\pi}{4}=1$

(vi) $\text{LHS}=\dfrac{\cos\theta}{\sin\theta}-\dfrac{\sin\theta}{\cos\theta}=\dfrac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta}=\dfrac{2\cos 2\theta}{\sin 2\theta}=2\cot 2\theta$

Illustration - 5 *Calculate :*

(i) $\sin 15°, \cos 15°, \tan 15°, \cot 15°$ (ii) $\tan 22.5°, \cot 22.5°$ (iii) $\tan 7.5°, \cot 7.5°$

SOLUTION :

(i) $\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° \Rightarrow \sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° \Rightarrow \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$\tan 15° = \dfrac{1-\cos 30°}{\sin 30°}$ $\left(\text{using } \tan\dfrac{\theta}{2}=\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow \tan 15° = 2-\sqrt{3}$

and also, $\cot 15° = \dfrac{1}{\tan 15°} = 2 + \sqrt{3}$

(ii) $\tan 22\frac{1}{2}° = \dfrac{1-\cos 45°}{\sin 45°} = \sqrt{2}-1$

$\cot 22\frac{1}{2}° = \dfrac{1}{\tan 22\frac{1}{2}°} = \sqrt{2}+1$

(iii) $\tan 7\frac{1}{2}° = \dfrac{1-\cos 15°}{\sin 15°} = \dfrac{1-\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \dfrac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$

$\Rightarrow \tan 7\frac{1}{2}° = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$

and also, $\cot 7\frac{1}{2}° = \dfrac{1}{\tan 7\frac{1}{2}°} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

Illustration - 6 *Show that :* $\dfrac{\sin 2x - \sin 3x + \sin 4x}{\cos 2x - \cos 3x + \cos 4x} = \tan 3x$

SOLUTION :

$$\text{LHS} = \frac{\sin 2x - \sin 3x + \sin 4x}{\cos 2x - \cos 3x + \cos 4x}$$

$$= \frac{(\sin 2x + \sin 4x) - \sin 3x}{(\cos 2x + \cos 4x) - \cos 3x} \qquad \left[\text{note that } \frac{2x+4x}{2} = 3x\right]$$

$$= \frac{2\sin\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right) - \sin 3x}{2\cos\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right) - \cos 3x} = \frac{2\sin 3x\cos(-x) - \sin 3x}{2\cos 3x\cos(-x) - \cos 3x}$$

$$= \frac{\sin 3x\,[2\cos x - 1]}{\cos 3x\,[2\cos x - 1]} \quad [\because \cos(-\theta) = \cos\theta]$$

$= \tan 3x = \text{RHS}$

Illustration - 7 *Simplify:* $\dfrac{\sin 300° \tan 330° \sec 420°}{\tan 135° \sin 210° \sec 315°}$

SOLUTION : Given expression

$$= \frac{\sin 300° \tan 330° \sec 420°}{\tan 135° \sin 210° \ \sec 315°} = \frac{\sin(360° - 60°)\tan(360 - 30°)\sec(360° + 60°)}{\tan(180° - 45°)\sin(180° + 30°)\sec(360° - 45°)}$$

$$= \frac{(-\sin 60^o) \times (-\tan 30^o) \times \sec 60^o}{(-\tan 45^o) \times (-\sin 30^o) \times \sec 45^o} = \frac{\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} \times 2}{1 \times \frac{1}{2} \times \sqrt{2}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

Illustration - 8

If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha \cos\beta}$, *then show that* $\tan\dfrac{\theta}{2} = \pm \tan\dfrac{\alpha}{2} \cot\dfrac{\beta}{2}$.

SOLUTION :

Manipulating the given condition as follows : $\quad \dfrac{1}{\cos\theta} = \dfrac{1 - \cos\alpha \cos\beta}{\cos\alpha - \cos\beta}$

$$\Rightarrow \quad \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{(1 - \cos\alpha \cos\beta) - (\cos\alpha - \cos\beta)}{(1 - \cos\alpha \cos\beta) + (\cos\alpha - \cos\beta)} \qquad \text{[Apply 'C' and 'D']}$$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} = \frac{(1 - \cos\alpha) + \cos\beta\,(1 - \cos\alpha)}{(1 + \cos\alpha) - \cos\beta\,(1 + \cos\alpha)} = \frac{(1 - \cos\alpha)\ (1 + \cos\beta)}{(1 + \cos\alpha)\ (1 - \cos\beta)}$$

$$\Rightarrow \quad \tan^2\frac{\theta}{2} \frac{\frac{1 - \cos\alpha}{1 + \cos\alpha}}{\frac{1 - \cos\beta}{1 + \cos\beta}} = \frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\beta}{2}} \quad \Rightarrow \quad \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2} \cdot \cot^2\frac{\beta}{2} \quad \Rightarrow \quad \tan\frac{\theta}{2} = \pm \tan\frac{\alpha}{2} \cdot \cot\frac{\beta}{2}$$

Illustration - 9

If $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1 - e}{1 + e}} \tan\dfrac{\phi}{2}$, *show that* $\cos\phi = \dfrac{\cos\theta - e}{1 - e\cos\theta}$.

SOLUTION : We have to find cosϕ in terms of e and cos θ, so try to convert tanθ/2 to cosϕ.

$$\tan^2\frac{\theta}{2} = \frac{1 - e}{1 + e} \tan^2\frac{\phi}{2}$$

$$\Rightarrow \quad \tan^2\frac{\phi}{2} = \frac{1 + e}{1 - e} \tan^2\frac{\theta}{2} = \frac{1 + e}{1 - e}\left(\frac{1 - \cos\theta}{1 + \cos\theta}\right)$$

$$\Rightarrow \quad \frac{\tan^2\frac{\phi}{2}}{1} = \frac{1 + e - \cos\theta - e\cos\theta}{1 - e + \cos\theta - e\cos\theta}$$

$$\Rightarrow \quad \frac{1-\tan^2\phi/2}{1+\tan^2\phi/2} = \frac{(1-e+\cos\theta-e\cos\theta)-(1+e-\cos\theta-e\cos\theta)}{(1-e+\cos\theta-e\cos\theta)+(1+e-\cos\theta-e\cos\theta)} \quad \text{[Apply C and D]}$$

$$\Rightarrow \quad \cos\phi = \frac{-2e+2\cos\theta}{2-2e\cos\theta} = \frac{\cos\theta-e}{1-e\cos\theta}$$

Illustration - 10 *If* $\tan\beta = \dfrac{\tan\alpha+\tan\gamma}{1+\tan\alpha\tan\gamma}$, *prove that :* $\sin 2\beta = \dfrac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$.

SOLUTION : We have tan β in terms of α and γ, so and we have to express sin 2β in terms of α, γ. Hence we will start with $\sin 2\beta = \dfrac{2\tan\beta}{1+\tan^2\beta}$ and substitute for tan β in R.H.S. Also, as the final expression does not contain tan α and tan γ, so express tanβ in terms of sine and cosine.

$$\tan\beta = \frac{\sin\alpha\cos\gamma+\cos\gamma\sin\alpha}{\cos\alpha\cos\gamma+\sin\alpha\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}$$

Now $\quad \sin\beta = \dfrac{2\tan\beta}{1+\tan^2\beta}$

$$\Rightarrow \quad \sin 2\beta = \frac{2\dfrac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1+\dfrac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}} = \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$= \frac{\sin[\overline{\alpha+\gamma}+\overline{\alpha-\gamma}]+\sin[\overline{\alpha+\gamma}-\overline{\alpha-\gamma})}{1+\sin^2(\alpha+\gamma)-\sin^2(\alpha-\gamma)}$$

$$= \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin[\overline{\alpha+\gamma}+\overline{\alpha-\gamma}]\sin[\overline{\alpha+\gamma}-\overline{\alpha-\gamma}]}$$

$$\Rightarrow \quad \sin 2\beta = \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$$

Illustration - 11 *If* $2\tan\alpha = 3\tan\beta$, *then show that :*

$$\tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}.$$

SOLUTION :

We have to express tan (α – β) in terms of β only. Starting with standard result of tan (α – β) and substituting for tan α = 3/2 tan β in R.H.S. we have :

$$\Rightarrow \quad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\tan\beta}=\frac{3/2\tan\beta-\tan\beta}{1+3/2\tan^2\beta}$$

$$\Rightarrow \quad \tan(\alpha-\beta)=\frac{\tan\beta}{2+3\tan^2\beta}=\frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}$$

$$\Rightarrow \quad =\frac{2\sin\beta\cos\beta}{4\cos^2\beta+6\sin^2\beta}=\frac{\sin 2\beta}{2(1+\cos 2\beta)+3(1-\cos 2\beta)}$$

$$\Rightarrow \quad \tan(\alpha-\beta)=\frac{\sin 2\beta}{5-\cos 2\beta}$$

Illustration - 12 *If* $\alpha+\beta=90^\circ$ *and* $\beta+\gamma=\alpha$ *then prove that* $\tan\alpha=\tan\beta+2\tan\gamma$.

SOLUTION : $\alpha+\beta=90^\circ$

Taking tan on both sides

Now $\quad \tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan 90^\circ=\dfrac{\text{finite}}{0} \quad \Rightarrow \quad \tan\alpha\tan\beta=1$

Now $\quad 2n\pi\le x\le\dfrac{\pi}{3}+2n\pi \text{ and } 2n\pi+\dfrac{5\pi}{3}\le x\le 2\pi+2n\pi$

$\Rightarrow \quad \tan\alpha-\tan\alpha\tan\beta\tan\gamma=\tan\beta+\tan\gamma$

$\Rightarrow \quad \tan\alpha-\tan\gamma=\tan\beta+\tan\gamma \qquad (\because \tan\alpha\tan\beta=1)$

$\Rightarrow \quad \tan\alpha=\tan\beta+2\tan\gamma$

Illustration - 13 $a\sin^2 x+b\cos^2 x=c,\ b\sin^2 y+a\cos^2 y=d$ *and* $a\tan x=b\tan y$, *then* $\dfrac{a^2}{b^2}$ *is*

(A) $\dfrac{(b-c)(d-b)}{(a-d)(c-d)}$ (B) $\dfrac{(a-d)(c-a)}{(b-c)(d-b)}$ (C) $\dfrac{(d-a)(c-a)}{(b-c)(d-b)}$ (D) $\dfrac{(b-c)(b-d)}{(a-c)(a-d)}$

SOLUTION : (B)

We have to find value of $\dfrac{a^2}{b^2}$, i.e. $\dfrac{\tan^2 y}{\tan^2 x}$. $[\because a\tan x=b\tan y]$

Given that: $a\sin^2 x+b\cos^2 x=c \Rightarrow a\sin^2 x+b\left(1-\sin^2 x\right)=c$

$$\Rightarrow \quad \sin^2 x(a-b)=c-b \quad \Rightarrow \quad \sin^2 x=\frac{c-b}{a-b}$$

$\therefore \quad \cos^2 x = 1 - \sin^2 x \quad \Rightarrow \quad \cos^2 x = \dfrac{a-c}{a-b}$

So, $\quad \tan^2 x = \dfrac{c-b}{a-c} = \dfrac{b-c}{c-a} \quad \ldots .(i)$

Similarly, we can find $\tan^2 y$

$\Rightarrow \quad \tan^2 y = \dfrac{a-d}{d-b}$ [Replace c by d, b by a, a by b in (i)]

$\Rightarrow \quad \dfrac{a^2}{b^2} = \dfrac{\tan^2 y}{\tan^2 x} = \dfrac{a-d}{d-b} \times \dfrac{c-a}{b-c}$

Illustration - 14 *If $x = \sin(\alpha-\beta)\cdot\sin(\gamma-\delta)$; $y = \sin(\beta-\gamma)\cdot\sin(\alpha-\delta)$ and $z = \sin(\gamma-\alpha)\cdot\sin(\beta-\delta)$, then*

(A) $x + y + z = 0$ **(B)** $x + y - z = 0$ **(C)** $y + z - x = 0$ **(D)** *None of these*

SOLUTION : (A)

Consider $x = \sin(\alpha-\beta)\sin(\gamma-\delta)$, multiply both side by '2'. We get :

$\because \quad 2x = 2\sin(\alpha-\beta)\sin(\gamma-\delta)$

$2x = \cos(\alpha-\beta-\gamma+\delta) - \cos(\alpha-\beta+\gamma-\delta) \quad \ldots .(i)$

And similarly

$2y = \cos(\beta-\gamma-\alpha+\delta) - \cos(\beta-\gamma+\alpha-\delta) \quad \ldots .(ii)$

$2z = \cos(\gamma-\alpha-\beta+\delta) - \cos(\gamma-\alpha+\beta-\delta) \quad \ldots .(iii)$

Now adding at (i), (ii) and (iii) to get :

$$2x + 2y + 2z = \cos(\alpha-\beta-\gamma+\delta) - \cos(\alpha-\beta+\gamma-\delta) + \cos(\beta-\gamma-\alpha-\delta)$$
$$- \cos(\beta-\gamma-\alpha+\delta) + \cos(\gamma-\alpha-\beta+\delta) - \cos(\gamma-\alpha+\beta-\delta)$$

As $\quad \cos(\alpha-\beta-\gamma+\delta) = \cos(\gamma-\alpha+\beta-\delta)$

$\cos(\beta-\gamma-\alpha+\delta) = \cos(\alpha-\beta+\gamma-\delta)$

$\cos(\gamma-\alpha-\beta+\delta) = \cos(\beta-\gamma-\alpha-\delta)$

$\Rightarrow \quad 2x + 2y + 2z = 0 \quad \Rightarrow \quad x + y + z = 0$ **[As $\cos(-\theta) = \cos\theta$]**

Illustration - 15 *The value of $\sqrt{3}$ cosec 20° – sec 20° is :*

(A) 1 **(B)** -1 **(C)** 0 **(D)** 4

SOLUTION :

Consider L.H.S. $= \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°} = \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cdot \cos 20°}$

Multiply and divided by

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°}$$

$$= 4 \cdot \frac{(\sin 60° \cdot \cos 20° - \cos 60° \cdot \sin 20°)}{\sin 40°} = 4\,\frac{\sin(60° - 20°)}{\sin 40°} = 4 \cdot \frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Illustration - 16 *The value of expression $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A$ is :*

(A) $\tan A$ **(B)** $\cot A$ **(C)** $\tan\dfrac{A}{2}$ **(D)** $\cot\dfrac{A}{2}$

SOLUTION : (B)

L.H.S. $= \tan A + 2\tan 2A + 4\tan 4A + 8\left(\dfrac{1-\tan^2 4A}{2\tan 4A}\right)$ $\left[\text{By using } \cot 2\theta = \left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right)\right]$

$= \tan A + 2\tan 2A + \left(\dfrac{4\tan^2 4A + 4 - 4\tan^2 4A}{\tan 4A}\right)$

$= \tan A + 2\tan 2A + 4\cot 4A$

$= \tan A + 2\tan 2A + 4\left(\dfrac{1-\tan^2 2A}{2\tan 2A}\right)$ $\left[\text{By using } \cot 2\theta = \left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right)\right]$

$= \tan A + \left[\dfrac{2\tan^2 2A + 2 - 2\tan^2 2A}{\tan 2A}\right]$

$= \tan A + 2\cot 2A$

$= \tan A + 2\left(\dfrac{1-\tan^2 A}{2\tan A}\right)$ $\left[\text{By using } \cot 2\theta = \left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right)\right]$

$= \dfrac{\tan^2 A + 1 - \tan^2 A}{\tan A} = \cot A = \text{R.H.S.}$

> ***Note :*** Students are advised to learn above result as formulae.
> *i.e.*, $\tan A + 2\cot 2A = \cot A$

Illustration - 17

Find set of all possible values of α in $[-\pi, \pi]$ such that $\sqrt{\dfrac{1-\sin\alpha}{1+\sin\alpha}}$ is equal to $(\sec\alpha - \tan\alpha)$.

(A) $0 < \alpha < \dfrac{\pi}{3}$ **(B)** $-\pi < \alpha < \dfrac{\pi}{4}$ **(C)** $-\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$ **(D)** $-\pi < \alpha < \pi$

SOLUTION : (C)

Clearly $\alpha \neq \pm\dfrac{\pi}{2}$

as, $\sec\alpha - \tan\alpha = \dfrac{1-\sin\alpha}{\cos\alpha}$ . . . .(i)

and $\sqrt{\dfrac{1-\sin\alpha}{1+\sin\alpha}} = \sqrt{\dfrac{(1-\sin\alpha)^2}{\cos^2\alpha}} = \left|\dfrac{1-\sin\alpha}{\cos\alpha}\right| = \dfrac{1-\sin\alpha}{|\cos\alpha|}$ . . . .(ii) As [$1-\sin\alpha$ is always +ve]

From (i) and (ii) two expressions are equal only if $\cos\alpha > 0$, i.e., $-\pi/2 < \alpha < \pi/2$

$\therefore$ $\sqrt{\dfrac{1-\sin\alpha}{1+\sin\alpha}}$ and $\sec\alpha - \tan\alpha$ are equal only where $\alpha = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Illustration - 18

If $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta + \dfrac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta - \dfrac{2\pi}{3}\right)}$, then $x + y + z$ is equal to :

(A) -1 **(B)** 1 **(C)** 0 **(D)** *None of these*

SOLUTION : (C)

Given $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta + \dfrac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta - \dfrac{2\pi}{3}\right)} = \lambda$ (say)

$$\Rightarrow \quad x + y + z = \lambda\left\{\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right)\right\} = \lambda\left\{\cos\theta + 2\cos\theta\cos\frac{2\pi}{3}\right\}$$

Illustration - 19 *Match the column :*

	Column I		***Column II***
(A)	$cos^2 52\frac{1}{2}^\circ - sin^2 22\frac{1}{2}^\circ$	**1.**	1
(B)	$cos^2 \frac{3\pi}{5} + cos^2 \frac{4\pi}{5}$	**2.**	$\frac{3-\sqrt{3}}{4\sqrt{2}}$
(C)	sin 24° + cos 6°	**3.**	$\frac{3}{4}$
(D)	$\sin^2 50° + \cos^2 130°$	**4.**	$\frac{\sqrt{15}+\sqrt{3}}{4}$

SOLUTION : (A-Q) [B-R] [C-S] [D-P]

(i) $\cos^2 52\frac{1}{2} - \sin^2 22\frac{1}{2} = \cos 75° \cos 30° = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{3-\sqrt{3}}{4\sqrt{2}}$

(ii) $\cos^2 \frac{3\pi}{5} + \cos^2 \frac{4\pi}{5} = \left(\frac{\sqrt{5}-1}{2}\right)^2 + \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{3}{4}$

(iii) $\sin 24° + \cos 6° = 2 \sin 54 \cos 30 = \frac{\sqrt{15}+\sqrt{3}}{4}$

(iv) $\sin^2 50° + \cos^2 130° = 1$

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRICAL EXPRESSIONS Section - 4

In this section, we shall discuss problems on finding the maximum and minimum values of various trigonometrical expresions.

As discussed that $-1 \le \sin x \le 1, -1 \le \cos x \le 1, -\infty < \tan x < \infty, |\sec x| \ge 1$ and $|\operatorname{cosec} x| \ge 1$

Consider the expression $a \cos\theta \pm b \sin\theta$, where θ is a variable.

Let $y = a\cos\theta \pm b\sin\theta$

Further, let $a = r\cos\alpha$ and $b = r\sin\alpha$. Then, $r = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$

$\therefore \quad y = r\cos\alpha\cos\theta \pm r\sin\alpha\sin\theta \quad \Rightarrow \quad y = r\cos(\theta \mp \alpha)$

We know that $-1 \le \cos(\theta \mp \alpha) \le 1$ for all θ

$\Rightarrow \quad -r \le r\cos(\theta \mp \alpha) \le r$ for all θ

$\Rightarrow \quad -\sqrt{a^2+b^2} \le y \le \sqrt{a^2+b^2}$ for all θ

$\Rightarrow \quad -\sqrt{a^2+b^2} \le a\cos\theta \pm b\cos\theta \le \sqrt{a^2+b^2}$ for all θ

It follows from the above discussion that $-\sqrt{a^2+b^2}$ and $\sqrt{a^2+b^2}$ are minimum and maximum values of $a\cos\theta \pm b\sin\theta$ for varying values of θ.

***Note* :** Above result can also be derived by taking $a = r\sin\alpha$ and $r\cos\alpha$.

Important : The maximum and minimum values of $a\cos\theta \pm b\sin\theta + c$ are

$c+\sqrt{a^2+b^2}$ and $c-\sqrt{a^2+b^2}$, respectively.

i.e., $c-\sqrt{a^2+b^2} \le a\cos\theta \pm b\sin\theta + c \le c+\sqrt{a^2+b^2}$.

Illustrating the Concepts:

Find the maximum and minimum value of :

(i) $\sin\theta + \cos\theta$ **(ii)** $\sqrt{3}\sin\theta - \cos\theta$ **(iii)** $5\sin\theta + 12\cos\theta + 7$

Given expressions are in the form of :

$a\sin\theta + b\cos\theta$.

Express this in terms of one T-ratio by dividing and multiplying by $(a^2 + b^2)^{1/2}$

(i) $\sin\theta + \cos\theta = 1\cdot\sin\theta + 1\cdot\cos\theta$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$$

$$= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

Now sine of an angle must be between –1 and 1.

$$\Rightarrow \quad -1 \le \sin\left(\theta + \frac{\pi}{4}\right) \le 1$$

$$\Rightarrow \quad -\sqrt{2} \le \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \le \sqrt{2}$$

So maximum value of $\sin\theta + \cos\theta$ is $\sqrt{2}$ and minimum value of $\sin\theta$ + coθ is $-\sqrt{2}$.

(ii)

$$\sqrt{3}\sin\theta - \cos\theta = 2\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$$

[Multiplying and divide by $\sqrt{(\sqrt{3})^2 + (1)^2}$]

$$= 2\left(\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6}\right)$$

$$= 2\sin\left(\theta - \frac{\pi}{6}\right)$$

as $-1 \le \sin\left(\theta - \frac{\pi}{6}\right) \le 1$

$$\Rightarrow \quad -2 \le 2\sin\left(\theta - \frac{\pi}{6}\right) \le 2$$

So maximum value is 2 and minimum value is –2.

(iii) Consider 5 sinθ + 12 cosθ

= 13 [5/13 sinθ + 12/13 cosθ]

[Multiplying and divide by $\sqrt{5^2+12^2}$]

Construct a triangle with sides, 5, 12, 13.

If α is an angle of triangle,

then cosθ = 5/13, sinθ = 12/13,

5 sinθ + 12 cosθ

=13[sin θ cosα + cosθ sinα]

5 sinθ + 12 cosθ + 7

= 13 [sin (θ + α)] + 7

as $-1 \le \sin(\theta+\alpha) \le 1$

$\Rightarrow \quad -13 \le 13\sin(\theta+\alpha) \le 13$

$-13+7 \le 13\sin(\theta+\alpha)+7 \le 13+7$

So maximum value is 20 and minimum value is –6.

Note : Above questions can be solved using formula given in section -4.

Illustration - 20 *The expression 5 cosx + 3 cos (x + π/3) + 3 lies between*

(A) 4 and 10 **(B)** -4 and 10 **(C)** (0,4) and (5, 10) **(D)** None of these

SOLUTION : (B)

$$5\cos x + 3\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right) + 3$$

$$= \cos x\left(5+\frac{3}{2}\right) - \sin x\cdot\frac{3\sqrt{3}}{2} + 3$$

$$= \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$= \sqrt{\frac{169}{4}+\frac{27}{4}}\left\{\frac{13/2}{\sqrt{\frac{169}{4}+\frac{27}{4}}}\cos x - \frac{3\sqrt{3}/2}{\sqrt{\frac{169}{4}+\frac{27}{4}}}\sin x\right\} + 3$$ [Multiplying and divide by $\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$]

= 7 (cosα cos$x$ – sinα sin$x$) + 3 [where $\tan\alpha = \frac{3\sqrt{3}}{13}$]

= 7 cos (α + x) + 3

As $-1 \le \cos(\alpha+x) \le 1$

i.e., $-7+3 \le 7\cos(\alpha+x)+3 \le 7+3$

i.e., $-4 \le 7\cos(\alpha+x)+3 \le 10$

Illustration - 21 *If $\theta \in R$, the expression $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$ lies between*

(A) $\frac{-1}{2}\sqrt{b^2+(a-c)^2}$ and $\frac{1}{2}\sqrt{b^2+(a-c)^2}$ (B) $\frac{a+c}{2}$ and $\frac{a-c}{2}$

(C) $\frac{a+c}{2}-\frac{1}{2}\sqrt{b^2+(a-c)^2}$ and $\frac{a+c}{2}+\frac{1}{2}\sqrt{b^2+(a-c)^2}$ (D) *None of these*

SOLUTION : (C)

Let $f(\theta) = a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$

$$= \frac{a(1-\cos 2\theta)}{2} + \frac{b}{2}\sin 2\theta + \frac{c(1+\cos 2\theta)}{2} = \left(\frac{c}{2}-\frac{a}{2}\right)\cos 2\theta + \frac{b}{2}\sin 2\theta + \frac{a}{2} + \frac{c}{2}$$

$$= \frac{1}{2}\left\{(a+c) + \sqrt{b^2+(a-c)^2}\right\}(\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha)$$

$$= \frac{1}{2}(a+c) + \frac{\sqrt{b^2+(a-c)^2}}{2}\sin(2\theta-\alpha)$$

As $-1 \le \sin(2\theta-\alpha) \le 1$

$$\therefore \quad \left\{\frac{a+c}{2}\right\} - \frac{\sqrt{b^2+(a-c)^2}}{2} \le f(\theta) \le \left\{\frac{a+c}{2}\right\} + \frac{\sqrt{b^2+(a-c)^2}}{2}$$

Note : Above questions can also be solved by directly applying result given in Section -4.

Illustration - 22 *Find the maximum and minimum values of $\sin^6 x + \cos^6 x$.*

(A) 1 *and* $\frac{1}{4}$ (B) 1 *and* $\frac{3}{4}$ (C) 0 *and* $\frac{1}{4}$ (D) 0 *and* $\frac{3}{4}$

SOLUTION : (C)

If the function contains only even powers of sine and cosine of the same angle, use the following propeties

(i) $\sin^2 x + \cos^2 x = 1$ **(ii)** $2\sin x\cos x = \sin 2x$.

Let $f(x) = \sin^6 x + \cos^6 x$

$= (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)$

$= 1 - \sin^2 x\cos^2 x$

$= 1 - \frac{3}{4}(\sin 2x)^2$

When $(\sin 2x)^2$ is minimum *i.e.* 0, then y will be maximum and when $(\sin 2x)^2$ is maximum *i.e.* 1, then y will be minimum.

Hence $f(x)_{max} = 1 - \frac{3}{4} \times 0 = 1$ and $f(x)_{max} = 1 - \frac{3}{4} \times 1 = \frac{1}{4}$.

TRIGNOMETRIC SERIES IN WHICH ANGLES ARE IN A.P. — Section - 5

5.1 TYPE-1

Problems based on finding the sum of series of sines or cosines whose angles are in A.P.

Following two results are very useful in solving such problems.

Result I : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots.. + \sin(\alpha + \overline{n-1}\beta) = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$.

Result II : $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots\ldots\ldots + \cos(\alpha + \overline{n-1}\beta) = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \cos\left[\alpha + (n-1)\frac{\beta}{2}\right]$.

Proof for Result I :

Let $S = \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots.. + \sin(\alpha + \overline{n-1}\,\beta)$

Here angles are in A.P. and common difference of angles = β

Note : In a series of sine and cosine whenever angles are in A.P. and power of sine and cosine is one then we multiply each term by $2\sin\left(\dfrac{\text{Common difference of angles}}{2}\right)$, then express each term as a difference of two terms and add.

So, multiplying both sides by $2\sin\frac{\beta}{2}$, we get :

$$S.\,2\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\cdot\sin\frac{\beta}{2} + \ldots.. + 2\sin(\alpha + \overline{n-1}\beta)\sin\frac{\beta}{2} \quad \ldots. \text{(i)}$$

Now, first term of above R.H.S. can be written as $2\sin\alpha\cdot\sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$

Similarly, the second term R.H.S. can be written as $2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$

… … … … … … … … … … … …
… … … … … … … … … … … …

and also the last term of R.H.S. can be written as

$$2\sin(\alpha+\overline{n-1}\beta)\cdot\sin\frac{\beta}{2}=\cos\left[\alpha+(2n-3)\frac{\beta}{2}\right]-\cos\left[\alpha+(2n-1)\frac{\beta}{2}\right]$$

Now, adding above all terms, we get R.H.S. of (1) as $\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left[\alpha+(2n-1)\frac{\beta}{2}\right]$.

From (i), we get :

$$2\sin\frac{\beta}{2}\cdot S=2\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]\cdot\sin\frac{n\beta}{2} \quad \Rightarrow \quad S=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cdot\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

Students are advised to prove the result – 2 themselves.

Illustration - 23 *The value of expression : cos $2\pi/7$ + cos $4\pi/7$ + cos $6\pi/7$ is :*

(A) $\frac{1}{2}$ **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** 1

SOLUTION : (B)

$$\text{L.H.S.} = \frac{2\sin\frac{\pi}{7}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)}{2\sin\frac{\pi}{7}}$$

$$=\frac{1}{2\sin\frac{\pi}{7}}\left[\left(\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}\right)+\left(\sin\frac{5\pi}{7}-\sin\frac{3\pi}{7}\right)+\left(\sin\frac{7\pi}{7}-\sin\frac{5\pi}{7}\right)\right]$$

$$=\frac{\sin\pi-\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}=-\frac{1}{2}$$

Alternative Method :

We can also use the relation :

$$\cos a+\cos(a+d)+\ldots\ldots\cos(a+\overline{n-1}d)=\frac{\sin nd/2}{\sin d/2}\cos\left(\frac{2a+\overline{n-1}d}{2}\right)$$

[where d is common difference of .AP]

$$\Rightarrow \quad \text{L.H.S.} = \frac{\sin 3\left(\frac{2\pi/7}{2}\right)}{\sin\frac{2\pi/7}{2}}\cos\left(\frac{\frac{4\pi}{7}+2\left(\frac{2\pi}{7}\right)}{2}\right)$$

$$= \frac{\sin\frac{3\pi}{7}\cdot\cos\frac{4\pi}{7}}{\sin\frac{\pi}{7}} = \frac{\cos\left(\frac{\pi}{2}-\frac{3\pi}{7}\right)\sin\left(\frac{\pi}{2}-\frac{4\pi}{7}\right)}{\sin\frac{\pi}{7}} = \frac{\cos\frac{\pi}{14}\sin\left(\frac{-\pi}{14}\right)}{\sin\frac{\pi}{7}} = \frac{-1}{2}\frac{\sin\frac{\pi}{7}}{\sin\frac{\pi}{7}} = \frac{-1}{2}$$

5.2 TYPE - 2

If angles are in A.P. and sum of the first and the last angles is π or $\pi/2$. Then we will use the following working rule and power of sine and cosine may or may not be one.

Working Rule

If angles are in A.P. and sum of the sum first and last angles is $\pi/2$ or π etc., then group equidistant terms from both ends, express last angle in terms of the first in each group and simplify.

Illustration - 24 *The value of expression :* $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$ *is :*

(A) $\frac{1}{2}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{4}$

SOLUTION : (B)

$$\text{L.H.S.} = \left(\cos^4\frac{\pi}{8}+\cos^4\frac{7\pi}{8}\right)+\left(\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}\right)$$

$$= \left[\cos^4\frac{\pi}{8}+\cos^4\left(\pi-\frac{\pi}{8}\right)\right]+\left[\cos^4\frac{3\pi}{8}+\cos^4\left(\pi-\frac{3\pi}{8}\right)\right]$$

$$= 2\cos^4\frac{\pi}{8}+2\cos^4\frac{3\pi}{8} \qquad [\text{As } \cos(\pi-\theta)=-\cos\theta]$$

$$= 2\left[\left(\cos^2\frac{\pi}{8}\right)^2+\left(\cos^2\frac{3\pi}{8}\right)^2\right] = 2\left[\left(\frac{1+\cos\frac{\pi}{4}}{2}\right)^2+\left(\frac{1+\cos\frac{3\pi}{4}}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}\left[1+\frac{1}{2}+1+\frac{1}{2}\right] = \frac{3}{2}$$

5.3 TYPE - 3

If angles are in G.P. having common ratio 2 or 1/2, terms of sine or cosine are in power one and in product.

Illustrating the Concepts:

Prove that : $\cos A \cos 2A \cos 4A \cos 8A \ldots\ldots \cos 2^{n-1}A = \dfrac{1}{2^n \sin A}\sin(2^n A)$

Multiply above and below by $2^n \sin A$

$$\therefore \quad \text{L.H.S.} = \frac{2^{n-1}}{2^n \sin A}[2 \sin A \cos A \cos 2A \cos 4A \ldots. \cos 2^{n-1} A]$$

$$= \frac{2^{n-2}}{2^n \sin A}[2 \sin 2A \cos 2A \cos 4A \ldots.. \cos 2^{n-1} A]$$

$$= \frac{2^{n-3}}{2^n \sin A}[2 \sin 4A \cos 4A \ldots. \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}[2 \sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}\sin(2 \,.\, 2^{n-1} A) = \frac{\sin(2^n A)}{2^n \sin A}.$$

Illustration - 25 *The value of expression cos 20° cos 40° cos 60° cos 80° is:*

(A) 1/8 **(B)** 1/4 **(C)** 1/16 **(D)** *None of these*

SOLUTION : (C)

In given expression : $\cos 60° = \dfrac{1}{2}$ and $n = 3$

by using Type 3 to get :

$$\Rightarrow \quad \text{L.H.S.} = \frac{1}{2}\left[\frac{1}{2^3 \sin 2\theta}\right]\sin(2^3 \cdot 20°)$$

$$= \frac{1}{16} \times \frac{1}{\sin 20°} \cdot \sin 160° = \frac{1}{16 \sin 20°} \cdot \sin(180 - 20°) = \frac{1}{16}\left(\frac{\sin 20°}{\sin 20°}\right) = \frac{1}{16}$$

Illustration - 26 *The value of expression sin 10° sin 30° sin 50° sin 70° is:*

(A) 1/8 **(B)** 1/16 **(C)** 1/4 **(D)** 3/4

SOLUTION : (B)

By complementry rule

$\sin\theta = \cos(90 - \theta)$

The given expression reduces to cos 20° cos 40° cos 60° cos 80°

which is same as above illustration.

5.4 TYPE-4

If angles are in A.P., terms in sine or cosine having power one, and in product and sum of the first and last angles is not $\pi/2$ or π etc. then

Working Rule :

(i) Change the last angle using formula for $\pi - \theta$ etc.

(ii) rearrange the terms in ascending order of angles.

(iii) group the terms in two parts: One part consisting of terms having angles in G.P. and the other part consisting of remaining terms.

(iv) Simplify

Illustration - 27 *The value of expression :* $\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}\cdot\cos\frac{6\pi}{7}$ *is :*

(A) 1 **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{16}$

SOLUTION : (C)

Let $y = \cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}\cdot\cos\frac{6\pi}{7}$

$= \cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}\cdot\cos\left(\pi - \frac{\pi}{7}\right)$

$= -\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}$.

[Here there is only one group consisting of terms having angles in G.P. So, we multiply both sides by $2\sin\pi/7$ and simplify].

$\therefore \quad 2y\sin\frac{\pi}{7} =$

$-\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

$\Rightarrow \quad 2y\sin\frac{\pi}{7} = -\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

$\Rightarrow \quad 4y\sin\frac{\pi}{7} =$

$-\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\cdot\cos\frac{4\pi}{7}$

[Multiplying both side by 2]

$\Rightarrow \quad 4y\sin\frac{\pi}{7} = -\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}$

$\Rightarrow \quad 8y\sin\frac{\pi}{7} = -2\sin\frac{4\pi}{7}\cdot\cos\frac{4\pi}{7}$

[Multiplying both side by 2]

$\Rightarrow \quad 8y\sin\frac{\pi}{7} = -\sin\frac{8\pi}{7} = -\sin\left(\pi + \frac{\pi}{7}\right)$

$\Rightarrow \quad 8y\sin\frac{\pi}{7} = +\sin\frac{\pi}{7} \Rightarrow y = \frac{1}{8}$

Illustration - 28 *The value of expression is* $\sin^2 12° + \sin^2 21° + \sin^2 39° + \sin^2 48° - \sin^2 9° - \sin^2 18°$ *is:*

(A) 1 **(B)** 0 **(C)** -1 **(D)** None

SOLUTION : (A)

$\sin^2 12° + \sin^2 21° + \sin^2 39° + \sin^2 48° - \sin^2 9° - \sin^2 18°$

$\sin^2 12° + \sin^2 21° + (\sin^2 39° - \sin^2 9°) + (\sin^2 48° - \sin^2 18°)$

Multiplying and divide by 2,

$$\frac{1}{2}[\,2\sin^2 12° + 2\sin^2 21° + 2(\sin^2 39° - \sin^2 9°) + 2(\sin^2 48° - \sin^2 18°)]$$

By using $2\sin^2\theta = 1 - \cos 2\theta$ and $\sin^2\theta - \sin^2\phi = \sin(\theta + \phi)\cdot\sin(\theta - \phi)$

$$\Rightarrow \frac{1}{2}[1 - \cos 24° + 1 - \cos 42° + 2\sin 48°\sin 30° + 2\sin 66°\sin 30°] = 1$$

As $\sin 30° = \frac{1}{2}$ and by complementary rule $\sin 66° = \cos 24°$; $\sin 48° = \cos 42°$.

Illustration - 29 *The value of expression :* $\sin 6°\sin 42°\sin 66°\sin 78°$ *is :*

(A) $\frac{3}{4}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{1}{2}$

SOLUTION :

Note that (66 + 6)/2 = 36 and (66 – 6)/2 = 30. Hence sin 6° and sin 66° should be combined.

L.H.S. $= 1/4\,[2\sin 6°\sin 66°]\,[2\sin 42°\sin 78°]$

$= 1/4\,[\cos(6° - 66°) - \cos(6° + 66°)]\,[\cos(42° - 78°) - \cos(42° + 78°)]$

$= 1/4\,[\cos 60° - \cos 72°]\,[\cos 36° - \cos 120°]$

Substituting the values, we get :

$$\text{L.H.S.} = \frac{1}{4}\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}+\frac{1}{2}\right)=\frac{1}{4}\left(\frac{2-\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}+1+2}{4}\right)$$

$$= \frac{1}{64}\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)=\frac{1}{16}=\text{R.H.S.}$$

Illustration - 30 *The value of expression : sin 20 sin 40 sin 80 is: (where angles are in degrees)*

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{\sqrt{3}}{8}$

SOLUTION : (D)

Consider L.H.S. = sin 20 sin 40 sin 80

= sin 20 sin (60 – 20) sin (60 + 20)

$$= \frac{1}{4}\sin 3(20) = \frac{1}{4}\sin 60 = \frac{\sqrt{3}}{8}$$

Illustration - 31 *The value of expression :* $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}$ *is :*

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{16}$

SOLUTION : (A)

(i) $\sin\frac{\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{6\pi}{14}\right)=\cos\frac{6\pi}{14}=\cos\left(\pi-\frac{8\pi}{14}\right)=-\cos\frac{8\pi}{14}$

$$\sin\frac{3\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right)=\cos\frac{4\pi}{14}$$

$$\sin\frac{5\pi}{14}=\sin\left(\frac{\pi}{2}-\frac{2\pi}{14}\right)=\cos\frac{2\pi}{14}$$

$\therefore$ L.H.S. $= -\cos\frac{2\pi}{14}\cos\frac{4\pi}{14}\cos\frac{8\pi}{14}$

$$= -\frac{1}{2^3\sin A}\cdot\sin\left(2^3 A\right)$$ [where $A=\frac{2\pi}{14}$]

$$= -\frac{1}{8\sin\frac{\pi}{7}}\sin\frac{8\pi}{7}$$

$$= -\frac{1}{8\sin\frac{\pi}{7}}\sin\left(\pi+\frac{\pi}{7}\right) = -\frac{1}{8}(-1) = \frac{1}{8}$$ [sin (π + θ) = – sin θ]

Illustration - 32 *The value of expression :* $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$ *is:*

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{\sqrt{3}}{8}$

SOLUTION : (C)

By supplementary rule : cos (π – *A*) = – cos *A*

$$\text{L.H.S.} = \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{\pi}{10}\right)$$

$$= \left(1-\cos^2\frac{\pi}{10}\right)\left(1-\cos^2\frac{3\pi}{10}\right) = \sin^2 18^\circ \sin^2 54^\circ = \left(\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{5}+1}{4}\right)^2 = \frac{1}{16}$$

Illustration - 33 *The value of expression : cos 60° cos 36° cos 42° cos 78° is:*

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{\sqrt{3}}{8}$

SOLUTION : (C)

Consider L.H.S. :

$$= \frac{1}{2}\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{1}{2}\cdot(2\cos 42^\circ\cos 78^\circ) = \frac{1}{16}(\sqrt{5}+1)(\cos 120^\circ + \cos 36^\circ)$$

$$= \frac{1}{16}(\sqrt{5}+1)\left(-\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right) = \frac{1}{16}(\sqrt{5}+1)\cdot\frac{\sqrt{5}-1}{4} = \frac{1}{16}\cdot\frac{5-1}{4} = \frac{1}{16}.$$

Illustration - 34 *For a positive integer n, let*

$$f_n(\theta) = \left(\tan\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta).$$ *Then :*

(A) $f_2\left(\frac{\pi}{16}\right)=1$ **(B)** $f_3\left(\frac{\pi}{32}\right)=1$ **(C)** $f_4\left(\frac{\pi}{64}\right)=1$ **(D)** $f_5\left(\frac{\pi}{128}\right)=1$

SOLUTION : (ABCD)

We have $\left(\tan\frac{\theta}{2}\right)(1+\sec\theta)=\frac{\sin(\theta/2)}{\cos(\theta/2)}\left[1+\frac{1}{\cos\theta}\right]=\frac{\sin(\theta/2)}{\cos(\theta/2)}\cdot\frac{2\cos^2(\theta/2)}{\cos\theta}$

$$=\frac{2\sin(\theta/2)\cos(\theta/2)}{\cos\theta}=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

Thus, $f_n(\theta)=\left(\tan\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= (\tan\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= (\tan 2\theta)(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= (\tan 4\theta)(1+\sec 8\theta)\ldots.(1+\sec 2^n\theta)$

$= \ldots..$

$= \tan(2^n\theta)$

Now, $f_2\left(\frac{\pi}{16}\right)=\tan\left(2^2\frac{\pi}{16}\right)=\tan\left(\frac{\pi}{4}\right)=1$

$f_3\left(\frac{\pi}{32}\right)=\tan\left(2^3\frac{\pi}{32}\right)=\tan\left(\frac{\pi}{4}\right)=1$

$f_4\left(\frac{\pi}{64}\right)=\tan\left(2^4\frac{\pi}{64}\right)=\tan\left(\frac{\pi}{4}\right)=1$

and $f_5\left(\frac{\pi}{128}\right)=\tan\left(2^5\frac{\pi}{128}\right)=\tan\left(\frac{\pi}{4}\right)=1$

CONDITIONAL IDENTITIES ★★ Section - 6

TYPE-I

Problems based on transformation of the plus form of sine or cosine in product form :

Working Rule :

(i) Simplify the terms containing A and B using the formula of $\sin C \pm \sin D$ and $\cos C \pm \cos D$ whichever is applicable.

(ii) Simplify the term containing C by using $\sin 2\theta = 2\sin\theta\cos\theta$ or $\cos 2\theta = 2\cos^2\theta - 1$ or cos

$2\theta = 1 - 2\sin^2\theta$. and write $A + B$ in terms of C and then take out the common factor (which will be a term containing C).

(iii) Then transform C in $A + B$ (not in the factor which has been taken common).

(iv) Then simplify to get the desired result.

Illustrating the Concepts :

If $A + B + C = \pi$, then show that :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iii) $\cos A + \cos B + \cos C = 1 + 4 \sin A/2 \sin B/2 \sin C/2$

(iv) $\sin A + \sin B + \sin C = 4 \cos A/2 \cos B/2 \cos C/2$

(i) LHS $= \sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A + B)\cos(A - B) + \sin 2C$

$= 2\sin(\pi - C)\cos(A - B) + \sin 2C$ [Convert $A + B$ to $\pi - C$]

$= 2\sin C\cos(A - B) + 2\sin C\cos C$

$= 2\sin C[\cos(A - B) + \cos C]$ [Take the term involving C as factor]

Convert the expression with-in brackets in A and B only.

$= 2\sin C[\cos(A - B) + \cos(\pi - A - B)]$

$= 2\sin C[\cos(A - B) - \cos(A + B)]$

$= 2\sin C(2\sin A\sin B)$

$= 4\sin A\sin B\sin C =$ RHS

(ii) LHS $= \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + \cos 2C$

$= -2\cos C\cos(A - B) + 2\cos^2 C - 1$

$= -2\cos C[\cos(A - B) - \cos C] - 1$

$= -2\cos C[\cos(A - B) + \cos(A + B)] - 1$

$= -2\cos C[2\cos A\cos B] - 1$

$= -1 - 4\cos A\cos B\cos C =$ RHS

(iii) LHS $= \cos A + \cos B + \cos C$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + \left(1 - 2\sin^2\frac{C}{2}\right)$$ [Express cos C in terms of sin $\frac{C}{2}$]

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right] + 1$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right] + 1$$

$$= 2\sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right] + 1$$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \text{RHS}$$

(iv) Try On The Same Pattern. (As in part 3 above)

TYPE-II

Problems involving squares of sines or cosines of angles.

Working Rule :

(i) First, rearrange the terms in L.H.S. so that either $\sin^2 A - \sin^2 B = \sin(A+B)\cdot\sin(A-B)$ or $\cos^2 A - \sin^2 B = \cos(A+B)\cdot\cos(A-B)$ can be applied.

(ii) Simplify the term containing C and then take out the common factor (which will be a term containing C).

(iii) Then transform C in $A + B$ (not in the factor which has been taken common).

(iv) Then simplify to get the desired result.

Illustrating the Concepts :

If $A + B + C = \pi$, then show that :

(i) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

(ii) $\cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2 = 2 + 2\sin A/2 \sin B/2 \sin C/2$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

(i) Starting from L.H.S. :

$= \sin^2 A + \sin^2 B - \sin^2 C$

$= \sin^2 A + \sin(B + C)\sin(B - C)$

$= \sin^2 A + \sin(\pi - A)\sin(B - C)$

$= \sin A\,[\sin A + \sin(B - C)]$

$= \sin A\,\{\sin[\pi - (B + C)] + \sin(B - C)\}$

$= \sin A\,[\sin(B + C) + \sin(B - C)]$

$= \sin A\,[2\sin B\cos C] = 2\sin A\sin B\cos C$

(ii) L.H.S. $= \cos^2 A/2 + (1 - \sin^2 B/2) + \cos^2 C/2$

$= 1 + (\cos^2 A/2 - \sin^2 B/2) + \cos^2 C/2$

$= 1 + \cos(A + B)/2\cos(A - B)/2 + \cos^2 C/2$

$= 1 + \sin C/2\cos(A - B)/2 + 1 - \sin^2 C/2$

$= 2 + \sin C/2\,[\cos(A - B)/2 - \sin C/2]$

$= 2 + \sin C/2\,[\cos(A - B)/2 - \cos(A + B)/2]$

$= 2 + 2\sin C/2\sin A/2\sin B/2$

(iii) L.H.S. $= \sin^2 A + \sin^2 B + \sin^2 C$

$= 1 - (\cos^2 A - \sin^2 B) + \sin^2 C$

$= 1 - \cos(A + B)\cos(A - B) + \sin^2 C$

$= 1 + \cos C\cos(A - B) + 1 - \cos^2 C$

$= 2 + \cos C\,[\cos(A - B) - \cos C]$

$= 2 + \cos C\,[\cos(A - B) + \cos(A + B)]$

$= 2 + 2\cos C\cos A\cos B$

TYPE III

Problems involving only tangents or cotangents of angles.

Working Rule :

(i) Write the given relation between angles keeping two angles on one side and remaining on the other side.

(ii) Take tangent or cotangent of both sides and simplify to get the desired result.

Illustrating the Concepts :

(i) *In a ΔABC, prove that $\tan A + \tan B + \tan C = \tan A\tan B\tan C$*

(ii) *If $A + B + C = \pi$, prove that : $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$.*

(iii) *If $A + B + C = \pi$, prove that $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.*

(i) In ΔABC, $\because A + B + C = \pi$ $\therefore A + B = \pi - C$

$\Rightarrow \tan(A + B) = \tan(\pi - C)$ [By taking tan on both side]

$\Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$ [$\because \tan(\pi - C) = -\tan C$]

$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $\because A + B + C = \pi$

$\therefore \dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} = \dfrac{\pi}{2}$ or, $\dfrac{A}{2} + \dfrac{B}{2} = \dfrac{\pi}{2} - \dfrac{C}{2}$

$\therefore \tan\left(\dfrac{A}{2} + \dfrac{B}{2}\right) = \tan\left(\dfrac{\pi}{2} - \dfrac{C}{2}\right) = \cot\dfrac{C}{2}$ [By taking tan on both side]

$\Rightarrow \dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \dfrac{1}{\tan\frac{C}{2}}$

$\Rightarrow \tan\dfrac{A}{2}\tan\dfrac{C}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} = 1 - \tan\dfrac{A}{2}\tan\dfrac{B}{2}$

$\Rightarrow \tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1.$

(iii) $\because A + B + C = \pi$ $\therefore A + B = \pi - C$

$\Rightarrow \cot(A + B) = \cot(\pi - C)$ [By taking cot on both side]

$\Rightarrow \dfrac{\cot A \cdot \cot B - 1}{\cot B + \cot A} = -\cot C$

$\Rightarrow \cot A \cot B - 1 = -\cot B \cot C - \cot C \cot A$

$\Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

TYPE IV

Miscellaneous Type

Working Rule :

(i) If it is given that $x + y + z = xyz$ or, $xy + yz + zx = 1$ put $x = \tan A$, $y = \tan B$ and $z = \tan C$ in it and simplify it to find the value of $A + B + C$.

(ii) Then put $\tan A$, $\tan B$, $\tan C$ in place of x, y, z respectively in the L.H.S. of the given question and simplify it to get the desired identify using the value of $A + B + C$.

Illustrating the Concepts :

If $xy + yz + zx = 1$, prove that $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$

Let $x = \tan A$, $y = \tan B$, $z = \tan C$

Given, $xy + yz + zx = 1$

$\therefore \quad \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

$\Rightarrow \quad \tan C (\tan A + \tan B) = 1 - \tan A \tan B$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{\tan C} = \cot C = \tan\left(\frac{\pi}{2} - C\right)$

$\Rightarrow \quad \tan (A + B) = \tan\left(\frac{\pi}{2} - C\right)$

$\therefore \quad A + B = \frac{\pi}{2} - C$ [Taking principal value]

$\Rightarrow \quad A + B + C = \frac{\pi}{2}$ ....(i)

$$\text{Now, L.H.S.} = \frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}$$

$$= \frac{\tan A}{1-\tan^2 A}+\frac{\tan B}{1-\tan^2 B}+\frac{\tan C}{1-\tan^2 C} = \frac{1}{2}\left[\frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}\right]$$

$$= \frac{1}{2}(\tan 2A + \tan 2B + \tan 2C) \quad \text{....(ii)}$$

Now from (i), $A + B + C = \frac{\pi}{2}$ $\Rightarrow$ $2A + 2B + 2C = \pi$

or, $2A + 2B = \pi - 2C$ $\Rightarrow$ $\tan (2A + 2B) = \tan (\pi - 2C)$

or, $\frac{\tan 2A + \tan 2B}{1 - \tan 2A \tan 2B} = -\tan 2C$ or, $\tan 2A + \tan 2B = -\tan 2C + \tan 2A \tan 2B \tan 2C$

or, $\tan 2A + \tan 2C + \tan 2B = \tan 2A \tan 2B \tan 2C$. ....(iii)

From (ii), L.H.S. $= \frac{1}{2}(\tan 2A + \tan 2B + \tan 2C)$

$$= \frac{1}{2}(\tan 2A \tan 2B \tan 2C) \quad \text{[From (iii)]}$$

$$= \frac{1}{2} \cdot \frac{2\tan A}{1-\tan^2 A} \cdot \frac{2\tan B}{1-\tan^2 B} \cdot \frac{2\tan C}{1-\tan^2 C}$$

$$= \frac{1}{2} \cdot \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Illustration - 35 *If $\alpha + \beta + \gamma = \pi$, then : $\tan(\beta + \gamma - \alpha) + \tan(\gamma + \alpha - \beta) + \tan(\alpha + \beta - \gamma)$ is :*

(A) $\tan(\beta - \gamma - \alpha).\tan(\gamma + \alpha - \beta).\tan(\alpha + \beta - \gamma)$

(B) $\cot(\beta + \gamma - \alpha).\cot(\alpha + \gamma - \beta).\cot(\alpha + \beta - \gamma)$

(C) $\cot(\beta + \gamma - \alpha).\cot(\alpha + \gamma - \beta).\tan(\alpha + \beta - \gamma)$

(D) None of these

SOLUTION : (A)

Let $\beta + \gamma - \alpha = A$, $\gamma + \alpha - \beta = B$, and $\alpha + \beta - \gamma = C$

Now, $A + B + C = \beta + \gamma - \alpha + \gamma + \alpha - \beta + \alpha + \beta - \gamma$

or, $A + B + C = \alpha + \beta + \gamma = \pi$ [As $\alpha + \beta + \gamma = \pi$]

or, $A + B = \pi - C$

$\therefore$ $\tan(A + B) = \tan(\pi - C)$

or, $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$ [As $\tan(\pi - C) = -\tan C$]

or, $\tan A + \tan B = -\tan C + \tan A \tan B \tan C$ or, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Putting the values of A, B and C, we get :

$\tan(\beta + \gamma - \alpha) + \tan(\gamma + \alpha - \beta) + \tan(\alpha + \beta - \gamma)$

$= \tan(\beta + \gamma - \alpha)\tan(\gamma + \alpha - \beta)\tan(\alpha + \beta - \gamma)$.

TRIGONOMETRIC EQUATIONS — Section - 7

7.1 Some Basic Results

(i) If $\sin\theta = 0$, then $\theta = 0, \pm\pi, \pm 2\pi, \ldots\ldots\ldots$

$\Rightarrow$ $\theta = n\pi$ where $n \in I$.

(ii) If $\cos\theta = 0$, then $\theta = \pm\pi/2, \pm 3\pi/2 \ldots\ldots.$

$\Rightarrow$ $\theta = (2n + 1)\pi/2$ where $n \in I$.

(iii) If $\tan\theta = 0$ then, $\theta = 0, \pm\pi, \pm 2\pi, \ldots\ldots\ldots$

$\Rightarrow$ $\theta = n\pi$ where $n \in I$.

(iv) If $\sin\theta = 1$ then, $\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, ...$

$\Rightarrow \quad \theta = (4n+1)\frac{\pi}{2}$ where $n \in I$.

(v) If $\cos\theta = 1$ then, $\theta = 0, 2\pi, 4\pi, ...$

$\Rightarrow \quad \theta = 2n\pi$ where $n \in I$.

(vi) If $\tan\theta = 1$ then, $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$

$\Rightarrow \quad \theta = (4n+1)\frac{\pi}{4}$ where $n \in I$.

7.2 Some More Basic Results

(i) If $\sin\theta = \sin\alpha$

then $\theta = n\pi + (-1)^n \alpha$, where $n \in I$

(ii) If $\cos\theta = \cos\alpha$

then $\theta = 2n\pi \pm \alpha$, where $n \in I$

(iii) If $\tan\theta = \tan\alpha$

then $\theta = n\pi + \alpha$, where $n \in I$

(iv) If $\tan^2\theta = \tan^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$

(v) If $\cos^2\theta = \cos^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$

(vi) If $\sin^2\theta = \sin^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$

Illustrating the Concepts :

(i) *Find the values of θ satisfying $\sin\theta = \sin\alpha$*

(ii) *Find the values of θ satisfying $\cos\theta = \cos\alpha$ in the interval $0 \le \theta \le \pi$*

(iii) *Find the values of θ satisfying the equation : $\tan\theta = \tan\alpha$.*

(i) $\sin\theta = \sin\alpha$

$\Rightarrow \quad \sin\theta - \sin\alpha = 0$

$$\Rightarrow \quad 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

Either : $\cos\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2l+1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = n\pi$ [where l, n are integers]

$\theta = (2l+1)\pi - \alpha$ or $\theta = 2n\pi + \alpha$

$\theta = (\text{odd no.})\ \pi - \alpha$ or $\theta = (\text{even no.})\ \pi + \alpha$

$\theta = n\pi + (-1)^n \alpha,\ n \in \text{I}$. [where, $n \in \text{I}$]

(ii) $\cos\theta = \cos\alpha$

$\Rightarrow \cos\theta - \cos\alpha = 0$

$\Rightarrow -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

Either $\sin\frac{\theta+\alpha}{2} = 0$

or $\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow \frac{\theta+\alpha}{2} = n\pi$

or $\frac{\theta-\alpha}{2} = n\pi$ [where, $n \in$ I]

$\Rightarrow \theta = 2n\pi - \alpha$

or $\theta = 2n\pi + \alpha$ [where, $n \in$ I]

Combining the two values, we get :

$\theta = 2n\pi \pm \alpha \quad n \in I$

(iii) $\tan\theta = \tan\alpha$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

$\Rightarrow \sin(\theta - \alpha) = 0$

$\Rightarrow \theta - \alpha = n\pi, n \in I$

$\Rightarrow \theta = n\pi + \alpha, n \in I$

Note : The following results should be committed to memory before proceeding further.

(i) $\sin\theta = \sin\alpha$

$\Rightarrow \theta = n\pi + (-1)^n\alpha, n \in I$

(ii) $\cos\theta = \cos\alpha$

$\Rightarrow \theta = 2n\pi \pm \alpha, n \in I$

(iii) $\tan\theta = \tan\alpha$

$\Rightarrow \theta = n\pi + \alpha, n \in I$

Important : Every trigonometric equation should be manipulated so that it reduces to any of the above results.

Illustration - 36 *The number of value of x lying between 0 and 2π satisfying the equation : sin x + sin 3x = 0 is are:*

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5

SOLUTION : (B)

The given equation is $\sin x + \sin 3x = 0$

$$2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2} = 0$$

$\Rightarrow 2\sin 2x\cos x = 0$

Either $\sin 2x = 0$ or $\cos x = 0$

$\Rightarrow 2x = n\pi$ or $x = (2n+1)\pi/2$, [where, $n \in 1$]

$\Rightarrow x = n\pi/2$ or $x = (2n+1)\pi/2$, [where, $n \in 1$]

This is the general solution of the equation. To get particular solution satisfying $0 < x < 2\pi$, we will substitute integral values of *n*.

(i) $n = 0 \Rightarrow x = 0, \pi/2$ **(ii)** $n = 1 \Rightarrow x = \pi/2, 3\pi/2$

(iii) $n = 2 \Rightarrow x = \pi, 5\pi/2$ **(iv)** $n = 3 \Rightarrow x = 3\pi/2, 7\pi/2$

Hence for $0 < x < 2\pi$, the solution is $x = \pi/2, \pi, 3\pi/2$.

HOW TO SOLVE TRIGONOMETRIC EQUATIONS Section - 8

8.1 While solving equations following points must be kept in mind.

(i) Squaring should be avoided as far as possible, If squaring is done check for the extraneous roots.

(ii) Never cancel equal terms containing 'unknown or variable' on two sides which are in product. It may cause root loss.

(iii) The answer should not contain such values of θ which make any of the terms underfined.

(iv) Domain should not change. If it changes, necessary corrections must be made.

(v) Check that denominator is not zero at any stage while solving equations.

TYPE-I

Problems based on method of factorisation :

Working Rule :

Step I. Take all the terms to L.H.S. i.e. make R.H.S. zero, then factorise L.H.S.

Step II. Equate each factor to zero and solve.

Step III. Check for root loss and extraneous roots.

Illustrating the Concepts :

(i) *Solve : $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ for general values of θ.*

(ii) *Find the values of x satisfying $3 - 2\cos x - 4\sin x - \cos 2x + \sin 2x = 0$.*

(i) Using $\tan(A + B)$, $\tan\theta + \tan 2\theta$

$= \tan 3\theta\,(1 - \tan\theta \tan 2\theta)$

Hence the equation can be written as :

$\tan 3\theta\,(1 - \tan\theta \tan 2\theta) + \tan 3\theta = 0$

$\tan 3\theta\,(2 - \tan\theta \tan 2\theta) = 0$

$\Rightarrow \tan 3\theta = 0$ or $\tan\theta \tan 2\theta = 2$

$\Rightarrow 3\theta = n\pi$ or $2\tan^2\theta = 2(1 - \tan^2\theta)$

$\Rightarrow \theta = n\pi/3,$

or $\tan\theta = \pm \dfrac{1}{\sqrt{2}}$ [where, $n \in$ I]

$\Rightarrow \theta = n\pi/3,$

or $\theta = n\pi \pm \tan^{-1}\dfrac{1}{\sqrt{2}}$ [where, $n \in$ I]

(ii) Put $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow$ $3 - 2\cos x - 4\sin x - (2\cos^2 x - 1) + \sin 2x = 0$

$\Rightarrow$ $(4 - 4\sin x) - 2\cos^2 x - 2\cos x + \sin 2x = 0$

$\Rightarrow$ $4(1 - \sin x) - 2(1 - \sin^2 x) - 2\cos x(1 - \sin x) = 0$

$\Rightarrow$ $(1 - \sin x)(2 - 2\sin x - 2\cos x) = 0$

Either $\sin x = 1$ or $\sin x + \cos x = 1$

$\Rightarrow$ $\sin x = \sin \pi/2$ or $\sqrt{2}\cos(x - \pi/4) = 1$

$\Rightarrow$ $x = n\pi + (-1)^n \pi/2$ or $x - \pi/4 = 2n\pi \pm \pi/4$ **[where, $n \in$ I]**

$\Rightarrow$ $x = n\pi + (-1)^n \pi/2$ or $x = 2n\pi \pm \pi/4 + \pi/4$ **[where, $n \in$ I]**

$\Rightarrow$ $x = n\pi + (-1)^n \pi/2$ or $x = 2n\pi,\ 2n\pi + \pi/2$ **[where, $n \in$ I]**

Combining the two, we get :

$x = 2n\pi,\ 2n\pi + \pi/2.$

Illustration - 37 *The number of solution of $\cos x + \cos 2x + \cos 4x = 0$, where $0 \le x \le \pi$.*

(A) 2 **(B)** 3 **(C)** 4 **(D)** *None of these*

SOLUTION : (C)

$\cos x + (\cos 2x + \cos 4x) = 0$

$\Rightarrow$ $\cos x + 2\cos 3x \cos x = 0$

$\Rightarrow$ $\cos x(1 + 2\cos 3x) = 0$

$\Rightarrow$ $\cos x = 0$

or $1 + 2\cos 3x = 0$

$\Rightarrow$ $\cos x = 0$

or $\cos 3x = -1/2 = \cos 2\pi/3$

$\Rightarrow$ $x = (2n + 1)\pi/2$

or $3x = 2n\pi \pm 2\pi/3$ [where, $n \in$ I]

$\Rightarrow$ $x = (2n + 1)\pi/2,$

or $x = 2n\pi/3 \pm 2\pi/9,$ [where, $n \in$ I]

This is the general solution of the equation. To get particular solution satisfying $0 \le x \le \pi$, we will substitute integral values of n.

(i) $n = 0 \Rightarrow x = \pi/2,\ \pm 2\pi/9$

(ii) $n = 1 \Rightarrow x = 3\pi/2,\ 8\pi/9,\ 4\pi/9$

(iii) $n = 2 \Rightarrow x = 5\pi/2,\ 14\pi/9,\ 10\pi/9$ (greater than π)

(iv) $n = -1 \Rightarrow x = -\pi/2,\ -2\pi/3 \pm 2\pi/9$ (less than 0)

Hence the values for $0 \le x \le \pi$ are $x = \pi/2,\ 2\pi/9,\ 4\pi/9,\ 8\pi/9.$

Illustration - 38 *Number of solution of the equation sinx = cos4x for* $0 \le x \le \pi$.

(A) 3 **(B)** 2 **(C)** 4 **(D)** None of these

SOLUTION : (C)

$\sin x = \cos 4x$

$\Rightarrow \quad \cos 4x = \cos(\pi/2 - x)$

$\Rightarrow \quad 4x = 2n\pi \pm (\pi/2 - x)$

Either $4x = 2n\pi + \pi/2 - x$

or $\quad 4x = 2n\pi - \pi/2 + x$

$\Rightarrow \quad x = 2n\pi/5 + \pi/10$

or $\quad x = 2n\pi/3 - \pi/6$

This is the general solution of the equation. To get particular solution satisfying $0 \le x \le \pi$, we will substitute integral values of n.

(i) $n = 0 \Rightarrow x = \pi/10, -\pi/6$

(ii) $n = 1 \Rightarrow x = \pi/2$

(iii) $n = 2 \Rightarrow x = 9\pi/10, 7\pi/6$

(iv) $n = 3 \Rightarrow x = 13\pi/10, 11\pi/6$ (greater than π)

(v) $n = -1 \Rightarrow x = -3\pi/10, -5\pi/6$ (less than 0)

Hence the required solution for $0 \le x \le \pi$ is $x = \pi/10, \pi/2, 9\pi/10$.

Illustration - 39 *Number of solution of the equation are* $\sec 4x - \sec 2x = 2$; *in* $-\pi \le x \le \pi$.

(A) 8 **(B)** 10 **(C)** 12 **(D)** 14

SOLUTION : (C)

$\sec 4x - \sec 2x = 2$

$$\Rightarrow \quad \frac{1}{\cos 4x} - \frac{1}{\cos 2x} = 2$$

$\Rightarrow \quad \cos 2x - \cos 4x = 2\cos 2x \cos 4x$

$\Rightarrow \quad \cos 2x - \cos 4x = \cos 6x + \cos 2x$

$\Rightarrow \quad \cos 6x + \cos 4x = 0$

$\Rightarrow \quad 2\cos 5x \cos x = 0$

Either $\cos 5x = 0$ or $\cos x = 0$

$\Rightarrow \quad 5x = 2n\pi \pm \pi/2$

or $\quad x = 2n\pi \pm \pi/2$ [where, $n \in$ I]

$\Rightarrow \quad x = 2n\pi/5 \pm \pi/10$

or $\quad x = 2n\pi \pm \pi/2$ [where, $n \in$ I]

This is the general solution of the equation. To get particular solution satisfying $-\pi \le x \le \pi$, we will substitute integral values of n.

Consider $x = 2n\pi/5 \pm \pi/10$:

$n = 0 \Rightarrow x = \pm\pi/10$

$n = \pm 1 \Rightarrow x = \pm\pi/2, -3\pi/10$

$n = \pm 2 \Rightarrow x = \pm 9\pi/10, \pm 7\pi/10$

Consider $x = 2n\pi \pm \pi/2$:

$n = 0 \Rightarrow x = \pm\pi/2$

These are the only values of x in $[-\pi, \pi]$.

Illustration - 40 *The solution of the given equation $\cos\theta\cos 2\theta\cos 3\theta = 1/4$ are $\theta = (2n+1)A$ and $n\pi + B$, then 'A and B' are :*

(A) $\pi, \frac{\pi}{3}$ **(B)** $\frac{\pi}{8}, \frac{\pi}{3}$ **(C)** $\frac{\pi}{2}, 0$ **(D)** None of these

SOLUTION : (B)

$4\cos\theta\cos 2\theta\cos 3\theta = 1$

or, $(2\cos 3\theta\cos\theta)(2\cos 2\theta) = 1$

or, $(\cos 4\theta + \cos 2\theta)(2\cos 2\theta) - 1 = 0$

or, $2\cos 4\theta\cos 2\theta + 2\cos^2 2\theta - 1 = 0$

or, $2\cos 4\theta\cos 2\theta + \cos 4\theta = 0$

or, $\cos 4\theta\,[2\cos 2\theta + 1] = 0$

Either $\cos 4\theta = 0$,

or $2\cos 2\theta + 1 = 0$.

$4\theta = (2n+1)\frac{\pi}{2}$

or $\cos 2\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\Rightarrow \theta = (2n+1)\frac{\pi}{8}$

or $2\theta = 2n\pi \pm \frac{2\pi}{3}$

$\theta = n\pi \pm \frac{\pi}{3}$ [where, $n \in I$]

Hence, $\theta =$

$(2n+1)\frac{\pi}{8}, n\pi \pm \frac{\pi}{3}$ *where* $n = 0, \pm 1, \pm 2, \ldots$

TYPE-II

Equations of the form $a\cos\theta + b\sin\theta = c$

Working Rule :

(i) Divide by $\sqrt{a^2+b^2}$ on both sides and get $\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$.

(ii) Write L.H.S. as $\sqrt{a^2+b^2}\cos(\theta - \alpha)$ [where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$]

(iii) Simplify the equation.

Illustration - 41 *The number of solution of the equation $\sqrt{3}\sin x + \cos x = 1$ in the interval $0 \le x \le 2\pi$.*

(A) 3 **(B)** 2 **(C)** 4 **(D)** None of these

SOLUTION : (A)

$\sqrt{3}\sin x + \cos x = 1$

$\Rightarrow 2(\sqrt{3}/2\sin x + 1/2\cos x) = 1$

[Multiplying and divide by $\sqrt{(\sqrt{3})^2 + (1)^2}$]

$\Rightarrow 2(\cos\pi/3\cos x + \sin\pi/3\sin x) = 1$

$\Rightarrow 2\cos(x - \pi/3) = 1$

$\Rightarrow \cos(x - \pi/3) = \cos\pi/3$

$\Rightarrow x - \pi/3 = 2n\pi \pm \pi/3$

$\Rightarrow \quad x = 2n\pi + 2\pi/3,\ x = 2n\pi$ [where, $n \in I$]

This is the general solution of the equation. To get particular solution satisfying $0 \le x \le 2\pi$ we will substitute integral values of n.

(i) $n = 0 \Rightarrow x = 0$ or $2\pi/3$

(ii) $n = 1 \Rightarrow x = 2\pi + 2\pi/3$ or 2π

(iii) $n = 2 \Rightarrow x = 4\pi + 2\pi/3$

or 4π (greater than π)

(iv) $n = -1 \Rightarrow x = -4\pi/3$

or -2π (less than 0)

Hence the required values of x are 0, $2\pi/3$, 2π.

Illustration - 42

Solve the equation : $\sin x + \cos x = \sin 2x - 1$.

SOLUTION :

Let $t = \sin x + \cos x$

$\Rightarrow \quad t^2 = 1 + 2 \sin x \cos x$

$\Rightarrow \quad \sin 2x = t^2 - 1$

Hence the given equation is $t = (t^2 - 1) - 1$

$\Rightarrow \quad t^2 - t - 2 = 0$

Solving the equation, to get :

$(t - 2)(t + 1) = 0$

$\Rightarrow \quad t = 2$ or $t = -1$

$\Rightarrow \quad \sin x + \cos x = 2$

or $\sin x + \cos x = -1$

$\Rightarrow \quad \sqrt{2}\cos(x - \pi/4) = 2$

or $\sqrt{2}\cos(x - \pi/4) = -1$

$\Rightarrow \quad \cos(x - \pi/4) = \sqrt{2}$

or $\cos(x - \pi/4) = -\dfrac{1}{\sqrt{2}}$

As $-1 \le \cos\theta \le 1$, $\cos(x - \pi/4) = \sqrt{2}$ is impossible.

$\Rightarrow \quad \cos(x - \pi/4) = -\dfrac{1}{\sqrt{2}}$ is the only possibiilty.

$\Rightarrow \quad \cos(x - \pi/4) = \cos(\pi - \pi/4)$

$\Rightarrow \quad x - \pi/4 = 2n\pi \pm 3\pi/4$ [where, $n \in I$]

$\Rightarrow \quad x = 2n\pi \pm 3\pi/4 + \pi/4$ is the general solution.

TYPE-III

Trigonometric Inequality

Working Rule :

While solving inequations involving trigonometric functions, it is best to use graphs of trigonometric functions.

Illustrating the Concepts :

Solve $\cos x \ge \dfrac{1}{2}$

We construct the graph $y_1 = \cos x$, $y_2 = +\frac{1}{2}$.

Now, on this graph we want those values of x for which graph of $y = \cos x$ is more than the graph of $y = \frac{1}{2}$ which is shown by color part of the graph.

We get solution as $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$ or general solution is

$2n\pi \le x \le \frac{\pi}{3} + 2n\pi$ and $2n\pi + \frac{5\pi}{3} \le x \le 2\pi + 2n\pi$.

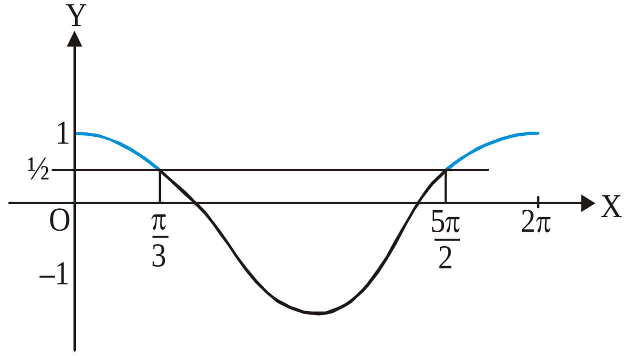


Illustration - 43 *The solution of the inequality sin x + cos 2x > 1 if $0 \le x \le \pi/2$.*

(A) $0 < x < \frac{\pi}{4}$ **(B)** $\frac{\pi}{3} < x < \frac{\pi}{2}$ **(C)** $0 < x < \frac{\pi}{6}$ **(D)** $\frac{\pi}{4} < x < \frac{\pi}{2}$

SOLUTION : (C)

Let $\sin x = t \Rightarrow \cos 2x = 1 - 2t^2$

$\Rightarrow$ The inequality is : $t + 1 - 2t^2 > 1$

$\Rightarrow 2t^2 - t < 0$

$\Rightarrow t(2t - 1) < 0$

$\Rightarrow (t - 0)(t - 1/2) < 0$

$\Rightarrow 0 < t < 1/2$

$\Rightarrow 0 < \sin x < 1/2$

In $0 \le x \le \pi/2$, this means that $0 < x < \pi/6$ is the solution.

TYPE-IV

Simultaneous Equations

(i) Two equations in one variable.

Working Rule :

(i) If two equations in one unknown (say x) are given, then solve given equations separately for x lying between 0 and 2π.

(ii) Select the value of x for which both the equations are satisfied.

(iii) Add $2n\pi$ to get general solution.

Illustrating the Concepts :

Solve $\cos x = \frac{1}{\sqrt{2}}$*, and* $\tan x = -1$

$\cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}, \frac{7\pi}{4}$ in $(0, 2\pi)$

$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$ in $(0, 2\pi)$

$\therefore$ Common value of $x = \frac{7\pi}{4}$

Now for general solution add a factor of ***np***.

$\therefore$ General solution is $x = 2n\pi + \frac{7\pi}{4}$.

★

TYPE-IV

(ii) Two equations in two variable.

Working Rule :

Step I. Find $A + B$ and $A - B$ between 0 and 2π from the two given equations. Impose the condition that A and B must be smallest and positive, since A and B are positive angles, $A + B > A - B$. Find $A + B$ and $A - B$ accordingly.

Step II. Solve the two equation to get A and B.

Step III. Then generalised it with the help of common period. i.e. adding a factor of ***np***.

Illustrating the Concepts :

If tan (A – B) = 1, sec (A + B) = 2/√3, calculate the smallest positive values and the most general values of A and B.

Smallest Positive Values

Let $A, B \in (0, 2\pi)$

$\Rightarrow \quad (A + B) > (A - B)$

Now $\tan (A - B) = 1 \quad \Rightarrow \quad (A - B) = \pi/4, 5\pi/4$

$\sec (A + B) = 2/\sqrt{3} \quad \Rightarrow \quad (A + B) = \pi/6, 11\pi/6$

As $(A + B) > (A - B)$, there are two possibilities :

(i) $A - B = \pi/4$ and $A + B = 11\pi/6$

(ii) $A - B = 5\pi/4$ and $A + B = 11\pi/6$

From (i), we get :

$$A = \frac{25\pi}{24} \quad \text{and} \quad B = \frac{19\pi}{24}$$

From (ii), we get : $A = \frac{37\pi}{24}$ and $B = \frac{7\pi}{24}$

General Values

$\tan(A - B) = 1 \quad \Rightarrow \quad A - B = n\pi + \pi/4$

$\sec(A + B) = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad A + B = 2k\pi \pm \frac{\pi}{6}$

Taking $A - B = n\pi + \frac{\pi}{4}$ and $A + B = 2k\pi + \frac{\pi}{6}$ we get :

$A = \frac{(2k+n)\pi}{2} + \frac{5\pi}{24}$ and $B = \frac{(2k-n)\pi}{2} - \frac{\pi}{24}$

Taking $A - B = n\pi + \frac{\pi}{4}$ and $A + B = 2k\pi - \frac{\pi}{6}$ we get :

$A = \frac{(2k+n)}{2} + \frac{\pi}{24}$ and $B = \frac{(2k-n)}{2} + \frac{5\pi}{24}$

TYPE-V

Miscellaneous Type

(i) Whenever the terms on the two sides (L.H.S. and R.H.S.) of the equation are of different nature, We use inequality method for testing whether the equation has any real solution or not.

Working Rule :

(i) Let y = each side of the equation and break the equation in two parts.

(ii) Form the inequality for y taking L.H.S. of the equation and also for R.H.S. of the equation. If there is any value of y satisfying both the inequalities, then the equation will have real solution and if there is no such y, the equation will have no real solution.

Illustrating the Concepts :

Show tht the equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$ *for* $0 < x \le \frac{\pi}{2}$ *has no real solution.*

Let $y_1 = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x$ ....(i)

and $y_2 = x^2 + x^{-2}$ ....(ii)

From (i), $y_1 = 2\cos^2\frac{x}{2}\cdot\sin^2 x$

$= (1 + \cos x)\cdot\sin^2 x$

$= (< 2) \times (\le 1)$

< 2 $\left[\text{For } 0 < x \le \frac{\pi}{2}, 0 \le \cos x < 1 \text{ and } 0 < \sin x \le 1\right]$

i.e. $y < 2$ . . . .(iii)

From (ii), $y_2 = x^2 + x^{-2} = x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 \ge 2$ $[\because x > 0]$

i.e. $y_2 \ge 2$ . . . .(iv)

No value of y can be obtained satisfying (iii) and (iv) simultaneously,

$\Rightarrow$ no real solution of the equation exists.

(ii) Whenever the equation cotains power terms.

Working Rule :

(i)★ Equate the base if possible.

(ii) If not possible to equate the base then take log of both side

Illustration - 44 *Find the values of x in $(-\pi, \pi)$ which satisfy the equation*

$$8^{1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \ldots \text{ to infinity}} = 4^3.$$

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8

SOLUTION : (B)

$$8^{1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \ldots \text{ to infinity}} = 8^2$$

$\Rightarrow$ $1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \ldots.$ to infinity $= 2$

This is an infinite geometric series with first term 1 and common ratio $|\cos x|$.

$\Rightarrow$ $\frac{1}{1 - |\cos x|} = 2$ [by using sum of infinite geometric series with first term a and common ration r is $\frac{a}{1-r}$ if $|r| < 1$]

or, $1 - |\cos x| = \frac{1}{2}$ $\Rightarrow$ $|\cos x| = \frac{1}{2}$ $\Rightarrow$ $\cos x = \pm\frac{1}{2}$

When $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}, x = 2n\pi \pm \frac{\pi}{3}$

When $\cos x = -\frac{1}{2} = \cos\frac{2\pi}{3}, x = 2n\pi \pm \frac{2\pi}{3}$

The value of x in the given interval $(-\pi, \pi) = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

TYPE-VI

Whenever terms are in sin, cos in power 1, all terms connected with plus sign and number of terms in L.H.S. is equal to the number in R.H.S. [with (+) or (–) sign] then each term must have its exteremum value.

Working Rule :

In such problems each term will be (+1) when the number in R.H.S. is (+)ve and each term will be (–1) when the number in R.H.S. is (–)ve.

Illustrating the Concepts :

Solve the equation sin6x + cos4x + 2 = 0.

$\sin 6x + \cos 4x + 2 = 0$ or $\sin 6x + \cos 4x = -2$ . . . .(i)

$\Rightarrow$ $\sin 6x = -1$ and $\cos 4x = -1$ both satisfied simultaneously

Now, $\sin 6x = -1 = \sin\frac{3\pi}{2} \Rightarrow 6x = 2n\pi + \frac{3\pi}{2} \Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{4}$, where $n \in I$.

$\Rightarrow$ Values of x between 0 and 2π are $\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$

Also, $\cos 4x = -1 = \cos\pi \Rightarrow 4x = 2n\pi + \pi$

$\therefore$ $x = \frac{n\pi}{2} + \frac{\pi}{4}$, where $n \in I$.

$\Rightarrow$ Values of x between 0 and 2π are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Hence, values of x lying between 0 and 2π satisfying both the equations are $\frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore$ General solution will be given by

$x = 2n\pi + \frac{\pi}{4}$ or, $2n\pi + \frac{5\pi}{4}$ [where $n \in I$]

i.e. $x = 2n\pi + \frac{\pi}{4}$ or, $(2n+1)\pi + \frac{\pi}{4}$

Combining these two results, to get :

$x = m\pi + \frac{\pi}{4}$, [where $n \in I$]

THINGS TO REMEMBER

1. Some standard identities for trignometric ratios are :

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \text{cosec}^2\theta$$

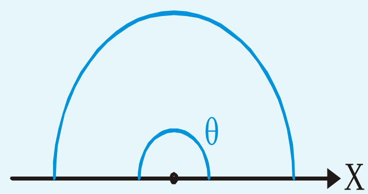
$$1 + \tan^2\theta = \sec^2\theta$$

For acute angles, all T-ratios are positive.

2. Relation between degree and radians :

Let θ be the angle subtended at centre by a semicircle = 180°.

From figure, $\theta = 180°$



In radians : $\theta = \frac{\text{arc}}{\text{radius}} = \frac{\pi r}{r} = \pi$

$\Rightarrow$ 180° degrees = π radian

3. Signs of Trigonometrical Functions

In First quadrant, we have

$x > 0, y > 0$

$\therefore$ $\sin\theta = \frac{y}{r} > 0$ $\cos\theta = \frac{x}{r} > 0,$ $\tan\theta = \frac{y}{x} > 0$

$\text{cosec}\theta = \frac{r}{y} > 0,$ $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} > 0$

Thus, in the first quadrant all trigonometric functions are positive.

In second quadrant, we have

$x < 0, y > 0$

$\therefore$ $\sin\theta = \frac{y}{r} > 0$ $\cos\theta = \frac{x}{r} < 0,$ $\tan\theta = \frac{y}{x} < 0$

$\text{cosec}\theta = \frac{r}{y} > 0,$ $\sec\theta = \frac{x}{r} < 0$ and $\cot\theta = \frac{x}{y} < 0$

Thus, in the second quadrant all trigonometric function are negative other than sine and cosecent.

In third quadrant, we have

$x < 0, y < 0$

$\therefore$ $\sin\theta = \frac{y}{r} < 0$ $\cos\theta = \frac{x}{r} < 0,$ $\tan\theta = \frac{y}{x} > 0$

$\text{cosec}\theta = \frac{r}{y} < 0$, $\sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} > 0$

Thus, in the third quadrant all trigonometric function are negative other than tangent and contangent.

In fourth quadrant, we have

$x > 0,\ y < 0$

$\therefore$ $\sin\theta = \frac{y}{r} < 0$, $\cos\theta = \frac{x}{r} > 0$, $\tan\theta = \frac{y}{x} < 0$

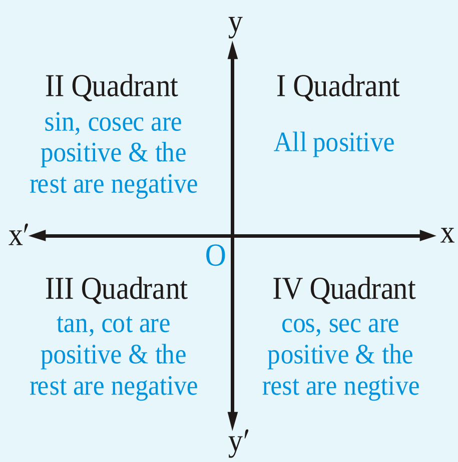
$\text{cosec}\theta = \frac{r}{y} < 0$, $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} < 0$.

Thus, in the fourth quadrant all trigonometric functions are negative other than cosine and secant.

It follows from the above discussion that the signs of the trigonometric ratios in different quadrants are as follows:

4. Sign of T-ratios in four Quadrants :

- If revolving ray lies in Q-I, x and y are positive, hence all T-ratios are positive.
- If revolving ray lies in Q-II, x is negative and y is positive, hence only sinθ and cosecθ are positive.
- If revolving ray lies in Q-III, x is negative and y is negative, hence only tanθ and cotθ positive.
- If revolving ray lies in Q-IV, x is positive and y is negative, hence only cosθ and secθ are positive.



5. Graph and Properties of Trignometric Functions

I. Properties of y = sin$x$:

(i) Domain of y = sin$x$ is $x \in R$

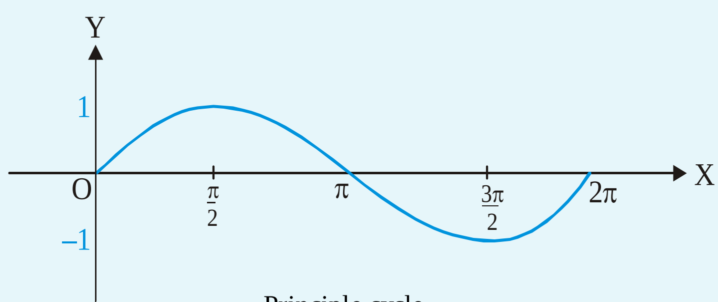
(ii) Range of y = sin$x$ is $y \in [-1, 1]$

(iii) It is periodic function with fundamental period of 2π.

(iv) variation of y = sin$x$

$y \in [-1, 1] \Rightarrow -1 \leq \sin x \leq 1$

$\Rightarrow$ $y_{max} = 1$ and $y_{min} = -1$



Principle cycle

(v) Variation of $y = A\sin(mx)$

as $y \in [-A, A] \Rightarrow -A \leq A\sin(mx) \leq A$

$\Rightarrow$ $A\sin(mx)$ can never be greater than A or less than $-A$

$\Rightarrow$ $y_{max} = A$ and $y_{min} = -A$

II. Properties of $y = \cos x$:

(i) Domain of $y = \cos x$ is $x \in R$

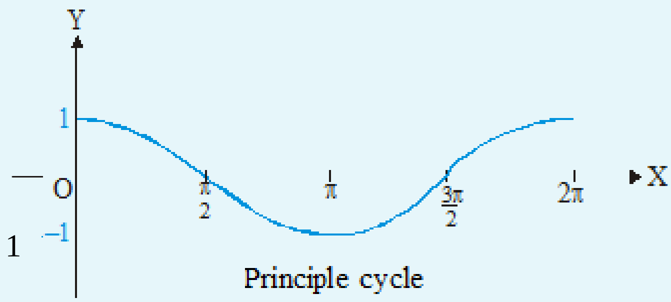
(ii) Range of $y = \cos x$ is $y \in [-1, 1]$

(iii) It is periodic function with fundamental period of 2π.

(iv) variation of $y = \cos x$

$y \in [-1, 1] \quad \Rightarrow \quad -1 \leq \cos x \leq 1$

$\Rightarrow \quad y_{max} = 1$ and $y_{min} = -1$



Principle cycle

(v) Variation of $y = A\cos(mx)$

as $y \in [-A, A] \quad \Rightarrow \quad -A \leq A\cos(mx) \leq A$

$\Rightarrow \quad A\cos(mx)$ can never be greater than A or less than $-A \Rightarrow y_{max} = A$ and $y_{min} = -A$

III. Properties of $y = \tan x$:

(i) Domain of $y = \tan x$ is $x \in R - (2n+1)\dfrac{\pi}{2}$

(ii) Range of $y = \tan x$ is $y \in (-\infty, \infty)$

(iii) It is periodic function with fundamental period of π.

(iv) variation of $y = \tan x$

$y \in (-\infty, \infty) \quad \Rightarrow \quad -\infty < \tan(mx) < \infty$

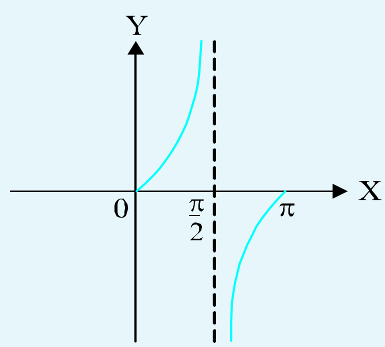
$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty$

(v) Variation of $y = A\tan(mx)$ as $y \in (-\infty, \infty)$

$\Rightarrow \quad -\infty < A\tan(mx) < \infty$

$\Rightarrow \quad A\tan(mx)$ can take any positive or negative value.

$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty \quad \Rightarrow \quad$ Not defined.



y = tan x
Principle cycle

IV. $y = A\cot(mx)$

Properties of $y = A\cot(mx)$

(i) Domain of the $y = A\cot(mx)$ is $x \in R - (n\pi)$

(ii) Range of the $y = A\cot(mx)$ is $y \in (-\infty, \infty)$

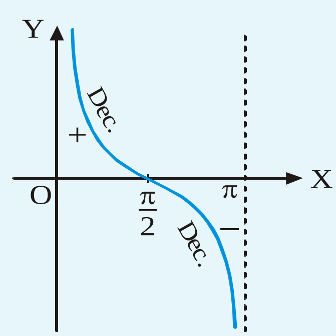
(iii) It is a periodic function with period of π and it is Denoted by 'T' and Period of $y = A\cot(mx)$ is $T = \pi/m$

(iv) Variation of $y = \cot(mx)$

as $\quad y \in (-\infty, \infty) \quad \Rightarrow \quad -\infty < \cot(mx) < \infty$

$\Rightarrow \quad \cot(mx)$ can take any positive or negative value

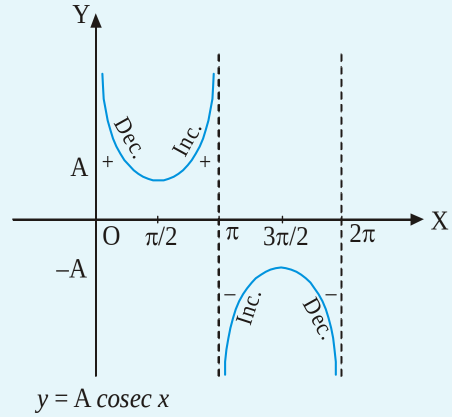
$\Rightarrow \quad y_{max} = \infty$ and $y_{min} = -\infty \quad \Rightarrow \quad$ Not defined



$y = A\cot x$
Principle cycle

V $y = A$ cosec (mx)

Properties of $y = A$ cosec (mx)



Principle cycle

(i) Domain of the $y = A$ cosec (mx) is $x \in R - (n\pi)$

(ii) Range of the $y = A$ cosec (mx) is $y \in (-\infty, -A] \cup [A, \infty)$

(iii) It is periodic function with period of 2π and it is denoted by 'T' and Period of $y = A$ cosec (mx) is $T = 2\pi/m$.

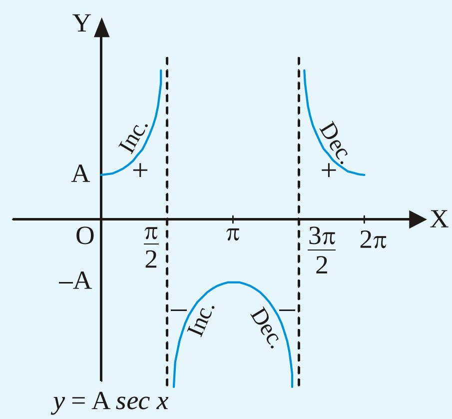
(iv) Variation of $y = A$ cosec (mx) as $y \in (-\infty, -A] \cup [A, \infty)$

$\Rightarrow$ A cosec (mx) can be greater than A or less than $-A$

$\Rightarrow$ $y_{max} = \infty$ and $y_{min} = -\infty$ $\Rightarrow$ Not defined

VI $y = A$ sec (mx)

Properties of $y = A$ sec (mx)



(i) Domain of the $y = A$ sec (mx) is $x \in R - (2n + 1)\dfrac{\pi}{2}$

(ii) Range of the $y = A$ sec (mx) is $y \in (-\infty, -A] \cup [A, \infty)$

(iii) It is periodic function with period of 2π and it is denoted by 'T' and Period of $y = A$ sec (mx) is $T = 2\pi/m$.

(iv) Variation of $y = A$ sec (mx) as $y \in (-\infty, -A] \cup [A, \infty)$

$\Rightarrow$ A sec (mx) can be greater than A or less than $-A$

$\Rightarrow$ $y_{max} = \infty$ and $y_{min} = -\infty$

6. I. Trigonometric Ratios for sum and difference of angles:

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $$\left.\begin{aligned} \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ \tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}\right] \begin{aligned} &\text{where } A \neq n\pi + \frac{\pi}{2}, B \neq n\pi + \frac{\pi}{2} \\ &\text{and } A \pm B \neq m\pi + \frac{\pi}{2} \end{aligned}$$

- $\cot(A+B) = \dfrac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$, $\cot(A-B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$ where $A \neq n\pi, B \neq n\pi$ and $A \pm B \neq m\pi$

- $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

- $\cot(A+B+C) = \dfrac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}$

- $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

  or,

  $\sin(A+B+C) = \cos A \cos B \cos C(\tan A + \tan B + \tan C - \tan A \tan B \tan C)$

- $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

  or,

  $\cos(A+B+C) = \cos A \cos B \cos C(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

- $\tan(A_1 + A_2 + \ldots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$, where

  $S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$ = the sum of the tangents of the separate angles,

  $S_2 = \tan A_1 \tan A_2 + \tan A_2 \tan A_3 + \ldots$ = the sum of the tangents taken two at a time,

  $S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$ = the sum of the tangents taken three at a time, and so on.

II. Trigonometric Ratios of Multiple and Submultiple Angles

(i) $\sin 2A = 2 \sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

(iii) $\cos 2A = 2\cos^2 A - 1$ or, $1 + \cos 2A = 2\cos^2 A$

(iv) $\cos 2A = 1 - 2\sin^2 A$ or, $1 - \cos 2A = 2\sin^2 A$

(v) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

(vi) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

(vii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(ix) $\sin 3A = 3\sin A - 4\sin^3 A$

(x) $\cos 3A = 4\cos^3 A - 3\cos A$ (xi) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

7. Tranformation Formulae

I. Expressing Product of Trigonometric Functions as Sum or Difference

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

II. Expressing Sum or Difference of Two Sines or Two Cosines as a Product

In the formulae derived in the earlier section if we put $A + B = C$ and $A - B = D$, then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$, these formulae can be rewritten as

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\sin\frac{C-D}{2}\cdot\cos\frac{C+D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2} \quad \text{or} \quad 2\sin\frac{C+D}{2}\cdot\sin\frac{D-C}{2}$$

8. General formulae

- $\tan A \pm \tan B = \left.\dfrac{\sin(A \pm B)}{\cos A\cos B}\right]$ where $A, B \neq n\pi + \dfrac{\pi}{2}, n \in Z$
- $\cot A \pm \cot B = \left.\dfrac{\sin(B \pm A)}{\sin A\sin B}\right]$ where $A, B \neq n\pi, n \in Z$
- $1 \pm \tan A\cdot\tan B = \left.\dfrac{\cos(A \mp B)}{\cos A\cos B}\right]$ where $A, B \neq n\pi + \dfrac{\pi}{2}, n \in Z$
- $1 \pm \cot A\cdot\cot B = \pm\left.\dfrac{\cos(A \mp B)}{\sin A\sin B}\right]$ where $A, B \neq n\pi, n \in Z$

- $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta}$ where $\theta \neq 2n\pi$
- $\dfrac{1+\cos\theta}{\sin\theta} = \cot\dfrac{\theta}{2}$, where $\theta \neq (2n+1)\pi$
- $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\dfrac{\theta}{2}$, where $\theta \neq (2n+1)\pi$
- $\dfrac{1+\cos\theta}{1-\cos\theta} = \cot^2\dfrac{\theta}{2}$, where $\theta \neq 2n\pi$
- $\tan\left(\dfrac{\pi}{4}+\theta\right) = \dfrac{1+\tan\theta}{1-\tan\theta} = \dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \dfrac{1+\sin 2\theta}{\cos 2\theta}$
- $\tan\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{1-\sin 2\theta}{\cos 2\theta}$

9. Values of Trigonometrical Ratios of Some Important Angles and Some Important Results

- $\sin 15° = \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$
- $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$
- $\tan 15° = 2-\sqrt{3} = \cot 75°$
- $\cot 15° = 2+\sqrt{3} = \tan 75°$
- $\sin 22\dfrac{1°}{2} = \dfrac{1}{2}\left(\sqrt{2-\sqrt{2}}\right)$
- $\cos 22\dfrac{1°}{2} = \dfrac{1}{2}\left(\sqrt{2+\sqrt{2}}\right)$
- $\tan 22\dfrac{1°}{2} = \sqrt{2}-1$
- $\cot 22\dfrac{1°}{2} = \sqrt{2}+1$
- $\sin 18° = \dfrac{\sqrt{5}-1}{4} = \cos 72°$
- $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$
- $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$
- $\cos 36° = \dfrac{\sqrt{5}+1}{4} = \sin 54°$
- $\sin 9° = \dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4} = \cos 81°$
- $\cos 9° = \dfrac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4} = \sin 81°$
- $\cos 36° - \cos 72° = \dfrac{1}{2}$
- $\cos 36° \cos 72° = \dfrac{1}{4}$
- $\sin\theta \sin(60°-\theta)\sin(60°+\theta) = 1/4 \sin 3\theta$

- $\cos\theta \cos(60° - \theta) \cos(60° + \theta) = 1/4 \cos 3\theta$
- $\tan\theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta$

10. **Maximum and minimum value of $a\cos\theta \pm b\sin\theta$**

$$-\sqrt{a^2+b^2} \le a\cos\theta \pm b\cos\theta \le \sqrt{a^2+b^2} \quad \text{for all } \theta$$

It follows that $-\sqrt{a^2+b^2}$ and $\sqrt{a^2+b^2}$ are minimum and maximum values of $a\cos\theta \pm b\sin\theta$ for varying values of θ.

11. **The maximum and minimum values of $a\cos\theta \pm b\sin\theta + c$ are**

$c+\sqrt{a^2+b^2}$ and $c-\sqrt{a^2+b^2}$, respectively.

i.e., $c-\sqrt{a^2+b^2} \le a\cos\theta \pm b\sin\theta + c \le c+\sqrt{a^2+b^2}$.

12. **Result I :**

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots\ldots + \sin(\alpha + \overline{n-1}\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \sin\left[\alpha + (n-1)\frac{\beta}{2}\right].$$

Result II :

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots\ldots + \cos(\alpha + \overline{n-1}\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \cos\left[\alpha + (n-1)\frac{\beta}{2}\right].$$

13. **Some Basic Results**

(i) If $\sin\theta = 0$, then $\theta = 0, \pm\pi, \pm 2\pi, \ldots\ldots$

$\Rightarrow \quad \theta = n\pi$ where $n \in I$.

(ii) If $\cos\theta = 0$, then $\theta = \pm\pi/2, \pm 3\pi/2 \ldots\ldots$

$\Rightarrow \quad \theta = (2n+1)\pi/2$ where $n \in I$.

(iii) If $\tan\theta = 0$ then, $\theta = 0, \pm\pi, \pm 2\pi, \ldots\ldots$

$\Rightarrow \quad \theta = n\pi$ where $n \in I$.

(iv) If $\sin\theta = 1$ then, $\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\ldots$

$\Rightarrow \quad \theta = (4n+1)\frac{\pi}{2}$ where $n \in I$.

(v) If $\cos\theta = 1$ then, $\theta = 0, 2\pi, 4\pi, \ldots\ldots$

$\Rightarrow \quad \theta = 2n\pi$ where $n \in I$.

(vi) If $\tan\theta = 1$ then, $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots\ldots$

$\Rightarrow \quad \theta = (4n+1)\frac{\pi}{4}$ where $n \in I$.

14. Some More Basic Results

(i) If $\sin\theta = \sin\alpha$

then $\theta = n\pi + (-1)^n \alpha$, where $n \in I$

(ii) If $\cos\theta = \cos\alpha$

then $\theta = 2n\pi \pm \alpha$, where $n \in I$

(iii) If $\tan\theta = \tan\alpha$

then $\theta = n\pi + \alpha$, where $n \in I$

(iv) If $\tan^2\theta = \tan^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$

(v) If $\sec^2\theta = \sec^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$

(vi) If $\cos^2\theta = \cos^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$

(vii) If $\sin^2\theta = \sin^2\alpha$

then $\theta = n\pi \pm \alpha$, where $n \in I$