

**BUREAU'S**  
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**BUSINESS MATHEMATICS**  
**AND STATISTICS**

**Class - XII**

**(For Class XII Commerce Students)**

*(Prescribed by the Council of Higher Secondary Education, Odisha)*

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# **FOREWORD**

The Odisha State Bureau of Textbook Preparation and Production, Bhubaneswar has made a pioneer attempt to publish text books for Commerce Stream with an excellent team of teachers in different subjects.

The present book “**Business Mathematics and Statistics - Class - XII**” is meant for Class - XII Higher Secondary Commerce students. This book has been written by a team of learned academicians namely Dr. Ranjit Mishra, Dr. R.C Jena, Dr. Banamali Panda, and Dr. P.K Parida and reviewed by Prof. A. P. Nayak. I would like to record my sincere gratitude to all of them for accomplishing this maiden venture in time. The main purpose of developing this text book is to provide a thorough exposure to the students of Commerce in this subject. The book prepared according to the new syllabus prescribed by the CHSE, Odisha, and as per new pattern of questions, shall cater to the needs of young students. I owe my special thanks to Prof. A.P. Nayak for making necessary revision in the text and questions according to new pattern.

I believe that the students and teachers of commerce stream shall welcome and appreciate the book. I would also like to welcome constructive suggestions for further improvement of the book.

**Sri Umakanta Tripathy**  
**Director**  
Orissa State Bureau of Textbook  
Preparation and Production,  
Pustak Bhavan, Bhubaneswar



# **PREFACE**

Elementary knowledge in Mathematics & Statistics has assumed immense importance in every sphere of human activity. Its application is found in various diversified fields such as business, industry, management, economics, planning, agriculture, &- insurance, sociology, biometry etc. and also in many professional courses. The Odisha State Bureau of Textbook Preparation and Production, Bhubaneswar through its Board of Writers and Reviewers, has presented this new edition of the book, "**Business Mathematics and Statistics Class-XII** for Commerce students to have them a simple and comprehensive exposition to the application of quantitative techniques.

We, the writers are pleased to commend the readers the new edition of this book which is commensurate with the latest syllabus of CHSE, Odisha. The textbook is prepared to fully cater to the needs of the students both in terms of the course content as well as the level of knowledge required to excel in the examination. Further the textbook is thoroughly reviewed by an expert to incorporate the requirements of the students.

The main features of the book are :

- The book has two parts, namely Mathematics and Statistics -
- Simplicity of expression
- Sufficient illustrations to tackle practical problems
- Systematic presentation of subject matter
- Latest course content
- Meaningful focus on new question pattern

The book is unique in its presentation because steps have been taken to keep pace with new syllabus and new pattern of questions. Sufficient Multiple Choice Questions and other objective type questions are provided along with their answers in addition to long questions.

We wish to thank the Odisha State Bureau of Textbook Preparation and Production, Bhubaneswar for its efforts and co-operation in the publication of the book in time.

Any suggestion for improvement of the book will be highly appreciated.

**Board of Writers**

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# SYLLABUS

## Business Mathematics and Statistics

### Class-XII

#### Objectives :

- ❖ To enable the students to learn basic concepts of determinants & Matrices;
- ❖ To learn the concept, features, types of Set Theory;
- ❖ To develop an understanding about concept, relations, types and application of functions;
- ❖ To enable the students to learn the concept, methods and applications of Limit, continuity, Derivation and Integration;
- ❖ To help the students in learning the concept, types and calculation of Average and
- ❖ To enable the students to understand the concept, objectives, features and applicationsof measures of dispersion.

#### COURSE INPUTS :

##### Unit - I Business Mathematics:

**Determinants :-** Upto third order, Minors, Co-factors, properties and Cramer's rule,

**Matrices :-** Meaning, Definition, Types, Algebra of matrices, Solving Linear Equation Problems through Matrics.

**Set Theory :-** Meaning, Definition, Types and Operations (Union & Intersection)

**Functions :-** Meaning and Relations of Functions, Types of Functions and Classification of Functions (excluding Trigonometric Functions)

##### Unit -II Calculus :

**Calculus -1** Limit & Continuity - Meaning, Definition, Methods of Finding Limits, Differentiation

**Calculus - II** Integration up to substitution

### **Unit-III Measure of Central Tendency :-**

Meaning, Objectives, Types, of Averages (Mathematical & Positional Averages)

Mathematical Averages : AM, GM, HM (Simple & Weighted)

Positional Averages : Median, Mode, Quartile, Deciles and Percentiles

Relationship of AM, GM, HM, Median, Mode

### **Unit - IV Measure of Dispersion :-**

Meaning, Objectives, Characteristics of dispersion, Measures of Dispersion, (Absolute and Relative) Positional Dispersion : Range, Inter Quartile Range, Quartile Deviation.

Mathematical Dispersion : Mean Deviation, Standard Deviation & Co-efficient of variation.

### **Unit-V Project work and viva :-**

#### **Suggested outline for Project Work**

- ❖ Case study on Application of Matrix for solving real life business problems.
- ❖ Report on concept and Rules regarding Matrix, Determining inverse of a matrix by using Elementary Operation Methods and Co-factor Method.
- ❖ Find out the Averages (Mean, Median and Mode) by taking sample from Number of students from any class and Marks secured by them in their annual examination as two variables.
- ❖ Calculate the Mean or Average deviation and co-efficient of Variation from Mean / Median / Mode from by taking example of Life Time in No. of years of two different T.V. sets namely Model A and Model B to find out the average life of each model of these TV and the model having greater uniformly or variability.
- ❖ Calculating Standard Deviation and its co-efficient alongwith its coefficient of variation from the data relating to the Profit or Loss made by Engineering Companies in Odisha during the year 201-15.



## **CHAPTER - 1**

# ***DETERMINANTS***

### **STRUCTURE**

- 1.1 Meaning of determinant
- 1.2 Expansion of determinants up to third order
- 1.3 Minors and co-factors
- 1.4 Properties of determinants
- 1.5 Cramer's Rule
- 1.6 Questions

## 1.1 Meaning of determinant

The concept of matrices and determinant has a wide applicability not only in mathematics but also in other disciplines like sociology, Genetics, Engineering etc. Therefore for their wide applicability it is essential to study the fundamentals of matrices and determinants. To learn about determinant one must know the meaning of matrix.

A matrix is a rectangular array of numbers, real or complex. It is usually denoted by capital letters. The numbers inside the rectangular array are known as elements of the matrix. The horizontal lines of elements are called rows and vertical lines of elements are called columns. Let us have an example.

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 1 & 5 \\ 2 & 0 & 4 \end{bmatrix} \begin{array}{l} \text{First row} \\ \text{Second row} \\ \text{Third row} \end{array}$$

1st    2nd    3rd  
Col.   Col.   Col.

In general, if a matrix A has m rows and n columns, it is expressed as below :

$$A = \begin{bmatrix} a_{11} & a_{12} \dots a_{1j} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2j} \dots a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mj} \dots a_{mn} \end{bmatrix}$$

To understand determinant let us solve the following two linear equation.

We have learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Can be represented as  $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Now this system of equations has a unique solution or not, is determined by the number  $a_1b_2 - a_2b_1$  (Recall that if  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$  or,  $a_1b_2 - a_2b_1 \neq 0$ , then the system of linear equations has a unique solution). The number  $a_1b_2 - a_2b_1$  which determines uniqueness of solution is associated with the matrix

$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$  and is called the determinant of A or  $\det A$ . Determinants have wide applications in Engineering, Science, Economics, Social Science etc.

In this chapter, we shall study determinants upto order three only with real entries. Also, we will study various properties of determinants, minors, cofactors & applications of determinants in finding the area of a triangles, adjoint & inverse of a square matrix, consistency and inconsistency of system of linear equations and solutions of linear equation in two or three variables using inverse of a matrix.

### **Determinant :**

To every square matrix  $A = [a_{ij}]$  of order n, we can associate a number (real or complex) called determinant of the square matrix A, Where  $a_{ij} = (i, j)^{\text{th}}$  element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex) called determinant of the square matrix  $A$ , where  $a_{ij} = (i, j)^{\text{th}}$  element of  $A$ . This may be thought of as a function which associate each square matrix with a unique number (real or complex). If  $M$  is the set of square matrices,  $K$  is the set of numbers (real or complex) and  $f : M \rightarrow K$  is defined by  $f(A) = k$ , where  $A \in M$  and  $k \in K$ , then  $f(A)$  is called the determinant of  $A$ . It is also denoted by  $|A|$  or  $\det A$  or  $\Delta$ .

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then determinant of  $A$  is

Written as  $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

**Remarks :**

- (i) For matrix  $A$ ,  $|A|$  is read as determinant of  $A$  and not modules of  $A$ .
- (ii) Only square matrices have determinants.

**Determinant of a matrix of order one .**

Let  $A = [a]$  be the matrix of order 1, then determinant of  $A$  is defined to be equal to  $a$ .

**Determinant of a matrix of order two :**

Let  $A = \begin{bmatrix} a_{11} & b_{12} \\ c_{21} & d_{22} \end{bmatrix}$  be a matrix of

order  $2 \times 2$ , then the determinant of  $A$  is

defined as :

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

**Example 1 :** Evaluate  $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

**Solution :** We have  $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1)$   
 $= 4+4$   
 $= 8$

**Example 2 :**

**Evaluate**  $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

**Solution :**

**We have,**

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2-1)$$

$$= x^2 - x^2 + 1 = 1$$

## 1.2 EXPANSION OF DETERMINANTS :

### Determinant of a matrix of order 3×3 :

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows ( $R_1$ ,  $R_2$  and  $R_3$ ) and three columns ( $C_1$ ,  $C_2$  and  $C_3$ ) giving the same value as shown below :

Consider the determinant of square matrix

$$A = [a_{ij}]_{3 \times 3}$$

$$\text{i.e., } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**Expansion along first Row ( $R_1$ ):**

**Step-1:** Multiply first element  $a_{11}$  of  $R_1$  by  $(-1)^{(1+1)}[(-1)^{\text{sum of suffixes in } a_{11}}]$  and with the second order determinant obtained by deleting the elements of first row ( $R_1$ ) and first column ( $C_1$ ) of  $|A|$  as  $a_{11}$  lies in  $R_1$  and  $C_1$ ,

$$\text{i.e., } (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

**Step-2:** Multiply 2nd element  $a_{12}$  of  $R_1$  by  $(-1)^{1+2} [(-1)^{\text{sum of suffixes in } a_{12}}]$  and the second order determinant obtained by deleting elements of first row ( $R_1$ ) and 2nd column ( $C_2$ ) of  $|A|$  as  $a_{12}$  lies in  $R_1$  and  $C_2$ ,

$$\text{i.e., } (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

**Step-3:** Multiply third element  $a_{13}$  of  $R$  by  $(-1)^{1+3} [(-1)^{\text{sum of suffixes in } a_{13}}]$  & the second order determinant obtained by deleting elements of first row ( $R_1$ ) and third column ( $C_3$ ) of  $|A|$  as  $a_{13}$  lies in  $R_1$  and  $C_3$ .

$$\text{i.e., } (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix}$$

**Step-4:** Now the expansion of determinant of  $A$ , that is,  $|A|$  written as sum of all three terms obtained in steps 1, 2 and 3 above is given by :

$$\text{Let } A = |A| = (-1)^{1+1} \cdot a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} +$$

$$(-1)^{1+2} \cdot a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +$$

$$(-1)^{1+3} \cdot a_{13} \begin{vmatrix} a_{21} & b_{22} \\ c_{31} & d_{32} \end{vmatrix}$$

$$\begin{aligned} \text{or, } |A| &= a_{11} (a_{22}a_{33} - a_{32}a_{23}) - a_{12} (a_{21}a_{33} - a_{31}a_{23}) + a_{13} (a_{21}a_{32} - a_{31}a_{22}) \\ &\quad + a_{13} (a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} - (1) \end{aligned}$$

N.B : We shall apply all four steps together.

**Expansion along second row (R<sub>2</sub>) :**

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R<sub>2</sub> use get :-

$$\begin{aligned} |A| &= (-1)^{2+1} \cdot a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} \cdot a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + (-1)^{2+3} \cdot a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -a_{21} (a_{12}a_{33} - a_{32}a_{13}) + a_{22} (a_{11}a_{33} - a_{31}a_{13}) \\ &\quad - a_{23} (a_{11}a_{32} - a_{31}a_{12}) \\ \Rightarrow |A| &= -a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13} \\ &\quad - a_{23}a_{11}a_{32} + a_{23}a_{31}a_{12} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ &\quad + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \dots \dots \dots (2) \end{aligned}$$

**Expansion along first column ( $C_1$ ) :**

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

**By expanding along  $C_1$ , we get :-**

$$|A| = a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ + a_{31}(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} \dots \dots \dots (3)$$

Clearly, Value of  $|A|$  in (1), (2) and (3) are equal. It's left as an exercise to the reader to verify that the value of  $|A|$  by expanding along  $R_3$ ,  $C_2$  and  $C_3$  are equal to the value of  $|A|$ , Obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

### Remarks

- (i) For easier calculation, We shall expand the determinant along the row or column, which contains maximum number of zeroes.
- (ii) While expanding, instead of multiplying by  $(-1)^{i+j}$ , we can multiply by  $(+1)$  or  $(-1)$  according as  $(i+j)$  is even or odd.

(iii) Let  $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$  then

it is easy to verify that  $A = 2B$ . Also,

$$|A| = 0 - 8 = -8 \text{ and } |B| = 0 - 2 = -2$$

Observe that,  $|A| = \{(4) \cdot 2\}$  and  $= 2^2 |B|$  or

$$|A| = 2^n |B|, \text{ where } n=2 \text{ is the order of square matrices } A \text{ and } B$$

In general, if  $A = KB$ , where  $A$  and  $B$  are square matrices of order  $n$ , then  $|A| = K^n |B|$ , where  $n=1,2,3$

**Example 3 :** Evaluate the determinant.

$$\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$$

**Solution :** Note that in the third column, two entries are zero. So expanding along third column ( $C_3$ ), we get :-

$$\begin{aligned} \Delta &= 4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \\ &= 4 (-1-12) - 0 + 0 \\ &= -52 \end{aligned}$$

**Remark -** For solving problems on determinant of 3rd or higher order, expansion method is applied.

**Example 5 :** Find values of  $x$  for which  $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Solution :- We have,  $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

i.e,  $3-x^2= 3-8$

$\Rightarrow x^2 = 8$

Hence,  $x = \pm 2\sqrt{2}$

### 1.3 Minors And Co-Factors :

The minor of an element  $a_{ij}$  of a determinant A is the determinant obtained from A by deleting the row and column in which the given element occurs. The minor of an element  $a_{ij}$  is denoted by  $M_{ij}$ . The minor of an element of third order determinant is a second order determinant.

For Example :- If  $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Then, the minor of the element :

$$a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, minor of the element  $a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$$= a_{11}a_{23} - a_{21}a_{13}$$

$$\therefore M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ and}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

We find that the minors, are second order determinants.

### **CO-FACTOR :**

The co-factor of an element  $a_{ij}$  of determinant A is defined as :

$A_{ij} = (-1)^{i+j} \cdot M_{ij}$ ; where i is the number of the row & j is the number of the column, in which the given element is present;  $M_{ij}$  is the minor of the element  $a_{ij}$  and  $A_{ij}$  is the co factor of the element  $a_{ij}$ . Usually, the co-factor is denoted by the corresponding capital letter of the given element. For example :

$$\text{If } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

### **Illustration - 1**

**Evaluate the determinant :**

$$(i) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad (ii) \begin{vmatrix} -2 & 4 \\ 3 & -7 \end{vmatrix} \quad (iii) \begin{vmatrix} 4 & 2 \\ 1/2 & 6 \end{vmatrix}$$

$$(iv) \begin{vmatrix} 4 & -17 \\ 0 & 5 \end{vmatrix}$$

**Solution :**

$$(i) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$(ii) \begin{vmatrix} -2 & 4 \\ 3 & -7 \end{vmatrix} = -2 \times (-7) - 3 \times 4 = 14 - 12 = 2$$

$$(iii) \begin{vmatrix} 4 & 2 \\ \frac{1}{2} & 6 \end{vmatrix} = 4 \times 6 - \frac{1}{2} \times 2 = 24 - 1 = 23$$

$$(iv) \begin{vmatrix} 4 & -17 \\ 0 & 5 \end{vmatrix} = 4 \times 5 - 0 \times (-17) = 20 - 0 = 20$$

**Illustration 2 :** Find the minors and co-factors of elements of the following determinant.

$$|A| = \begin{vmatrix} 2 & -3 & 4 \\ 5 & 0 & 6 \\ 7 & 5 & 1 \end{vmatrix}$$

$$\text{Solution : } M_{11} = \begin{vmatrix} 0 & 6 \\ 5 & 1 \end{vmatrix} = 0 \times 1 - 5 \times 6 = -30;$$

$$A_{11} = (-1)^{1+1} (-30) = -30$$

$$m_{12} = \begin{vmatrix} 5 & 6 \\ 7 & 1 \end{vmatrix} = 5 \times 1 - 7 \times 6 = -37;$$

$$A_{12} = (-1)^{1+2} (-37) = 37$$

$$M_{13} = \begin{vmatrix} 5 & 0 \\ 7 & 5 \end{vmatrix} = 5 \times 5 - 7 \times 0 = 25; A_{13} = (-1)^{1+3} 25 = 25$$

$$M_{21} = \begin{vmatrix} -3 & 4 \\ 5 & 1 \end{vmatrix} = -3 \times 1 - 5 \times 4 = -23; A_{21} = (-1)^{2+1} (-23) = 23$$

$$M_{22} = \begin{vmatrix} 2 & 4 \\ 7 & 1 \end{vmatrix} = 2 \times 1 - 7 \times 4 = -26; A_{22} = (-1)^{2+2}(-26) = -26$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 7 & 5 \end{vmatrix} = 2 \times 5 - 7 \times (-3) = 31; A_{23} = (-1)^{2+3}(31) = -31$$

$$M_{31} = \begin{vmatrix} -3 & 4 \\ 0 & 6 \end{vmatrix} = -3 \times 6 - 0 \times 4 = -18; A_{31} = (-1)^{3+1}(-18) = -18$$

$$M_{32} = \begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} = 2 \times 6 - 5 \times 4 = -8; A_{32} = (-1)^{3+2}(-8) = 8$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 5 & 0 \end{vmatrix} = 2 \times 0 - 5 \times (-3) = 15; A_{33} = (-1)^{3+3}15 = 15$$

**Illustration 3 :** Find minors & co-factors of elements of the determinant

$$\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$$

Solution:  $\det A = \begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$

$$M_{11} = \text{Minor of the element } 3 = |4| = 4$$

$$M_{12} = \text{Minor of the element } 5 = |-2| = -2$$

$$M_{21} = \text{Minor of the element } (-2) = |5| = 5$$

$$M_{22} = \text{Minor of the element } 4 = |3| = 3$$

$$A_{11} = \text{Co-factor of } 3 = (-1)^{1+1} 4 = 4$$

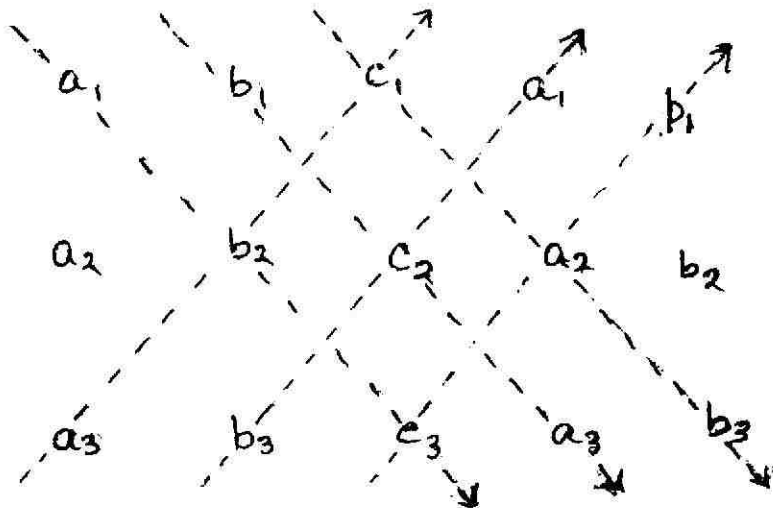
$$A_{12} = \text{Co-factor of } 5 = (-1)^{1+2}(-2) = 2$$

$$A_{21} = \text{Co-factor of } -2 = (-1)^{2+1} 5 = -5$$

$$A_{22} = \text{Co-factor of } 4 = (-1)^{2+2} 3 = 3$$

### Expansion of Determinant with the Help of Sarrus Diagram :

We can also evaluate a third order determinant by drawing a sarrus diagram. In sarrus diagram first two column are added to the given matrix of the determinant & the elements are connected through arrows heading downwards & are assigned plus sign (+) & those connected upwards are assigned minus sign (-). Let's take the following example and find out the value of determinant through sarrus diagram.

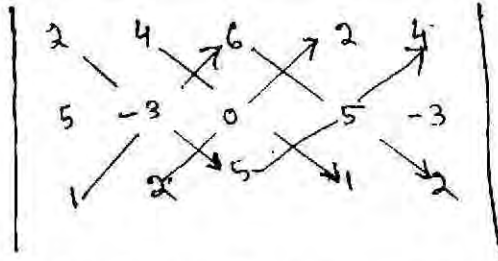


Now, according to the sarrus diagram :-

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

**Illustration - 4 :** Use sarrus Diagram and evaluate :-

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 5 & -3 & 0 \\ 1 & 2 & 5 \end{vmatrix}$$



**Solution :** With the help of sarrus diagram are as follows. i.e, the product of the elements which are connected by down-word arrow are calculated first, out of which the product of the elements of the up-word arrow are subtracted. The resulting figure is the value of determinant.

Therefore,

$$\begin{aligned}
 |A| &= [2 \times (-3) \times 5] + (4 \times 0 \times 1) + (6 \times 5 \times 2) \\
 &\quad - 1(-3) \times 6 - 2 \times 0 \times 2 - 5 \times 5 \times 4 \\
 &= -30 + 0 + 60 + 18 - 0 - 100 \\
 &= -130 + 78 \\
 &= -52
 \end{aligned}$$

#### 1.4. Properties of Determinants :

Its often difficult to simplify certain determinants. The properties of determinants mentioned below enable us to simplify the determinants & quickly obtain the results. So, it becomes necessary on our part to carefully observe some important properties of determinants, & apply it to situations intelligently. Though the properties are true for determinants of any order, we restrict its use up to determinants of order three only.

1. The value of a determinant remains unchanged, if its rows and columns are interchanged.

$$\text{Proof : } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding  $\det A$  along first column, we have

$$\det A = a_{11}(a_{22} \cdot a_{33} - a_{32} \cdot a_{23}) - a_{21}(a_{12} \cdot a_{33} - a_{32} \cdot a_{13}) + a_{31}(a_{12} \cdot a_{23} - a_{22} \cdot a_{13})$$

Changing the rows and columns of  $\det A$ , we get that :-

$$\det A_1 = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Now, expanding  $\det A$ , along first row, we have

$$\det A_1 = a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{21}(a_{12} a_{33} - a_{13} a_{32}) + a_{31}(a_{12} a_{23} - a_{13} a_{22})$$

Now  $\det A = \det A_1$ , hence, there's no change in the value of determinant.

2. If two rows or columns of a determinant are interchangeable, then the value of determinant doesn't change but its sign change.

$$\text{Proof : Let } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding the above along first column, we have :-

$$\det A_1 = a_{11}(a_{22} a_{33} - a_{32} a_{23}) - a_{21}(a_{12} a_{33} - a_{32} a_{13}) + a_{31}(a_{12} a_{23} - a_{22} a_{13})$$

Interchanging 1st and 3rd column, we have :-

$$\det A_1 = \begin{vmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{vmatrix}$$

Expanding  $\det A$ , along third column, we have

$$\begin{aligned}\det A_1 &= a_{11}(a_{32}a_{23} - a_{22}a_{33}) - a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{22}a_{13} - a_{12}a_{23}) \\ &= -[a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{13}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})] \\ &= -\det A\end{aligned}$$

Hence, the value of the determinant doesnot change, but its sign changes.

3. If any two rows/columns of a determinant are identical, then the value of the deternimant is zero.

$$\text{Proof : } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along first row, we have:-

$$\begin{aligned}\det A &= a_{11}(a_{12}a_{33} - a_{32}a_{13}) - a_{12}(a_{11}a_{33} - a_{31}a_{13}) + a_{13}(a_{11}a_{32} - a_{31}a_{12}) \\ &= a_{11}a_{12}a_{33} - a_{11}a_{32}a_{13} - a_{12}a_{11}a_{33} + a_{12}a_{31}a_{13} + a_{13}a_{11}a_{32} - a_{13}a_{31}a_{12} \\ &= 0\end{aligned}$$

4. If each element of a particular row / column of a determinant is multiplied by a constant K, then the value of the determinant gets multiplied by k.

Proof :

$$\text{Let } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det A_1 = \begin{vmatrix} ka_{11} & Ka_{12} & Ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding  $\det A_1$  along first row, we have :-

$$\begin{aligned}\det A_1 &= ka_{11}(a_{22}a_{33} - a_{32}a_{23}) - ka_{12}(a_{21}a_{33} - a_{31}a_{23}) + ka_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= K \{a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})\} \\ &= K \det A\end{aligned}$$

5. If every element of a row / column of a determinant is expressed as the sum or difference of two terms, then the determinant can be expressed as sum or difference of two determinants, thus,

$$\begin{vmatrix} x_1 + a_1 & x_2 + a_2 & x_3 + a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Proof : Expanding the LHS, we have :-

$$\begin{aligned}& (x_1 + a_1)(y_2z_3 - z_2y_3) - (x_2 + a_2)(y_1z_3 - z_1y_3) + (x_3 + a_3)(y_1z_2 - z_1y_2) \\ &= x_1(y_2z_3 - z_2y_3) - x_2(y_1z_3 - z_1y_3) + x_3(y_1z_2 - z_1y_2) + a_1(y_2z_3 - z_2y_3) - a_2(y_1z_3 - z_1y_3) + \\ & \quad a_3(y_1z_2 - z_1y_2)\end{aligned}$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

6. If any row / column of a determinants, a multiple of any other row / column is added or subtracted the value of the original determinant remains the same.

$$\text{Let } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let us multiply a constant  $k$  to third row & add to the first row :-

$$\therefore \det A_1 = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

As per previous properties, we have :

$$\det A_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + 0$$

=  $\det A$  ( $\because$  two rows are identical in the 2nd determinant)

**Illustration 5 :** Show that (i)  $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 5 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix}$

ii)  $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 2 \end{vmatrix}$

**Solution (i)**  $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = 4(2-4) - 5(4-8) + 6(2-2)$

$$= -8 + 20 + 0$$

$$= 12.$$

Now changing  $c_2$  to  $c_3$  and vice - verse, we have :

$$\text{and } \begin{vmatrix} 4 & 6 & 5 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 4(4-2) - 6(2-2) + 5(4-8)$$

$$= 8 + 0 - 20$$

$$= -12.$$

$$\therefore \begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 4 & 6 & 5 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix}$$

$$\text{(iii) } \begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = 4(2-4) - 5(4-8) + 6(2-2)$$

$$= -8 + 20 + 0 = 12$$

Now changing columns into Rows, we have

$$\text{and } \begin{vmatrix} 4 & 2 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 2 \end{vmatrix} = 4(2-4) - 2(10-6) + 2(20-6)$$

$$= -8 - 8 + 28 = 12.$$

**Illustration 6 :** Show that :-  $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$

**Solution :**  $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix}$

$$=1(0-0) - 2(3-0) + 3(2-0)$$

$$= 0 - 6 + 6 = 0$$

Again since Row 1 and Row 3 are identical, the value of the determinant = 0

**Illustration 7 :** Show that 
$$\begin{vmatrix} 1 \times 8 & 2 \times 8 & 3 \times 8 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix} = 8 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$$

**Solution :-** 
$$\begin{vmatrix} 1 \times 8 & 2 \times 8 & 3 \times 8 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$$

$$= 8(24-14) - 16(20-56) + 24(10-48)$$

$$= (8 \times 10) + 5 + 6 - 912$$

$$= -256$$

$$= 8 \times (-32)$$

Now, 
$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$$

$$= 1(24-14) - 2(20-56) +$$

$$3(10-48) = 10 + 72 - 114$$

$$= -32$$

$$\therefore \begin{vmatrix} 1 \times 8 & 2 \times 8 & 3 \times 8 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix} = 8 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$$

**Illustration 8 :**

Prove that :

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -b(b-a)(b-c)(c-a)$$

Solution - L H S

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

Replacing  $c_1 - c_2$  and  $c_2 - c_3$  :-

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) [(0-0) + \{1(b+c) - (a+b).1\}]$$

$$= (a-b)(b-c) [0-0+b+c-a-b]$$

$$= -(b-a)(b-c)(c-a) = \text{RHS. (Hence Proved)}$$

**Illustration - 9 :** Show that

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$$

$$= (1+abc)(b-a)(c-a)(c-b)$$

**Solution :**

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Now applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have :

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (1+abc)(b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (1+abc)(b-a)(c-a) \{(c+a)-(b+a)\}$$

$$= (1+abc)(b-a)(c-a)(c-b) \text{ (proved)}$$

**Illustration - 10 :** Show that :

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 \quad (\text{where } \omega \text{ is one of the imaginary cube roots of unity})$$

**Solution :** 
$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Replacing  $R_1$  by  $R_1+R_2+R_3$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad (\text{As } 1+\omega+\omega^2 = 0)$$

$= 0+0+0$  (On expanding along first row)

$= 0$

**Illustration 11 :** Solve the following :

a) 
$$\begin{vmatrix} 5 & 2 \\ x+1 & 4 \end{vmatrix} = 0$$

b) 
$$\begin{vmatrix} 2 & 2 & x \\ -1 & x & 4 \\ 3 & 3 & 3 \end{vmatrix} = 0$$

$$\text{Solutation (a) } \begin{vmatrix} 5 & 2 \\ x+1 & 4 \end{vmatrix} = 0$$

$$\Rightarrow 20 - (x+1)x2$$

$$\Rightarrow 20 - (2x+2) = 0$$

$$\Rightarrow 20 - 2x - 2 = 0$$

$$\Rightarrow -2x = -18$$

$$\therefore x = 9$$

$$(b) \begin{vmatrix} 2 & 2 & x \\ -1 & x & 4 \\ 3 & 3 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 2(3x-12) - 2(-3-12) + x(-3-3x) = 0$$

$$\Rightarrow 6x - 24 + 6 + 24 - 3x - 3x^2 = 0$$

$$\Rightarrow 6x - 24 + 30 - 3x - 3x^2 = 0$$

$$\Rightarrow -3x^2 + 3x + 6 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\therefore x = 2 \text{ or } x = -1$$

**Illustration 12 :** Show that :

$$\begin{vmatrix} x & y & z \\ p & q & r \\ m & n & o \end{vmatrix} = \begin{vmatrix} y & n & q \\ x & m & p \\ z & o & r \end{vmatrix}$$

$$\text{Solution : LHS} = \begin{vmatrix} x & y & z \\ p & q & r \\ m & n & o \end{vmatrix}$$

$$= \begin{vmatrix} x & p & m \\ y & q & n \\ z & r & o \end{vmatrix} \text{ (by changing rows to columns)}$$

$$= - \begin{vmatrix} x & m & p \\ y & n & q \\ z & o & r \end{vmatrix} \text{ (by interchanging } C_3 \text{ and } C_2)$$

$$= \begin{vmatrix} y & n & q \\ x & m & p \\ z & o & r \end{vmatrix} \text{ (by interchanging } R_1 \text{ and } R_2)$$

$$= \text{RHS}$$

$$\therefore \begin{vmatrix} x & y & z \\ p & q & r \\ m & n & o \end{vmatrix} = \begin{vmatrix} y & n & q \\ x & m & p \\ z & o & r \end{vmatrix}$$

### 1.5 Cramer's Rule :-

Swiss mathematician, Gabriel Cramer provided a simple method to solve a system of linear equations with the help of determinants, which is known as Cramer's rule. Thus, Cramer's rule is a rule, using determinants, to express the solution of a system of linear equations, for which the number of equations is equal to the number of variables.

Let us consider the following two linear equations, with two unknown,  $x$  and  $y$  :-

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

As we have seen earlier,

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}; a_1b_2 - a_2b_1 \neq 0$$

If we denote  $D = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$$D_x = b_2c_1 - b_1c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$D_y = a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

then,  $x = \frac{D_x}{D}$  and  $y = \frac{D_y}{D}$ , where  $D \neq 0$

The above method shows how to solve a system of linear equations using Cramer's rule. The method can be extended to solve a system of 'n' equation with 'n' variables (unknowns). In other words, for solving a system of three linear equations, with three unknowns, we may proceed as below :

Let  $a_1x + b_1y + c_1z = d_1 \dots \dots \dots (i)$

$a_2x + b_2y + c_2z = d_2 \dots \dots \dots (ii)$

$a_3x + b_3y + c_3z = d_3 \dots \dots \dots (iii)$

be a system of three linear equations with three unknown variable x, y and z. On solving the equations simultaneously (i.e. by addition and subtraction), we get.

$$x = \left[ \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2} \right]$$

and similar other values of y and x. if we denote,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} Dx = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = Dy = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \dots\dots\dots (iv)$$

$$D_z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

then, as per equation (iv),

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dx}{D}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dy}{D}$$

Similarly,

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_z}{D}$$

So, we use Cramer's rule & find out the values of variables as below :-

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = -\frac{D_z}{D}$$

Let us illustrate application of Cramer's rule to linear equations.

**Illustration - 13 :** Solve the equations using Cramer's rule.

$$3x + 2y = 7$$

$$5x - y = 3$$

Solution : Using Cramer's rule, we have :-

$$D = \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} = -3 - 10 = -13$$

$$D_x = \begin{vmatrix} 7 & 2 \\ 3 & -1 \end{vmatrix} = -7 - 6 = -13$$

$$D_y = \begin{vmatrix} 3 & 7 \\ 5 & 3 \end{vmatrix} = 9 - 35 = -26$$

$$\therefore x = \frac{D_x}{D} = \frac{-13}{-13} = 1, y = \frac{D_y}{D} = \frac{-26}{-13} = 2$$

**Illustration 14 :-** Solve the equations using determinants.

$$x + 2y + 3z = 6$$

$$2x + y + z = 4$$

$$x + y + 2z = 4$$

$$\text{We have } D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-1) - 2(4-1) + 3(2-1) = 1 - 6 + 3 = -2$$

$$D_x = \begin{vmatrix} 6 & 2 & 3 \\ 4 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = 6(2-1) - 2(8-4) + 3(4-4) = 6 - 8 = -2$$

$$D_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 4 & 1 \\ 1 & 4 & 2 \end{vmatrix} = 1(8-4) - 6(4-1) + 3(8-4) = 4 - 18 + 12 = -2$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 1 & 4 \\ 1 & 1 & 4 \end{vmatrix} = 1(4-4) - 2(8-4) + 6(2-1) = -8 + 6 = -2$$

Applying Cramer's rule, we have :-

$$x = \frac{D_x}{D}; y = \frac{D_y}{D}; z = \frac{D_z}{D}$$

$$\text{i.e., } x = \frac{-2}{-2}, y = \frac{-2}{-2} \text{ and } z = \frac{-2}{-2}$$

$$\therefore x = 1, y = 1 \text{ and } z = 1$$

**Illustration - 15 :** Examine and solve the following homogeneous equations :-

$$x+y+z = 0$$

$$9x+3y+2z=0$$

$$2x+y+2z = 0$$

**Solution :** In the homogeneous equations :-

$$x+y+z = 0$$

$$9x+3y+2z=0$$

$$2x+y+2z = 0$$

$$\text{Where, } D = \begin{vmatrix} 1 & 1 & 1 \\ 9 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -7 \text{ i.e., } D \neq 0$$

$$D_x = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 0, D_y = \begin{vmatrix} 1 & 0 & 1 \\ 9 & 0 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$$

$$\text{and } D_z = \begin{vmatrix} 1 & 1 & 0 \\ 9 & 3 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

As  $D_x=D_y=D_z=0$ , the system of equations is called system of homogeneous equations.

$$\text{Obviously, } x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

$$\text{(ii) } x = \frac{0}{-7}, y = \frac{0}{-7}, z = \frac{0}{-7}$$

$$\therefore x = 0, y = 0 \text{ and } z = 0$$

**Illustration - 16 :** The cost of 2 books, 1 pen and 2 notebooks is ₹ 375. The cost of 1 book, 1 pen and 1 notebook is ₹ 225. The cost of 9 books, 3 pens and 2 notebooks is ₹ 875, using Cramer's rule, find out the price of each item.

**Solution :** Let the price of each book, each pen and each notebook be ₹  $x$ , ₹  $y$ , and ₹  $z$  respectively. According to the question, we have :-

$$2x+y+2z=375$$

$$x+y+z = 225$$

$$x+y+z=225$$

$$\text{or} \quad 9x+3y+2z = 875$$

$$9x+3y+2z=875$$

$$2x+y+2z = 375$$

Now the determinant of the coefficients of  $x, y$  and  $z$  is :-

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 9 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -7$$

$$D_x = \begin{vmatrix} 225 & 1 & 1 \\ 875 & 3 & 2 \\ 375 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 200 & 3 & -1 \\ 150 & 1 & 1 \end{vmatrix}$$

Replacing  $c_1$  by  $c_1 - 225c_2$  and  $c_3 - c_2$  :-

$$= -1(200+150)$$

$$= -350 \dots\dots\dots (i)$$

$$D_y = \begin{vmatrix} 1 & 225 & 1 \\ 9 & 875 & 2 \\ 2 & 375 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 9 & -1150 & -7 \\ 2 & -75 & 0 \end{vmatrix}$$

Replacing  $c_2$  by  $c_2 - 225c_1$  and  $c_3$  by  $c_3 - c_1$

$$= 1(0-525)$$

$$= -525 \dots\dots\dots (ii)$$

$$Dz = \begin{vmatrix} 1 & 1 & 225 \\ 9 & 3 & 875 \\ 2 & 1 & 375 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 9 & -6 & 200 \\ 2 & -1 & 150 \end{vmatrix}$$

Replacing  $c_3$  by  $c_3 - 225c_2$  and  $c_2$  by  $c_2 - c_1$

$$= (-900 + 200)$$

$$= -700 \dots\dots\dots (iii)$$

By Cramer's rule we have :-

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D};$$

$$\therefore x = \frac{-350}{-7}, y = \frac{-525}{-7} \text{ and } z = \frac{-700}{-7}$$

So,  $x = 50, y = 75$  and  $z = 100$

Ans - The costs of each book, each pen and each note book is ₹ 50, ₹ 75 and ₹ 100 respectively.

**1.6. Questions**

**Multiple choice Questions**

1. Choose the correct answer from the following alternatives.

(i) The value of  $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$  is

- (a) 14      (b) -10      (c) 10      (d) 12

(ii) Cramare's rule solve a system of liner equations using.

- (a) Matrices (b) differentiation (c) Integration (d) determinant

(iii) Under Cramer's rule  $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$

Where D is not equal to :

- (a) One (b) two (c) three (d) Zero

(iv) The determinant  $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$  is a determinant of order :

- (a) one (b) two (c) three (d) four

(v) The matrix associated with determinant is a

- (a) Rectangular matrix (b) Square matrix

(c) Both square and rectangular matrix.

(d) Triangular Matrix

(vi) The minor of the element  $a_{32}$  is

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$  is :

(a)  $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$  (b)  $-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$  (c)  $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

(d)  $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(vii) The co-factor of the element  $a_{32}$  in

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$  is :-

a)  $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$  (b)  $-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$  (c)  $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

d)  $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(viii) The minor and co-factor of 5 in  $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

are respectively :-

- (a) 10, -10    (b) -10, -10    (c) 10, 10    (d) -10, 10

(ix) If all the elements of a row / column of a determinant are zeroes, the value of the determinant is -

- (a) One    (b) Two    (c) Three    (d) Zero

(x) If two rows / column of a determinant are identical, i.e.,  $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

the value of the determinant is :

- (a) k    (b) a    (c) zero    (d) Two

(xi) The expansion of the determinant

$\begin{vmatrix} 4 & 5 & 6 \\ -1 & 7 & 3 \\ 2 & 3 & 8 \end{vmatrix}$  along second row is

a)  $-1 \begin{vmatrix} 5 & 6 \\ 3 & 8 \end{vmatrix} - 7 \begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$

b)  $1 \begin{vmatrix} 5 & 6 \\ 3 & 8 \end{vmatrix} + 7 \begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} - 3 \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$

$$c) \quad 5 \begin{vmatrix} -1 & 3 \\ 2 & 8 \end{vmatrix} - 7 \begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} + 3 \begin{vmatrix} 4 & 6 \\ -1 & 7 \end{vmatrix}$$

$$d) \quad 3 \begin{vmatrix} 5 & 6 \\ 3 & 8 \end{vmatrix} + 7 \begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} - 1 \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$$

(xii)  $\omega$  is :

- (a) Cube-root of unity (b) Square root of unity  
 (c) Imaginary cube-root of unity (d) Unity

(xiii) The minor of the elements of the determinant  $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{vmatrix}$  are

- (a) 5, 0, -5, 7, 0, -7, -2, 0, 2.  
 (b) -5, 0, 5, 7, 0, -7, 1, 0, 1.  
 (c) 5, 0, -5, 7, 0, -7, -1, 0, 1.  
 (d) 5, 0, 5, 7, 0, 7, 1, 0, 1.

xiv) The co-factor of the following determinant

$$\begin{vmatrix} 7 & 5 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} \text{ are :-}$$

- (a) -6, 3, 4, -8, -4, -9, 15, -13, -10  
 (b) 6, 3, 4, 8, 4, 9, 15, 13, 10  
 (c) -6, -3, 4, -8, -4, 9, 15, -13, -10  
 (d) -6, -3, -4, -9

**Q.2 Answer the following questions in one word / term.**

- (i) The value of the determinant  $\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix}$  is :
- (ii) If two rows / columns of a determinant are identical then, the value of the determinant is :
- (iii) If rows of determinant are changed to columns or vice-versa the value of determinant remain :
- (iv) The value of  $1 + \omega + \omega^2$  is always equal to :
- (v) The number of rows and number of column of a determinant is always :
- (vi) Cramer's rule can be applied only when the value of determinant is :
- (vii) If two adjacent rows or column of a determinant are interchanged, the sign of the determinant is :
- (viii) If all the element of a row/column of a determinant are zero, then the value of determinant would be.

**Q. 3. Fill in the blanks of the following :**

- (i) The value of the determinant  $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$  is \_\_\_\_\_.
- (ii) The value of the determinant  $\begin{vmatrix} \sec x & \tan x \\ \tan x & \sec x \end{vmatrix}$  is \_\_\_\_\_.
- (iii) The value of  $\omega$  is equal to \_\_\_\_\_.
- (iv) To evaluate the value of determinant under sarrus diagram method, the product of all the elements lies in down-ward arrow are \_\_\_\_\_.
- (v) We generally expand the rows columns having maximum number of \_\_\_\_\_ for finding the value of determinant.
- (vi) Given  $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

∴ The co factor of element 3 is \_\_\_\_\_.

(vii) ∴ The value of determinant  $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$  is \_\_\_\_\_.

(viii) Sarrus diagram can be applied for determinant of order \_\_\_\_\_ only.

(ix) The determinant which has two rows and two columns, is called a determinant of \_\_\_\_\_ two.

**Q.4 Correct the under lined portion of the following :**

(i) A determinant of order 3 contains 6 elements.

(ii) A matrix having one element is called a determinant of order two.

(iii) If every element of a third order determinant of  $|A|$  is multiplied by 5, then the value of the new determinant is  $5|A|$ .

(iv) The determinant which is obtained by changing rows into the corresponding columns or the columns into the corresponding rows is called inverse of a determinant.

(v) The product of the minor and  $(-1)^{i+j}$  of an element is called the minor of that element.

**Q.5 Answer the following in one sentence each.**

(i) Minor

(ii) Co-factor

(iii) Cramer's rule

(iv) Sarrus expansion

(v) Consistent system of equation.

(vi) Inconsistent system of equation.

6. Evaluate (Answer the following within 30 words each)

$$(i) \begin{vmatrix} 4 & 4 \\ 8 & 8 \end{vmatrix} \quad (ii) \begin{vmatrix} 7 & 9 & 15 \\ 0 & 0 & 0 \\ 8 & 7 & 5 \end{vmatrix} \quad (iii) \begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{vmatrix}$$

$$(iv) \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+ca & c^2 \end{vmatrix} \quad (v) \begin{vmatrix} 4 & 0 & 0 \\ 2 & 5 & 0 \\ 10 & 0 & 9 \end{vmatrix}$$

$$(vi) \begin{vmatrix} a & a+1 \\ a+2 & a+3 \end{vmatrix} \quad (vii) \begin{vmatrix} i^{103} & 3 & i^{101} \\ i^{56} & 5 & i^{54} \\ i^{23} & 7 & i^{21} \end{vmatrix} \quad \text{where } i = \sqrt{-1}$$

7. Answer the following within 50 words :-

Write the minors and co factors of each element of the following determinants.

a)

$$(i) \begin{vmatrix} 3 & -2 \\ 1 & 7 \end{vmatrix} \quad (ii) \begin{vmatrix} 3 & -2 & 5 \\ 4 & 0 & 4 \\ 1 & 7 & 6 \end{vmatrix}$$

b. Write the minors and co-factors of each element of the second column of each determinant.

$$(i) \begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} \quad (ii) \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

c. (i) Evaluate the determinant by expanding along the first row.

$$\begin{vmatrix} 1 & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$

- (ii) Evaluate the determinant by expanding along the second row.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

- (iii) Evaluate the determinant by expanding along the third row.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

8. Evaluate the determinant by expanding along second column and along second row and show that both the expansions give the same result.

$$\begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix}$$

9. Evaluate the determinant with the help of surrus diagram

$$(i) \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix} \quad (ii) \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix}$$

- 10.(a) Show that the value of determinant becomes zero, if any row / column of the determinant consists of zeros only.

b) Show that 
$$\begin{vmatrix} 3+2 & 2 & 5 \\ 2+3 & 1 & 4 \\ 1+4 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 4 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 4 \\ 4 & 2 & 3 \end{vmatrix}$$

11. Expand the determinant without using properties of determinant.

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+ca & c^2 \end{vmatrix}$$

12. Using properties of determinants, prove that :-

(a) 
$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$$

(b) 
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(c) 
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

(d) 
$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

$$(e) \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2 + b^2 + c^2)(a+b+c)(b-c)(c-a)(a-b)$$

(Hints : take  $c_1 = c_1 + c_2 - 2c_3$ )

$$(f) \begin{vmatrix} a^3 - x^3 & a^2 & a \\ b^3 - x^3 & b^2 & b \\ c^3 - x^3 & c^2 & c \end{vmatrix} = (abc - x^3)(a-b)(b-c)(a-c)$$

(Hints : Express it as the sum of two sets)

13. Using properties of determinants, prove that :

$$(a) \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

$$(b) \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3 \quad (\text{Hints : take common } a, b \text{ and } c)$$

$$(c) \begin{vmatrix} x^2 & 2xy & y^2 \\ y^2 & x^2 & 2xy \\ 2xy & b^2 & a^2 \end{vmatrix} = (a^3 + b^3)^2$$

$$(d) \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2$$

$$(e) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$f) \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3 \quad (\text{Hints : apply } c_1+c_2+c_3=c_1)$$

$$14. \text{ Prove that : } \begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+3y+2z \\ 3x & 6x+by & 10x+6y+2z \end{vmatrix} = x^3$$

$$15. \text{ Prove that :- (a) } \begin{vmatrix} 18 & 9 & 6 \\ 12 & 4 & 4 \\ 6 & 8 & 2 \end{vmatrix} = 0 \quad (b) \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$$

$$(c) \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0 \quad (d) \begin{vmatrix} a+1 & \omega & \omega^2 \\ \omega & a+\omega^2 & 1 \\ \omega^2 & 1 & a+\omega \end{vmatrix} = 0$$

$$16. \text{ Prove that :- } \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab+bc+ca+abc$$

$$17. \text{ Prove that : } \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

(Hints : take common a from  $R_1$  &  $C_1$ , b from  $R_2$  &  $C_2$  and c from  $R_3$  &  $C_3$ )

18. Find the minor and co-factor of the elements  $a_{11}$ ,  $a_{13}$  and  $a_{31}$  of the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

19. Prove that: 
$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

20. Prove that: 
$$a \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$$

21. Solve the following question of linear equation using determinants.

(i) $3x+4y=11$	(ii) $2x+y=10$	(iii) $6x-5y=0$
$4x+3y=10$	$4x-3y=0$	$y-x=1$
(iv) $x+y=3$	(v) $3x+y=7$	(vi) $4x+3y=8$
$3x+4y=10$	$2x+3y=21$	$6x+7y=17$

22. Solve the following by using Cramer's rule :

(i) $x+2y=249$	(ii) $2x+y-25=0$
$2x-y=103$	$y+3-2x=0$

23. Solve the following system of linear equations by using Cramer's rule.

(i) $6x+y-3z=5$	(ii) $x+2y+3z=6$
$x+3y-2z=5$	$2x+y+z=4$
$2x+y+4z=8$	$x+y+2z=4$

24. Using determinants, solve the following :

(i)  $y+z=0$   
 $x+y-2z=2$   
 $2x-5y+z=4$

$$(ii) \quad \frac{x}{2} + \frac{y}{3} + \frac{z}{5} = 3$$

$$\frac{x}{3} + \frac{2y}{5} + \frac{z}{3} = 27$$

$$\frac{x}{10} + \frac{y}{10} + \frac{z}{10} = 9$$

25. Using Cramer's rule, solve the following :

$$a^2x + b^2y + c^2z = d^2$$

$$ax + by + cz = d$$

$$x + y + z = 1$$

26. Solve the following by using Cramer's rule :

$$(i) \quad x + y + 2z = 5 \qquad (ii) \quad 7x + y + z = 4$$

$$x + 2y + z = 6 \qquad 2x + 2y - z = 2$$

$$-x - y + z = -2 \qquad 7x - 3z = 13$$

27. Solve using Cramer's rule

$$5x + y + z = 10,$$

$$7x - y + 5z = 21$$

$$2x + 5y + 2z = 18$$

28. The sum of three numbers is 14. If we multiply the first number by 2 and add the second number we get 13. If we add the first number and second to two times the third number, we get 18, find the number.

29. Cost of 110g of sugar, 1 kg of rice and 1 kg of ghee is Rs. 180; cost of 2 kgs of sugar, 1 kg of rice and 1 kg of ghee is Rs. 230. Similarly the cost of 1 kg of sugar, 2 kgs of rice and 2 kg of ghee is Rs. 310. Find the cost of each per kg.

30. The cost of 9 tables and 12 chairs is Rs. 279 and that of 8 tables and 16 chairs is Rs. 312. Find the cost of a chair and a table.

31. The perimeter of a triangle is 45 cm. The longest side exceeds the shortest by 8 cm and the sum of the length of the longest and shortest side is twice the length of the other side. Find the length of the sides.



**Answer**

1. (i) c      (ii) d      (iii) d      (iv) c      (v) b      (vi) a  
 (vii) b      (viii) a      (ix) d      (x) c  
 (xi) b      (xii) c      (xiii) c      (xiv) a
2. (i) 3      (ii) Zero      (iii) same      (iv) Zero  
 (v) equal      (vi) Non-singular.
3. (i)  $a_{11}a_{22} - a_{21}a_{12}$       (ii)  $\sec^2x - \tan^2x$   
 (iii)  $\sqrt[3]{1}$       (iv) added      (v) zeroes  
 (vi) -2      (vii) zero      (viii) Three      (ix) Order
4. (i) 9      (ii) One      (iii) 125A      (iv) transpose      (v) co-factor
5. (i) The minor of any element is the determinant obtained by deleting the row and the column in which that element lies.
- (ii) The product of  $M_{ij}$  and  $(-1)^{i+j}$  is called the co-factor of that element, where  $M_{ij}$  is the minor.
- (iii) Cramer's rule is a rule which is applied to solve linear equations under determinant.
- (iv) A simple method used for evaluating the determinant of order three with help of arrows is called sarrus diagram.
- (v) When there is solution to the system of equations, the system is called consistent, i.e.,  $D \neq 0$ .
- (vi) When there is no solution to the system of equations, it is called inconstant i.e.,  $(D = 0)$



## CHAPTER - 2

# ***MATRICES***

### **STRUCTURE :**

- 2.1 Meaning of matrix
- 2.2 Types of matrices
- 2.3 Algebra of matrices
- 2.4 Applications
- 2.5 Adjoint of a Matrix
- 2.6 Inverse of Matrix
- 2.7 Solution of system of linear equations using matrix
- 2.8 Questions

## 2.1 Meaning of Matrix

When we have to express a single value, there is hardly any difficulty. But when we have to express several values of several variables in a single arrangement, it becomes complicated. The study of matrices simplifies this difficulty to a great extent.

In a simpler sense, a matrix is a rectangular array (arrangement) of numbers or symbols, arranged in rows and columns. For example,

$$(i) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (ii) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (iii) \begin{pmatrix} 7 & 2 \\ 8 & 1 \\ 9 & 0 \end{pmatrix}$$

Suppose we have to express the following information in matrix form, then,

	Ram	Shyam	Hari	
Age	17	18	16	=
Marks	90	95	97	
				$\begin{bmatrix} 17 & 18 & 16 \\ 90 & 95 & 97 \end{bmatrix}$

The numbers of a matrix are called elements of a matrix or entries of a matrix.

Let us analyze the features of the following matrix.

		First Column	Second Column
		↓	↓
First row	→	$\begin{bmatrix} 7 & 2 \\ 8 & 1 \\ 9 & 0 \end{bmatrix}$	
Second row	→		
Third row	→		

Numbers in this same vertical line are called a column, numbers in the same horizontal line are called a row.

The number of rows and columns of a matrix determines the order of a matrix. In the above example, as there are 3 rows and 2 columns, the order of the matrix is  $3 \times 2$ . The first number of the order represents the number of rows, whereas the latter represents the number of columns.

Each element is an intersectional placement of a particular row & a particular column. In the above example, the element 2 is placed at the intersection of 1st row and 2nd column. Hence, the element may be represented by  $a_{12}$ . For general term  $i$  is row and  $j$  is column, we write the element as  $a_{ij}$ .

It is customary to denote a matrix by a capital letter & the matrix by parantheses ( ) or by square [ ] or by curly brackets { }. It should not be expressed by two straight lines like | |, which will otherwise mean a determinant. A matrix consists of 'm' rows 'n' columns is called a matrix of order  $m \times n$  (read as a m by n matrix). In general, the different elements of the matrix will be represented as below :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{bmatrix}$$

The element  $a_{ij}$  appears in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. The matrix may be noted as  $A = [a_{ij}]_{m \times n}$  where,  $m \times n$  indicates the order of the matrix x.

**Illustration 1 :**

$$A = \begin{pmatrix} 2 & 4 & 9 & 7 \\ 8 & 9 & 2 & 8 \\ 3 & 2 & 0 & 1 \end{pmatrix}$$

From the above matrix, find :-

- (i) Number of rows and columns
- (ii) Order of the matrix
- (iii) Rows and Columns separately
- (iv)  $a_{24}$ ,  $a_{32}$ ,  $a_{12}$ , and  $a_{23}$

**Solution :**

- (i) There are 3 rows and 4 columns.
- (ii) The order of matrix is  $3 \times 4$ .
- (iii) Rows are (2 4 9 7), (8 9 2 8) and (3 2 0 1)

and columns are  $\begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} 4 \\ 9 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 9 \\ 2 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 7 \\ 8 \\ 1 \end{pmatrix}$ .

(iv)  $a_{24} = 8$ ,  $a_{32} = 2$ ,  $a_{12} = 4$  and  $a_{23} = 2$ ,

**Illustration 2 :** Construct a  $3 \times 2$  matrix whose elements are  $a_{ij} = 3_i + 2_j$ .

**Solution :**

The  $3 \times 2$  matrix is represented by :

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

As  $a_{ij} = 3_i + 2_j$ , the respective elements are :

$$a_{11} = 3 \times 1 + 2 \times 1 = 5$$

$$a_{12} = 3 \times 1 + 2 \times 2 = 7$$

$$a_{21} = 3 \times 2 + 2 \times 1 = 8$$

$$a_{22} = 3 \times 2 + 2 \times 2 = 10$$

$$a_{31} = 3 \times 3 + 2 \times 1 = 11$$

$$a_{32} = 3 \times 3 + 2 \times 2 = 13$$

Hence, the required matrix is  $\begin{pmatrix} 5 & 7 \\ 8 & 10 \\ 11 & 13 \end{pmatrix}$

## 2.2 TYPES OF MATRICES :

The different types of matrix may be defined as below :

**(I) Row matrix :** A matrix is said to be a row matrix, if it has only one row and any number of columns. For example  $(2 \ 7 \ 0)$  is a row matrix of order  $1 \times 3$ .

**(II) Column matrix :** A matrix is said to be a column matrix, if it has only one column

and any number of rows. For example  $\begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$  is column matrix of order  $(3 \times 2)$

**(III) Zero matrix or null matrix :** A matrix is said to be a zero or null matrix, if all its elements are zero.

For example,  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  are null matrices of different order. Usually, null

matrices are denoted by 0.

**(IV) Square matrix :** A matrix is said to be a square matrix, if its number of rows is equal to its number of columns.

For example,  $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}$  is a square matrix of order  $3 \times 3$ .

**(V) Diagonal matrix :** A square matrix  $A = [a_{ij}]$  is said to be a diagonal matrix if all its elements other than that of principal diagonal are zero, i.e. if  $a_{ij} = 0$  for  $i \neq j$ .

For example :  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

where the main diagonal elements of A and B are 2, 3, and 1, 2, 3 respectively & other elements are zero.

**(VI) Identity or Unit matrix :** A square matrix where elements in the main diagonal are all '1' (one) & rest are all zero, is called an identity matrix or unit matrix. For example :

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is an identity matrix of order  $3 \times 3$ .

Such identity matrix is denoted as I.

**(VII) Scalar matrix :** A square matrix whose diagonal elements are equal & all non-diagonal elements are zero is called a scalar matrix. Thus unit matrix & null matrix are also called scalar matrix.

For example,  $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$  is a scalar matrix.

**(VIII) Triangular matrix :** A square matrix in which all elements above the principal diagonal are zero and rest are non-zero, is called a lower triangular matrix.

For example  $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 7 & 2 & 3 \end{bmatrix}$ . Similarly, a square matrix in which all elements below the principal

diagonal are zero and rest are non-zero, is called an upper triangular matrix.

For example,  $\begin{bmatrix} 1 & 4 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$  is an upper triangular matrix.

**(IX) Singular matrix :** If the determinants of a matrix is zero, the matrix is called singular matrix. Similarly, if the determinant of the matrix is not zero, the matrix is known as non-singular matrix.

**(X) Sub-matrix :** A matrix obtained by deleting any row/rows or column/columns of the matrix is known as a sub-matrix. For example,

if  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ , then  $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$  is a sub-matrix.

**Illustration 3 : Match the columns.**

Column A	Column B
(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	(i) Zero matrix
(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	(ii) Unit matrix
(c) $[2 \ 3 \ 4]$	(iii) Upper triangular matrix

$$(d) \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

(iv) Diagonal matrix

$$(e) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(v) Row matrix

$$(f) \begin{bmatrix} 1 & 7 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

(vi) Column matrix

**Solution :** (a) ii (d) vi  
 (b) i (e) iv  
 (c) v (f) iii

### 2.3. Algebra of Matrices :-

In this section, we shall deal with the following.

- 1) Equivalent matrix and equality of matrices.
- 2) Multiplication of matrix by a scalar.
- 3) Addition and subtraction of matrices.
- 4) Multiplication of Matrices.
- 5) Transpose of a matrix.
- 6) Adjoint of a matrix.
- 7) Inverse of a matrix.

Adjoint of a matrix and inverse of a matrix will be dealt with towards the end of this chapter.

#### 2.3.1 Equivalent of a matrix and Equality of matrices :-

A matrix is said to be equivalent to another matrix, if its number of rows & column

are equal to that of the other matrix. For example,  $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}_{2 \times 3}$  is equivalent to

$\begin{pmatrix} 8 & 11 & 12 \\ 10 & 15 & 17 \end{pmatrix}_{2 \times 3}$  as both have 2 rows and 3 columns. It is denoted by the symbol  $\simeq$ . The matrices are said to be equal if (i) they are of same order and (ii) each element of a matrix is equal to the corresponding element of the other, i.e., let  $A = [a_{ij}]$ ,  $B = [b_{ij}]$ , then matrix A is equal to matrix B, if  $a_{ij} = b_{ij}$ , for all values of i and j.

For example, if  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$ , then  $a = 4$ ,  $b = 7$ ,  $c = 5$  and  $d = 6$ .

In other words, if  $a = 4$ ,  $b = 7$ ,  $c = 5$  and  $d = 6$ , then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$$

**Example 3 :** Find the values of x, y, z, r, if:

$$\begin{bmatrix} z & 2x+y \\ r & 3x-2y \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 0 \end{bmatrix}$$

**Solution :**

$$\text{As } \begin{bmatrix} z & 2x+y \\ r & 3x-2y \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 0 \end{bmatrix}$$

Equating the corresponding elements on both sides

we have,  $z = 3$

$$r = 5$$

$$2x + y = 7$$

and  $3x - 2y = 0$

Solving  $2x + y = 7$  and  $3x - 2y = 0$ , we get  $x = 2$  and  $y = 3$ .

### 2.3.2 Multiplication of a matrix by a scalar :

If  $A = [a_{ij}]$  is a matrix & 'k' is a scalar, then  $kA = [ka_{ij}]$ , that is, when a matrix is multiplied by a scalar, each element of the matrix gets multiplied by the same scalar. For example, if

$$A = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}, \text{ then } 5A = \begin{bmatrix} 4 \times 5 & 5 \times 5 \\ 7 \times 5 & 8 \times 5 \end{bmatrix} = \begin{bmatrix} 20 & 25 \\ 35 & 40 \end{bmatrix}$$

**Example 4 :** If  $A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & 1 \\ 3 & 5 & -7 \end{bmatrix}$ , find  $-A$ .

**Solution :**  $A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & 1 \\ 3 & 5 & -7 \end{bmatrix}$

$$-A = (-1) A = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & -1 \\ -3 & -5 & 7 \end{bmatrix}.$$

### 2.3.3 Addition and subtraction of matrices :

If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be two matrices of same order, their sum or difference is another matrix, whose  $(ij)^{\text{th}}$  element is  $a_{ij} + b_{ij}$  or  $a_{ij} - b_{ij}$ . That is, on addition or subtraction, the corresponding elements are added or subtracted. For example,

If  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ , and  $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ , then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = [a_{ij} + b_{ij}] \text{ and}$$

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix} = [a_{ij} - b_{ij}]$$

### Illustration 4 :

If  $A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \\ -2 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$ , find  $A + B$  and  $A - B$ .

**Solution :**

$$A + B = \begin{bmatrix} 2 & 3 \\ 4 & 0 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2+1 & 3+2 \\ 4+3 & 0+1 \\ -2-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 1 \\ -4 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 3 \\ 4 & 0 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2-1 & 3-2 \\ 4-3 & 0-1 \\ -2+2 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -3 \end{bmatrix}.$$

**Properties of matrix addition :**

(i) Matrix addition is commutative.

If A and B are matrices of same order, then,  $A + B = B + A$

(ii) Matrix addition is associative.

If A, B and C are three matrices of same order, then,  $(A + B) + C = A + (B + C)$

(iii) Existence of addition identity.

If O is a null matrix of same order as that of matrix A, then :

$$A + 0 = A = 0 + A$$

(iv) Existence of additive inverse.

To each matrix A, there correspond a matrix B, such that,

$A + B = 0 = B + A$ , where 0 is a null matrix i.e.  $B = (-A)$ .

For example, if  $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 2 \end{pmatrix}$

$$\text{then, } A + (-A) = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ -3 & -1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

$$\text{Also, } (-A) + A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

$$\therefore A + (-A) = 0 = -A + A.$$

**Illustration 5 :**

$$\text{If } A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 1 & 2 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix}, \text{ find } 5A - 3B.$$

$$\begin{aligned} \text{Solution : } 5A - 3B &= 5 \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} + (-3) \begin{bmatrix} 7 & 1 & 2 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -5 & 15 \\ 10 & 20 & 25 \\ 10 & 15 & 30 \end{bmatrix} + \begin{bmatrix} -21 & -3 & -6 \\ 3 & 6 & 9 \\ -12 & -15 & -18 \end{bmatrix} \\ &= \begin{bmatrix} -16 & -8 & -9 \\ 13 & 26 & 34 \\ -2 & 0 & 12 \end{bmatrix} \end{aligned}$$

**2.3.4 Multiplication of matrices :**

The product of a row (row matrix) and a column (column matrix) is defined by :

$$(a_1, a_2, a_3, \dots, a_n)_{1 \times 4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}_{4 \times 1} = (a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n)_{1 \times 1}$$

which is the sum of products of corresponding elements. For example,

$$(1 \ 2 \ 3 \ 4)_{(1 \times 4)} \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}_{(4 \times 1)} = (1 \cdot 2 + 2 \cdot 3 + 3 \cdot 0 + 4 \cdot 1) = (12)_{(1 \times 1)}$$

Two matrices A and B can be multiplied only if **number of columns of the first matrix is equal to number of rows of second matrix**. If A is an  $(m \times n)$  matrix and B is an  $(n \times z)$  matrix, then the two matrices are multiplicable and AB is an  $m \times z$  matrix. When the number of columns of the first matrix is not equal to the number of rows of the second matrix, their product is not defined. Let us take an example to understand the computation.

**Illustration 6 :** If  $A = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{pmatrix}$  find AB and BA.

**Solution :**  $AB = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}_{3 \times 3} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{pmatrix}_{3 \times 2}$

$$= \begin{pmatrix} 2 \cdot 1 + 3 \cdot 2 + 0 \cdot 3 & 2 \cdot 2 + 3 \cdot 4 + 0 \cdot 0 \\ 4 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 & 4 \cdot 2 + 1 \cdot 4 + 2 \cdot 0 \\ 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 0 \end{pmatrix}_{3 \times 2}$$

$$= \begin{pmatrix} 8 & 16 \\ 12 & 12 \\ 14 & 10 \end{pmatrix}_{3 \times 2}$$

$$B \cdot A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{pmatrix}_{3 \times 2} \cdot \begin{pmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}_{3 \times 3}$$

B.A is not defined as the number of columns of the first matrix B is not equal to the numbers of rows of the second matrix A.

**Properties of matrix multiplication :**

(i) Matrix multiplication is associative, that is  $(AB) \cdot C = A \cdot (BC)$ , whenever all products are defined.

(ii) Matrix multiplication distributes over addition.

That is,  $A \cdot (B + C) = AB + AC$ , provided the products and sum exist.

(iii) Matrix multiplication is not commutative.

If A and B are two matrices, then  $AB \neq BA$ .

**Illustration 7:** If  $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 7 \end{pmatrix}$ ,  $B = \begin{pmatrix} 4 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$  and  $C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix}$

Show that  $(AB) \cdot C = A \cdot (BC)$ .

**Solution :**  $AB = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 7 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{pmatrix}$

Now,  $(AB) \cdot C = \begin{pmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 6 & 23 \\ 14 & 4 & 17 \\ 46 & 8 & 43 \end{pmatrix}$

$BC = \begin{pmatrix} 4 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 9 \\ 4 & 2 & 7 \\ 2 & 0 & 1 \end{pmatrix}$

Now  $A \cdot (BC) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 7 \end{pmatrix} \begin{pmatrix} 8 & 2 & 9 \\ 4 & 2 & 7 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 6 & 23 \\ 14 & 4 & 17 \\ 46 & 8 & 43 \end{pmatrix}$

$\therefore (AB) \cdot C = A \cdot (BC)$ .

**Illustration 8 :** If  $[3x \ 1] \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$ , find  $x$ .

**Solution :**  $[3x \ 1] \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$ , (Null matrix)

$$\Rightarrow [3x+2 \ 6x]_{1 \times 2} \begin{bmatrix} x \\ 3 \end{bmatrix}_{2 \times 1} = [0]_{1 \times 1}$$

$$\Rightarrow [3x^2 + 2x + 18x] = [0]$$

$$\Rightarrow 3x^2 + 20x = 0 \text{ (Equating the elements on both sides)}$$

$$\Rightarrow x(3x+20) = 0$$

$$\therefore x = 0 \text{ or } x = -\frac{20}{3}$$

**Illustration 9 :** If  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 8 & 4 \end{bmatrix}$

find  $AB$  and  $BA$ . Are they equal ?

**Solution :** As  $A$  and  $B$  are  $3 \times 2$  and  $2 \times 3$  matrices,  $AB$  and  $BA$  will be  $3 \times 3$  and  $2 \times 2$  matrices and hence, both  $AB$  and  $BA$  are defined.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 7 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 3 + 2 \times 7 & 1 \times 2 + 2 \times 8 & 1 \times 1 + 2 \times 4 \\ 3 \times 3 + 4 \times 7 & 3 \times 2 + 4 \times 8 & 3 \times 1 + 4 \times 4 \\ 5 \times 3 + 6 \times 7 & 5 \times 2 + 6 \times 8 & 5 \times 1 + 6 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 18 & 9 \\ 37 & 38 & 19 \\ 57 & 58 & 29 \end{bmatrix} \dots\dots\dots(1)$$

$$\text{Now, } BA = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 2 \times 3 + 1 \times 5 & 3 \times 2 + 2 \times 4 + 1 \times 6 \\ 7 \times 1 + 8 \times 3 + 4 \times 5 & 7 \times 2 + 8 \times 4 + 4 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 20 \\ 51 & 70 \end{bmatrix}$$

Now  $AB \neq BA$ .

**2.3.5 Transpose of a matrix and symmetric and skew symmetric matrix :**

The matrix obtained by interchanging the rows and columns of a matrix  $A$ , is called transpose of the matrix  $A$  and is denoted by  $A'$  or  $A^T$ . That is, if  $A = [a_{ij}]_{m \times n}$ , then  $A^T = [a_{ji}]_{n \times m}$ .

Let us take a numerical example.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ -1 & 2 \end{bmatrix}$$

If the transpose of a matrix is equal to the matrix itself, then the matrix is called symmetric matrix. For example,

$$\text{If } A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 4 \\ 1 & 4 & 7 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 4 \\ 1 & 4 & 7 \end{bmatrix}$$

As  $A = A^T$ ,  $A$  is called symmetric matrix. If  $A^T = -A$ , then the matrix  $A$  is said to be Skew-symmetric matrix. For example,

$$\text{if } A = \begin{bmatrix} 0 & 4 & 9 \\ -4 & 0 & 2 \\ -9 & -2 & 0 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} 0 & -4 & -9 \\ 4 & 0 & -2 \\ 9 & 2 & 0 \end{bmatrix}$$

$$\text{and } A^{-1} = \begin{bmatrix} 0 & -4 & -9 \\ 4 & 0 & -2 \\ 9 & 2 & 0 \end{bmatrix}$$

As  $A^{-1} = -A$ ,  $A$  is a skew-symmetric matrix. Similarly, when a matrix  $A$  is multiplied with its transpose  $A^t$ , if it gives an identity matrix  $I$ , the matrix  $A$  is known as an orthogonal matrix.

## 2.4 APPLICATIONS

**Illustration 10 :** If  $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ , find

$$2A + 3AB + B^2.$$

**Solution :**

$$2A = 2 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 2 \end{bmatrix}$$

$$3AB = 3 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = 3 \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 3 & 6 \end{bmatrix}$$

$$B^2 = B \times B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}$$

$$\text{Now } 2A + 3AB + B^2 = \begin{bmatrix} 4 & 6 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 18 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 24 \\ 11 & 12 \end{bmatrix}.$$

**Illustration 11 :** Show that matrix  $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 4 & 0 \end{bmatrix}$  satisfies the equation

$$A^3 - A^2 - 8A + 8I = 0, \text{ where } I \text{ and } 0 \text{ are identity and null matrix of the order } 3.$$

**Solution :**

$$8I = 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$8A = 8 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 16 & 0 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 10 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 10 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 10 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 18 & 4 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix}$$

Now,  $A^3 - A^2 - 8A + 8I$

$$= \begin{bmatrix} 1 & 18 & 4 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 8 & 16 & 0 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1-8+8 & 18-2-16+0 & 4-4+0+0 \\ 0+0+0+0 & 0-8+0+8 & 16+0-16+0 \\ 0+0+0+0 & 32+0-32+0 & 0-8+0+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0.

Thus, matrix A satisfies the equation  $A^3 - A^2 - 8A + 8I = 0$ .

**Illustration 12 :** If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$ , then show that  $(AB)^1 = B^1A^1$ .

**Solution :**

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 17 \\ 6 & 21 \end{bmatrix}$$

$$AB^1 = \begin{bmatrix} 10 & 6 \\ 17 & 21 \end{bmatrix} \dots\dots\dots (i)$$

$$B^1A^1 = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 17 & 21 \end{bmatrix} \dots\dots\dots (ii)$$

$$\therefore (AB)^1 = B^1A^1.$$

**Illustration 13 :** If  $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ , find x, y and z.

**Solution :**

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}_{(3 \times 3)} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{(3 \times 1)} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}_{(3 \times 1)}$$

$$\Rightarrow \begin{bmatrix} x - y + z \\ 2x + y - 3z \\ x + y + z \end{bmatrix}_{(3 \times 1)} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}_{(3 \times 1)}$$

$$\therefore x - y + z = 4 \dots\dots\dots (i)$$

$$2x + y - 3z = 0 \dots\dots\dots (ii)$$

$$x + y + z = 2 \dots\dots\dots (iii)$$

Solving the equations (i) and (iii) simultaneously, we have

$$x - y + z = 4$$

$$x + y + z = 2$$

$$2x + 2z = 6$$

$$\Rightarrow x + z = 3$$

Putting  $x + z = 3$ , in equation (iii), we have :

$$y + 3 = 2$$

$$\Rightarrow y = -1$$

Multiplying 3 in eqn. (iii) & solving simultaneously with eqn. (ii) we have :

$$3x + 3y + 3z = 6$$

$$2x + y - 3z = 0$$

$$5x + 4y = 6 \dots\dots\dots (iv)$$

Putting  $y = -1$  in eqn. (iv) we have :  $5x + 4(-1) = 6$

$$\Rightarrow 5x = 10$$

$$\Rightarrow x = 2$$

Now,  $x + y + z = 2$

$$\Rightarrow 2 - 1 + z = 2$$

$$\Rightarrow z = 1$$

$\therefore x = 2, y = -1$  and  $z = 1$ .

**2.5 ADJOINT OF A MATRIX :**

The adjoint of a square matrix  $A [a_{ij}]$  is defined a matrix obtained by taking the transpose of the matrix  $(a_{ij})$ , where  $A_{ij}$  is the co-factor of the element  $a_{ij}$ , it is denoted by  $\text{adj } A$ , i.e.,  $\text{Adj } A = [A_{ij}]$ .

For example, If  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ , then

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \text{ where } A_{11}, A_{12}, \dots\dots\dots \text{ are cofactors of } a_{ij}.$$

Note : If  $A$  is a square matrix, then  $A(\text{adj } A) = |A|I = (\text{adj } A).A$

But if  $A$  is a singular matrix, the product is zero matrix since  $|A| = 0$ .

**Illustration 14 :** Find the adjoint of the following matrix,  $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 5 & 2 \\ -10 & 4 & 3 \end{bmatrix}$ .

**Solution :** Let  $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 5 & 2 \\ -10 & 4 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Now co-factor  $A_{ij} = (-1)^{i+j}$  minor of  $a_{ij}$

$$\therefore A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix} = 7$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ -10 & 3 \end{vmatrix} = -20$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 5 \\ -10 & 4 \end{vmatrix} = 50$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} = -3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 0 \\ -10 & 3 \end{vmatrix} = 9$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 1 \\ -10 & 4 \end{vmatrix} = -22$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = -6$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ 0 & 5 \end{vmatrix} = 15$$

Now  $\text{adj } A = \text{transpose of co-factor matrix.}$

$$= \text{transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \text{transpose of } \begin{bmatrix} 7 & -20 & 50 \\ -3 & 9 & -22 \\ 2 & -6 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & 2 \\ -20 & 9 & -6 \\ 50 & -22 & 15 \end{bmatrix}.$$

**Illustration 15 :** Find the adjoint of the matrix,  $\begin{bmatrix} 2 & -1 \\ -4 & -3 \end{bmatrix}$

**Solution :** Let  $A = \begin{bmatrix} 2 & -1 \\ -4 & -3 \end{bmatrix}$ , then

$\text{adj } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  where  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$  and  $A_{22}$  are co-factors of the element of the matrix  $A$ .

$$\text{Now } A_{11} = (-1)^{1+1}(-3) = -3$$

$$A_{12} = (-1)^{1+2}(-4) = 4$$

$$A_{21} = (-1)^{2+1}(-1) = 1$$

$$A_{22} = (-1)^{2+2}(2) = 2$$

$$\therefore \text{adj } A = \text{transpose of } \begin{bmatrix} -3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}.$$

**Illustration 16 :** Show that  $(AB)^{-1} = B^{-1} \cdot A^{-1}$  with the help of the following matrices.

$$A = \begin{bmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

**Solution :**  $AB = \begin{bmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 6 & 23 \\ 14 & 4 & 17 \\ 46 & 8 & 43 \end{bmatrix}$

$$(AB)^{-1} = \begin{bmatrix} 16 & 14 & 46 \\ 6 & 4 & 8 \\ 23 & 17 & 43 \end{bmatrix} \dots\dots\dots (i)$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 8 & 7 & 23 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 14 & 46 \\ 6 & 4 & 8 \\ 23 & 17 & 43 \end{bmatrix} \dots\dots\dots (ii)$$

## 2.6 INVERSE OF MATRIX :

A square matrix is said to be invertible if there exists another square matrix B such that  $AB=BA=I$  (where I is the identity matrix).

The matrix B is called the inverse of A and is denoted by  $A^{-1}$ . In other words,  $A \cdot A^{-1} = A^{-1} \cdot A = I$ .

If A is a non-singular square matrix then inverse of A is given by  $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$ .

**Proof :** We know that :

$$A \cdot (\text{adj } A) = (\text{Adj } A)A = |A| \cdot I \text{ where } |A| \neq 0$$

$$\Rightarrow A \left( \frac{1}{|A|} \text{adj } A \right) = \left( \frac{1}{|A|} \text{adj } A \right) A = I$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \cdot \text{adj } A.$$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \text{adj } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{Illustration 17 : If } A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \text{ show that } A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Solution : Assuming } B = A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{we have } AB = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \text{ (identity matrix of order 3).}$$

$$\text{Now } BA = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$\therefore AB = I = BA$$

Thus  $A^{-1}B = B = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ .

**Illustration 18 :** If  $A = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$ , prove that  $A^{-1} = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$

**Solution :** Let  $B = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -20+21 & 35-35 \\ -12+12 & 21-20 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

As  $AB = I = BA$ , B is the inverse of A.

i.e.,  $A^{-1} = B = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$ .

**Illustration 19 :** Find the inverse of the matrix,  $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$ .

**Solution :**

$$\begin{aligned} \det A = |A| &= \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{vmatrix} \\ &= 1(3-0) - (-1)(2-4) + 0(0+6) \\ &= 3-2 \\ &= 1 \dots\dots\dots (i) \end{aligned}$$

The co-factors of the elements of the Matrix A are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 3$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 0 \\ 3 & -2 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} = 2$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

$$\text{adj } A = \text{transpose of } \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix} \dots\dots\dots (ii)$$

$$\text{Therefore } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{1} \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}.$$

## 2.7 SOLUTION OF SYSTEM OF LINEAR EQUATIONS USING MATRIX :

In the following system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3.$$

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then this system of equations can be expressed

$$\text{as } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

this is  $AX = B$

$\Rightarrow X = A^{-1}B$ . (where  $A$  is non-singular matrix.)

Then equating the corresponding elements of the above matrix equation, the values of the variable can be found out.

**Illustration 20 :** Solve the system of equation using matrix method.

$$3x + 2y = 7$$

$$4x - y = 2$$

**Solution :** Expressing the equations in matrix form, we have

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

It can also be written as  $AX = B$

$$\text{where } A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\det A = -3 - 8 = -11 \dots\dots\dots (i)$$

As  $|A| = -11$ , it is non-singular and its inverse exists. The co-factors of the matrix  $A$  are

$$A_{11} = -1 \qquad A_{21} = -2$$

$$A_{12} = -4 \qquad A_{22} = 3$$

$$\text{Adjoint } A = \text{transpose of } \begin{bmatrix} -1 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix} \dots\dots\dots (ii)$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{-11} \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix} \dots\dots\dots (iii)$$

$$\begin{aligned} \text{again } X = A^{-1}B &= \frac{1}{-11} \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} \\ &= \frac{1}{-11} \begin{bmatrix} -7 - 4 \\ -28 + 6 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{-11} \begin{bmatrix} -11 \\ -22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Equating the corresponding elements on both sides, we have  $x = 1$  and  $y = 2$ .

**Illustration 21 :** The following shown the monthly salary of an undertaking for the month of May 2016.

Month	No. of employees			Total monthly Salary (in ₹)
	Clerks	Peons	Labourers	
May	1	3	1	2,075
June	2	3	1	2,625
July	2	4	2	3,350

Compute the monthly salary of each type of employees.

**Solution :** The matrix expression of the table data will be :

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2075 \\ 2625 \\ 3350 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 4 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2075 \\ 2625 \\ 3350 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 4 & 2 \end{vmatrix} = -2.$$

Co-factors of the elements of the matrix A are :

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = -2$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3$$

$$\text{adj } A = \text{transpose} \begin{bmatrix} 2 & -2 & 2 \\ -2 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 2 & -3 \end{bmatrix} \dots\dots\dots \text{(ii)}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-2} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 2 & -3 \end{bmatrix} \dots\dots\dots \text{(iii)}$$

Now  $AX = B$   
 $\Rightarrow X = A^{-1}B$

$$= \frac{1}{-2} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2075 \\ 2625 \\ 3350 \end{bmatrix} = \begin{bmatrix} -2075 + 2625 + 0 \\ 2075 + 0 - \frac{3350}{2} \\ -2075 - 2625 + \frac{3}{2} \cdot 3350 \end{bmatrix}$$

$$= \begin{bmatrix} 550 \\ 2075 - 1675 \\ -4700 + 5025 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 550 \\ 400 \\ 325 \end{bmatrix}$$

$x = 550$ ,  $y = 400$  and  $z = 325$

The monthly salary of each clerk, peon and labourer are ₹550, ₹400 and ₹325 respectively.

## 2.5 QUESTIONS :

1. Choose the correct answer from the alternatives given below :

- (i) The inverse of the matrix  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  can not be found as A is a :
- (a) singular matrix (b) non-singular matrix  
(c) square matrix (d) Identity matrix
- (ii) A matrix with single column with any number of rows is known as :
- (a) Singular matrix (b) Row matrix (c) Column matrix (d) Diagonal matrix
- (iii) A row matrix has only :
- (a) one element (b) one row with one or more columns  
(c) one column with one or more rows (d) one row and one column

- (iv) The co-factor of 3 in  $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$  is :

- (a) 11      (b) -11      (c) 22      (d) -22

(v) If  $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$  then  $A^2$  is equal to :

- (a)  $\begin{bmatrix} 9 & 7 \\ 4 & 11 \end{bmatrix}$       (b)  $\begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$       (c)  $\begin{bmatrix} 19 & 17 \\ 14 & 11 \end{bmatrix}$       (d)  $\begin{bmatrix} 17 & 19 \\ 11 & 14 \end{bmatrix}$

(vi) The solutions of the following equations, if solved simultaneously are :

$$2x + 3y = 1$$

$$3x + 4y = 1$$

- (a) -1, -1      (b) 1, 1      (c) 1, -1      (d) -1, 1.

(vii) The matrix expression of the following system of linear equation is :

$$7x - 3y - z = 3$$

$$4x + y - 2z = 3$$

$$5x + y + z = 7$$

(a)  $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

(b)  $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & -2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

(c)  $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -7 \end{bmatrix}$

(d)  $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

(viii) If  $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ,  $A^{-1}B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $x = A^{-1}B$ , the value of x, y and z respectively are :

- (i) 1, 2, 3      (b) 3, 2, 1      (c) 2, 1, 3      (d) 2, 3, 1

(ix) The inverse of  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  is :

- (a)  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$       (b)  $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$       (c)  $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$       (d)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- (x) If the matrix  $A$  is both symmetric and skew symmetric, then  
 (a)  $A$  is a diagonal matrix. (b)  $A$  is a zero matrix.  
 (c)  $A$  is a square matrix. (c) Identity matrix.
- (xi) If  $A$  is square matrix such that  $A^2 = A$ , then  $(I+A)^3 - 7A$  is equal to :  
 (a)  $A$  (b)  $I-A$  (c)  $I$  (d)  $3A$

**2. Answer the following in one word/ term each :**

- (i) A matrix that comprises one element only.  
 (ii) A matrix that consists of zero elements.  
 (iii) A square matrix in which all the principal diagonal elements are non-zeroes and all other elements are zeroes.  
 (iv) A diagonal matrix in which all the leading diagonal elements are equal.  
 (v) A square matrix in which all the leading diagonal elements are unity or one, and other elements are zeroes.  
 (vi) A square matrix in which all the elements above or below the principal diagonal are zeros.  
 (vii) A matrix that appears with equal number of rows and columns.

**3. Correct the underlined portion of the following sentences :**

- (i) If the transpose of a matrix is equal to the matrix itself, then the matrix is called a skew-symmetric matrix.  
 (ii) If  $A$  and  $B$  are two matrices then  $A \times B$  is equal to  $B \times A$ ,  
 (iii) If  $AX = B$ , then  $X = B^{-1}A$ .  
 (iv) A matrix whose determinant value is zero is called a non-singular matrix.  
 (v) A matrix  $A = [a_{ij}]_{m \times n}$  is said to be a square matrix if  $m > n$ .  
 (vi) A matrix obtained by deleting the rows or columns or both of a matrix is called its super-matrix.

**4. Fill in the blanks :**

- (i) A square matrix, which when multiplied by its transpose amounts to an identity matrix is called a/an \_\_\_\_\_ matrix.  
 (ii) A symmetric matrix that reproduces itself is termed as a/an \_\_\_\_\_ matrix.  
 (iii) A square matrix  $A$  is said to be symmetric if  $A^1 =$  \_\_\_\_\_.  
 (iv) A square matrix  $A$  is said to be skew-symmetric, if  $A^1 =$  \_\_\_\_\_.  
 (v) A square matrix with 1's on the main diagonal and 0's elsewhere, is called a \_\_\_\_\_ matrix.

**5. Answer the following questions in one sentences each.**

- (i) What is a matrix ?  
 (ii) Give a numerical example of a matrix.

- (iii) Define a null matrix.
- (iv) What do you mean by a square matrix ?
- (v) What is meant by a singleton matrix ?
- (vi) What do you mean by transpose of a matrix ?

**6. Answer the following questions in within 30 words.**

- (i) What do you mean by Equal matrices ? Explain with examples.
- (ii) Explain orthogonal matrix with examples.
- (iii) What is skew symmetric matrix ? Explain with examples.
- (iv) Explain and illustrate triangular matrices.
- (v) What is adjoint matrix ? Explain with example.
- (vi) Explain the condition necessary for addition of matrices.

**7. Answer the following questions within 50 words.**

- (i) What is equal matrix ? How it is different from equivalent matrix, explain with example.
- (ii) What is subtraction of matrices state the conditions necessary for subtraction and explain the procedure of subtraction.
- (iii) What is multiplication of matrices ? Explain the condition and procedure necessary for multiplication.
- (iv) What is transpose of a matrix ? Prove that transpose of a transposed matrix reproduces the original matrix.
- (v) Explain the procedure of finding the Adjoint of a matrix with an example.
- (vi) What is singular and non-singular matrix ? Explain with example.

8. If  $A = \begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 7 \\ 5 & 3 \end{bmatrix}$  find

- (a)  $A + B$       (b)  $B - A$       (c)  $-2A$       (d)  $3B$

9. Find the transpose of (a)  $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$       (b)  $\begin{bmatrix} 3 & -2 & 5 \\ 7 & 2 & 8 \\ 1 & 1 & 0 \end{bmatrix}$

10. Find the sum of the matrices given below :

(a)  $\begin{bmatrix} 4 & 2 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}$       (b)  $\begin{bmatrix} 17 \\ 12 \end{bmatrix} + \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

$$(c) \begin{bmatrix} \frac{1}{5} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{4}{5} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$(d) \begin{bmatrix} x & y \\ 2x & 3y \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 5 & 0 \end{bmatrix}$$

11. Evaluate:  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ -2 & -2 & 5 \\ -4 & -5 & -7 \end{bmatrix}$

12. Solve the following matrix equations :

$$(a) \begin{bmatrix} 2x+y & x \\ z & p \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

$$(b) \begin{bmatrix} x & 2 & y \\ 4 & z & 9 \end{bmatrix} + \begin{bmatrix} y & 4 & -x \\ 4 & 2 & p \end{bmatrix} = \begin{bmatrix} 10 & 6 & 6 \\ 8 & 10 & 7 \end{bmatrix}$$

$$(c) \begin{bmatrix} x+9 & y \\ z & 4 \end{bmatrix} + 2 \begin{bmatrix} x & 1 \\ y & p \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(d) \begin{bmatrix} 2x+1 & z \\ z-3 & 3y \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 4 & 9 \end{bmatrix}$$

13. If  $A = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$

find (a)  $(A)'$  (b)  $(A+A)'$  (c)  $3A' - B$   
 (d)  $2A + 4B$  (e)  $(2A+B)'$  (f)  $4A + 9B + A'$

14. Find the product of the following matrices :

$$(a) \begin{pmatrix} 2 & 4 & 9 & 8 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \\ 9 \\ 7 \end{pmatrix}$$

$$(b) \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

$$(c) \begin{pmatrix} 2 & 4 & 9 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 2 & 3 & 0 \end{pmatrix} \quad (d) \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 2 & 0 \\ 5 & 0 & 1 \\ 1 & 0 & 7 \end{pmatrix}$$

$$15. \text{ If } A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 3 & 7 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 6 & 7 & 8 \\ 0 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

Show that :

$$(a) \quad A + (B + C) = (A + B) + C$$

$$(b) \quad A(B + C) = AB + AC$$

$$16. \text{ Prove that the matrix } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \text{ is a root of the polynomial } f(x) = x^3 - x^2 - 8x + 8$$

(Hints : Prove  $f(A) = A^3 - A^2 - 8A + 8.I = 0$ )

$$17. \text{ Verify that } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ satisfies the equation } x^2 - (a + d)x + (ad - bc) = 0$$

where  $I$  and  $0$  are  $2 \times 2$  matrices.

$$18. \text{ If } \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ find the value of } x \text{ and } y.$$

$$19. \text{ If } \begin{bmatrix} x+y & x-z \\ 2x-y & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \text{ find the value of } x, y, z.$$

$$20. \text{ If } A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & 5 \end{bmatrix} \text{ then find } A - 3B + 2C.$$

$$21. \text{ Find the matrix which when added to } \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \text{ gives } \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}.$$

$$22. \text{ If } A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}, \text{ then show that } (A + B)^2 \neq A^2 + 2AB + B^2.$$

23. Determine the value of  $k$  for which the matrix  $\begin{bmatrix} k & k \\ 2 & k \end{bmatrix}$  has no inverse.
24. Determine the value of  $k$  for which the matrix  $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 4-k & 4 \\ k & 3 & 4 \end{bmatrix}$  has no inverse.
25. Give one example of identity matrix and one example of null matrix each of the order 3.

26. If  $\begin{bmatrix} 1 & 1 & 1 \\ -3 & 4 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 11 \end{bmatrix}$  find the value of  $x$ ,  $y$  and  $z$ .

27. Find the value of  $x$  and  $y$  if  $\begin{bmatrix} 3 & 2 \\ 7 & x \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

28. If  $A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & -1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & -2 \end{bmatrix}$  show that  $(AB)' = B' - A'$ .

29. Find  $A$  if  $A^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$ .

30. Find the adjoint of the matrices given below :

(a)  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  (b)  $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \\ 1 & 5 & 1 \end{bmatrix}$  (c)  $\begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ 0 & -7 & -8 \end{bmatrix}$

31. Find the inverse of the following matrices :

(a)  $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$

32. Find the inverse of the matrices given below :

$$(a) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

33. Find the determinant, adjoint and inverse of the matrix.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -5 \\ 1 & 1 & -2 \end{bmatrix}$$

34. If  $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ , find  $(AB)^{-1}$ .

35. If  $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$ , show that  $(AB)^{-1} = B^{-1}A^{-1}$ .

36. If  $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ , prove that  $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & -3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ .

37. If  $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$ , show that  $AA^{-1} = I$  where  $I$  is the identity matrix of order 3.

38. Matrix  $x$  and  $y$  are such that  $3x + 4y = I$  and  $x - 2y = 2I$  where  $I$  denotes

the identity  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . (Hints : Solve the two equations.)

Find  $x$  and  $y$ .

Solve the following system of equations by matrix method :

39.  $3x - 2y = 12$

$5x - 3y = 1$

40.  $x + 2y = 7$

$3x + 2y = 13$

41.  $x + y = 10$

$2x + 5y - 29 = 0$

42.  $5x + 7y = 12$

$11y + 3x = 14$

43.  $x + 3y - 2z = 1$

$3x + 4y + z = 5$

$5x + 5y - 3z = 5$

44.  $3x - 2y + z = 1$

$2x + y - 5z = 2$

$x - y - 2z = 3$

45.  $x + y - 3z = -6$

$2x + y + z = 7$

$x - y + 2z = 5$

46. If  $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 4 & 7 \end{pmatrix}$ , find  $A^{-1}$  and using  $A^{-1}$  solve the following system of linear equations :

$x + y + z = 5$

$2x + y + z = 7$

$3x + 4y + 7z = 21.$

47. If the cost of 5 cutlets and 7 chops is ₹12 and cost of 3 cutlets and 11 chops is ₹14. Find the cost of each chops and cutlets by using matrices.

48. A man purchases a chair and a table for ₹13,000, another man purchases at the same price 4 chairs and a table for ₹28,000. What is the cost of each chair and each table ? Use matrix to get the answer.

49. There are skilled, semi-skilled labours in 3 factories. Their remuneration is stated below ?

Factory	No. of labourers			Total remuneration (₹)
	Skilled	Semi-skilled	Unskilled	
A	2	4	10	1780
B	2	5	10	1900
C	1	3	5	1010

Find the salary of each skilled, semi-skilled and un-skilled labourer.

50. If  $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

show that  $AB = I$ , where  $I$  is the identity matrix of order 3. Also show that  $B = A^{-1}$ .



### ANSWER

1. (i) a (ii) c (iii) b (iv) b (v) b (vi) d (vii) b (viii) a (ix) c (x) b (xi) c.
2. (i) single-ton matrix, (ii) zero or null matrix (iii) Diagonal matrix (iv) scalar matrix, (v) unity or identity matrix (vi) Triangular matrix, (vii) Square matrix.
3. (i) symmetric, (ii) Not equal, (iii)  $A^{-1}B$ , (iv) singular (v)  $m = n$  (vi) Sub-matrix
4. (i) orthogonal (ii) idempotent (iii)  $A$  (iv)  $-A$  (v) unity/Identity.
5. (i) Matrix is an ordered array of elements in rows and columns.

(ii)  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

- (iii) When all the elements contained by a matrix are zeros, is called a null matrix.
- (iv) When all the number of elements of rows and number of elements of columns are equal, the matrix is termed as a square matrix.
- (v) A matrix having one element is called a single ton matrix.
- (vi) When the row of a matrix are changed to columns or vice-verse, such resultant matrix is called transpose of the matrix.



## **CHAPTER - 3**

# ***SET THEORY***

### **Structure**

- 3.1 Introduction
- 3.2 Finite and infinite sets
- 3.3 Set notation
- 3.4 Single member set
- 3.5 Empty or null set
- 3.6 Disjoint sets
- 3.7 One-to-one correspondence
- 3.8 Equivalent sets
- 3.9 Equal sets
- 3.10 Subsets
- 3.11 Power sets
- 3.12 Universal sets
- 3.13 Complement of a set
- 3.14 Venn diagrams
- 3.15 Union of sets
- 3.16 Intersection of sets
- 3.17 properties of set operation
- 3.18 Application of sets
- 3.19 Questions

### 3.1 Introduction

In everyday life, we usually speak of “a pack of cards” ‘a bunch of keys’, ‘a herd of cattle’, ‘a crowd at a cricket match’ and collection of articles etc. to denote group of different objects. In mathematics, we use the word set to denote such collections.

Thus, a set is a collection of well-defined objects. By “well-defined”, we mean that, it must be possible to tell beyond doubt, whether or not a given object belongs to the collection that we are considering. If we are asked to write the set of all intelligent students of a class, it is not possible to do so. No two persons will have the common list. Thus the collection of intelligent students in the class is not a set. It is not necessary that a set contains only the same type of objects. If we have a book, a football, a glass & a table, then these objects also form a set although, they are different from each other. These objects are called elements or members of the set.

In mathematics, we have set of numbers of various kinds, sets of points on a line, sets of lines in a plane etc.

A few examples of the sets are given below:-

- (i) The set of pupils in a school
- (ii) The set of teachers in a school
- (iii) The set of natural numbers between 2 and 20
- (iv) The set of vowels in English alphabet
- (v) The set of odd numbers between 4 and 16
- (vi) The set of a pen, a pencil, a notebook, a tennis ball and a racket.

### 3.2 Finite and Infinite sets

A set is called a finite set, if we can count the number of elements of the set.

A set in which the number of elements cannot be counted is called an infinite set.

Set of students in a school, set of people living in Delhi, a set of natural numbers from 3 to 19 are some of the examples of finite sets.

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Set of all natural numbers, set of points on a line & set of circles with a fixed centre are some of the examples of infinite sets.

### 3.3 Set Notation

We use capital letters A, B, C, D,..... to describe a set. The elements of a set are denoted by small alphabets a, b, c, x, y etc.

A set can be expressed in the following two ways :

- (i) Tabulation or Roster Method.
- (ii) Rule method or set Builder form.
- (i) Roster method :** In this method, the elements of the set are listed together & are placed within braces { }. For example, set of natural numbers less than 5 is written as {1, 2, 3, 4} whereas, the set of vowels in English alphabet is {a, e, i, o, u}. In the case of Infinite set, as we cannot list all the elements, we write some elements & then put 3 or 4 dots before putting the braces. Thus, set of all natural numbers is written as {1, 2, 3, 4.....}.

The set of whole members is written as : {0, 1, 2, 3, .....}

The set of prime members is written as : {2, 3, 5, 7, 11, ....}

- (ii) Rule Method :** In this method, we write a set by some special property that has to be satisfied by the elements to become the member of that set. Thus, the set {1, 2, 3, 4, .....} can be rewritten as {All natural numbers less than 5}.

Set of integers {..., -3, -2, -1, 0, 1, 2, 3, .....} can be written as {set of integers}.

We also write the set by rule method in the form {x; x is a natural number <5} and {x:x is and integer}. Here x denotes all the elements satisfying the given property.

Thus, set P= { x:x is prime} is read as "P is the set of numbers x such that x is a prime number: Sometimes, we also use symbol. 'E' in expressing a set. Thus, set of even numbers is written as { 2x:x ∈ N}

Set of natural numbers between 5 and 20.

$$= \{ x : x \in \mathbb{N} \text{ and } 5 < x < 20 \}$$

**Example 1:** Use the roster method to describe each of the following sets:

- (i) Set of integers greater than 10 and less than 15.

(ii) Set of positive integers divisible by 3 and less than 18.

**Solution :**

(i) Let the given set be denoted by A. then  $A = \{11, 12, 13, 14\}$ .

(ii) Let the given set be denoted by B.

Then  $B = \{3, 6, 9, 12, 15\}$ .

**Examples 2 :** Use the rule method to describe each of the following.

(i)  $\{1, 3, 5, 7, 9, 11, \dots\}$

(ii)  $\{5, 10, 15, 20, \dots\}$

**Solution :**

(i)  $\{1, 3, 5, 7, 9, 11, \dots\} = \text{set of all odd numbers.}$

$$= \{x : x \text{ is an odd number}\}$$

This set can also be written as :  $\{2x - 1 : x \in \mathbb{N}\}$

(ii)  $\{5, 10, 15, 20, \dots\} = \text{set of multiples of 5.}$

$$= \{5x : x \in \mathbb{N}\}$$

**Example 3 :** Use roster method to express the set,

$$A = \{x : x = 5y - 1, y \in \mathbb{N}, y > 3\}$$

**Solution :** As  $y > 3$ , we start substituting  $y = 4, 5, 6, \dots$  and we get,

$$A = \{19, 24, 29, 34, \dots\}$$

**Example 4 :** Write the set containing all days of the week beginning with S.

**Solution :** let S be the required set.

$$\text{Then } S = \{\text{Sunday, Saturday}\}.$$

### 3.4 Single Member set

A set which has only one member is known as a single member set. For example, a set of natural numbers between 7 and 9 is a single member set having only one element 8. i.e.,  $\{8\}$  it is also called a 'singleton'.

### 3.5 Empty or Null set

Can you think of a natural number less than 1? Certainly, there is no natural number less than 1. Thus, set of all natural numbers less than 1, will not contain any element. Hence, a set which contains no element is called an empty or null set. It is also called a void set. In Listing elements of an empty set, we have braces { } and having nothing within the braces. Null set is usually denoted by the Greek letter ' $\phi$ ' (phi). Clearly the set {0} is not an empty set as it is a set which has one element 'zero' belonging to it. The set of triangles with two obtuse angles is { } & the set of natural numbers between 5 and 6 is also  $\phi$ .

### 3.6 Disjoint Sets

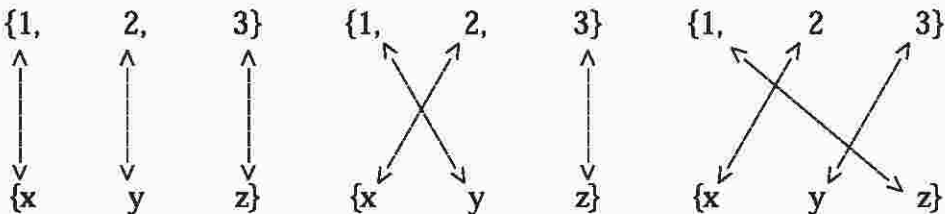
If no element of A is in B and no element of B is in A, then A and B are called disjoint sets. For example, sets {1, 3, 5} and {2, 4, 6} are disjoint sets as there is no common element in them. In other words, two sets which have no common members are called disjoint sets.

### 3.7 One-to-one correspondence

Two sets are said to be in one to-one correspondence, if they can be matched in such a way, that each element of one set is associated with a single element of the other.

Considers the sets  $X = \{1, 2, 3\}$  and  $Y = \{x, y, z\}$

We can pair the elements of one set with the elements of the other in different ways.



and so on.

Thus, we observe that, each member of one set is matched with one and only one member of the other set. so, X and Y have one-to-one correspondence between them.

Let  $A = \{a,b,c\}$  and  $B = \{p,q,r,s\}$ , then it is clear that A and B are not in one-to-one correspondence, If the elements of A and B are matched, then one element of B remains

unmatched. Thus, we say that A has fewer elements than B or B has more elements than A. In other words, set b is larger than set A.

### 3.8 Equivalent sets

Two sets X and Y are said to be equivalent sets, if the number of elements in X is equal to the numbers of elements in Y i.e , there is one-to-one correspondence between the elements of X and Y. For example, {1,2,3} and {x,y,z} are equivalent sets. The symbol " $\sim$ " is used to denote equivalence. Thus  $A \sim B$  is read as 'A is equivalent to B'.

### 3.9 Equal sets

Two sets A and B are called equal sets if every element of A is also an element of B & every element of B is also an element of A. For example {r,s,t} and {t, r, s} are equal sets.

Obviously, if two sets are equal, they are equivalent too, but if two sets are equivalent they may not be equal. Corresponding to a set, there are infinite number of equivalent sets.

**Example :** Given  $A = \{2,4,6,8\}$  and  $B = \{2,8, 4,6\}$  we see that every element of A is a member of B and each element of B is also a member of A.

Therefore, set A = set B

Further, if X is a set of letters in the word 'ronak' and y is a set of letters in the word 'kanor' then  $x = \{\text{ronak}\}$  and  $y = \{\text{kanor}\}$

$\therefore x = y.$

From the above, we observe that the order of elements and repetition of elements do not change a set.

### 3.10 Subsets

Let A and B be two sets, such that :

$A = \{\text{all months of the year}\}$

$B = \{\text{all months of the year starting with j}\}$

We see that each element of B is also a member of A. Thus, B is Included in the set A. B is called subset of A. Thus, we define a subset as :

If every element of a set A is also an element of another set B, then A is called a subset of B. and we write it as  $A \subset B$  or  $B \supset A$ . Symbol  $\subset$  is used to denote "is a subset of" or "is

contained in” and symbol  $\supset$  is used to denote “is a superset of” or “contains”. Thus,  $A \subset B$  is read as A is contained in B and  $B \supset A$  is read as B contains A.

For A to be a subset of B, every element of A must be in B, but every element of B may or may not be in A. If every element of B is also in A, then  $B \subset A$ , too. This shows that A and B are same sets.

Thus,  $A \subset B$  and  $B \subset A \Leftrightarrow A = B$ .

In case every element of A is in B, but every element of B is not in A i.e., if  $A \subset B$  but  $B \not\subset A$ , then A is said to be a proper subset of B. For example, the set of odd natural numbers is a proper subset of the set of natural numbers.

Clearly, every set is a subset of itself and null set or  $\phi$  is a subset of every set.

**Example :** Consider a set  $A = \{1\}$ . It has two possible subsets  $\{1\}$  and  $\phi$ .

Again, in set  $B = \{1,2\}$ , the possible subsets are  $\phi$ ,  $\{1\}$ ,  $\{2\}$ ,  $\{1,2\}$

Further the set  $\{1,2,3\}$  has  $\phi$ ,  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{1,2\}$ ,  $\{1,3\}$ ,  $\{2,3\}$ ,  $\{1,2,3\}$  as its sub sets.

Thus, we observe that :

A set with 1 element has 2 sub sets.

A set with 2 elements has  $2^2$  subsets.

A set with 3 elements has  $2^3$  subsets.

A set with n elements has  $2^n$  subsets.

Proceeding in the same way, it can easily be verified that the number of proper subsets of a set with ‘n’ elements is  $2^n - 1$ .

### 3.11 Power Set

Elements of a set can also be some sets. Such sets are called sets of sets. For example,  $\{\phi, \{1\}, \{2\}, \{1,2\}\}$  is a set whose elements are four sets  $\phi$ ,  $\{1\}$ ,  $\{2\}$ ,  $\{1,2\}$ . The set of all subsets of a given set X is called the power set of X and is denoted by  $P(X)$ .

For example, if  $X = \{x,y,z\}$ , then,

$P(X) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$ .

### 3.12 Universal Set

Every set is likely to be a subset of another set. A set which is such that all the sets under consideration are its subset is called the universal set or universe. It is denoted by  $U$ . Set of natural numbers is a universal set for all the finite set of natural number. The set of students of Class X is the universal set for the set of students of Class X A, the set of students of Class X B, the set of students of Class X C, etc. For plane geometry, the universal set is the set of all points of the plane and so on.

### 3.13 Complement of a set

Complement of a set  $A$  is the set which contains all those elements of the universal set which are not in  $A$ . It is denoted by  $A'$  or  $\bar{A}$ .

Thus,  $A' = \{x : x \in U, x \notin A\}$

For example, if  $U = \{1, 2, 3, 4, 5, 6, 7\}$  and  $A = \{2, 4, 3\}$

then  $A' = \{1, 5, 6, 7\}$

Clearly, complement of the complement of a set  $A$  is set itself, i.e.  $(A')' = A$

## SET OPERATIONS

### 3.14 Venn Diagrams

The idea of sets can also be represented diagrammatically. These representation of sets are called Venn Diagrams.

In Venn diagrams, universal set  $U$  is usually shown by a rectangular region and the subsets of  $U$  by circular regions inside it. The elements of  $U$  are represented as points inside the rectangle, whereas the points inside the circle denote the elements of the subsets. Note that the size of the region has nothing to do with the number of elements of set.

In Fig. 3.1,  $U$  represents the set of natural number  $< 15$  and  $A = \{\text{odd numbers} < 15\}$

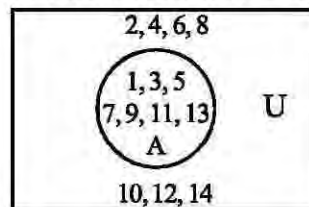


Fig. 3.1

### 3.15 : Union of Sets

Let A and B be two given sets. A set in which elements of A or B or both are included is called union of A and B and is denoted by  $A \cup B$  i.e.  $A \cup B = \{x : x \in A \text{ or } x \in B\}$ .

In other words, union is described as an 'either' 'or' idea i.e.  $A \cup B$  contains all those elements which are either in A or in B. For example, if  $A = \{1, 2, 3, 4\}$  and  $B = \{5, 6, 7\}$  then,

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

$$\text{and if } X = \{2, 4, 6\} \text{ and } Y = \{1, 4, 6\}$$

$$\text{then, } X \cup Y = \{1, 2, 4, 6\}$$

Here, one element is written only once.

Diagrammatically,  $A \cup B$  is represented by the shaded portion in the Venn diagram given below :

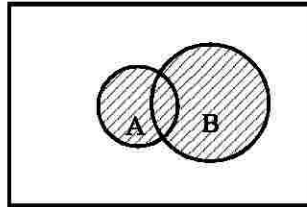


Fig. 3.2

### 3.16 : Intersection of Sets

Let A and B be two given sets. The set of those elements which are common to both A and B is called intersection of A and B and is denoted by  $A \cap B$ .

$$\text{Thus, } A \cap B = \{x : x \in A \text{ or } x \in B\}.$$

In other words, intersection is described as an 'and' idea i.e.  $A \cap B$  contains those elements which are in A as well as in B. For example, if  $A = \{1, 2, 3, 4\}$  and  $B = \{2, 4, 5, 6, 9\}$  then,

$$A \cap B = \{2, 4\}$$

Diagrammatically,  $A \cap B$  is represented by the shaded portion in the Venn diagram given below :

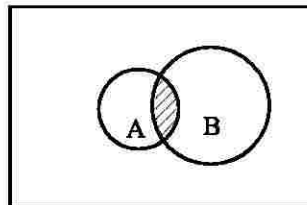


Fig. 3.3

**3.17 : Properties of Set operations****(i) Commutative Property :**

(a) Union is commutative i.e. if A and B are two sets, then  $A \cup B = B \cup A$ .

(b) Intersection is commutative i.e. If A and B are two sets, then :  $A \cap B = B \cap A$ .

For example, if  $A = \{a, b, c, d\}$  and  $B = \{p, q, b, c, \}$ , then

(a)  $A \cup B = \{a, b, c, d, p, q\}$  and  $B \cup A = \{p, q, a, c, b, d\}$  clearly  $A \cup B = B \cup A$ .

(b)  $A \cap B = \{b, c\}$  and  $B \cap A = \{b, c\}$

$$\therefore A \cap B = B \cap A$$

**(ii) Associative Property :**

(a) Union is associative i.e. if A, B, C are three sets, then

$$(A \cup B) \cup C = A \cup (B \cup C).$$

(b) Intersection is associative, i.e., if A, B, C are three sets, then

$$\therefore (A \cap B) \cap C = A \cap (B \cap C).$$

For example, if  $A = \{1, 2, 3, 4\}$  and  $B = \{2, 4, 6, 8\}$  and  $C = \{1, 2, 4, 5\}$ , then,

(a)  $(A \cup B) \cup C = \{1, 2, 3, 4, 6, 8\} \cup \{1, 2, 4, 5\} = \{1, 2, 3, 4, 5, 6, 8\}$  and

$$A \cup (B \cup C) = \{1, 2, 3, 4\} \cup \{1, 2, 4, 5, 6, 8\} = \{1, 2, 3, 4, 5, 6, 8\}$$

$$\therefore (A \cup B) \cup C = A \cup (B \cup C)$$

(b)  $(A \cap B) \cap C = \{2, 4\} \cap \{1, 2, 4, 5\} = \{2, 4\}$  and

$$A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{2, 4\} = \{2, 4\}$$

$$\text{Thus, } (A \cap B) \cap C = A \cap (B \cap C).$$

**(iii) Distributive Properties :**

If A, B, C are three sets, then :

(a)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  i.e. intersection is distributive over union.

(b)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  i.e. union is distributive over intersection.

For example, if  $A = \{1, 2, 3, 4, 5\}$ ,  $B = \{2, 3, 5, 7, 9\}$  and  $C = \{3, 4, 6, 8, 10\}$  then,

$$(a) \quad A \cap (B \cup C) = \{1, 2, 3, 4, 5\} \cap (\{2, 3, 5, 7, 9\} \cup \{3, 4, 6, 8, 10\})$$

$$= \{1, 2, 3, 4, 5\} \cap \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$= \{2, 3, 4, 5\}$$

$$\text{and } (A \cap B) \cup (A \cap C) = (\{1, 2, 3, 4, 5\} \cap \{2, 3, 5, 7, 9\})$$

$$\cup (\{1, 2, 3, 4, 5\} \cap \{3, 4, 6, 8, 10\})$$

$$= \{2, 3, 5\} \cup \{3, 4\} = \{2, 3, 4, 5\}$$

$$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(b) \quad A \cup (B \cap C) = \{1, 2, 3, 4, 5\} \cup (\{1, 3, 5, 7, 9\} \cap \{3, 4, 6, 8, 10\})$$

$$= \{1, 2, 3, 4, 5\} \cup \{3\} = \{1, 2, 3, 4, 5\}$$

$$\text{and } (A \cup B) \cap (A \cup C) = (\{1, 2, 3, 4, 5\} \cup \{2, 3, 5, 7, 9\})$$

$$\cap (\{1, 2, 3, 4, 5\} \cup \{3, 4, 6, 8, 10\})$$

$$= \{1, 2, 3, 4, 5, 7, 9\} \cap \{1, 2, 3, 4, 5, 6, 8, 10\}$$

$$= \{1, 2, 3, 4, 5\}$$

$$\text{Thus, } A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(iv) \quad (a) \quad A \cup \phi = A \quad (b) \quad A \cap \phi = \phi$$

Let  $A = \{p, q, r, s, t\}$ , then,

$$(a) \quad A \cup \phi = \{p, q, r, s, t\} \cup \phi = \{p, q, r, s, t\} = A$$

$$(b) \quad A \cap \phi = \{p, q, r, s, t\} \cap \phi = \phi.$$

(v) If  $U$  is a universal set, then

$$(a) \quad A \cup U = U \quad (b) \quad A \cap U = A$$

Let  $U = \{a, b, c, d, e, f\}$  and  $A = \{b, d, f\}$ , then

$$(a) \quad A \cup U = \{b, d, f\} \cup \{a, b, c, d, e, f\}$$

$$= \{a, b, c, d, e, f\} = U$$

$$(b) \quad A \cap U = \{b, d, f\} \cap \{a, b, c, d, e, f\} = \{b, d, f\} = A$$

$$(vi) \quad A \cup A = A$$

$$(vii) \quad A \subset A \cup B, B \subset A \cup B$$

(viii)  $A \cap B \subset A$  and  $A \cap B \subset B$  [ $A \cap B$  contains only those elements which are in  $A$  as well as in  $B$ ]

(ix) If  $A \cup B = \phi$ , then  $A = \phi = A$ .

(x) If A and B are disjoint sets, then

$$A \cap B = \phi$$

(xi)  $A \cap A' = \phi$  i.e., A and A' are disjoint sets. For example Let  $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

and  $A = \{1, 3, 5, 7\}$ , then  $A' = \{2, 4, 6, 8\}$

$$\therefore A \cap A' = \{1, 3, 5, 7\} \cap \{2, 4, 6, 8\} = \phi.$$

(xii)  $A \cup A' = U$

Let  $A = \{1, 2, 3, 8\}$  and  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

then,  $A' = \{4, 5, 6, 7, 9, 10\}$

$$A \cup A' = \{1, 2, 3, 8\} \cup \{4, 5, 6, 7, 9, 10\}$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$= U.$$

(xiii)  $(A')' = A$  i.e., the complement of the complement of a set is the set itself.

Let  $U = \{a, b, c, d, e\}$

and  $A = \{a, c, e\}$

then,  $A' = \{b, d\}$

and  $(A')' = \{a, c, e\} = A$

(xiv) If  $A = U$ , then,  $A' = \phi$ .

(xv) Demorgan's Laws :

$$(i) (A \cup B)' = A' \cap B'$$

$$(ii) (A \cap B)' = A' \cup B'.$$

i.e., the complement of union is the intersection of complements and complement of an intersection is the union of the complements.

**Example 1 :** Let  $U = \{1, 2, 3, 4, 5, 6, 7\}$ ,  $A = \{2, 4, 6\}$

$B = \{1, 3, 5, 7\}$  and  $C = \{1, 2, 4, 7\}$

find : (i)  $A \cup B$  (ii)  $B \cup C$  (iii)  $B \cap A$  (iv)  $A \cap C$  (v)  $A'$  (vi)  $A' \cap B'$

**Solution :**

- (i)  $A \cup B = \{2, 4, 6\} \cup \{1, 3, 5, 7\}$   
 $= \{1, 2, 3, 4, 5, 6, 7\}$
- (ii)  $B \cup C = \{1, 3, 5, 7\} \cup \{1, 2, 4, 7\}$   
 $= \{1, 2, 3, 4, 5, 7\}$
- (iii)  $B \cap A = \{1, 3, 5, 7\} \cap \{2, 4, 6\}$   
 $= \{ \}$ .
- (iv)  $A \cap C = \{2, 4, 6\} \cap \{1, 2, 4, 7\} = \{2, 4\}$
- (v)  $A' = \{1, 3, 5, 7\}$
- (vi)  $A' \cap B' = \{1, 3, 5, 7\} \cap \{2, 4, 6\}$   
 $= \{ \}$

**Example 2 :** If  $A = \{2, 6, 8\}$ ,  $B = \{1, 3, 7, 8\}$ ,  $C = \{3, 6, 8, 9\}$  and  $U = \{1, 2, 3, 6, 7, 8, 9\}$ , then verify that :

- (i)  $(A \cup B)' = A' \cap B'$
- (ii)  $(A \cap B)' = A' \cup B'$

**Solution :** (i)  $A \cup B = \{1, 2, 3, 6, 7, 8\}$

$$\therefore (A \cup B)' = \{9\}$$

Also,  $A' = \{1, 3, 7, 9\}$  and  $B' = \{2, 6, 9\}$ .

$$\therefore (A' \cap B') = \{9\}.$$

Thus,  $(A \cup B)' = A' \cap B'$ .

- (ii)  $(A \cap B) = \{8\}$

$$\therefore (A \cap B)' = \{1, 2, 3, 6, 7, 9\}$$

$$A' \cup B' = \{1, 3, 7, 9\} \cup \{2, 6, 9\} = \{1, 2, 3, 6, 7, 9\}$$

$$(A \cap B)' = A' \cup B'$$

**Example 3 :** If  $A = \{2x; x \in N\}$ ,  $B = \{3x; x \in N\}$  and  $C = \{5x; x \in N\}$ , then find

- (i)  $A \cap B$  (ii)  $A \cap C$  (iii)  $(A \cap B) \cap C$ .

**Solution :** Here  $A = \{2, 4, 6, 8, 10, \dots\}$

$$B = \{3, 6, 9, 12, 15, \dots\}$$

$$C = \{5, 10, 15, 20, 25, \dots\}$$

and

$$(i) \quad A \cap B = \{6, 12, 18, \dots\}$$

$$(ii) \quad A \cap C = \{10, 20, 30, \dots\}$$

$$(iii) \quad (A \cap B) \cap C = \{6, 12, 18, 24, 30, \dots\} \cap \{5, 10, 15, 20, 25, 30, \dots\} \\ = \{30, 60, 90, \dots\}$$

### 3.18. Applications of Sets

If  $A$  is a finite set, then the number of elements of  $A$  can be counted & is denoted by  $n(A)$ . We state below two results which will be used in solving the problems on applications of sets.

(i) If  $A$  and  $B$  are two sets, then :

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

For example, if  $A = \{1, 2, 3, 5, 6\}$  and  $B = \{2, 3, 6, 8, 9, 10, 12\}$ ,

then,  $A \cup B = \{1, 2, 3, 5, 6\} \cup \{2, 3, 6, 8, 9, 10, 12\}$

$$= \{1, 2, 3, 5, 6, 8, 9, 10, 12\}$$

and  $A \cap B = \{1, 2, 3, 5, 6\} \cap \{2, 3, 6, 8, 9, 10, 12\}$

$$= \{2, 3, 6\}$$

Now,  $n(A) = 5$ ,  $n(B) = 7$ ,  $n(A \cup B) = 9$  and  $n(A \cap B) = 3$ .

$$\therefore n(A) + n(B) - n(A \cap B) = 5 + 7 - 3 = 9 = n(A \cup B)$$

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(ii) If  $A$  and  $B$  are two disjoint sets, then  $n(A \cup B) = n(A) + n(B)$ .

If  $A$  and  $B$  are disjoint sets, then :

$$n(A \cap B) = 0$$

$$\therefore n(A \cup B) = n(A) + n(B) \quad 0 = n(A) + n(B).$$

Now, let us consider some examples.

**Example 4 :** The population of a town is 6000. Out of these 3400 persons read Hindustan Times and 2700 persons read Times of India. There are 700 persons, who read both the papers. Find the number of persons who do not read either of the two papers.

**Solution :** Let A be the set of persons who read Hindustan Times and B, the set of persons who read Times of India. Clearly, we have to find  $n(A \cup B)'$ .

We are given that  $n(A) = 3400$

$$n(B) = 2700$$

and  $n(A \cap B) = 700$

$$\begin{aligned} \therefore n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 3400 + 2700 - 700 \\ &= 5400. \end{aligned}$$

$$\therefore n(A \cup B)' = 6000 - 5400 = 600$$

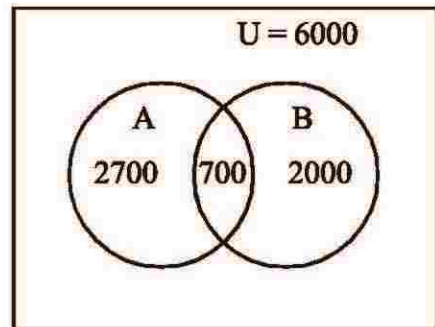


Fig. 3.4

So, the number of persons who do not read either of the two papers = 600.

**Example 5 :** In an examination 80 students secured first class marks in English or Mathematics. Out of these 50 students, obtained first class marks in Mathematics & 10 students in English & Mathematics both. How many students secured first class marks in English only ?

**Solution :** Let E be the set of students who secured first class marks in English and M the set of students who secured first class marks in Mathematics.

The given information may be written as :

$$n(E \cup M) = 80, n(M) = 50, n(E \cap M) = 10$$

We know that,

$$n(E \cup M) = n(E) + n(M) - n(E \cap M).$$

Substituting the values of  $n(E \cup M)$ ,  $n(M)$  and  $n(E \cap M)$ , we get,

$$80 = n(E) + 50 - 10$$

$$\Rightarrow n(E) = 80 - 40 = 40$$

$$\therefore n(E) = 40.$$

Thus the number of students who secured first class marks in English only

$$= n(E) - n(E \cap M)$$

$$= 40 - 10 = 30.$$



**3.19 Questions**

1. Choose the correct answer from the alternatives given in each bit :

(i) Which of the following is a set ?

- (a) A set of all rivers in India.
- (b) A group of 10 best sportsmen in B.J.B College.
- (c) A Group of eleven best cricket players in the world.
- (d) A set of all beautiful flowers.

(ii) Which of the following is a well defined set ?

- (a) All healthy students of a class
- (b) All good girls of a class.
- (c) All the excellent lecturers of a college
- (d) All natural numbers less than 20

(III) Which of the following is a void set ?

- (a) Set of even numbers between 10 and 19
- (b) Set of squares with five sides
- (c) All days of the week beginning with S.
- (d) All prime numbers less than 9

IV) Which of the following pair of sets are equal sets ?

- (a)  $A = \{m, n, p, q\}$  and  $B = \{p, n, m, q, o\}$
- (b)  $A = \{m, n, p, q\}$  and  $B = \{1, 2, 3, 4\}$
- (c)  $A = \{m, n, p, q\}$  and  $B = \{p, n, m, q, n\}$
- (d)  $A = \{0\}$  and  $B = \{\phi\}$

V. Which of the following pair of sets are equivalent ?

- (a)  $A = \{a, b, c, d\}$  and  $B = \{a, b, c, d, e\}$
  - (b)  $A = \{x, y, z\}$  and  $B = \{1, 2, 3, 4\}$
  - (c)  $A = \{0\}$  and  $B = \{\phi\}$
  - (d)  $A = \{\text{January, June, July}\}$  and  $B = \{\text{Sunday, Monday, Friday, Saturday}\}$
-

- VI) If  $\{p, i, a, e, o\} = \{x : x \text{ is a vowel in English Alphabet}\}$  then p is equal to :
- a) b
  - b) c
  - c) d
  - d) u
- VII) A is a set such that  $\{0, 1, 3\} \subset A \subset \{0, 3, 2, 1, 4\}$   
then set A is :
- (a)  $A = \{0, 1, 2, 3\}$
  - (b)  $A = \{0, 1, 3, 4\}$
  - (c) both (a) and (b)
  - (d)  $A = \{0, 3, 2, 5\}$
- (viii)  $A \cap (B \cup C) = (A \cap C) \cup (A \cap B)$  indicates the following property of set operations :
- (a) Commutative property
  - (b) Associative Property
  - (c) Distributive Property
  - (d) Idem potence property
- IX) A is a set of the factors of 6, therefore :
- (a)  $A = \{1, 2, 3\}$
  - (b)  $A = \{0, 1, 2, 3\}$
  - (c)  $A = \{2, 3\}$
  - (d)  $A = \{1, 2, 3, 6\}$
- X) The total number of subsets of a finite set containing n elements is :
- a)  $n^2$
  - b)  $2^n$
  - c)  $2n$
  - d)  $n+n^2$
- 
-

**2. Answer the following questions in one word or term :**

- (i) A set having only one element.
- (ii) Complement of set A
- (iii) A set containing no element.
- (iv) The set of sets
- (v) Name the set, whose number of elements can be counted.
- (vi) Name the sets, if and only if each one is a subset of the other.

**3. Fill in the blanks :**

- (i)  $A \cup \phi = \underline{\hspace{2cm}}$ .
- (ii)  $A \cap \phi = \underline{\hspace{2cm}}$ .
- (iii)  $(A \cup B)' = A' - B'$
- (iv) A single ton set is a set containing \_\_\_\_\_ element.
- (v) An empty set is denoted by \_\_\_\_\_.
- (vi) According to De Morgan's Law  $(A \cap B)' = \underline{\hspace{2cm}}$ .

**4. Correct the underlined portions of the following sentences :**

- (i) A set having only one zero as its element is called zero set.
- (ii) If set A = {All girls} and B = {All boys}, then  $A \cap B$  is an Equivalent set.
- (iii) The notation  $\supset$  is read as subset.
- (iv) For only two sets A and B;  $(A \cap B)' = \underline{A' \cap B'}$
- (v) Number of male students in a wo-men's college will constitute single ton set.
- (vi) Intersection of set A and set  $\cup$  is equal to \_\_\_\_\_  $\cup$  \_\_\_\_\_

**5. Answer the following questions in one sentence each.**

- (i) Write the tabular form of  $\{x : x \text{ is an odd number and } 0 \leq x \leq 10\}$
- (ii) Write the set builder form of  $A = \{a, e, i, o, u\}$
- (iii) Define set.

(iv) What is a disjoint set ?

(v) What is a subset ?

(vi) What is an empty set ?

**6. Answer the following questions within 30 words.**

(i) If  $A = \{x : x^2 - 4x + 3 = 0\}$  and  $B = \{0, 1, 2, 3\}$  find  $A \cap B$

(ii) Define the following terms and give two example of each :

(a) Empty set                      (b) Subset

(iii) What is a power set ? Find all subsets of the set A, where  $A = \{3, 5, m, n\}$

(iv) Explain universal set with examples.

(v) If  $A = \{3, 5, a, b\}$  and  $B = \{1, 5, b, d\}$  verify that.

(a)  $A \cap B = B \cap A$

(b)  $A \cup B = B \cup A$

**7. Answer the following question within 50 words.**

(i) What is a venn-diagram ? Does venn-diagram constitute a proof ? Draw a venn-diagram to show a subset of set.

(ii) What is complement law ? Explain this law with an example. Also prove that  $A \cap A' = \phi$

(iii) What do you mean by De-Morgan's Law ?

Prove that  $(A \cap B)' = A' \cup B'$  and  $(A \cup B)' = A' \cap B'$

(iv) Define intersection of two sets with examples.

(v) What is union of two sets ? Draw a venn-diagram to show the union with numeral figure.

(vi) If  $A = \{1, 3, 5, 7, 9\}$ ,  $B = \{2, 4, 6, 8, 10\}$  and

$C = \{3, 4, 7, 8, 11, 12\}$

Show that : (a)  $(A \cup B) \cup C = A \cup (B \cup C)$

(b)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8. If  $A = \{1,5,7,10\}$ ,  $B = \{5,10,12,16\}$  and  $C = \{1,10,12,18,21\}$  Draw venn -diagram to show.
- $A \cup B$
  - $B \cup C$
  - $A \cap C$
  - $(A \cup C) \cap B$
  - $(A \cap B) \cap C$
9. If  $A = \{p, q, r\}$ ,  $B = \{x: x \text{ is a vowel}\}$  find the cartesian product of set A and B and count the number of elements in it.

Ans :  $A \times B = \{(P,a), (P,e), (P,i), (P,o), (P,u), (q,a), (q,e), (q,i), (q,o), (q,u), (r,a), (r,e), (r,i), (r,o), (r,u)\}$  and  $n(A \times B) = 15$

10.

- Identify the elements of A if  $B = \{1,2,3\}$  and  $A \times B = \{(4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (6,1), (6,2), (6,3)\}$   
[Ans.  $\{4,5,6\}$ ]

- If  $A = \{x: x^2 - 4x + 3\}$  and  $B = \{0,1,2,3\}$ , find  $A \times B$ .  
[Ans.  $A \times B = \{(3,0), (3,1), (3,2), (3,3), (1,0), (1,1), (1,2), (1,3)\}$ ]

11. Prove that  $A - B = A \cap B$  and hence show that :

- $B \cap (A - B) = \phi$  and  $B \cup (A - B) = A \cup B$

12. Out of total 150 students who appeared +2 Arts examination from a College, 45 failed in English, 50 failed in Economics and 30 in mathematics. Those who failed both in English and Economics were 30 those who failed both in Economics & Mathematics were 32 and those who failed both in English and mathematics were 35. The students who failed in all the three subjects were 25. Find the number of students who failed at least in any one of the subjects.

[Ans.  $n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) = 53$

Where  $A = \text{English}$ ,  $B = \text{Economics}$ ,  $C = \text{Mathematics}$ .

13. In a test there are 500 candidates. Out of them 360 candidates can write Hindi and 130 candidates can write English only. How many of them can write both English and Hindi ?  
[Ans-10]
14. In a survey of 500 people, it was found that 280 read Times of India and 230 read The Samaj while 40 read both the papers.
- (a) How many read at least one paper ?  
 (b) How many read neither times of India nor the Samaj ?  
 (c) How many read exactly one paper ?  
 [Ans. (a) 470 (b) 30 (c) 190]
15. A, B, C be any three sets, then prove that :  $A \times (B \cup C) = (A \times B) \cup (A \times C)$
16. In a school, 60% students contributed towards National Defence Fund and 70% Students towards the National Savings Scheme. What percentage of students contributed towards both the schemes ? Assume that all the students have contributed to one or both the schemes.  
[Ans - 30%]
17. A survey was conducted on the T.V programmes watched by 120 students of a school hostel. It was revealed that 70 students watched 'sports' programme and 56 students watched 'Youth Forum' where, as 24 students watched both the programmes. Find the number of students who did not watch T.V on that day. [Ans. 18]

### Answer

1. (i) a (ii) d (iii) b (iv) c (v) c (vi) d (viii) c (viii) c (ix) d (x) b
2. (i) Singleton (ii)  $A'$  (iii) Null / Void / empty (iv) Power set (v) Finite Set  
(vi) Equal Set
3. (i)  $A$  (ii)  $\phi$  (iii)  $\cap$  (iv) One (v)  $\phi$  (vi)  $A' \cup B'$
4. (i) Single-ton set (ii) Disjoint Set (iii) Power Set (iv)  $A' \cup B'$  (v) Null Set (vi)  $A$

5. (i)  $\{1,3,5,7,9\}$
- (ii)  $\{x : x \text{ is set of vowel in English alphabet } \}$
- (iii) A set is a collection of well-defined objects.
- (iv) If no element of set A is in set B and no element of set B is in set A then set A and B are called disjoint sets.
- (v) If every element of a set A is also an element of another set B then A is called a subset of B.
- (vi) A set which contains no element is called an empty or null set.



## **CHAPTER - 4**

# ***FUNCTIONS***

### **Structure**

- 4.1 Introduction
- 4.2 Relation
- 4.3 Meaning of function
- 4.4 Classification of functions
- 4.5 Composite function
- 4.6 Inverse function
- 4.7 Algebra of function
- 4.8 Applications
- 4.9 Questions

## 4.1 Introduction

One of the most important concepts in mathematics is that of a function. The word 'function' is derived from the Latin word meaning operation. If we square a real number  $x$ , we get another real number  $x^2$ .

Let us take another example. If we consider the marks obtained by different students, we can form pair of numbers to express this. In the example of squaring of numbers we take the pair  $(x, x^2)$ . In the example of students and marks, we can have pairs (Madan, 21), (Mohan, 13), (Arun, 37). etc..... Thus we see that we can associate the members of one set with the members of second set by taking ordered pairs.

## 4.2 Relation

A relation is a subset of cartesian product set which also contains ordered pair of elements within it. For example, a polygon has several vertices, a number can have several factors, but a real number has only one number as its square, a circle has one centre so on. If a given element in an ordered pair, is associated with exactly one element, it is a special type of relation which is called a function. Thus, a relation from a set  $A$  to  $B$  is denoted by:

$$R = \{ (a, b) \mid a \in A \text{ and } b \in B \text{ and } aRb \}$$

For example :  $R = \{ (1, 2), (2, 2), (1, 6), (2, 6), (3, 6), (6, 6) \}$

Here, the first element is a factor of the second element in each pair.

It cannot be accepted as a function as the first elements in the pairs are associated with more than one second element of the pair.

## 4.3 Meaning of function :

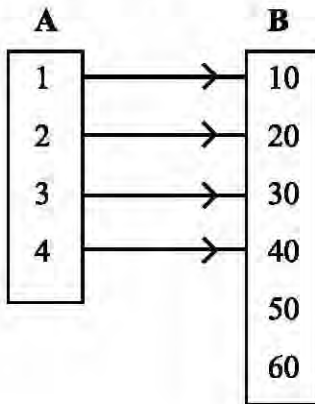
A function from  $A$  to  $B$  is a set of ordered pair of elements, in which every element of  $A$  is associated with exactly one element of  $B$ . If  $f: A \rightarrow B$ ,

$$f = \{ (a, b) \mid a \in A \text{ and } b \in B \text{ and has unique assignment with } b \}$$

For example :  $f = \{ (1, 10), (2, 20), (3, 30), (4, 40) \}$ , which are taken from the sets.

$A = \{ 1, 2, 3, 4 \}$  and  $B = \{ 10, 20, 30, 40, 50, 60 \}$ .

Diagrammatically,



In the above diagram, the elements of set A i.e., 1, 2, 3, 4 associated with only one element of set B i.e., 10, 20, 30, 40. The set of first elements of i.e., set A is called domain of the function, whereas the set of second elements i.e., set B is called the co-domain and the set of second elements which are associated with the elements of the first is called range. In the above example,

$$\text{Domain} = \{1, 2, 3, 4\}$$

$$\text{co-domain} = \{10, 20, 30, 40, 50, 60\}$$

$$\text{Range} = \{10, 20, 30, 40\}$$

In the above function in a given pair. (1, 10), 10 is called image of 1 or 1 is called pre-image of 10.

A function may be expressed in several ways. Take, for example, a function where (x,y) are ordered pair and  $y = 3x + 1$ .

Now, it can be written as :

$$(a) f = \{ (x,y) \mid y = 3x + 1 \}$$

$$(b) f: x \rightarrow y, \text{ where } y = 3x + 1$$

$$(c) f = \{ (x, 3x + 1) \mid x \in \mathbb{R} \}$$

$$(d) f(x) = 3x + 1$$

$$(e) y = 4x - 7$$

$$(f) f: \mathbb{R} \rightarrow \mathbb{R}, \text{ where } f(x) = 3x + 1$$

**(i) Equal functions :**

Two functions  $F : A \rightarrow B$  and  $g : A \rightarrow B$  are said to be equal, if  $f(a) = g(a)$ , for all  $a \in A$ .

Example :  $A = \{ 1, 2, 3 \}$  and  $B = \{ 1, 4, 9 \}$  and  $f = \{ (1, 1), (2, 4), (3, 9) \}$

Similarly,  $g : A \rightarrow B$ , where 'b' is the square of 'a'.

$f = \{ (1, 1), (2, 4), (3, 9) \}$

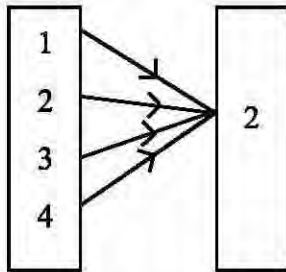
In both the functions :  $f(1) = 1, f(2) = 4$  and  $f(3) = 9$ .

Hence,  $f = g$  or both functions are equal.

**(ii) Constant function :**

A function  $f : R \rightarrow R$  is defined by  $f(x) = c$ , for all  $x \in R$ .

Example :  $f = \{ (1, 2), (2, 2), (3, 2), (4, 2) \}$

**(iii) Identify function :**

A function  $f : A \rightarrow A$  is defined by  $f(a) = a$  for all  $a \in A$  is called the identity function of  $A$ .

Example :  $f = \{ (1, 1), (2, 2), (3, 3) \}$  is an identity function of the set  $A = \{ 1, 2, 3 \}$ .

**(iv) Real function :**

A function  $f : A \rightarrow B$  is called a real function, if  $A$  and  $B$  are subsets of the set of real numbers.

**(v) Polynomial function :**

Let 'n' be a positive integer and  $a_0, a_1, a_2, a_3, \dots, a_n$  be any real numbers except that  $a_n \neq 0$

A function  $f : R \rightarrow R$  is defined by the formula :  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  is called a polynomial function of degree  $n$ .

Example :  $f(x) = 4x + 1$ ,  $f(x) = x^2 + 5$ ,  $f(x) = 2x^3 + 7$  are polynomial functions of degree 1, 2, 3 respectively & are called linear, quadratic & cubic functions respectively. Symbolically,  $f = \{x, f(x) \mid f(x) \text{ is a polynomial function}\}$ .

**(vi) Rational function :-**

A rational function is defined the quotient of two polynomial functions as :-

$$f(x) = \frac{g(x)}{h(x)}, \text{ where } g(x) \text{ and } h(x) \text{ are two polynomial functions and } h(x) \neq 0$$

$$\text{Example : } f(x) = \frac{x^2 + 3x + 5}{x^4 + 7}$$

**(vii) Even and odd function :-**

A real function is said to be an even function, if  $f(-x) = f(x)$ .

$$\text{Example : } \therefore f(x) = x^2$$

$$\therefore f(-x) = (-x)^2 = x^2 = f(x)$$

Hence,  $f(x) = x^2$  is an even function.

A real function  $f(x)$  is said to be odd function if  $f(-x) = -f(x)$

$$\text{Example : If } f(x) = x^3$$

$$\text{Then } f(-x) = (-x)^3 = -x^3 = -f(x)$$

Hence,  $f(x) = x^3$  is an odd function.

**(viii) Absolute value function :**

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by :

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called absolute value function. It is also known as modulus function & written as  $y = f(x) = |x|$ .

**(ix) Signum function :**

A function  $f(x)$  is said to be signum function, if:

$$Y = f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

**(x) Greatest Integer function :-**

A function  $f(x)$  is said to be greatest integer function, if  $f(x) = [x]$ , which implies  $[x]$  is the greatest integer less than or equal to  $x$ .

Example :  $f = \{ (3.2, 3), (4.7, 4), (-3.4, -4), (-5, -5) \}$

**4.4 CLASSIFICATION OF FUNCTIONS :**

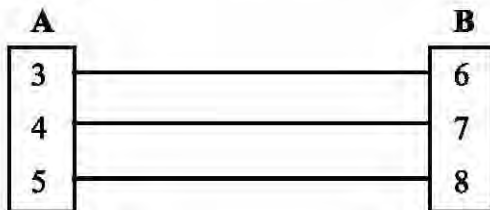
**(i) One-One function :**

A function  $f: A \rightarrow B$  is said to be one-one function if distinct elements of  $A$  have distinct images. It is also called injective function.

Example :  $A = \{ 3, 4, 5 \}$  and  $B = \{ 6, 7, 8 \}$  and  $f = \{ (3, 6), (4, 7), (5, 8) \}$

Hence each element of domain have different and distinct image in the co-domain.

Diagrammatically,



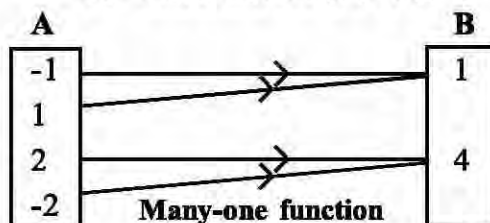
One-one function

**(ii) Many-one function:**

A function  $f: A \rightarrow B$  is said to be many-one function if more than one element of  $A$  have the same image in  $B$ .

Example : Let  $A = \{ -1, 1, 2, -2 \}$  and  $B = \{ 1, 4 \}$  and

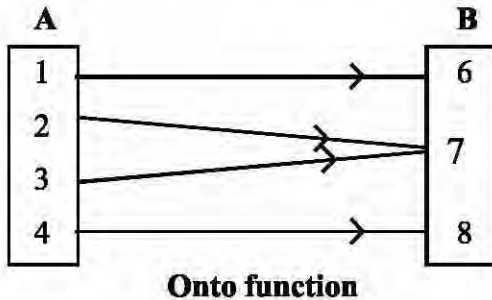
$F = \{ (-1, 1), (1, 1), (2, 4), (-2, 4) \}$



**(iii) Onto function :**

A function  $f: A \rightarrow B$  is said to be onto if every element of  $B$  is an image of at least one element of  $A$ . Onto function is also called subjective function.

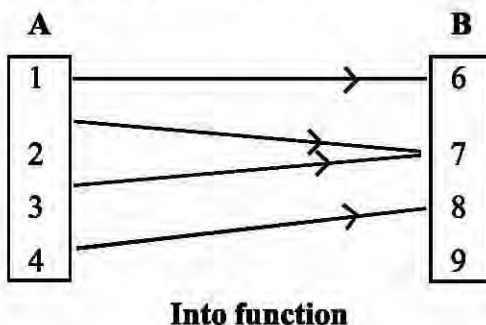
Example : Let  $A = \{1, 2, 3, 4\}$  and  $B = \{6, 7, 8\}$  and  $f: A \rightarrow B$  is such that  $f(1) = 6, f(2) = 7, f(3) = 7$  and  $f(4) = 8$ . Then  $f$  is an onto function. All the elements of  $B$  are images of elements of  $A$ .

**(iv) Into function :**

A function  $f: A \rightarrow B$  is called into if there is at least one element in  $B$  which is not an image of elements of  $A$ .

Example : Let  $A = \{1, 2, 3, 4\}$  and  $B = \{6, 7, 8, 9\}$  and  $f: A \rightarrow B$  is such that  $f(1) = 6, f(2) = 7, f(3) = 7$  and  $f(4) = 8$ , but 9 is an element of  $B$  which is not the image of the elements of  $A$ .

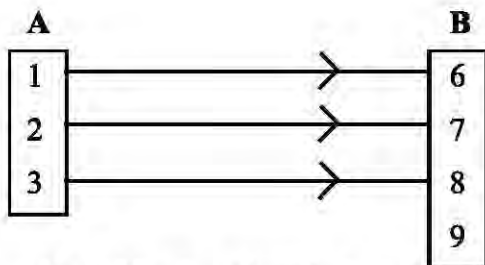
Then, the function  $f$  is an into function.



**(v) One - One into function :-**

A function which is one - one & into is called one-one into function.

Example :

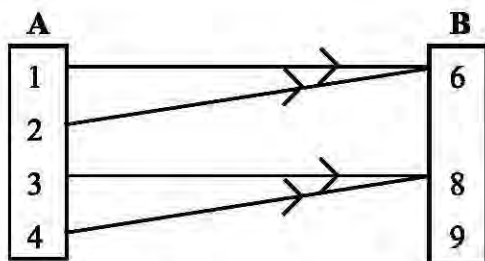


**One - One into function**

**(vi) Many - one into function :-**

A function which is many-one, as well as, into is called many-one into function.

Example :

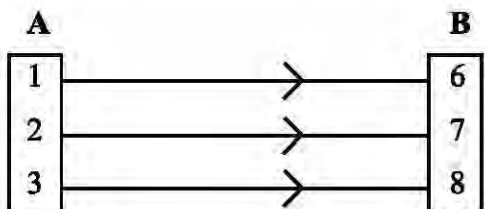


**Many - one into function**

**(vii) One-One onto function :**

A function which one-one as well as onto is called one-one onto function.

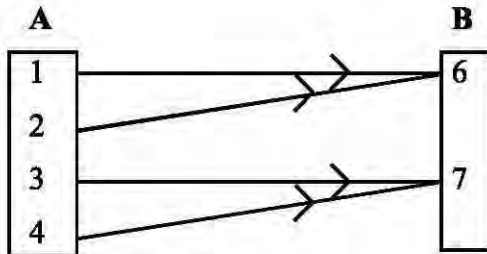
Example :



**(viii) Many - one onto function :**

A function which is many-one as well as onto is called many-one onto function.

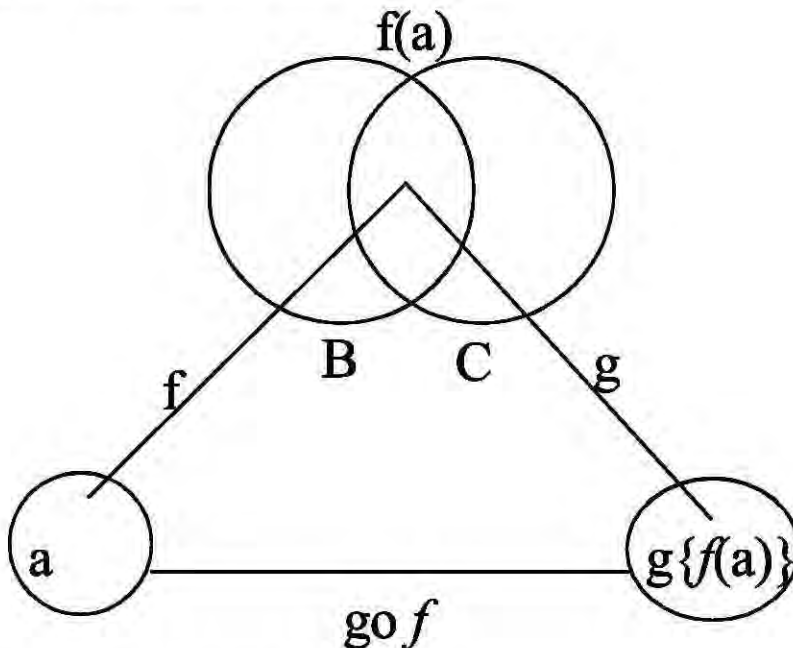
Example :



**Many- one onto function.**

**4.5 COMPOSITE FUNCTION**

Let  $f: A \rightarrow B$  and  $g: C \rightarrow D$  be two functions, such that the range of  $f \subset C$ . The set of ordered pairs  $\{a, d \mid a \in A \text{ and } d \in g(f(a))\}$  is a function from the set  $A$  into set  $D$ . This function is called composite of  $f$  and  $g$  and is denoted by  $g \circ f$ .



Similarly,  $f \circ g$  can be obtained

Example : Let  $f(x) = x^2 + 5$  and  $g(x) = x + 2$

Then,  $g \circ f = g[f(x)]$

$$\begin{aligned}
 &= g [x^2+5] \\
 &= x^2 + 5 + 2 \quad (\because g(x) = x+2) \\
 &= x^2 + 7
 \end{aligned}$$

and  $f \circ g = f[g(x)]$

$$\begin{aligned}
 &= f[x+2] \\
 &= (x + 2)^2 + 5 \quad (\because f(x) = x^2 + 5) \\
 &= x^2 + 4 + 4x + 5 \\
 &= x^2 + 4x + 9
 \end{aligned}$$

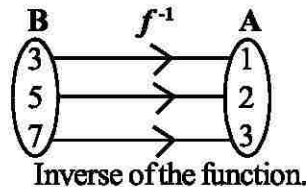
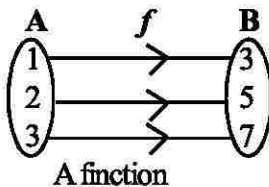
**4.6 INVERSE FUNCTION :-**

If  $f: A \rightarrow B$  is one-one onto function, then a function  $g: B \rightarrow A$ , which associates each element of  $B$  with a unique element of  $A$ , such that,  $b = f(a)$ , is called an inverse function. Inverse function is denoted by  $f^{-1}$ .

$$\begin{aligned}
 \therefore f^{-1} &= \{ (b,a) \mid b = f(a) \} \\
 &= \{ (b,a) \mid (a,b) \in f \}
 \end{aligned}$$

Example : If  $y = f(x) = 2x + 1$ , then  $x = \frac{y-1}{2}$

Thus,  $x = \frac{y-1}{2}$  can be written as  $f^{-1}(x)$



Thus,  $f = \{ (x,y) \mid y = 2x + 1 \}$  and  $f^{-1} = \{ (y,x) \mid x = \frac{y-1}{2} \}$

**4.7 ALGEBRA OF FUNCIONS :-**

Addition, subtraction, multiplication and division on function can be defined as below :

Let  $f$  and  $g$  be two function of  $x$ , such that,

$$y = f(x) \text{ and } z = g(x)$$

Then, the basic operations of addition, subtraction, multiplication and division can be stated as follows.

- (a)  $(f + g) : x \rightarrow y + z$       or       $y + z = f(x) + g(x)$
- (b)  $(f - g) : x \rightarrow y - z$       or       $y - z = f(x) - g(x)$
- (c)  $(fg) : x \rightarrow yz$               or       $yz = f(x) \cdot g(x)$

$$(d) \left( \frac{f}{g} \right) : x \rightarrow \frac{y}{z} \quad \text{or} \quad \frac{y}{z} = \frac{f(x)}{g(x)}, \quad g(x) \neq 0.$$

Example : If  $f(x) = x + 2$  and  $g(x) = x^2 + 1$ , then find  $f + g, f - g, fg$  and  $\frac{f}{g}$ .

Solution : Let  $f(x) = x + z = y$ ,  $g(x) = x^2 + 1 = z$

$$(i) f + g = y + z = x + z + x^2 + 1 = x^2 + x + 3$$

$$(ii) f - g = y - z = x + 2 - (x^2 + 1) = x - x^2 + 1$$

$$(iii) fg = y \times z = (x + 2)(x^2 + 1) = x^3 + 2x^2 + x + 2$$

$$(iv) \frac{f}{g} = \frac{y}{z} = \frac{x + 2}{x^2 + 1}$$

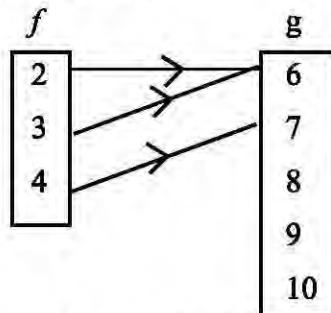
#### 4.8 APPLICATIONS :-

Example 1 = If  $A = \{2, 3, 4\}$  and  $B = \{6, 7, 8, 9, 10\}$ , verify whether the following are functions or not.

$$(a) f = \{(2, 6), (3, 6), (4, 7)\}$$

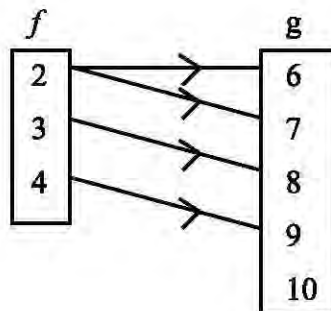
$$(b) g = \{(2, 6), (2, 7), (3, 8), (4, 9)\}$$

Solution : (a)



In the above diagram, every element 2 of  $f$  is associated with only one element of  $g$ , hence,  $f$  is a function.

(b)



In the above diagram, the element 2 of  $f$  is associated with 6 and 7 (i.e) more than one element of  $(g)$ , hence  $g$  is not a function.

**Example 2 :** find the value of the function, if  $f(x) = \frac{x+2}{x^2-1}$

(a)  $f(2) + f(3)$                       (b)  $f(0) \times f(4)$

**Solution :**  $f(x) = \frac{x+2}{x^2-1}$

$$\therefore f(2) = \frac{2+2}{4-1} = \frac{4}{3}, f(0) = \frac{0+2}{0-1} = -2$$

$$f(4) + \frac{4+2}{16-1} = \frac{6}{15} = \frac{2}{5} \text{ and } f(3) = \frac{3+2}{9-1} = \frac{5}{8}$$

$$\therefore \text{(a) } f(2) + f(3) = \frac{4}{3} + \frac{5}{8} = \frac{32+15}{24} = \frac{47}{24}$$

$$\text{(b) } f(0) \times f(4) = -2 \times \frac{2}{5} = -\frac{4}{5}$$

**Example 3 :** If  $f(x) = x+2$  and  $g(x) = x^2$ , find  $f \circ g$  and  $g \circ f$  for the functions.

**Solution :-**  $f(x) = x+2$

$$g(x) = x^2$$

Then,  $f \circ g(x) = f[g(x)] = f[x^2] = x^2 + 2$ .

Similarly,  $g \circ f(x) = g[f(x)] = g[x+2] = (x+2)^2 = x^2 + 4x + 4$

**Example 4 :** Identify the even and odd function from the following:-

(a)  $x^6 + 2x^4 + 7x^2 + 4$

(b)  $x^7 - 5x^5 + 2x^3 - x$

**Solution :** (a) Let  $f(x) = x^6 + 2x^4 + 7x^2 + 4$

$$\begin{aligned} \text{Then, } f(-x) &= (-x)^6 + 2(-x)^4 + 7(-x)^2 + 4 \\ &= x^6 + 2x^4 + 7x^2 + 4 \\ &= f(x) \end{aligned}$$

As  $f(-x) = f(x)$ , it is an even function.

(b) Let  $f(x) = x^7 - 5x^5 + 2x^3 - x$

$$\begin{aligned} \text{Then } f(-x) &= (-x)^7 - 5(-x)^5 + 2(-x)^3 - (-x) \\ &= -x^7 + 5x^5 - 2x^3 + x \\ &= -(x^7 - 5x^5 + 2x^3 - x) \\ &= -f(x) \end{aligned}$$

As  $f(-x) = -f(x)$ , it is an odd function.

**Example 5 :** Find the inverse of the function  $f(x) = 5x + 1$

**Solution :** The function  $f(x) = 5x + 1$  is a linear function and it is 1-1 function.

Let  $y = 5x + 1$

$\Rightarrow 5x = y - 1$

$\Rightarrow x = \frac{y-1}{5}$

Now, if,  $f = \{ (x, y) \mid y = 5x + 1 \}$

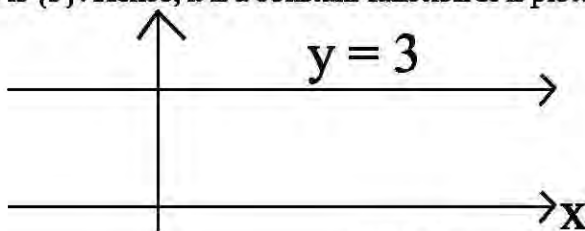
Then  $f^{-1} = \left\{ (y, x) \mid x = \frac{y-1}{5} \right\}$

**Example 6 :** Draw a graph of the function  $f(x) = 3$ .

**Solution :** Let us take several sets of ordered pairs, then given function  $f(x) = y = x^2$

x	0	1	2	3	-3	-2	-1
y	3	3	3	3	3	3	3

From the above table, we find that the domain of the function is a set of real numbers and the range is  $\{3\}$ . Hence, it is a constant function & is plotted in the graph paper as below.



□□□

**4.9. Questions :-**

1. From the alternatives given below, choose and write the correct answer along with its serial number.

**(i) A is a function if :**

- (a)  $A = \{(a,1), (a,2), (c,1), (d,4)\}$
- (b)  $A = \{a,1), (a,2), (b,3), (c,d)\}$
- (c)  $A = \{(a,2), (b,2), (c,2), (d,3)\}$
- (d)  $A = \{(a-1), (b,2), (b,3), (c,4)\}$

**(ii) A is a constant function if :**

- (a)  $A = \{a,2), (a,3), (b,3), (b,4)\}$
- (b)  $A = \{(a,3), (b,3), (c,3), (d,3)\}$
- (c)  $A = \{(a,2), (b,3), (c,4), (d,4)\}$

**(iii) If  $A = \{1,2,3\}$ ,  $B = \{4,5\}$  and  $f = (1,4), (2,4), (3,5)$  is a function**

- (a) 1-1 into
- (b) 1-1 onto
- (c) Many-one into
- (d) Many-one onto

**(iv) If  $A = \{a, b, c\}$ ,  $B = \{1,2,3\}$  and  $f = \{(a,1), (b,2), (c,3)\}$  is a function :**

- (a) Constant function
  - (b) Onto function
  - (c) Into function
  - (d) Manyone into function
  - (v)  $Y^2 = x^2 + 2xy + y$  is a/an
  - (a) Implicit function
  - (b) Explicit function
-

- (c) Linear function
- (d) Even function
- (vi)  $f(x) = x^5 - 1$  is
- (a) an even function
- (b) an odd function
- (c) both odd and even
- (d) Neither even nor odd
- (vii) If  $f(x) = \frac{2}{x^2 - 4}$  and  $g(x) = x + 2$ , then  $\frac{2}{x - 2}$  is
- (a)  $f + g$
- (b)  $f - g$
- (c)  $fg$
- (d)  $\frac{f}{g}$
- (viii) If  $f(x) = 3$ , then the function is :
- (a) Identity function
- (b) Constant function
- (c) Odd function
- (d) Even function
- (ix) If  $f(x) = x^3$  and  $g(x) = 2x + 1$ , then  $f \circ g(x)$  would be :
- (a)  $(2x + 1)^2$
- (b)  $(2x + 1)^3$
- (c)  $(2x + 1)^4$
- (d)  $(2x + 1)^5$
- (x) If the selling price per unit and cost per unit are ₹  $x$  and ₹  $y$  respectively and 5 units of the product are sold, then the profit function  $P$  is :
- (a)  $P = 5x + y$
- 
-

- (b)  $P = 5x - y$   
 (c)  $P = 5x - 5y$   
 (d)  $P = 5x + 5y$

2. Answer the following questions in one word or term.

- (i) What is the name of a function, when at least one element of the codomain which does not correspond to any element of the domain ?  
 (ii) What would be the name of a function, when its range is equal to its codomain ?  
 (iii) Name the function, when all elements of A are associated with the same elements of B.  
 (iv) What is the name of the function, when  $f(-x) = f(x)$   
 (v) Name the function, when  $f(x) = 1$ , for  $x > 0$   
 $f(x) = 0$ , for  $x = 0$   
 $f(x) = -1$ , for  $x < 0$ .

3. Fill in the Blanks :

- (i) If  $f(-x) = -f(x)$ , then  $f(x)$  is an \_\_\_\_\_ function  
 (ii) The function  $f(x) = Ka^x$ , where  $k \neq 0$   
 (iii) The function  $f(x) = x$  is called \_\_\_\_\_ function  
 (iv) The domain of the function  $f = \{(1,2), (2,3), (3,4)\}$  is \_\_\_\_\_  
 (v) The range of the function  $f = \{(2,1), (3,2), (3,4)\}$  is \_\_\_\_\_  
 (vi) if  $f(x) = 3x + 2$ ,  $g(x) = x^2$  then  $f \circ g =$  \_\_\_\_\_.

4. Correct the underlined portion of the following sentences .

- (i) The function  $f(x) = 2^x$  is called a logarithmic function.  
 (ii) The function  $f(x) = 3x^4 - 3x^2 + 1$  is an odd function.  
 (iii) If  $f$  and  $g$  are two functions and defined as  $f(x) = 3x + 1$  and  $g(x) = x^2 + 2$ , then  $f \circ g$  is equal to  $x^2 - 3x + 1$   
 (iv) If in a function, there is at least one extra element in set B which is not associated with any element of set A, the function is called onto.  
 (v) If all the elements of set A are associated with one element of Set B, it is called identity function.

5. Answer the following questions in one sentence each.

- (i) What is a function ?
- (ii) Find the domain of the function,  $f = \{(1,2), (2,4), (3,9)\}$
- (iii) What is an odd function ?
- (iv) What is an even function.
- (v) Find the range of the function  $f(x)=2$
- (vi) What do you mean by inverse function ?
- (vii) What is a composite function

6. If a function  $f$  is defined by  $f(x) = 9x^2 + 1$ , find  $f(0)$ ,  $f\left(\frac{1}{2}\right)$ ,  $f(\sqrt{3})$ ,  $f(-2)$  and  $f\left(-\frac{2}{3}\right)$ .

7. Given  $A = \{2,4,6,8\}$  and  $B = \{6,8,10,12\}$  clarify the following ordered pair set as function or not a function.

- (i)  $C_1 = \{(2,6), (4,8), (6,10), (8,12)\}$
- (ii)  $C_2 = \{(2,6), (4,8), (6,10)\}$
- (iii)  $C_3 = \{(2,6), (4,6), (6,8), (8,8)\}$
- (iv)  $C_4 = \{(2,10), (4,10), (4,10), (6,10), (8,10)\}$
- (v)  $C_5 = \{(2,6), (2,8), (4,10), (6,12), (8,12)\}$

8. Find the domain and range of the following functions.

- (i)  $f = \{(1,2), (2,4), (3,6), (4,8)\}$
- (ii)  $f = \{x, x+2\} : x \in \{1,2,3\}$

(iii)  $f(x) = \frac{x-2}{x-2}$

(iv)  $f(x) = 1$

9. From the following, find out which are even and which are odd functions.

- (i)  $f(x) = x^2 + 5$
  - (ii)  $f(x) = x^3 - x$
  - (iii)  $f(x) = 19x^4 - 7x^2 + 1$
-

(iv)  $f(x) = 3x + 5x^7$

10. If  $f(x) = 3x + 2$ ,  $g(x) = x^2$ ,  $h(x) = 2x^3 + 1$

Find (i) fog (ii) foh (iii) fof (iv) hof (v) gof (vi) gog

11. Find the inverse of the following functions :

(i)  $f(x) = 5x + 2$

(ii)  $f(x) = \frac{2x-1}{x-1}$  and domain of f is  $\{x : x > 1\}$

(iii)  $f = \{(1,2), (2,3), (4,6), (5,7), (8,10)\}$

(iv)  $f(x) = \frac{x+1}{x-1}, x \neq 1$

12. If  $g(x) = \frac{1}{x}$  and  $x \neq 0$ , prove that  $g \circ g^{-1} = g^{-1} \circ g$

13. Draw the graph for the following functions :

(i)  $f(x) = 5$

(ii)  $f(x) = x + 1$

(iii)  $f(x) = x^2$

(iv)  $f(x) = I \times I$

(v)  $f(x) = [x]$

14. Show that  $e^x + e^{-x}$  is an even function.

15. Explain the meaning of 1-1 into and 1-1 onto function.

16. Show diagrammatically many-one into and many one onto function.

17. Classify the following functions as one-one onto, one-one into, many-one onto, many-one into.

(i)  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 5x^2 + 8$

(ii)  $g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = 5x + 3$

(iii)  $f = \{(1,2), (2,3), (3,3)\}$

(iv)  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 1 \times 1$

- (v)  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = [x]$
18. Find the inverse of the function  $f(x) = 2x + 1$  and draw a graph for the inverse function.
19. Find the quotient of an identify function and modulus function.  
 {Hints :  $f(x) = x$  and  $g(x) = |x|$ }
20. Find  $f+g, f-g, f/g, \frac{1}{f}, \frac{f}{g}$ , if  $f(x) = x^3+1$
21. Show that  $\lim_{x \rightarrow 0} \frac{e^{2+x} - e^2}{x} = e^2$
22. Show that  $\lim_{x \rightarrow 0} \frac{3^{2+x} - 9}{x} = 9 \text{Log } e^3$
23. Draw a graph of the function  $f(x) = x^2$
24. Find the values of functions
- (i)  $f(x) = \text{Log}_e x$ , for  $x = e$
- (ii)  $f(x) = \text{Log}_{10} x$ , for  $x = 10$
25. If  $f(x) = \text{Log} \left( \frac{1-x}{1+x} \right)$ , show that  $f(b) + f(c) = f \left( \frac{b+c}{1+bc} \right)$ .
26. Explain into and onto function with help of example.

### Answer

1. (i) c, (ii) b (iii) d (iv) b (v) a (vi) d (vii) c (viii) b, (ix) b (x) c
2. (i) into function (ii) Onto function (iii) constant function  
 (iv) Even function (v) Signum function.
3. (i) Odd (ii) exponential (iii) Identity (vi) {1,2,3}  
 (v) {1,2,4} (vi)  $3x^2 + 2$
- (4) (i) Exponential (ii) Even (iii)  $-x^2 + 3x - 1$   
 (iv) into (v) Contant.

- 
5. (i) A function  $f$  from  $A$  to  $B$  written as  $f: A \rightarrow B$  is a relation in which to every element of  $A$  corresponds a unique element of  $B$ .
- (ii) Domain  $\{1,2,3\}$
- (iii) A function is said to be odd, when  $f(-x) = -f(x)$
- (iv) A function is said to be even, when  $f(-x) = f(x)$
- (v) Range =  $\{2\}$
- (vi) A composite function may be called as a function of a function.



## CHAPTER - 5

# *Limit and Continuity*

### Structure

- 5.1 Limit of a function
- 5.2 Some theorems of limit
- 5.3 Evaluation of limits
- 5.4 Some standard limits
- 5.5 Continuity of a function
- 5.6 Questions.

### 5.1 Limit of a function

In the last chapter the meaning of a function is clearly explained. This chapter will explain the limits of such functions. Let us take an example,  $f(x) = 2x - 1$ . In this function the values of the independent variable,  $x$  constitute domain of the function and the corresponding values of the dependent variable,  $y$  or  $f(x)$  constitute the range of the function. The function assumes different values for the different values of  $x$ . That is,  $f(x) = 1$  as  $x = 1$ ,  $f(x) = 3$  as  $x = 2$ , so on. Sometimes it becomes difficult to determine the values of the range ( $y$ ) of the function for given values of the

domain( $x$ ). In the function,  $f(x) = \frac{x^2 - 1}{x - 1}$  the values of  $f(x) = \frac{0}{0}$  when  $x = 1$ , similarly in

$f(x) = \frac{x + 2}{x - 2}$ , the value of  $f(x) = \frac{4}{0}$  when  $x = 2$ . Such indeterminate forms of the function cause a great difficulty. In order to avoid such indeterminate values of the function, the use of limit has been introduced.

Let us try to understand the concept of limit. Observe the graph of a function  $f(x) = x$  or  $y = x$ . Let  $A(x, y)$  be any point of the straight line representing  $f(x) = x$ . Draw a perpendicular from the point  $A(x, y)$  on  $x$ -axis. Now  $AB = x$  and  $OB = x$  as  $y = x$ .

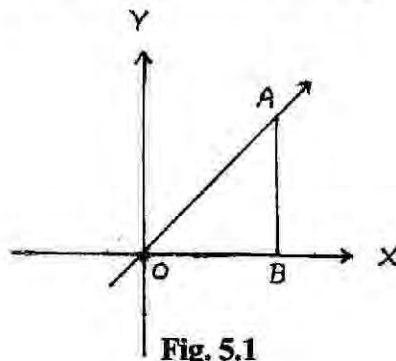


Fig. 5.1

As the point, 'A' approaches towards the origin, 'O' the length of  $AB$  decreases. Ultimately the length of  $AB$  becomes zero as the point reaches the origin,  $O$ . In this approach the function  $f(x)$  or  $y$  or the perpendicular,  $AB$  approaches towards zero as the value of  $x$  in  $A(x, y)$  or  $OB$  approaches towards zero. We write as  $x \rightarrow 0$ ,  $f(x) \rightarrow 0$ .

Let us take a numerical example to understand the meaning of 'approach'. Let us consider a function  $f(x) = 4x + 1$  and observe the values of the function as  $x$  approaches either from lower or higher side.

**Case I.** (When the value of  $x$  is less than one and then approaches 1).

**Table 5.1**

Value of $x$	Value of $f(x) = 4x + 1$
.95	4.8
.96	4.84
.97	4.88
.98	4.92
.99	4.96

**Case II.** (When the value of  $x$  is more than one and then approaches 1).

**Table 5.2**

Value of $x$	Value of $f(x) = 4x + 1$
1.05	5.2
1.005	5.02
1.0005	5.002
1.00005	5.0002
1.000005	5.00002

In case I we observed that as ' $x$ ' approaches from .95 to 1, the function  $f(x) = 4x + 1$  comes closer to 5. Thus we say, the limit of the function  $f(x) = 4x + 1$  is 5 as  $x$  approaches 1. It is also known as **left hand limit**. We write,

$$\lim_{x \rightarrow 1^-} 4x + 1 \text{ or, } \lim_{x \rightarrow 1} 4x + 1 = 5.$$

In case II, we observed that as ' $x$ ' approaches from 1.05 to 1, the function  $f(x) = 4x + 1$  comes closer to 5. Thus we say, the limit of the function  $f(x) = 4x + 1$  is 5 as  $x$  approaches 1. It is also known as **right hand limit**. We write,

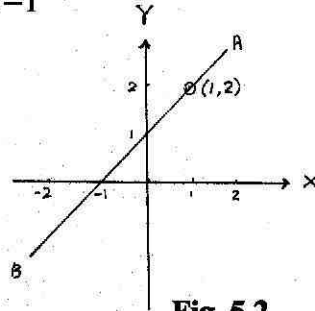
$$\lim_{x \rightarrow 1^+} 4x + 1 \text{ or, } \lim_{x \rightarrow 1} 4x + 1 = 5$$

**Example 1.** Find  $\lim_{x \rightarrow 1} f(x)$  and  $f(1)$ , where  $f(x) = \frac{x^2 - 1}{x - 1}$ .

**Solution :** Let us consider the behaviour of the function,  $f(x) = \frac{x^2 - 1}{x - 1}$  in the graph as well as in the numerical tables as  $x$  approaches 1.

The function  $\frac{x^2 - 1}{x - 1}$  does not exist as  $x$  approaches 1 since the division by zero

$(x-1)$  is not defined. But  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$  exists and is equal to 2 as shown below :



**Fig. 5.2**

On the above graph as  $x$  approaches 1 from lower side (left hand side), the value of  $f(x)$  comes closer to 2. Similarly as  $x$  approaches from higher side (right hand side), the function also comes closer to 2. Hence,

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

**Table 5.3**

$x$	$f(x) = \frac{x^2 - 1}{x - 1}$
.6	1.6
.7	1.7
.8	1.8
.9	1.9
.99	1.99

**Table 5.4**

$x$	$f(x) = \frac{x^2 - 1}{x - 1}$
1.5	2.5
1.2	2.2
1.1	2.1
1.01	2.01
1.001	2.001

In Table 5.3  $x$  approaches 1 from left i.e. from a lower value.

In Table 5.4  $x$  approaches 1 from right i.e. from a higher value.

In both the tables we find that as  $x$  approaches 1, the function  $\frac{x^2-1}{x-1}$  approaches 2.

Hence,

$$\lim_{x \rightarrow 1^-} \frac{x^2-1}{x-1} = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^2-1}{x-1} = 2.$$

### 5.2 Some theorems on limit :

Some theorems on the operation of addition, subtraction, multiplication, division and multiplication by a constant are given below :

Let  $f(x)$  and  $g(x)$  be two functions of  $x$  and both  $f(x)$  and  $g(x)$  exist.

$$(I) \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

(Limit of sum = Sum of limits)

$$(II) \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

(Limit of difference = Difference of limits)

$$(III) \quad \lim_{x \rightarrow a} [f(x) \cdot g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \cdot \left[ \lim_{x \rightarrow a} g(x) \right]$$

(Limit of product = Product of limits)

$$(IV) \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \left( \lim_{x \rightarrow a} g(x) \neq 0 \right)$$

(Limit of quotient = Quotient of limits)

$$(V) \quad \lim_{x \rightarrow a} k \cdot f(x) = k \cdot \lim_{x \rightarrow a} f(x) \quad (\text{where } k \text{ is a constant})$$

$$(VI) \quad \lim_{x \rightarrow a} k = k \quad (\text{where } k \text{ is a constant})$$

$$(VII) \quad \lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow a} f(x)}$$

### 5.3. Evaluation of limits :

While determining the limiting value of a function with respect to a particular value of the

independent variable we may come across indeterminate forms of the function such as  $\frac{0}{0}$ ,  $\frac{a}{0}$ ,  $\frac{\infty}{\infty}$ .

To overcome such situations we should make different approaches to find the limit of a function. We may **directly substitute** the value of the independent variable in the function or we may simplify the function by **cancelling the common factor** of the numerator and denominator of the rational function or we may **rationalise** the numerator or denominator of function. Let us examine such situations through various examples.

**Example 2.** Evaluate :

$$(i) \lim_{x \rightarrow 2} (x + 5), \quad (ii) \lim_{x \rightarrow 2} x(x + 5), \quad (iii) \lim_{x \rightarrow 2} x(x - 1)$$

**Solution :** (i) The limit of  $\lim_{x \rightarrow 2} (x + 5)$  as  $x \rightarrow 2$  is determined by :

$$\lim_{x \rightarrow 2} (x + 5) = 2 + 5 = 7.$$

$$(ii) \lim_{x \rightarrow 2} x(x + 5) = \lim_{x \rightarrow 2} (x^2 + 5x) = (2)^2 + 5 \times 2 = 14$$

$$(iii) \lim_{x \rightarrow 2} x(x - 1) = \lim_{x \rightarrow 2} (x^2 - x) = (2)^2 - 2 = 4 - 2 = 2$$

**Example 3.** Evaluate :

$$(i) \lim_{x \rightarrow 5} \frac{x^2 - 1}{x + 1}, \quad (ii) \lim_{x \rightarrow 3} \frac{x^2 - 5x}{x - 2}$$

$$\text{Solution : (i) } \lim_{x \rightarrow 5} \frac{x^2 - 1}{x + 1} = \frac{(5)^2 - 1}{5 + 1} = \frac{25 - 1}{6} = \frac{24}{6} = 4$$

$$(ii) \lim_{x \rightarrow 3} \frac{x^2 - 5x}{x - 2} = \frac{(3)^2 - 5 \times 3}{3 - 2} = \frac{9 - 15}{1} = -6$$

**Example 4.** Evaluate :

$$(i) \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1}, \quad (ii) \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9}$$

**Solution :** (i) The given function  $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1}$  at  $x = 1$  takes the form of  $\frac{a}{0}$  which is an indeterminate quantity. After factorising the numerator and denominator the common factor

$(x-1)$  can be cancelled from numerator and denominator to arrive at the limiting value.

$$\begin{aligned}\therefore \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{x^2 - 2x - x + 2}{x^2 - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 1} \frac{x-2}{x+1} \\ &= -\frac{1}{2}\end{aligned}$$

(ii) The given function  $\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9}$  at  $x = 3$  takes the form of  $\frac{0}{0}$  which is an indeterminate quantity. So by cancelling a common factor from numerator and denominator, the limit can be easily found out.

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9} &= \lim_{x \rightarrow 3} \frac{x^2 - 3x - 2x + 6}{x^2 - 9} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(x-2)}{(x+3)(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{x-2}{x+3} \\ &= \frac{1}{6}.\end{aligned}$$

**Example 5.** Evaluate :

$$(i) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}, \quad (ii) \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}, \quad (iii) \lim_{x \rightarrow \infty} \left[ \sqrt{x^2+x+1} - x \right]$$

**Solution :** (i) As the limit takes indeterminate form  $\frac{0}{0}$ , we rationalise the numerator.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \lim_{x \rightarrow 0} \left[ \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right]$$

$$= \lim_{x \rightarrow 0} \left[ \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} \right]$$

$$= \lim_{x \rightarrow 0} \left[ \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{2}$$

$$= 1.$$

$$(ii) \lim_{x \rightarrow 1} \left[ \frac{(2x-3)(\sqrt{x-1})}{2x^2+x-3} \right] = \lim_{x \rightarrow 1} \left[ \frac{(2x-3)(\sqrt{x-1})}{2x^2+3x-x-3} \right]$$

$$= \lim_{x \rightarrow 1} \left[ \frac{(2x-3)(\sqrt{x-1})}{(2x+3)(x-1)} \right]$$

$$= \lim_{x \rightarrow 1} \left[ \frac{(2x-3)(\sqrt{x-1})}{(2x+3)(\sqrt{x+1})(\sqrt{x-1})} \right]$$

$$= \lim_{x \rightarrow 1} \left[ \frac{(2x-3)}{(2x+3)(\sqrt{x+1})} \right]$$

$$= \frac{-1}{5 \times 2}$$

$$= -\frac{1}{10}.$$

$$(iii) \lim_{x \rightarrow \infty} \left[ \sqrt{x^2+x+1} - x \right]$$

$$= \lim_{x \rightarrow \infty} \left[ \frac{\left( \sqrt{x^2+x+1} - x \right) \left( \sqrt{x^2+x+1} + x \right)}{\left( \sqrt{x^2+x+1} + x \right)} \right] \text{ (Rationalising the denominator)}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(\sqrt{x^2 + x + 1}\right)^2 - x^2}{\left(\sqrt{x^2 + x + 1} + x\right)}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 + x + 1 - x^2}{\sqrt{x^2 + x + 1} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 + x + 1 - x^2}{\sqrt{x^2 + x + 1} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1} \quad (\text{Dividing both numerator and denominator by } x)$$

$$= \frac{1 + 0}{\sqrt{1 + 0 + 0} + 1} \left( \text{As } x \rightarrow \infty, \frac{1}{x} \text{ or } \frac{1}{x^2} \rightarrow 0 \right)$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}.$$

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**Example 6. Evaluate :**

$$(i) \lim_{x \rightarrow \infty} \frac{ax^3 + bx^2 + cx}{dx^3 + e} \quad (\text{where } a, b, c, d, e \text{ are constants})$$

$$(ii) \lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + 4}{x^3 - 4x + 4}$$

$$(iii) \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x}$$

**Solution :** (i) On direct substitution of the value of  $x$ , the function assumes the indeterminate form of  $\frac{\infty}{\infty}$ . Now, divide both numerator and denominator by  $x^3$  and then find the limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{ax^3 + bx^2 + cx}{dx^3 + e} &= \lim_{x \rightarrow \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x^3}} \quad (\text{dividing the numerator and denominator by } x^3) \\ &= \frac{a + 0 + 0}{d + 0} \left( \text{As } x \rightarrow \infty, \frac{1}{x} \text{ or } \frac{1}{x^2} \text{ or } \frac{1}{x^3} \rightarrow 0 \right) \\ &= \frac{a}{d}. \end{aligned}$$

(ii) On direct substitution of the values of  $x$ , the function assumes the form of  $\frac{0}{0}$ . So, that indeterminate situation may be avoided by factorising both the numerator and denominator.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + 4}{x^2 - 4x + 4} \\ &= \lim_{x \rightarrow 2} \frac{x^3 + x^2 - 4x^2 + 4}{x^2 - 4x + 4} \\ &= \lim_{x \rightarrow 2} \frac{x^2(x+1) - 4(x^2-1)}{x^2 - 4x + 4} \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 2} \frac{(x+1)\{x^2 - 4(x-1)\}}{x^2 - 4x + 4} \\
&= \lim_{x \rightarrow 2} \frac{x^2(x+1) - 4\{(x+1)(x-1)\}}{x^2 - 4x + 4} \\
&= \lim_{x \rightarrow 2} \frac{(x+1)\{x^2 - 4(x-1)\}}{x^2 - 4x + 4} \\
&= \lim_{x \rightarrow 2} \frac{(x+1)\{x^2 - 4x + 4\}}{x^2 - 4x + 4} \\
&= \lim_{x \rightarrow 2} (x+1) \\
&= 2+1 \text{ (Putting the value of } x) \\
&= 3.
\end{aligned}$$

(iii) On direct substitution of the value of  $x$ , the function assumes the indeterminate form of  $\frac{0}{0}$ . Hence, in order to avoid such situation, the function should be rationalised by multiplying  $(\sqrt{1+x} + 1)$  both to the numerator and denominator.

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)(\sqrt{1+x} + 1)}{x(\sqrt{1+x} + 1)} \\
\lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)(\sqrt{1+x} + 1)}{x(\sqrt{1+x} - 1)} \\
&= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x})^2 - (1)^2}{x(\sqrt{1+x} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{1+x-1}{x(\sqrt{1+x} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x} + 1)}
\end{aligned}$$

$$= \frac{1}{\sqrt{1+0+1}} \text{ (Putting the value of } x\text{)}$$

$$= \frac{1}{2}$$

**Example 7.** Evaluate :

(i)  $\lim_{x \rightarrow 1} \frac{x^3 + 1}{x + 1}$

(ii)  $\lim_{x \rightarrow 2} \frac{x^3 + 8}{x + 2}$

**Solution :** (i) On direct substitution the function becomes  $\frac{0}{0}$ . Hence after factorisation of the function, the limit may be found out.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3 + 1}{x + 1} &= \lim_{x \rightarrow 1} \frac{(x + 1)(x^2 - x + 1)}{x + 1} \text{ (As } a^3 + b^3 = (a + b)(a^2 - ab + b^2)\text{)} \\ &= (-1)^2 - (-1) + 1 \\ &= 1 + 1 + 1 \\ &= 3. \end{aligned}$$

(ii) On direct substitution the function becomes  $\frac{0}{0}$ . Hence the limit of the function may be found out after due factorisation.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^3 + 8}{x + 2} &= \lim_{x \rightarrow 2} \frac{(x + 2)(x^2 - 2x + 4)}{x + 2} \text{ (As } a^3 + b^3 = (a + b)(a^2 - ab + b^2)\text{)} \\ &= \lim_{x \rightarrow 2} (x^2 - 2x + 4) \\ &= (-2)^2 - 2(-2) + 4 \\ &= 4 + 4 + 4 \\ &= 12. \end{aligned}$$

**5.4 Some standard limits :**(I) If  $n$  be any positive integer, then

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \text{ (where } x > 0\text{)}$$

$$\text{(II) } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\text{(III) } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$$

$$\text{(IV) } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

Let us apply the standard limits in the following problems.

**Example 8.** Evaluate:  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$ 

$$\text{Solution : } \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x)^3 - (1)^3}{x - 1}$$

$$= 3(1)^{3-1} \left( \because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right)$$

$$= 3.$$

**Example 9.** Evaluate:  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4}$ 

$$\text{Solution : } \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \rightarrow 2} \left[ \frac{x^3 - 8}{x - 2} \div \frac{x^2 - 4}{x - 2} \right]$$

$$= \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 2} \div \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 2}$$

$$= 3(2)^{3-1} \div 2(2)^{2-1} \left( \because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right)$$

$$= 12 \div 4$$

$$= 3.$$

**Example 10.** Evaluate:  $\lim_{x \rightarrow a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x-a}$

**Solution :**

$$\begin{aligned} \lim_{x \rightarrow a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x-a} \\ &= \lim_{x+2 \rightarrow a+2} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{(x+2) - (a+2)} \quad (\text{when } x \rightarrow a, x+2 \rightarrow a+2) \\ &= \frac{5}{3} (a+2)^{5/3-1} \left( \because \lim_{x \rightarrow a} \frac{x^n - a^n}{x-a} = n a^{n-1} \right) \\ &= \frac{5}{3} (a+2)^{2/3}. \end{aligned}$$

**Example 11.** Find the value of the positive integer  $n$ , so that  $\lim_{x \rightarrow 2} \frac{x^n - (2)^n}{x-2} = 32$

**Solution :**

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^n - (2)^n}{x-2} &= 32 \\ \Rightarrow n(2)^{n-1} &= 32 \\ \Rightarrow n(2)^{n-1} &= 4 \times 2^{4-1} \\ \Rightarrow n &= 4 \end{aligned}$$

$\therefore$  The required positive integer  $n$  is 4.

**Example 12.** Evaluate:  $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$

**Solution :**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2x} \times 2 \\ &= 2 \left[ \lim_{2x \rightarrow 0} \frac{e^{2x} - 1}{2x} \right] \\ &= 2 \times 1 \left( \because \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right) \\ &= 2. \end{aligned}$$

**Example 13.** Find the value of  $\lim_{x \rightarrow 0} \frac{5^x - 2^x}{x}$ .

**Solution :** 
$$\lim_{x \rightarrow 0} \frac{5^x - 2^x}{x} = \lim_{x \rightarrow 0} \frac{(5^x - 1) - (2^x - 1)}{x}$$

$$= \log_e 5 - \log_e 2 \left( \because \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a \right)$$

$$= \log_e 5/2.$$

**Example 14.** Show that  $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$ .

**Solution :** Let  $y = \lim_{x \rightarrow 0} \frac{\log(1+x)}{x}$

$$\Rightarrow xy = \log_e(1+x)$$

$$\Rightarrow e^{xy} = (1+x)$$

$$\Rightarrow e^{xy} - 1 = x$$

$$\Rightarrow \frac{e^{xy} - 1}{x} = 1$$

$$\Rightarrow \frac{e^{xy} - 1}{xy} \cdot y = 1$$

$$\Rightarrow \lim_{xy \rightarrow 0} \frac{e^{xy} - 1}{xy} \cdot y = 1$$

$$\Rightarrow \lim_{xy \rightarrow 0} \frac{e^{xy} - 1}{xy} \cdot \lim_{xy \rightarrow 0} y = 1$$

$$\Rightarrow 1 \cdot \lim_{x \rightarrow 0} y = 1 \quad (\text{As } x \rightarrow 0 \text{ gives } xy \rightarrow 0) \ \& \ \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1 \quad (\text{Substituting the value of } y)$$

## 5.5 CONTINUITY OF A FUNCTION

A function  $f(x)$  is said to be continuous at  $x = a$  if and only if

- (i)  $f(x)$  is defined at  $x = a$
- (ii)  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$
- (iii)  $\lim_{x \rightarrow a} f(x) = f(a)$
- (iv)  $\lim_{h \rightarrow 0} f(a - h) = f(a) = \lim_{h \rightarrow 0} f(a + h)$

When any of the above conditions is not satisfied by a function, it is discontinuous at  $x = a$ .

### Properties of continuous functions

Let  $f(x)$  and  $g(x)$  be two real functions, continuous at  $a$ . Let  $c$  be a real number, then,

- (i)  $f(x) + g(x)$  is continuous at  $x = a$ .
- (ii)  $f(x) - g(x)$  is continuous at  $x = a$ .
- (iii)  $f(x) \cdot g(x)$  is continuous at  $x = a$ .
- (iv) is continuous at  $x = a$ .
- (v)  $c \cdot f(x)$  is continuous at  $x = a$ .
- (vi)  $\frac{1}{f(x)}$  is continuous at  $x = a$ , provided  $f(a) \neq 0$ .

The test of continuity of a function can also be made graphically. Let us take two functions :

(a)  $f(x) = 2x$

(b)  $f(x) = \begin{cases} 3 & x > 0 \\ 0 & x \leq 0 \end{cases}$

(a) Graphic representation of the function  $y = 2x$  where  $y = f(x)$

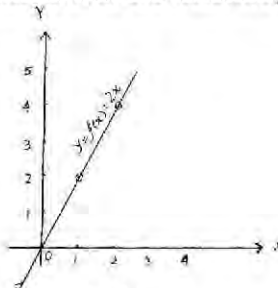


Fig. 5.3

The graphic line of the function is continuous.

- (b) Graphic representation of the function  $y = \begin{cases} 3 & x > 0 \\ 0 & x \leq 0 \end{cases}$

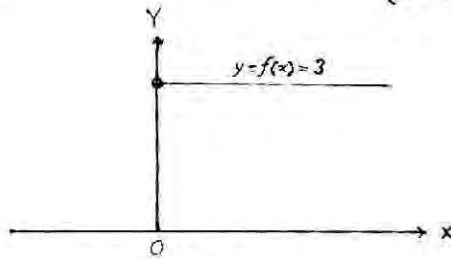


Fig 5.4

There is a jump in the graph  $x = 0$  to  $x = 1$ , so the function is discontinuous at  $x = 0$ .

**Example 15.** Examine the continuity of the following function at  $x = 0, 1$ .

$$f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 4x - 3 & \text{if } 0 < x \leq 1 \\ 4x^2 - 3^2 & \text{if } 1 < x \end{cases}$$

**Solution :**

- (i) At  $x = 0$

$$f(0) = -(0)^2 = 0$$

$$\text{L.H.L.} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x^2) = 0$$

$$\text{R.H.L.} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (4x - 3) = -3$$

As  $\text{LHL} \neq \text{RHL}$ ,  $\lim_{x \rightarrow 0} f(x)$  does not exist.

Hence the function is discontinuous.

- (ii) At  $x = 1$

$$f(1) = 4 \times 1 - 3 = 4 - 3 = 1$$

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (4x - 3) = 4 \times 1 - 3 = 4 - 3 = 1$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x^2 - 3x) = 4 \cdot 1^2 - 3 \times 1 = 4 - 3 = 1$$

As  $\lim_{x \rightarrow 1} f(x) = 1 = f(1)$ , the function is continuous at  $x = 1$ .

**Example 16.** A function is defined by

$$f(x) = \begin{cases} x-1 & \text{when } x \leq 0 \\ x^2 & \text{when } x > 0 \end{cases}$$

show that the function  $f(x)$  is discontinuous at  $x = 0$ .

**Solution :**

$$\begin{aligned} \text{Left Hand Limit (L.H.L.)} &= \lim_{x \rightarrow 0^-} f(x) \\ &= \lim_{x \rightarrow 0^-} (x-1) \\ &= \lim_{h \rightarrow 0} (0-h) - 1 \\ &= -1. \end{aligned}$$

$$\begin{aligned} \text{Right Hand Limit (R.H.L.)} &= \lim_{x \rightarrow 0^+} f(x) \\ &= \lim_{x \rightarrow 0^+} (x^2) \\ &= \lim_{h \rightarrow 0} (0+h)^2 \\ &= 0 \end{aligned}$$

As  $L.H.L \neq R.H.L$  the function  $f(x)$  is discontinuous at  $x = 0$ .



**5.6 QUESTIONS**

**1. From the given alternatives choose and write the correct answer against each bit along with its serial number :**

(i)  $\lim_{x \rightarrow 1} (3x + 2)$  is equal to

(a) 4

(b) 5

(c) 6

(d) 0

(ii)  $\lim_{x \rightarrow 2} \left( \frac{1}{x^2} + 7 \right)$  is

(a)  $\infty$

(b) 0

(c) 6

(d) 7

(iii)  $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8}$  is equal to

(a)  $\frac{20}{3}$

(b)  $\frac{3}{8}$

(c) 1

(d)  $\frac{3}{16}$

(iv)  $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x^2 - a^2}$  is equal to

(a)  $\frac{3}{2}$

(b)  $\frac{3t}{2}$

(c)  $\frac{2}{3t}$

(d)  $\frac{t}{3}$

(v)  $\lim_{x \rightarrow -4} \frac{x^2 - 16}{x + 4}$  is equal to

(a) 4

(b) 8

(c) -8

(d) 16

---

(vi)  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$  is equal to

- (a) 2 (b) 3  
(c) 0 (d) 1

(vii)  $\lim_{x \rightarrow -2} \frac{x^3 + 8}{x + 2}$  is equal to

- (a) 10 (b) 12  
(c) 16 (d) 8

(viii)  $\lim_{x \rightarrow \infty} \frac{ax^2 + bx + c}{cx^2 + d}$  is equal to

- (a)  $\frac{b}{c}$  (b)  $\frac{a}{c}$   
(c)  $\frac{c}{d}$  (d)  $\frac{a}{d}$

(ix)  $\lim_{x \rightarrow \infty} \left( \frac{x-1}{x+1} \right)$  is equal to

- (a) 1 (b) 2  
(c)  $\frac{1}{2}$  (d) 0

(x)  $\lim_{x \rightarrow \infty} \frac{9x^4 + 8x^3 + 7x^2 + 6x + 5}{4x^4 + 3x^3 + 2x^2 + x + 1}$  is equal to

- (a) 5 (b) 6  
(c)  $\frac{7}{2}$  (d)  $\frac{9}{4}$



**2. Express each of the following in one word /one term :**

(i)  $1 + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \dots\dots\dots$

(ii)  $\lim_{x \rightarrow a^+} f(x)$

(iii)  $\lim_{x \rightarrow 1} [(x + 2^2 - 6)]$

(iv) Is  $f(x) = \begin{cases} x-1 & x \leq 2 \\ 2x-3 & x > 0 \end{cases}$  continuous at  $x = 2$ .

(v)  $\lim_{x \rightarrow 3} \frac{3-3}{\sqrt{x-2} - \sqrt{4-x}}$

**3. Answer each of the following questions in one sentence :**

(i) Evaluate  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$

(ii) Evaluate  $\lim_{x \rightarrow 5} \frac{x^2 - 9x + 20}{x^2 - 6x + 5}$

(iii) Evaluate  $\lim_{x \rightarrow \infty} \left( \frac{5}{x^2} + 7 \right)$

(iv) Evaluate  $\lim_{x \rightarrow \infty} e^{x^2 + 2x + 1}$

(v) Write any one condition when a function  $f(x)$  will be said to be continuous at  $x = 1$ .**4. Rectify the underlined portions of the following sentences :**(i) In  $\lim_{x \rightarrow a} f(x) = l$ , the function  $f(x)$  is said to tend to a limit a.(ii) In  $\lim_{x \rightarrow a} f(x) = l$ ,  $l$  is called the left hand limit.

(iii)  $\lim_{x \rightarrow 4} \sqrt[3]{6x + 3} = \underline{8}$ .

$$(iv) \lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a} = \frac{5}{4} (a+2)^{3/2}$$

$$(v) \text{ In a continuous function } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \underline{f(x)}.$$

**5. Fill in the blanks :**

$$(i) \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x - 1} = \dots\dots\dots$$

$$(ii) \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1} = \dots\dots\dots$$

$$(iii) \text{ If } \lim_{x \rightarrow 2} \frac{x^n - (2)^n}{x - 2} = 32, \text{ then } n \text{ is equal to } \dots\dots\dots$$

(iv) A function is said to be continuous at  $x = a$  if and only if ..... is defined at  $x = a$

(v) If  $f(x) = \begin{cases} x-1 & \text{when } x \leq 0 \\ x^2 & \text{when } x > 0 \end{cases}$ , then  
the function is ..... at  $x = 0$

**6. Answer the following questions within 30 words each :**

(a) Evaluate :  $\lim_{x \rightarrow 2} (x + 3)$ .

(b) Evaluate :  $\lim_{x \rightarrow 5} (x^2 - 3x)$ .

(c) Evaluate :  $\lim_{x \rightarrow 3} \frac{x^3 - 4x}{x - 2}$ .

(d) Find the value of  $\lim_{x \rightarrow 2} x(x^2 - 1)$ .

(e) Evaluate :  $\lim_{x \rightarrow 2} \frac{x^3 - 2^3}{x - 2}$

(f) Evaluate:  $\lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

(g) Evaluate:  $\lim_{x \rightarrow a} \frac{x^8 - a^8}{x^3 - a^3}$

$$\left( \text{Hint s : } \frac{x^8 - a^8}{x^3 - a^3} = \frac{\frac{x^8 - a^8}{x - a}}{\frac{x^3 - a^3}{x - a}} \right)$$

(h) Evaluate:  $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x^3 - 27}$

(i) Evaluate:  $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}}$

(j) Evaluate:  $\lim_{x \rightarrow \infty} \sqrt{4x^2 + x + 1} - 2x$

(k) If  $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} = 12$ , then find the value of  $a$ .

(l) Find the values of 'a' if  $\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} = \lim_{x \rightarrow 2} x + 3$

(m) Find the value of  $\lim_{x \rightarrow 0} \frac{(1+x)^m - 1}{x}$  (Hint : take  $y = 1+x$  or  $x = y-1$ )

(n) Show that the following function is discontinuous at  $x = 2$

$$f(x) = \begin{cases} x-1 & \text{when } x \leq 2 \\ x^2 & \text{when } x > 0 \end{cases}$$

7. Answer each of the following questions within fifty words :

(a) Test the continuity of the following function at  $x = 0$

$$f(x) = \begin{cases} 4 & \text{when } x > 2 \\ 1 & \text{when } x \leq 0 \end{cases}$$

(b) Evaluate  $\lim_{x \rightarrow \infty} \frac{5x^2 + 4x + 1}{x^2 + 1}$

(c) Evaluate  $\lim_{x \rightarrow 0} 8$

(d) Evaluate  $\lim_{x \rightarrow -1} \frac{x^3 - 4x + 1}{x - 1}$

(e) Evaluate  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$

(f) Evaluate  $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$

(g) Evaluate  $\lim_{x \rightarrow 3} \frac{e^4 - e^3}{x - 3}$

(h) Find the value of  $n$ , if  $\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 32$

**Answer the following long questions.**

**A. Evaluate the following limits :**

8.  $\lim_{x \rightarrow 2} \frac{x + 1}{x - 1}$

9.  $\lim_{x \rightarrow 0} [3 - (x + 1)^2]$

10.  $\lim_{x \rightarrow -1} \frac{x^2 - 7x + 10}{x^2 - 5x + 6}$

11.  $\lim_{x \rightarrow 0} \frac{4x}{\sqrt{2 + x} - \sqrt{2 - x}}$

12.  $\lim_{x \rightarrow \infty} \frac{x^3 + x^2 + x + 2}{x^3 - 1}$

13.  $\lim_{x \rightarrow 2} 5$ 

---

$$14. \lim_{x \rightarrow -3} \frac{x^3 + 27}{x + 3}$$

$$15. \lim_{x \rightarrow a} \left[ \frac{x^m - a^m}{x^n - a^n} \right]$$

$$16. \lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - x^2}}{x}$$

$$17. \lim_{x \rightarrow 2} \frac{x^2 \sqrt{x} - 4\sqrt{2}}{x - 2}$$

$$18. \lim_{x \rightarrow a} \frac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$$

$$19. \lim_{h \rightarrow 0} \frac{3h^3 + 2h - h}{h^2 - h}$$

$$20. \lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 7}{\sqrt{x^4 + 2}}$$

$$21. \lim_{x \rightarrow a} \frac{(x + 2)^{7/3} - (a + 2)^{7/3}}{x - a}$$

$$22. \lim_{x \rightarrow 0} \frac{x}{\sqrt{2 + x} - \sqrt{2 - 3x}}$$

$$23. \lim_{x \rightarrow 1} \frac{x^3 - x^2 - 2x + 2}{x - 1}$$

**B. Test the continuity of each of the following functions :**

$$24. f(x) = 2x + 3$$

$$25. f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x > 0 \end{cases} \quad \text{at } x = 0$$

$$26. \quad f(x) = \begin{cases} 2x-3 & x < 2 \\ x-1 & x \geq 2 \end{cases} \quad \text{at } x = 2$$

$$27. \quad f(x) = \begin{cases} x, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -x, & \text{when } x < 0 \end{cases} \quad \text{at } x = 0$$

$$28. \quad f(x) = \begin{cases} x^2, & \text{when } x \text{ is greater than zero} \\ 2, & \text{when } x \text{ is equal to zero} \\ x-1, & \text{when } x \text{ is less than zero} \end{cases}$$

$$29. \quad f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{when } x \text{ is not equal to } 2 \\ 16 & \text{when } x \text{ is equal to } 2 \end{cases}$$

**C. Answer the following questions :**

$$30. \quad \text{Given, } f(x) = \begin{cases} 2x-1, & \text{when } x \leq 5 \\ 3c, & \text{when } x > 5 \end{cases}$$

If the function,  $f(x)$  is continuous at  $c = 5$ , find the value of  $c$ .

31. Show that the following function is continuous at  $x = 1$ .

$$f(x) = \begin{cases} 3x-2, & \text{if } 0 < x \leq 1 \\ 4x^2-3x, & \text{if } 1 < x \leq 2 \end{cases}$$

32. From the following function find,

- (a) Left-hand limit at  $x = 1$
- (b)  $f(x) = 2$
- (c) Right-hand limit at  $x = 1$

$$f(x) = \begin{cases} 4x+1, & \text{when } x > 1 \\ 3x+2, & \text{when } x = 1 \\ 5, & \text{when } x < 1 \end{cases}$$

Also test the continuity of the function at  $x = 1$ .

33. Show that the function  $f(x) = |x|$  is a continuous function.

**ANSWERS**

1. (i) b (ii) d (iii) a (iv) b (v) c (vi) d (vii) b (viii) b (ix) a  
 (x) d (xi) a (xii) d (xiii) d (xiv) a (xv) d
2. (i) e (ii) RHL (iii) 3 (iv) Continuous (v) 1
3. (i) 3 (ii)  $\frac{1}{4}$  (iii) 7 (iv)  $e^9$  (v)  $\lim_{x \rightarrow a^-} f(x) = f(a) = \lim_{x \rightarrow a^+} f(x)$
4. (i) e (ii) Right (iii) 3 (iv)  $\frac{5}{2}(a+2)$  (v)  $f(a)$
5. (i) 2 (ii)  $-\frac{1}{2}$  (iii) 4 (iv)  $f(x)$  (v) Discontinuous
6. (a) 5 (b) 10 (c) 15 (d) 6 (e) 12 (f)  $-\frac{1}{18}$  (g)  $\frac{8}{3}a^5$  (h) 4 (i) 1  
 (j)  $\frac{1}{4}$  (k) 6 (l)  $\pm 1$  (m) m
7. (a) discontinuous (b) 5 (c) 8 (d) -3 (e) 3 (f) 108 (g)  $e^3$  (h) 4
8. 3 9. 2 10. 8 11. 8 12. 1 13. 5 14. 13 15.  $\frac{m}{n}a^{m-n}$
16.  $\frac{1}{2}$  17.  $\frac{5}{2}$  18.  $\frac{2}{3\sqrt{3}}$  19. 1 20. 5 21.  $\frac{7}{3}(a+2)^{\frac{4}{3}}$
22.  $\frac{1}{\sqrt{2}}$  23. -1 24. continuous at every point
25. discontinuous 26. continuous 27. continuous 28. Discontinuous
29. Discontinuous at  $x=2$  30.  $c=3$ .



## CHAPTER - 6

# ***DIFFERENTIATION***

### **STRUCTURE**

- 6.1 Meaning of differential coefficient
- 6.2 Derivatives of some important functions
- 6.3 Derivative of a function of a function
- 6.4 Derivative of implicit function
- 6.5 Derivative of functions in parametric forms.
- 6.6 Derivative of higher order
- 6.7 Miscellaneous illustrations
- 6.8 Questions

### 6.1 Meaning of differential coefficient

The differential coefficient of a function is the rate of change of the function for a particular value of the variable. If  $y$  or  $f(x)$  is a function of  $x$  and  $h$  be any small increment to  $x$  which corresponds to  $f(a+h)-f(a)$  increase in  $y$  or  $f(x)$  at a point  $a$ , then the ratio of the two increments  $\frac{f(a+h)-f(a)}{h}$  represents the difference coefficient. It represents the average rate of change of  $f(x)$  in an interval of  $(a, a+h)$ .

The limiting value of the ratio as  $h$  tends to zero represents the instantaneous rate of change of  $f(x)$  at 'a'. This instantaneous rate of change is known as differential coefficient of the function with respect to the independent variable at a particular point i.e.  $\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$  or  $\frac{dy}{dx}$  is called the differential coefficient or derivative of  $y$  with respect to  $x$ . It is also denoted by  $f'(x)$ . Now

$$\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

#### Example 1.

If  $f(x) = x^2+4$ , find  $f'(4)$

**Solution :** Now  $f(x) = x^2+4$

$$\begin{aligned} \therefore f'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(4+h)^2 + 4] - [4^2 + 4]}{h} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{16 + h^2 + 8h + 4 - 16 - 4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 + 8h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(h + 8)}{h} \\
 &= \lim_{h \rightarrow 0} (h + 8) \\
 &= 8
 \end{aligned}$$

## 6.2 Derivatives of some important functions

The above method of finding out derivative is known as "differentiation from first principle or ab initio." To make computation more rapid and easier we may use some standard results of differentiation.

- (i) Derivative of  $x^n$ , where  $n$  is any integer.

$$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

- (ii) Derivative of a constant ( $c$ )

$$\frac{d(c)}{dx} = 0$$

- (iii) Derivative of a multiple of a function

$$\frac{d(k \cdot u)}{dx} = k \cdot \frac{du}{dx}$$

- (iv) Derivative of exponential function

$$\frac{d(e^x)}{dx} = e^x$$

- (v) Derivative of the logarithmic function

$$\frac{d(\log_e x)}{dx} = \frac{1}{x}$$

- (vi) Derivative of  $\log_a x$  ( $x > 0$ )

$$\frac{d(\log_a x)}{dx} = \frac{1}{x} \log_a e$$

(vii) Derivative of sum or difference two or more functions.

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \quad (\text{Where } u \text{ and } v \text{ are functions of } x)$$

(viii) Derivative of product of two functions

$$\frac{d(u \cdot v)}{dx} = v \cdot \frac{du}{dx} + u \cdot \frac{dv}{dx} \quad (\text{Where } u \text{ and } v \text{ are functions of } x)$$

(ix) Derivative of the quotient of two functions

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} \quad (\text{Where } u \text{ and } v \text{ are functions of } x)$$

### Example 2.

Find the differential coefficient of  $x^2$  from the first principle.

**Solution :** Let  $y = x^2$

$$\text{Now, } y + \Delta y = (x + \Delta x)^2$$

$$\Rightarrow \Delta y = (x + \Delta x)^2 - y$$

$$\Rightarrow \Delta y = x^2 + (\Delta x)^2 + 2 \cdot x \cdot \Delta x - x^2$$

$$\Rightarrow \Delta y = \Delta x (\Delta x + 2x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x (\Delta x + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (\Delta x + 2x)$$

$$= 2x$$

**Example 3.**

Find the derivative of  $x^7$  w.r.t.x

**Solution :** Let  $y = x^7$

$$\therefore \frac{dy}{dx} = \frac{d(x^7)}{dx} = 7x^{7-1} = 7x^6 \quad \left( \because \frac{d(x^n)}{dx} = nx^{n-1} \right)$$

**Example 4.**

Find the derivative of 8 w.r.t.x.

**Solution :** Let  $y = 8$

$$\therefore \frac{dy}{dx} = \frac{d(8)}{dx} = 0 \quad \left( \because \frac{d(c)}{dx} = 0 \right)$$

**Example 5.**

Find the derivative of  $7x^2$  w.r.t.x

**Solution :** Let  $y = 7x^2$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d(7x^2)}{dx} \\ &= 7 \cdot \frac{d(x^2)}{dx} \\ &= 7 \cdot 2x \\ &= 14x. \end{aligned}$$

**Example 6.**

Find the derivative of  $2e^x$  w.r.t.x

**Solution :** Let  $y = 2e^x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d(2e^x)}{dx} \\ &= 2 \cdot \frac{d(e^x)}{dx} \\ &= 2 e^x \end{aligned}$$

**Example 7.**

Find the derivative of  $5 \log x$  w.r.t.  $x$

**Solution :** Let  $y = 5 \log x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d(5 \log_e x)}{dx} \\ &= 5 \cdot \frac{d(\log_e x)}{dx} \\ &= 5 \cdot \frac{1}{x} \\ &= \frac{5}{x}\end{aligned}$$

**Example 8.**

Find the derivative of  $\log_5 x$  w.r.t.  $x$

**Solution :** Let  $y = \log_5 x$

$$\therefore \frac{dy}{dx} = \frac{d(\log_5 x)}{dx} = \frac{1}{x} \log_5 e$$

**Example 9.**

Find the derivative of  $x^3 + 2x^2 + 7$  w.r.t.  $x$

**Solution :** Let  $y = x^3 + 2x^2 + 7$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d(x^3 + 2x^2 + 7)}{dx} \\ &= \frac{d(x^3)}{dx} + \frac{d(2x^2)}{dx} + \frac{d(7)}{dx} \\ &= 3x^2 + 2 \cdot 2x + 0 \\ &= 3x^2 + 4x\end{aligned}$$

**Example 10.**

Find the derivative of  $x^4 - 3x^2 - 8$  w.r.t.x

**Solution :** Let  $y = x^4 - 3x^2 - 8$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d(x^4 - 3x^2 - 8)}{dx} \\ &= \frac{d(x^4)}{dx} - \frac{d(3x^2)}{dx} - \frac{d(8)}{dx} \\ &= 4x^3 - 3 \cdot 2x - 0 \\ &= 4x^3 - 6x \end{aligned}$$

**Example 11.**

Find the derivative of  $(x^2+3)(x^3-7)$  w.r.t.x

**Solution :** Let  $y = (x^2+3)(x^3-7)$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (x^3 - 7) \frac{d(x^2 + 3)}{dx} + (x^2 + 3) \frac{d(x^3 - 7)}{dx} \\ &= (x^3 - 7) \left\{ \frac{d(x^2)}{dx} + \frac{d(3)}{dx} \right\} + (x^2 + 3) \left\{ \frac{d(x^3)}{dx} - \frac{d(7)}{dx} \right\} \\ &= (x^3 - 7) \cdot 2x + (x^2 + 3) \cdot 3x^2 \\ &= 2x(x^3 - 7) + 3x^2(x^2 + 3) \\ &= 2x^4 - 14x + 3x^4 + 9x^2 \\ &= 5x^4 + 9x^2 - 14x \end{aligned}$$

**Example 12.**

Find the derivative of  $\frac{x^2+9}{x+2}$  w.r.t.x.

**Solution :** Let  $y = \frac{x^2 + 9}{x + 2}$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{d\left(\frac{x^2+9}{x+2}\right)}{dx} \\
&= \frac{(x+2)\frac{d(x^2+9)}{dx} - (x^2+9)\frac{d(x+2)}{dx}}{(x+2)^2} \\
&= \frac{(x+2)\left\{\frac{d(x^2)}{dx} + \frac{d(9)}{dx}\right\} - (x^2+9)\left\{\frac{d(x)}{dx} + \frac{d(2)}{dx}\right\}}{(x+2)^2} \\
&= \frac{(x+2) \cdot 2x - (x^2+9) \cdot 1}{(x+2)^2} \\
&= \frac{2x^2 + 4x - x^2 - 9}{(x+2)^2} \\
&= \frac{x^2 + 4x - 9}{(x+2)^2}
\end{aligned}$$

### 6.3 Derivative of a function of a function

In order to differentiate a composite of two or more functions we apply Chain Rule. The technique to find out the derivative of a composite function is explained below.

Let  $y$  be function of  $t$  and  $t$  be the function of  $x$ . Now  $y = f(t)$  and  $t = f(x)$ . Then

by Chain Rule, we have,  $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

The Chain Rule may also be extended to more than two functions.

#### Example 13.

Find the derivative of  $(x^3+8)^7$  w.r.t.  $x$

#### Solution :

Let  $t = x^3+8$  and  $y = (x^3+8)^7 = t^7$

Now  $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$\begin{aligned}
 &= \frac{d(t^7)}{dt} \cdot \frac{d(x^3 + 8)}{dx} \\
 &= 7t^6 \left\{ \frac{d(x^3)}{dx} + \frac{d(8)}{dx} \right\} \\
 &= 7(x^3 + 8)^6 (3x^2) \\
 &= 21 x^2 (x^3 + 8)^6
 \end{aligned}$$

#### 6.4. Derivative of implicit function

In equation  $x^2 + y^2 + 4xy = 16$ ,  $y$  is not explicitly defined. Hence it is an implicit function. The method to differentiate implicit is illustrated below. Let us take an implicit function :

$$x^2 + y^2 - 4xy = 16$$

Now differentiating both sides with respect to  $x$ , we have

$$\begin{aligned}
 \frac{d(x^2 + y^2 - 4xy)}{dx} &= \frac{d(16)}{dx} \\
 \Rightarrow \frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} - \frac{d(4xy)}{dx} &= 0 \\
 \Rightarrow 2x + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} - 4 \left\{ y \cdot \frac{dx}{dx} + x \frac{dy}{dx} \right\} &= 0 \\
 \Rightarrow 2x + 2y \cdot \frac{dy}{dx} - 4y - 4x \frac{dy}{dx} &= 0 \\
 \Rightarrow 2y \cdot \frac{dy}{dx} - 4x \frac{dy}{dx} &= 4y - 2x \\
 \Rightarrow \frac{dy}{dx} (2y - 4x) &= 4y - 2x \\
 \Rightarrow \frac{dy}{dx} = \frac{4y - 2x}{2y - 4x} &= \frac{2(2y - x)}{2(y - 2x)} \\
 \therefore \frac{dy}{dx} &= \frac{2y - x}{y - 2x}
 \end{aligned}$$

**6.5. Derivative of functions in parametric forms.**

Sometimes two variables may be related to be a third variable which show separate relation between the first two variables. The third variable in called the parameter. If  $y=f(t)$  and  $x=g(t)$ , then 't' is known as the parameter. To differentiate functions of parametric form we apply the following Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\text{or } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

**Example 14** If  $y = at^2$  and  $x = 5bt$ , find  $\frac{dy}{dx}$

Solution :

$$y = at^2 \text{ and } x = 5bt$$

Differentiating both the functions separately w.r.t.x we have,

$$\frac{dy}{dt} = \frac{d(at^2)}{dt} = a \frac{d(t^2)}{dt} = a.2t = 2at \dots\dots\dots (i)$$

$$\text{Similarly, } \frac{dx}{dt} = \frac{d(5bt)}{dt} = 5b. \frac{d(t)}{dt} = 5b \dots\dots\dots (ii)$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2at}{5b}$$

**6.6. Derivative of higher order.**

The differential Coefficient of a function is denoted  $\frac{dy}{dx}$  or  $f'(x)$  and is called

the first derivative. If this first derivative  $\frac{dy}{dx}$  or  $f'(x)$  is again differentiated

w.r.t.x, the resultant derivative is denoted by  $\frac{d^2y}{dx^2}$  or  $f''(x)$  and is called the second derivative. The successive derivatives are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n},$$

or  $f'(x), f''(x), f'''(x), \dots, f^n(x)$ .

### Example 15.

Find the second order derivative of  $x^3$  with respect to  $x$ .

**Solution :**

$$\text{Let } y = x^3$$

$$\therefore \frac{dy}{dx} = \frac{d(x^3)}{dx}$$

$$= 3x^2 \dots \dots \dots \text{(First order derivative)}$$

Again  $\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}$

$$= \frac{d(3x^2)}{dx}$$

$$= 3 \frac{d(x^2)}{dx}$$

$$= 3 \cdot 2x$$

$$= 6x \dots \dots \dots \text{( Second order derivative)}$$

## 6. Miscellaneous Illustrations :

### Example 16

Find the differential coefficient of the following functions w.r.t.x from the first principle.

(a)  $x$     (b)  $\sqrt{x+5}$     (c)  $\frac{1}{2x+3}$

**Solution :**

(a) Let  $y=x$

Then  $y + \Delta y = x + \Delta x$

$$\Rightarrow \Delta y = x + \Delta x - y$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{x + \Delta x - x}{\Delta x} = 1 \quad (\because y = x)$$

$$\text{Now } \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} 1 = 1$$

$$\therefore \frac{dy}{dx} = 1$$

(b) Let  $y = \sqrt{x+5}$

Then  $y + \Delta y = \sqrt{(x + \Delta x) + 5}$

$$\Rightarrow \Delta y = \sqrt{(x + \Delta x) + 5} - \sqrt{x + 5} \quad (\because y = \sqrt{x + 5})$$

$$\Rightarrow x_y = \frac{\sqrt{(x + \Delta x) + 5} - \sqrt{x + 5}}{1} \times \frac{\sqrt{(x + \Delta x) + 5} + \sqrt{x + 5}}{\sqrt{(x + \Delta x) + 5} + \sqrt{x + 5}}$$

$$\Rightarrow \Delta y = \frac{x + \Delta x + 5 - x - 5}{\sqrt{(x + \Delta x) + 5} + \sqrt{x + 5}}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{\Delta x}{\sqrt{x + \Delta x + 5} + \sqrt{x + 5}} \times \frac{1}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 5} + \sqrt{x + 5}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x+5}}$$

(c) Let  $y = \frac{1}{2x+3}$

Then  $y + \Delta y = \frac{1}{2(x + \Delta x) + 3}$

$$\Rightarrow \Delta y = \frac{1}{2(x + \Delta x) + 3} - \frac{1}{2x + 3}$$

$$\Rightarrow \Delta y = \frac{2x + 3 - 2x - 2\Delta x - 3}{[2(x + \Delta x) + 3][2x + 3]}$$

$$\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-2 \cdot \Delta x}{[2(x + \Delta x) + 3][2x + 3]} \times \frac{1}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{-2}{[2(x + \Delta x) + 3][2x + 3]}$$

$$= \frac{-2}{[2x + 3][2x + 3]}$$

$$= \frac{-2}{(2x + 3)^2}$$

**Example 17.**

Find the derivative of  $e^x + 2 \log_e x + \sqrt{x}$  w.r.t.x.

**Solution :**

Let  $y = e^x + 2 \log_e x + \sqrt{x}$

Now  $\frac{dy}{dx} = \frac{d(e^x + 2 \log_e x + \sqrt{x})}{dx}$

$$= \frac{d(e^x)}{dx} + \frac{d(2 \log_e x)}{dx} + \frac{d(x)^{1/2}}{dx}$$

$$= e^x + 2 \cdot \frac{d(\log_e x)}{dx} + \frac{1}{2} \cdot x^{\frac{1}{2}-1}$$

$$= e^x + 2 \cdot \frac{1}{x} + \frac{1}{2} x^{-\frac{1}{2}}$$

$$= e^x + \frac{2}{x} + \frac{1}{2\sqrt{x}}$$

**Example 18.**

If  $y = 7x^3 \cdot e^x + 3$ , then find  $\frac{dy}{dx}$ .

**Solution :**

$$\text{Now } y = 7x^3 \cdot e^x + 3$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d(7x^3 \cdot e^x + 3)}{dx} \\ &= \frac{d(7x^3 \cdot e^x)}{dx} + \frac{d(3)}{dx} \\ &= 7 \cdot \frac{d(x^3 \cdot e^x)}{dx} \\ &= 7 \cdot \left[ e^x \frac{d(x^3)}{dx} + x^3 \frac{d(e^x)}{dx} \right] \\ &= 7 \left[ e^x \cdot 3x^2 + x^3 e^x \right] \\ &= 7x^2 e^x (3 + x) \end{aligned}$$

**Example 19.**

Find the derivative of  $\frac{x^2 + 2x - 7}{x + 2}$  w.r.t.x.

**Solution :**

$$\text{Now } y = \frac{x^2 + 2x - 7}{x + 2}$$

$$\therefore \frac{dy}{dx} = \frac{d\left(\frac{x^2 + 2x - 7}{x + 2}\right)}{dx}$$

$$\begin{aligned}
 &= \frac{(x+2) \cdot \frac{d(x^2+2x-7)}{dx} - (x^2+2x-7) \cdot \frac{d(x+2)}{dx}}{(x+2)^2} \\
 &= \frac{(x+2) \left\{ \frac{d(x^2)}{dx} + \frac{d(2x)}{dx} - \frac{d(7)}{dx} \right\} - (x^2+2x-7) \left\{ \frac{d(x)}{dx} + \frac{d(2)}{dx} \right\}}{(x+2)^2} \\
 &= \frac{(x+2)(2x+2) - (x^2+2x-7)}{(x+2)^2} \\
 &= \frac{2x^2+2x+4x+4-x^2-2x+7}{(x+2)^2} \\
 &= \frac{x^2+4x+11}{(x+2)^2}
 \end{aligned}$$

**Example 20.**

Find the differential coefficient of the following functions :

(a)  $\frac{1}{(2x+7)^5}$                       (b)  $e^{\log x+x}$

**Solution :**

(a) Let  $u=(2x+7)$  and  $y = \frac{1}{(2x+7)^5} = (2x+7)^{-5}$

Now  $y = (2x+7)^{-5} = u^{-5}$

By Chain Rule,  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\begin{aligned}
 &= \frac{d(u^{-5})}{du} \times \frac{d(2x+7)}{dx} \\
 &= -5 \cdot u^{-5-1} \cdot \left\{ \frac{d(2x)}{dx} + \frac{d(7)}{dx} \right\}
 \end{aligned}$$

$$= -5 \frac{1}{u^6} \cdot 2$$

$$= -\frac{10}{(2x+7)^6}$$

(b) Let  $u = \log x + x$  and  $y = e^{\log x + x}$

$$\text{Now } y = e^{\log x + x}$$

$$\Rightarrow y = e^u$$

By Chain Rule,  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{d(e^u)}{du} \times \frac{d(\log x + x)}{dx}$$

$$= e^u \cdot \left\{ \frac{d(\log x)}{dx} + \frac{d(x)}{dx} \right\}$$

$$= e^u \cdot \left\{ \frac{1}{x} + 1 \right\}$$

$$= e^{\log x + x} \cdot \left\{ \frac{1+x}{x} \right\}$$

### Example 21.

If  $x^2 + 4y^2 = 12$ , find  $\frac{dy}{dx}$ .

#### Solution :

(a) The function  $x^2 + 4y^2 = 12$  is an implicit function. So differentiating both sides of the function w.r.t.  $x$ , we have,

$$\frac{d(x^2 + 4y^2)}{dx} = \frac{d(12)}{dx}$$

$$\Rightarrow \frac{d(x^2)}{dx} + \frac{d(4y^2)}{dx} = 1$$

$$\Rightarrow 2x + \frac{d(4y^2)}{dy} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow 2x + 4 \cdot 2y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow 8y \cdot \frac{dy}{dx} = 1 - 2x$$

$$\therefore \frac{dy}{dx} = \frac{1 - 2x}{8y}$$

**Example 22.**

If  $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$ , then find  $\frac{dy}{dx}$ .

**Solution :**

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$$

Squaring both sides of the function, we have,

$$\Rightarrow y^2 = x + \sqrt{x + \sqrt{x + \dots \infty}}$$

$$\Rightarrow y^2 = x + y$$

Differentiating both sides with respect to  $x$ , we have,

$$\frac{d(y^2)}{dx} = \frac{d(x + y)}{dx}$$

$$\Rightarrow \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(x)}{dx} + \frac{dy}{dx}$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx}(2y-1) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

**Example 23.**

If  $y = at^2$  and  $x = 2at$ , find  $\frac{dy}{dx}$ .

**Solution :**

Given that  $y = a \cdot t^2$  and  $x = 2a \cdot t$

$$\text{So, } \frac{dy}{dt} = \frac{d(at^2)}{dt} = 2a \cdot t \text{ and}$$

$$\frac{dx}{dt} = \frac{d(2a \cdot t)}{dt} = 2a$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2at}{2a} = t.$$

**Example 24.**

If  $x = a\left(t + \frac{1}{t}\right)$  and  $y = a\left(t - \frac{1}{t}\right)$ , then show that  $\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$ .

**Solution :**

Given that  $x = a\left(t + \frac{1}{t}\right)$

$$\Rightarrow \frac{dx}{dt} = \frac{d\left\{a\left(t + \frac{1}{t}\right)\right\}}{dt}$$

$$= a \left\{ \frac{d(t)}{dt} + d \left( \frac{1}{t} \right) \right\}$$

$$= a \left( 1 - \frac{1}{t^2} \right)$$

Similarly,  $y = a \left( t - \frac{1}{t} \right)$

$$\Rightarrow \frac{dy}{dx} = \frac{da \left( t - \frac{1}{t} \right)}{dx}$$

$$= a \left\{ \frac{d(t)}{dt} - \frac{d \left( \frac{1}{t} \right)}{dt} \right\}$$

$$= a \left( 1 + \frac{1}{t^2} \right)$$

Now,  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \left( 1 + \frac{1}{t^2} \right)}{a \left( 1 - \frac{1}{t^2} \right)}$

$$= \frac{\frac{t^2 + 1}{t^2}}{\frac{t^2 - 1}{t^2}}$$

$$= \frac{t^2 + 1}{t^2 - 1}$$

**Example 25.**

If  $f(x) = x^2 + 5x - 2$ , then find  $f'(x)$  and  $f'(20)$ .

**Solution :**

Given that  $f(x) = x^2 + 5x - 2$

$$\begin{aligned}\Rightarrow f'(x) &= \frac{d(x^2 + 5x - 2)}{dx} \\ &= \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} - \frac{d(2)}{dx} \\ &= 2x + 5\end{aligned}$$

Now  $f'(2) = 2(2) + 5 = 4 + 5 = 9$

and  $f'(20) = 2(20) + 5 = 40 + 5 = 45$ .

**Example 26.**

If  $f(x) = ax^2 + bx + 8$ ,  $f'(2) = 11$  and  $f'(7) = 31$ , then find the values of  $a$  and  $b$ .

**Solution :**

Given that  $f(x) = ax^2 + bx + 8$

$$\begin{aligned}\Rightarrow f'(x) &= \frac{d(ax^2 + bx + 8)}{dx} \\ &= \frac{d(ax^2)}{dx} + \frac{d(bx)}{dx} + \frac{d(8)}{dx} \\ &= 2ax + b. \quad (\because a \text{ and } b \text{ are constants})\end{aligned}$$

As  $f'(2) = 11$

$$\Rightarrow 2a(2) + b = 11$$

$$\Rightarrow 4a + b = 11 \dots\dots\dots (i)$$

$f'(7) = 31$

$$\Rightarrow 2a(7) + b = 31$$

$$\Rightarrow 14a + b = 31 \dots\dots\dots (ii)$$

Now  $14a + b = 31$  (Solving the two equations simultaneously)

$$\frac{4a + b = 11}{10a = 20}$$

$$a = 2$$

Putting  $a = 2$  in equation (i), we have,

$$4a + b = 11$$

$$\Rightarrow 4 \times 2 + b = 11$$

$$\Rightarrow b = 11 - 8$$

$$\Rightarrow b = 3$$

$$\therefore a = 2 \text{ and } b = 3.$$

**Example 27.**

Find the second order derivatives of the following functions :

(a)  $x^5$ ,      (b)  $e^x$       (c)  $\log_e x$

**Solution :**

(a) Let  $y = x^5$

$$\Rightarrow \frac{dy}{dx} = \frac{d(x^5)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 5x^4$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d(5x^4)}{dx}$$

$$= 5 \times 4x^3$$

$$= 20x^3.$$

(b) Let  $y = e^x$

Now  $f'(x) = \frac{dy}{dx} = \frac{d(e^x)}{dx}$

$$= e^x.$$

$$\therefore f''(x) = \frac{d^2y}{dx^2} = \frac{d(e^x)}{dx}$$

$$= e^x.$$

(c) Let  $y = \log_e x$

$$\text{Now } f'(x) = \frac{dy}{dx} = \frac{d(\log_e x)}{dx} = \frac{1}{x}$$

$$\begin{aligned} \text{and } f''(x) &= \frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{x}\right)}{dx} = \frac{d(x^{-1})}{dx} \\ &= -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}. \end{aligned}$$

□□□

### 6.8. Questions

1. Choose and write the correct answer from the alternatives given in each bit.

(i)  $\frac{dy}{dx}$  is equal to :

(a)  $\lim_{\Delta x \rightarrow 0} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$

(b)  $\lim_{\Delta x \rightarrow 1} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$

(c)  $\lim_{\Delta x \rightarrow \infty} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$

(d)  $\lim_{x \rightarrow 0} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$

(ii)  $f'(x)$  is equal to

(a)  $\frac{1}{\frac{dy}{dx}}$

(b)  $\frac{d[f(x)]}{dx}$

(c)  $\frac{1}{f(x)}$

(d)  $\sqrt{f(x)}$

(iii) If  $f(x)$  is equal to  $\left(2x + \frac{1}{2}\right)^2$ , then  $f'(2)$  is :

(a) 16

(b) 32

(c) 18

(d) 36

(iv) Under Chain Rule  $\frac{dy}{dx}$  is equal to

(a)  $\frac{dy}{du} \times \frac{dx}{du}$

(b)  $\frac{dy}{dx} \times \frac{dx}{du}$

(c)  $\frac{dy}{du} \times \frac{du}{dx}$

(d)  $\frac{dx}{dy} \times \frac{dy}{dx}$

(v) Derivative of  $a^x$  is equal to :

(a)  $a^x \log_e a$

(b)  $a^x \log_a e$

(c)  $a^x \cdot \log_a e$

(d)  $a^x \cdot \log_e a$

(vi) Derivative of  $e^x$  is equal to :

- a)  $a^x \cdot \log_e a$       (b)  $e^x \log_e e$       (c)  $\frac{1}{x \cdot \log_e a}$       (d)  $e^x \cdot \log_e a$

(vii) If  $f(x) = \left(x + \frac{1}{x}\right)^2$ , then  $f'$  is equal to :

- (a) 0      (b) 1      (c) 2      (d) -1

(viii) It is an implicit function if :

- (a)  $X^2 + 2x + 7 = y$       (b)  $X^3 + y^3 + 7xy = 12$   
 (c)  $y = 2x + 7$       (d)  $y = X^2 + 2x$

(ix) The differential coefficient of a constant  $\bar{K}$  is :

- (a) K      (b)  $\frac{1}{K}$       (c)  $K^2$       (d) 0

(x) The second order derivative of  $x^2$  w.r.t.  $x$  is :

- (a)  $2x$       (b)  $x$       (c)  $\frac{1}{x}$       (d) 2

2. Express / Answer each of the following in one word / term

(i) The alternative name of differential coefficient.

(ii) The rationalised value of the numerator of  $\frac{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2}}$

(iii) The Rule by which  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

(iv) The value of  $\frac{dy}{dx}$  in  $x^2 + y^2 = 16$

(v) The approximate value of  $e$  in natural logarithm.

**3. Answer each of the following in one sentence :**

- (i) Differentiate  $(x-2)(x+3)$  w.r.t.x.
- (ii) Differentiate  $x^3 + \log x + 5$
- (iii) Find  $f'(2)$  if  $f(x)$  is equal to  $x^3 + 2x^2 + 3x + 7$ .
- (iv) Find the differential coefficient of  $e^{3x}$
- (v) Differentiate  $x^4$  w.r.t. $x^2$ .

**4. Rectify the underlined portions of the following sentences :**

- (i) The differential coefficient of the sum of two functions is equal to the difference of their differential co-efficients.

(ii) It  $y = \frac{1}{x}$ , then  $\frac{dy}{dx} \cdot \frac{dx}{dy} = \underline{0}$

- (iii)  $x^2 - 5xy + xy^2 - 10 = 0$  is an explicit function.

(iv)  $\frac{d(\log_a x)}{dx} = \frac{1}{x \cdot \underline{\log e}}$

(v)  $n \cdot \underline{x}^{n+1} = x^n$

**5. Fill in the blanks :**

(i)  $\frac{d(x^5)}{dx} = 5 \underline{\hspace{2cm}}$

(ii)  $\frac{d\left(\frac{1}{x^5}\right)}{dx} = -5 \cdot \frac{1}{\underline{\hspace{2cm}}}$

- (iii) In derivative of a function of a function,

$$\frac{dy}{dx} = \frac{dy}{\underline{\hspace{2cm}}} \times \frac{du}{dx}$$

- (iv) The second derivative of  $x^4$  is ----- .

(v) If  $y = \sqrt{1+x^4}$ , then  $y \cdot \frac{dy}{dx}$  is equal to -----.

**6. Answer each of the following questions in 30 words :**

(i) Find the derivative of  $x^4 + 3x^3 + \log x + e^x - \frac{1}{x} + 7$

(ii) Find the differential coefficient of  $x^2$  w.r.t.x from first principle.

(iii) State the formula for finding the derivative of quotient of two functions.

(iv) If  $p = 4q^2 - 2q$ , find  $\frac{dp}{dq}$ .

(v) Find  $\frac{dy}{dx}$  for the function  $x^3 + y^3 + 3xy = 7$ .

(vi) Find the derivative of  $5^x$

(vii) Find the derivative of  $e^{3x}$

(viii) If  $x = 3t^2$  and  $y = 4t$  find  $\frac{dy}{dx}$

$$\left[ \text{Hints: } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right]$$

(ix) Find the differential coefficient of  $\log(\log x)$  w.r.t.x.

(x) If  $y = 2x^3 + 3x^2 + 4x + 5$ , find  $\frac{dy}{dx}$  at  $x = 2$

**7. Answer each of the following questions within 50 words.**

(i) Find  $\frac{dy}{dx}$  if,  $y = (x^2 - 5)(4x^3 - 3x^2 - 7)$ .

(ii) Find  $\frac{dy}{dx}$  if  $y = \frac{x^2 - 1}{x^2 + 1}$ .

- (iii) Find  $\frac{dy}{dx}$  if  $y = \sqrt{x} \cdot \log x^2$ .
- (iv) Differentiate  $\frac{x^2 + e^x}{\log x + 10}$  w.r.t.  $x$ .
- (v) Explain derivative of a function of a function with an example.
- (vi) If  $y = at^4$  and  $x = 2at^2$ , find  $\frac{dy}{dx}$ .
- (vii) Find the second order derivative of  $x^2 + 5x + 1$ .
- (viii) If  $x^2 - 3xy - 4y^2 = 0$ , find  $\frac{dy}{dx}$ .
- (ix) Differentiate  $(3x + 8)^7$  w.r.t.  $x$ .
- (x) If  $y = \left(\frac{2x}{x^2 + 1}\right)^3$ , find  $\frac{dy}{dx}$ .

**8. Answer the following long questions :**

(A) Differentiate the following functions with respect to  $x$  :

(i)  $5x^4 + 4x^3 + 3x^2 + 2x + 1$

(ii)  $x^2 \cdot e^{5x}$

(iii)  $x^x$

(iv)  $5^x + \log_5 x$

(v)  $\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

(vi)  $\frac{2x^2 + x + 1}{\sqrt{x}}$

---

(vii)  $\left(3x^2 + \frac{2}{x} + 5\right)^6$  (Hints : Apply chain rule)

(viii)  $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

(ix)  $\frac{x^3 - 2x^2 + 3}{x^4 + x^2 - 5}$

(x)  $x^2 + y^2 = 16$

(xi)  $x^2y - xy^2 = 4$

(xii)  $x + y = \frac{xy}{200}$

(B) (i) Differentiate  $x^2$  with respect to  $x$  from first principle.

(ii) Differentiate  $e^{2x}$  from first principle.

(iii) If  $f(x) = ax^2 + bx + 10$ ,  $f'(2) = 14$  and  $f'(3) = 20$ , then find the values of  $a$  and  $b$ .

(iv) If  $y = x^3e^x + 18$ , then find  $\frac{dy}{dx}$

(v) If  $y = at^2$  and  $x = 2bt$ , then show that  $\frac{dy}{dx} = \frac{at}{b}$ .

(vi) If  $y = e^{5\log x + 2x}$ , then find  $\frac{dy}{dx}$

(vii) If  $x^2 + y^2 = 9$ , then find  $\frac{dy}{dx}$

(viii) If  $y = \sqrt{\frac{(x-5)(x^2+3)}{(2x^2+4x+1)}}$ , then find  $\frac{dy}{dx}$ , (Hints : take log on both sides)

(ix) If  $x = 2at^2$ ,  $y = at^3$ , then find  $\frac{dy}{dx}$ .

(x) If  $y = \frac{e^x - 1}{e^x + 1}$ , then show that  $\frac{dy}{dx} = \frac{2e^x}{(e^x + 1)^2}$

### Answers

1. (i) a (ii) b (iii) c (iv) c (v) a (vi) b (vii) a (viii) b (ix) d (x) d

2. (i) Derivative (ii)  $2x^2$  (iii) Chain rule (iv)  $-\frac{x}{y}$  (v) 2.7183

3. (i)  $2x+1$  (ii)  $3x^2 + \frac{1}{2}$  (iii) 20 (iv)  $3e^{3x}$  (v)  $2x^2$

4. (i) Sum (ii) 1 (iii) Implicit (iv)  $\log a$  (v)  $x^{n-1}$

5 (i)  $x^4$  (ii)  $x^6$  (iii)  $du$  (iv)  $12x^2$  (v)  $2x^3$

□□□

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## **CHAPTER - 7**

# ***INTEGRATION***

### **STRUCTURE**

- 7.1 Introduction
- 7.2 Definition and meaning of integration
- 7.3 Properties of indefinite integrals.
- 7.4 Some standard results of indefinite integrals.
- 7.5 Integration by substitution
- 7.6 Comparison between differentiation and integration.
- 7.7 Questions.

## 7.1 Introduction :

In the 17th century major advance in integration came with the independent discovery of the fundamental theorem of calculus by Issac Newton and Gottfried Leibniz. In differentiation we find the rate of change between two or more variables from a given functional relationship existing between them. But in integration we find the functional relationship between two or more variables from a given rate of change between them. We observe that integration is a reverse process of differentiation. So the result of integration is termed as antiderivative. To better understand the technique of integration, let us define the term integration.

## 7.2 Definition and meaning of Integration

If  $f(x)$  is any differentiable function of  $x$  such that,  $\frac{d[f(x)]}{dx} = f'(x)$  then  $f(x)$  is called the antiderivative of  $f'(x)$ . That is, if we integrate  $f'(x)$  we get  $f(x)$  and if we differentiate  $f(x)$ , we get  $f'(x)$  Symbolically, we write,

$$\int f(x)dx = f'(x) + C$$

where  $C$  is called the constant of integration,  $f'(x)+C$  represents a set of functions called antiderivatives. The anti derivative is also called integral or primitive of  $f(x)$ .

Let us take an example :  $\frac{d\left(\frac{x^2}{2}\right)}{dx} = x$

In this example  $x$  is the derivative of  $\frac{x^2}{2}$ .  $\frac{x^2}{2}$  is the antiderivative of  $x$ . Hence the the integral of

$$x \text{ is } \frac{x^2}{2}. \text{ We write, } \int (x) dx = \frac{x^2}{2}$$

But the integral or antiderivative of  $x$  may be many other functions other than  $\frac{x^2}{2}$ . The possible antiderivatives may be many because,

$$\frac{d\left(\frac{x^2}{2} + 5\right)}{dx} = x$$

Also, 
$$\frac{d\left(\frac{x^2}{2} + 9\right)}{dx} = x \dots \dots \dots \text{so on.}$$

That is, 
$$\frac{d\left(\frac{x^2}{2} + C\right)}{dx} = x, \text{ where } C \text{ is a constant}$$

As there are indefinite number of functions which differ by the constant, but give the same derivative, the integration may provide an indefinite number of antiderivatives. That is why such type of integrals are called indefinite integrals. We define integration by,

$$\int f(x) dx = f'(x) + C$$

where the notations are as under :

$\int$  = Integral sign

$x$  = Variable of integration

$f(x)$  = Integrand

$f'(x)$  → Integral of  $f(x)$  w.r.t.  $x$

$C$  → Constant of integration

### 7.3 Properties of Indefinite Integrals

(i) The process of differentiation and integration are opposite to each other. That is,

$$\frac{d[f(x)]}{dx} = f'(x) \text{ and } \int f'(x)dx = f(x) + c$$

(ii) Two indefinite integrals with the same derivative are equivalent, That is,  
 $\int f(x)dx = \int g(x)dx$ , even though they differ by parameters of constants.

(iii)  $\int [f(x) + g(x)] = \int f(x)dx + \int g(x)dx$

(iv) For any real number k,

$$\int kf(x)dx = k \int f(x)dx$$

#### 7.4 Some standard results of indefinite integrals

We have listed below derivatives of some important functions and the corresponding integrals:

##### Derivatives

##### Integrals or Antiderivatives

1.  $\frac{d\left(\frac{x^{n+1}}{n+1}\right)}{dx} = x^n$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

2.  $\frac{d(\log x)}{dx} = \frac{1}{x}$

$$\int \left(\frac{1}{x}\right) dx = \log x + c$$

3.  $\frac{d(e^x)}{dx} = e^x$

$$\int e^x \cdot dx = e^x + c$$

4.  $\frac{d(x)}{dx} = 1$

$$\int 1 dx = x + c$$

5.  $\frac{d\left(\frac{a^x}{\log_e a}\right)}{dx} = a^x$

$$\int a^x dx = \frac{a^x}{\log_e a} + c$$

$$6. \quad \frac{d\left(\frac{e^{ax}}{a}\right)}{dx} = e^{ax}$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + c$$

$$7. \quad \frac{d\left(\frac{a^{mx}}{m \log_e a}\right)}{dx} = a^{mx}$$

$$\int a^{mx} dx = \frac{a^{mx}}{m \log_e a} + c$$

On the basis of the concept of integration and above -mentioned standard results of important indefinite integrals let us work out some practical problems.

**Example 1.** Write the antiderivative for each of the following functions :

(i)  $x^4$

(ii)  $x^{r-1}$

**Solution :**

(i) We have to find out a function whose derivative is  $x^4$

$$\text{Now, } \frac{d(x^5)}{dx} = 5x^{5-1}$$

$$\Rightarrow \frac{d(x^5)}{dx} = 5x^4$$

$$\Rightarrow x^4 = \frac{1}{5} \frac{d(x^5)}{dx}$$

$$\therefore \text{An antiderivative of } x^4 = \frac{1}{5} x^5$$

$$\text{(ii) Similarly, } \frac{d(x^r)}{dx} = r \cdot x^{r-1}$$

$$\Rightarrow x^{r-1} = \frac{1}{r} \frac{d(x^r)}{dx}$$

$$\Rightarrow x^{r-1} = \frac{d\left(\frac{1}{r} \cdot x^r\right)}{dx}$$

$\therefore$  An antiderivative of  $x^{r-1} = \frac{1}{r} \cdot x^r$

### Example 2.

Find the integrals of the following functions :

(i)  $5x^4$                       (ii)  $n \cdot x^{n-1}$

**Solution :**

(i)  $\int 5x^4 dx = 5 \int x^4 dx$

$$= 5x \frac{x^{4+1}}{4+1} + c \quad (\because \int x^n dx = \frac{x^{n+1}}{n+1} + c)$$

$$= 5x \frac{x^5}{5} + c$$

$$= x^5 + c$$

(ii)  $\int nx^{n-1} dx = n \int x^{n-1} dx$

$$= n \frac{x^{n-1+1}}{n-1+1} + c$$

$$= n \frac{x^n}{n} + c$$

$$= x^n + c$$

**Example 3.** Integrate the following functions w.r.t.x

(i)  $\int dx$

(ii)  $\int \frac{5}{x^2} dx$

(iii)  $\int 3\sqrt{x} dx$

(iv)  $\int \frac{e^x}{5} dx$

**Solution :**

$$(i) \int dx = \int 1 \cdot dx = \int x^0 \cdot dx = \frac{x^{0+1}}{0+1} + c = x + c$$

$$(ii) \int \frac{5}{x^2} dx = 5 \int x^{-2} dx = 5 \frac{x^{-2+1}}{-2+1} + c = 5 \frac{x^{-1}}{-1} + c = -5 \frac{1}{x} + c$$

$$(iii) \int 3\sqrt{x} dx = 3 \int x^{\frac{1}{2}} dx = 3 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = 3 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = 3 \times \frac{2}{3} \times x^{\frac{3}{2}} + c$$

$$= 2x^{\frac{3}{2}} + c$$

$$(iv) \int \frac{e^x}{5} dx = \frac{1}{5} \int e^x \cdot dx = \frac{1}{5} e^x + c$$

**Example 4.** Find the integrals of the following functions :

$$(i) x^2 + 5x + \frac{1}{x} - 5$$

$$(ii) \left(x - \frac{1}{x}\right)^2$$

**Solution :**

$$(i) \int \left(x^2 + 5x + \frac{1}{x} - 5\right) dx$$

$$= \int x^2 \cdot dx + \int 5x \cdot dx + \int \frac{1}{x} \cdot dx - \int 5 \cdot dx$$

$$= \frac{x^{2+1}}{2+1} + 5 \frac{x^{1+1}}{1+1} + \log x - 5 \frac{x}{0+1} + c$$

$$= \frac{x^3}{3} + \frac{5x^2}{2} + \log x - 5x + c$$

$$\begin{aligned}
 \text{(ii)} \quad \int \left(x - \frac{1}{x}\right)^2 dx &= \int \left(x^2 + \frac{1}{x^2} - 2 \cdot x \cdot \frac{1}{x}\right) dx \\
 &= \int x^2 dx + \int \frac{1}{x^2} dx - \int 2 dx \\
 &= \frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} - 2 \cdot \frac{x^{0+1}}{0+1} + c \\
 &= \frac{x^3}{3} - \frac{1}{x} - 2x + c
 \end{aligned}$$

**Example 5.** Evaluate the following integrals :

$$\text{(i)} \int (3e^{3x} + 1) dx \qquad \text{(ii)} \int x^3 \left(1 - \frac{1}{x^3}\right) dx$$

**Solution :** (i) Let us find a function whose derivative is  $3e^{3x} + 1$ .

$$\text{Now, } \frac{d(e^{3x})}{dx} = 3 \cdot e^{3x} \text{ and } \frac{d(x)}{dx} = 1$$

$$\therefore \frac{d(e^{3x} + x)}{dx} = 3e^{3x} + 1$$

As antiderivative of  $3e^{3x} + 1$  is  $e^{3x} + x$ ,

$$\int (3e^{3x} + 1) dx = e^{3x} + x + C$$

$$\begin{aligned}
 \text{(ii)} \quad \int x^3 \left(1 - \frac{1}{x^3}\right) dx &= \int (x^3 - 1) dx \\
 &= \int x^3 dx - \int 1 dx \\
 &= \frac{x^{3+1}}{3+1} - \frac{x^{0+1}}{0+1} + c
 \end{aligned}$$

$$= \frac{x^4}{4} - x + c$$

**Example 6 . Evaluate :**

$$(i) \int 5^x dx \quad (ii) \int e^{7x} dx \quad (iii) \int 5^{2x} dx \quad (iv) \int 3e^{2t} dt$$

**Solution :**

$$(i) \int 5^x dx = \frac{5^x}{\log 5} + c \quad (\because \int a^x dx = \frac{a^x}{\log_e a} + c)$$

$$(ii) \int e^{7x} dx = \frac{e^{7x}}{7} + c \quad (\because \int e^{ax} dx = \frac{e^{ax}}{a} + c)$$

$$(iii) \int 5^{2x} dx = \frac{5^{2x}}{2 \log 5} + c \quad (\because \int a^{mx} dx = \frac{a^{mx}}{m \log a} + c)$$

$$(iv) \int 3e^{2t} dt = 3 \int e^{2t} dt = 3 \left\{ \frac{e^{2t}}{2} + c \right\}$$

**Example 7 Evaluate :**

$$(i) \int (e^{3x} + 3^{7x} + x^{-3/2} - e^{x/2}) dx$$

$$(ii) \int \frac{ax^4 + bx^3 + cx^2 + dx + e}{x^2} dx, \text{ where } a, b, c, d, e \text{ are constants}$$

**Solution :**

$$(i) \int (e^{3x} + 3^{7x} + x^{-3/2} - e^{x/2}) dx$$

$$= \int e^{3x} dx + \int 3^{7x} dx + \int x^{-3/2} dx - \int e^{x/2} dx$$

$$= \frac{e^{3x}}{3} + \frac{3^7 x}{7 \log 3} + \frac{x^{-3/2+1}}{-3/2+1} - \frac{e^{x/2}}{1/2} + c$$

$$= \frac{e^{3x}}{3} + \frac{3^7 x}{7 \log 3} + \frac{x^{-1/2}}{-1/2} - 2e^{x/2} + c$$

$$= \frac{e^{3x}}{3} + \frac{3^7 x}{7 \log 3} - 2 \frac{1}{\sqrt{x}} - 2e^{x/2} + c$$

$$(ii) \int \frac{ax^4 + bx^3 + cx^2 + dx + e}{x^2} dx,$$

$$= \int \frac{ax^4}{x^2} dx + \int \frac{bx^3}{x^2} dx + \int \frac{cx^2}{x^2} dx + \int \frac{dx}{x^2} dx + \int \frac{e}{x^2} dx$$

$$= \int ax^2 dx + \int b.x dx + \int c.dx + \int d \frac{1}{x} dx + e \int x^{-2} dx$$

$$= a \frac{x^{2+1}}{2+1} + b \frac{x^{1+1}}{1+1} + c \cdot \frac{x^{0+1}}{0+1} + d \cdot \log x + e \frac{x^{-2+1}}{-2+1} + c$$

$$= a \frac{x^3}{3} + b \frac{x^2}{2} + c.x + d \cdot \log x + e \frac{x^{-1}}{-1} + c$$

$$= \frac{a}{3} x^3 + \frac{b}{2} x^2 + c.x + d \log x - e \frac{1}{x} + c$$

**Solution :**

$$\begin{aligned}
 \text{L.H.S} &= \int e^{x \log a} \cdot e^x dx \\
 &= \int e^{x \log a + x} dx \\
 &= \int e^{x(\log a + 1)} dx \\
 &= \int e^{x(\log_e a + \log_e e)} \cdot dx \\
 &= \int e^{x \cdot \log_e ae} \cdot dx \\
 &= \frac{e^{x \log ae}}{\log ae} + c \\
 &= \frac{\log_e (ae)^x}{\log_e ae} + c
 \end{aligned}$$

**Example 9.** If  $\frac{d[f(x)]}{dx} = 3x^2 - 2x$  and  $f(2) = 0$ , then find  $f(x)$ .

**Solution :**

$$\begin{aligned}
 \text{Given, } \frac{d[f(x)]}{dx} &= 3x^2 - 2x \\
 \Rightarrow f(x) &= \int (3x^2 - 2x) dx \\
 \Rightarrow f(x) &= 3 \frac{x^{2+1}}{2+1} - 2 \frac{x^{1+1}}{1+1} + c \\
 \Rightarrow f(x) &= x^3 - x^2 + c
 \end{aligned}$$

Putting  $x = 2$ , we have

$$f(2) = 2^3 - 2^2 + c$$

$$\Rightarrow 0 = 8 - 4 + c \quad (\text{Given } f(2) = 0)$$

$$\Rightarrow c = -4$$

Putting the value of  $c$  in  $f(x)$ , we have,

$$f(x) = x^3 - x^2 - 4.$$

**Example 10.** Evaluate  $\int \frac{1}{\sqrt{x+a} - \sqrt{x-b}} dx$

**Solution :**

$$\begin{aligned} \frac{1}{\sqrt{x+a} - \sqrt{x-b}} &= \frac{\sqrt{x+a} + \sqrt{x-b}}{(\sqrt{x+a} - \sqrt{x-b})(\sqrt{x+a} + \sqrt{x-b})} \\ &= \frac{\sqrt{x+a} + \sqrt{x-b}}{(x+a) - (x-b)} \\ &= \frac{\sqrt{x+a} + \sqrt{x-b}}{a+b} \\ &= \frac{1}{a+b} \left\{ (x+a)^{\frac{1}{2}} + (x-b)^{\frac{1}{2}} \right\} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{1}{\sqrt{x+a} - \sqrt{x-b}} dx &= \int \frac{1}{a+b} \left\{ (x+a)^{\frac{1}{2}} + (x-b)^{\frac{1}{2}} \right\} dx \\ &= \frac{1}{a+b} \left\{ \int (x+a)^{\frac{1}{2}} dx + \int (x-b)^{\frac{1}{2}} dx \right\} \\ &= \frac{1}{a+b} \left\{ \frac{2}{3} (x+a)^{\frac{3}{2}} + \frac{2}{3} (x-b)^{\frac{3}{2}} \right\} + C \\ &\quad \left\{ \because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \right\} \end{aligned}$$

## 7.5 Integration by Substitution

There are three important methods for finding integrals by reducing the functions into standard forms. They are :

1. Integration by substitution
2. Integration by parts
3. Integration by partial fractions.

We shall study the method of integration by substitution in this chapter. This method has no generalised rule to select any substitution. Under this method the integral  $\int f(x)dx$  is transformed into another form by changing the independent variable  $x$  to  $t$  by substituting  $x=g(t)$ . By this transformation, the integral can be easily evaluated.

Let us take an integral  $I$  which is equal to  $\int f(x)dx$ .

$$\therefore I = \int f(x)dx$$

Putting,  $x = g(t)$  ..... (i)

$$\text{or, } \frac{dx}{dt} = g'(t)$$

$$\text{or, } dx = g'(t)dt \text{ ..... (ii)}$$

$$\text{Now, } \therefore I = \int f(x)dx = \int f\{g(t)\}g'(t)dt$$

Under this substitution, the variable 'x' has changed to 't' which will enable in to evaluate the integral easily. Let us take examples and standard forms to understand the method clearly :

**Some important forms of Integrals :**

(i)  $\int f(x)dx = \log f(x) + c$ , where  $f'(x)$  is the derivative of  $f(x)$ .

(ii)  $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{(n + 1)a} + c$ , where  $a, b$  are constants and  $n \neq -1$ .

(iii)  $\int \frac{1}{x + 1}.dx = \log|x + 1| + c$

(iv)  $\int \frac{1}{ax + b}.dx = \frac{1}{a} \log|ax + b| + c$

**Proof :**

$$(i) \quad \int \frac{f'(x)}{f(x)} dx = \log\{f(x)\} + C$$

$$\text{Let } t = f(x)$$

$$\Rightarrow \frac{dt}{dx} = f'(x)$$

$$\Rightarrow dt = f'(x) \cdot dx$$

Substituting the above in  $\int \frac{f'(x)}{f(x)} dx$ , we have

$$\begin{aligned} \int \frac{f'(x)}{f(x)} dx &= \int \frac{1}{t} \cdot dt \quad (\text{As } f'(x)dx = dt \text{ and } f(x) = t) \\ &= \log t + c \\ &= \log \{f(x)\} + c \end{aligned}$$

Using this form of (i), other forms of integrals may be derived.

**Example 11.** Evaluate  $\int (2x + 5)^7 dx$

**Solution :**

$$\text{Let } 2x + 5 = t$$

$$\Rightarrow \frac{d(2x + 5)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(2x)}{dx} + \frac{d(5)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 2 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{2} \cdot dt$$

$$\begin{aligned}
 \text{Now, } \int (2x+5)^7 dx &= \int t^7 \cdot \frac{1}{2} dt \\
 &= \frac{1}{2} \cdot \frac{t^{7+1}}{7+1} + C \\
 &= \frac{1}{2} \cdot \frac{(2x+5)^{7+1}}{7+1} + C \\
 &= \frac{(2x+5)^8}{8 \times 2} + C \\
 &= \frac{(2x+5)^8}{16} + C.
 \end{aligned}$$

**Example : 12** Evaluate  $\int \frac{dx}{(5-3x)^4}$

**Solution :** Let  $I = \int \frac{dx}{(5-3x)^4}$  and  $5-3x = t$ .

$$5 - 3x = t.$$

$$\Rightarrow \frac{d(5-3x)}{dx} = \frac{dt}{dx} \quad (\text{Differentiating both sides})$$

$$\Rightarrow \frac{d(5)}{dx} - \frac{d(3x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -\frac{d(3x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -3.1 = \frac{dt}{dx}$$

$$\Rightarrow dx = -\frac{1}{3} dt.$$

Now,

$$I = \int \frac{dx}{(5-3x)^4}$$

$$= \int \frac{-\frac{1}{3} dt}{t^4}$$

$$= \int -\frac{1}{3} t^{-4} \cdot dt$$

$$= -\frac{1}{3} \int t^{-4} \cdot dt$$

$$= -\frac{1}{3} \frac{t^{-4+1}}{(-4+1)} + C$$

$$= -\frac{1}{3} \frac{t^{-3}}{(-3)} + C$$

$$= \frac{1}{9} \frac{1}{t^3} + C$$

$$= \frac{1}{9(5-3x)^3} + C.$$

**Example 13.** Integrate  $(x^3+9).3x^2$  with respect to  $x$ .

**Solution :**

Let  $(x^3+9) = t$

$$\Rightarrow \frac{d(x^3+9)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(x^3)}{dx} + \frac{d(9)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 3x^2 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{3x^2} \cdot dt$$

Now,

$$\int (x^3 + 9)3x^2 \cdot dx$$

$$= \int t \cdot 3x^2 \cdot \frac{1}{3x^2} \cdot dt \quad (\text{Substituting the values of } dx \text{ and } x^3 + 9)$$

$$= \int t \cdot dt$$

$$= \int \frac{t^{1+1}}{1+1} + C$$

$$= \frac{(x^3 + 9)^2}{2} + C.$$

**Example 14.** Find the integrals of the following.

(i)  $5^{2x}$       (ii)  $e^{5x}$

**Solution :** (i) Let  $2x = t$

$$\Rightarrow 2 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{2} \cdot dt$$

Now, 
$$\int 5^{2x} \cdot dx = \int 5^t \cdot \frac{1}{2} \cdot dt$$

$$= \frac{1}{2} \int 5^t \cdot dt$$

$$= \frac{1}{2} \frac{5^t}{\log_e 5} + c$$

$$= \frac{1}{2} \frac{5^{2x}}{\log_e 5} + c$$

(ii) Let  $5x = t$

$$\Rightarrow \frac{d(5x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 5 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{5} dt$$

Now,  $\int e^{5x} dx = \int e^t \cdot \frac{1}{5} dt$

$$= \frac{1}{5} \int e^t \cdot dt$$

$$= \frac{1}{5} e^t$$

$$= \frac{1}{5} e^{5x}$$

**Example 15.** Evaluate:  $\int \frac{(5 + \log x)^3}{x} dx$

**Solution :** Let  $5 + \log x = t$

$$\Rightarrow \frac{d(5 + \log x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(5)}{dx} + \frac{d(\log x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} = \frac{dt}{dx}$$

$$\Rightarrow dx = x \cdot dt$$

Now  $\int \frac{(5 + \log x)^3}{x} dx$

$$= \int \frac{(t)^3}{x} \cdot x dt$$

$$= \int t^3 \cdot dt$$

$$= \frac{t^{3+1}}{3+1} + c$$

$$= \frac{t^4}{4} + c$$

$$= \frac{(5 + \log x)^4}{4} + c$$

**Example 16.** Evaluate  $\int \frac{dx}{x + \sqrt{x}}$

**Solution :**

$$\int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x} + 1)}$$

Let  $\sqrt{x} + 1 = t$

$$\Rightarrow \frac{d(\sqrt{x}+1)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(\sqrt{x})}{dx} + \frac{d(1)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(x^{\frac{1}{2}})}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{2} x^{\frac{1}{2}-1} = \frac{dt}{dx}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}}}{2} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx}$$

$$\Rightarrow dx = 2\sqrt{x} dt$$

$$\text{Now } \int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x} + 1)}$$

$$= \int \frac{2\sqrt{x} \cdot dt}{\sqrt{x} \cdot t}$$

$$= \int \frac{2}{t} dt$$

$$= 2 \int \frac{1}{t} dt$$

$$= 2 \log t + c$$

$$= 2 \log(\sqrt{x} + 1) + c$$

**Example 17.** Evaluate  $\int (x^3 + 2x^2 + 4x - 5)^2 (3x^2 + 4x + 4) dx$

**Solution :** Let  $x^3 + 2x^2 + 4x - 5 = t$

$$\Rightarrow \frac{d(x^3 + 2x^2 + 4x - 5)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 3x^2 + 4x + 4 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{3x^2 + 4x + 4} dt$$

Now,  $\int (x^3 + 2x^2 + 4x - 5)^2 (3x^2 + 4x + 4) dx$

$$= \int t^2 (3x^2 + 4x + 4) \frac{1}{3x^2 + 4x + 4} dt$$

$$= \int t^2 dt$$

$$= \frac{t^{2+1}}{2+1} + c$$

$$= \frac{1}{3} (x^3 + 2x^2 + 4x - 5)^3 + c$$



**7.7 QUESTIONS**

**1. From the given alternatives in each bit, choose and write the correct answer against each bit :**

- (i) In  $\int f(x)dx$ ,  $f(x)$  is
- (a) Integral
  - (b) Integrand
  - (c) Integration
  - (d) Constant of integration
- (ii) Integration is the opposite process of
- (a) Logarithm
  - (b) Determinant
  - (c) Differentiation
  - (d) Matrix multiplication
- (iii) Constant of integration appears in
- (a) Indefinite Integrals
  - (b) Definite Integrals
  - (c) Derivatives
  - (d) All types of integrals.
- (iv)  $\int \log x \, dx = ?$
- (a)  $\frac{1}{x} + c$
  - (b)  $x(\log x + 1)$
  - (c)  $(\log x)^2 + c$
  - (d)  $\frac{1}{2}(\log x)^2 + c$
-

(v)  $\int \log_{10} x \, dx = ?$

(a)  $\frac{1}{x} + c$

(b)  $\frac{1}{x} \log_e x + c$

(c)  $x (\log x - 1) \log_{10} e + c$

(d)  $\frac{1}{x} \log_e 10 + c$

(vi) The integral of  $\int 2x(x^2 + 1) dx$  is

(a)  $(x^2 + 1)^2 + c$

(b)  $\frac{(x^2 + 1)^2}{2} + c$

(c)  $\frac{x^4}{4} + x^2 + c$

(d)  $x^4 + 2x^2 + c$

(vii)  $\int 6x^2(1 + x^3) dx = ?$

(a)  $(1 + x^3)^2 + C$

(b)  $\frac{(1 + x^3)^2}{2} + C$

(c)  $2x^3 + 6x^6 + C$

(d)  $12x^2 + 30x^4 + C$

(viii)  $\int (2x + 5)^6 dx = ?$

(a)  $\frac{1}{14} (2x + 5)^7 + C$

(b)  $(2x + 5)^7 + C$

(c)  $x^2 + 5x + C$

(d)  $\frac{1}{28} (2x + 5)^7 + C$

(ix)  $\int 5e^x \cdot dx = ?$

(a)  $5 e^{2x} + C$

(b)  $e^x + C$

(c)  $\frac{e^x}{2} + C$

(d)  $5 e^x + C$

(x)  $\int (3x^2 + 4x^3 + 5x^4 + 6) dx = ?$

(a)  $6x^3 + 12x^4 + 20 x^5 + C$

(b)  $x^5 + x^4 + x^3 + C$

(c)  $\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + C$

(d) Zero.

2. **Express / answer the following in one word/term :**

(i) A technique which is just opposite of the technique of differentiation.

(ii) A value by which two functions with same derivative differ.

(iii) The process by which an integral is found.

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(iv) Evaluate  $\int dx$ .

(v) What is  $f(x)$  in  $\int f(x) \cdot dx$  ?

**3. Answer each of the following in one sentence :**

(i) Show the derivative and antiderivative of  $x^{n+1}$  with respect to  $x$ .

(ii) Show that :  $\int (2x^2 + e^x) dx = \frac{2}{3} x^3 + e^x + C_1 + C_2$

(iii) Find the value of constant of integration if  $\frac{df(x)}{dx} = 2x$ .

(iv) Integrate  $(x+1)^3$  with respect to  $x$ .

(v) What do you understand by integration by substitution ?

**4. Rectify the underlined portion of the following :**

(i)  $\int k \cdot d(x) = \underline{k} + C$  ( where  $k$  is an integer)

(ii)  $\int e^{ax} dx = \underline{e^{ax}} + C$

(iii)  $\int a^x dx = \underline{a^x} + C$

(iv) The integral of the function  $e^{2x+3}$  is  $(\underline{e^{2x+3}} + C)$ .

(v)  $\int 4x(x^2 + 1) dx = 4(\underline{x^2 + 1})^2 + C$

**5. Fill in the blanks :**

(i)  $\int \frac{1}{x} dx = \dots\dots\dots + C$

(ii) If  $\frac{d[f(x)]}{dx} = f'(x)$ , then  $\int f'(x) dx = \dots\dots\dots + C$ .

(iii)  $\int \{f(x) + g(x)\} dx = \int f(x) dx + \dots\dots\dots$

(iv)  $\int K \cdot f(x) dx = Kx \dots\dots\dots$

(v) The process of differentiation and integration are ..... to each other.

**6. Answer each of the following questions within 30 words :**

(i) Evaluate :  $\int x^5 dx$

(ii) Evaluate :  $\int \left( e^x + \frac{1}{e^x} \right)^2 dx$

(iii) Evaluate :  $\int (x^2 + 3)^2 dx$

(iv) Evaluate :  $\int \frac{x^2}{x-1} dx$  (Hints :  $x^2 = x^2 - 1 + 1$ )

(v) Show that :  $\int \frac{dx}{x \cdot \log x} = \log |\log x| + C$  (Hints : Take  $\log x = t$ )

(vi) Find the value of  $\int 2x(1+x^2)^5 dx$

(vii) If  $\int f'(x) = 4x^3 + 3x^2 + 2x + 7$ , find  $f(x)$ .

(viii) Evaluate :  $\int 3^x dx$

(ix) Evaluate :  $\int 10^{2x} dx$

(x) Find the primitive of  $3 \cdot \sqrt{x}$

**7. Answer each of the following question within 50 words :**

(i) Evaluate :  $\int \left( x + \frac{1}{x} \right)^2 dx$

(ii) Evaluate :  $\int (x^3 + 2x^2 - 7x + 1) dx$

(iii) Evaluate:  $\int \frac{x^3}{x-1} dx$

(iv) Evaluate:  $\int \frac{1}{\sqrt{x+3} - \sqrt{x-2}} dx$

(v) Find the integral of  $(2^x - 3^x)^2$  with respect to  $x$ .

(vi) Find  $\int \frac{dx}{2x-1}$

(vii) Evaluate:  $\int \frac{x^3}{(1+x^2)^2} dx$

(viii) Evaluate:  $\int x \cdot e^{x^2} \cdot dx$

(ix) Evaluate:  $\int \frac{dx}{\sqrt{x+x}}$

(x) Integrate  $(x^4+3x^2+6x-10)(4x^3+6x+6)$  with respect to  $x$ .

**8. (a) Evaluate :**

(i)  $\int (2e^{3x} + 1) dx$

(ii)  $\int \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx$

(iii)  $\int \frac{x^3 + 2x^2 - 1}{x^2} dx$

(iv)  $\int \frac{x^2 + x + 1}{\sqrt{x}} dx$

(v)  $\int \sqrt{x}(4x^2 + 3x - 9) dx$

$$(vi) \int \frac{x^3 - 2x^2 + x - 2}{x - 2} dx$$

$$(vii) \int (ax^2 + bx + c) dx \text{ (where a, b, and c are constants)}$$

$$(viii) \int (5x^4 + 4x^3 + 3x^2 + 2x + 1) dx$$

$$(ix) \int (2^x + 3^x)^2 dx$$

$$(x) \int (x - 2)(x^2 - 4x + 3) dx$$

$$(xi) \int (x - 2)\sqrt{(x^2 - 4x + 5)} dx$$

$$(xii) \int x \cdot e^{x^2} dx$$

$$(xiii) \int \frac{(\log x)^2}{2} dx$$

$$(xiv) \int x\sqrt{2x^2 - 1} dx$$

$$(xv) \int \frac{x}{9 - 4x^2} dx$$

(b) (i) If  $\frac{d[f(x)]}{dx} = 4x^3 - 5x^4$  and  $f(2) = 0$ , then find  $f(x)$ .

(ii) If  $\frac{d[f(x)]}{dx} = 3x^2 - 2x$  and  $f(2) = 0$ , then the value of constant of integration.

(iii) Find a function whose derivative is  $(1 - x)\sqrt{x}$ .

(iv) Compare differentiation with integration.

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**ANSWERS**

1. (i) a (ii) c (iii) a (iv) a (v) c (vi) b (viii) a (viii) a (ix) d (x) c
2. (i) Integration (ii) Constant of integration (iii) Integration  
(iv)  $x + C$  (v) Integrand.
4. (i)  $kx$  (ii)  $\frac{e^{ax}}{a}$  (iii)  $\frac{a^x}{\log_e a}$  (iv)  $\frac{1}{2}e^{2x+3} + C$  (v)  $(x^2 + 1)^2$
5. (i)  $\log x$  (ii)  $f(x)$  (iii)  $\int g(x)dx$  (iv)  $\int f(x)dx$  (v) Opposite.



## CHAPTER - 8

# **MEASURE OF CENTRAL TENDENCY- MATHEMATICAL AVERAGES**

### **STRUCTURE**

- 8.1 Introduction
- 8.2 Meaning and Definition of Central Tendency
- 8.3 Objective
- 8.4 Requisites of a good measure of Central Tendency
- 8.5 Different measures of Central Tendency.
- 8.6 Mathematical Averages.
  - 8.6.1 Arithmetic Mean - its meaning, merits, demerits, uses, properties, computation and practical problems.
  - 8.6.2 Geometric Mean - its meaning, computation, mathematical properties, uses, merits, demerits and practical problems.
  - 8.6.3 Harmonic Mean - its meaning, computation, mathematical properties, uses, merits, demerits, practical problems, relationship between A.M., G.M. and H.M.
- 8.7 Questions.