Factorisation of polynomials

Remainder Theorem and Its Application

Remainder Theorem

Consider two polynomials $p(x)$ and $q(x)$, where $p(x) = 5x^4 - 4x^2 - 50$ and $q(x) = x - 2$. We know how to divide $p(x)$ by $q(x)$ using the long division method. The result of this division will give the quotient as $5x^3 + 10x^2 + 16x + 32$ and the remainder as 14.

The long division method of finding the remainder is quite tedious. There is a simpler way to find the above remainder. This method is generalized in the form of a theorem called the **remainder theorem**. This theorem helps us find the remainder when a polynomial is to be divided by a linear polynomial.

In this lesson, we will study the remainder theorem and some of its applications in the form of examples.

Understanding the Remainder Theorem

Consider the division of a polynomial $p(x)$ by a polynomial $q(x)$, where $p(x) = 5x^4 - 4x^2 - 50$ and $q(x) = x - 2$. In this case, we have:

Dividend = $p(x)$ and divisor = $q(x)$

On dividing $p(x)$ by $q(x)$ using the long division method, we get:

Quotient = $5x^3 + 10x^2 + 16x + 32$ and remainder = 14

Now, let us find the value of $p(x)$ at $x = 2$.

$p(2) = 5 \times 2^4 - 4 \times 2^2 - 50$

$= 5 \times 16 - 4 \times 4 - 50$

$= 80 - 16 - 50$

$= 14$

Note how the value of $p(2)$ is the same as the remainder obtained by the long division of $p(x)$ by $q(x)$. Also observe how $x = 2$ is a zero of the polynomial $q(x)$.

Thus, **if we replace *x* in the dividend with the zero (or root) of the divisor, then we get the remainder**.

This method of finding the remainder is called the remainder theorem. It can be stated as follows:

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number a, if $p(x)$ is divided by a linear polynomial $x - a$, then the remainder will be $p(a)$.

Proof of the Remainder Theorem

Statement

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number a, if $p(x)$ is divided by a linear polynomial $x - a$, then the remainder will be $p(a)$.

Proof

Let $p(x)$ be a polynomial of a degree greater than or equal to 1 and a be any real number. When divided by $x - a$, let $p(x)$ leave the remainder $r(x)$. Let $q(x)$ be the quotient obtained.

Then, $p(x) = (x - a)\, q(x) + r(x)$, where $r(x) = 0$ or degree $r(x) <$ degree $(x - a)$

Now, $x - a$ is a polynomial of degree 1; so, either $r(x) = 0$ or $r(x) =$ constant (since a polynomial of degree less than 1 is a constant).

Let $r(x) = \text{constant} = r$ (say). Then, $p(x) = (x - a)\, q(x) + r$

On putting $x = a$, we get $p(a) = (a - a)\, q(a) + r = 0 \times q(a) + r = r$

Thus, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

Notes:

1) If $p(x)$ is divided by $x + a$, then the remainder will be $p(-a)$.

2) If $p(x)$ is divided by $ax - b$, then the remainder will be $p\left(\frac{b}{a}\right)$.

3) If $p(x)$ is divided by $ax + b$, then the remainder will be $p\left(-\frac{b}{a}\right)$.

Solved Examples

Easy

Example 1:

Find the remainder when $x^3 - x^2a + 5xa$ is divided by $x - a$.

Solution:

Let $p(x) = x^3 - x^2a + 5xa$

According to the remainder theorem, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

On putting $x - a = 0$, we get $x = a$.

$\therefore$ Remainder = $p(a)$

$= a^3 - a^2 \times a + 5 \times a \times a$

$= a^3 - a^3 + 5a^2$

$= 5a^2$

Thus, when $x^3 - x^2a + 5xa$ is divided by $x - a$, we get $5a^2$ as the remainder.

Example 2:

What is the remainder when $81x^4 + 54x^3 - 9x^2 - 3x + 2$ is divided by $3x + 2$?

Solution:

Let $p(x) = 81x^4 + 54x^3 - 9x^2 - 3x + 2$

As per the remainder theorem, if $p(x)$ is divided by $ax + b$, then the remainder will be $p\left(-\frac{b}{a}\right)$.

On putting $3x + 2 = 0$, we get $x = -\frac{2}{3}$.

$$\therefore \text{Remainder} = p\left(-\frac{2}{3}\right)$$

$$= 81\left(-\frac{2}{3}\right)^4 + 54\left(-\frac{2}{3}\right)^3 - 9\left(-\frac{2}{3}\right)^2 - 3\left(-\frac{2}{3}\right) + 2$$

$$= 81 \times \frac{16}{81} - 54 \times \frac{8}{27} - 9 \times \frac{4}{9} + 3 \times \frac{2}{3} + 2$$

$$= 16 - 16 - 4 + 2 + 2$$

$$= 0$$

Thus, when $81x^4 + 54x^3 - 9x^2 - 3x + 2$ is divided by $3x + 2$, we get zero as the remainder.

Medium

Example 1:

Verify the remainder theorem for the division of $2x^3 - 3x^2 + 4$ by $x - 3$.

Solution:

Let $p(x) = 2x^3 - 3x^2 + 4$

Let us divide $p(x)$ by $x - 3$ using the long division method.

$$\begin{array}{r|l}
 & 2x^2 + 3x + 9 \\
x - 3 & 2x^3 - 3x^2 + 4 \\
 & 2x^3 - 6x^2 \\
 & - \quad + \\
\hline
 & 3x^2 + 4 \\
 & 3x^2 - 9x \\
 & - \quad + \\
\hline
 & 9x + 4 \\
 & 9x - 27 \\
 & - \quad + \\
\hline
 & 31 \\
\hline
\end{array}$$

Thus, the division of $2x^3 - 3x^2 + 4$ by $x - 3$ yields the remainder 31.

Let us now find the remainder using the remainder theorem. According to this theorem, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

On putting $x - 3 = 0$, we get $x = 3$.

∴ Remainder = $p(3)$

$= 2 \times 3^3 - 3 \times 3^2 + 4$

$= 54 - 27 + 4$

$= 31$

Clearly, the remainder obtained by using the remainder theorem is the same as that obtained via the long division method. Hence, the remainder theorem is verified.

Example 2:

For what value of m is $p(x) = mx^3 + 17x^2 - 31x - 2m$ completely divisible by $3x + 1$.

Solution:

It is given that $p(x) = mx^3 + 17x^2 - 31x - 2m$

If $p(x)$ is completely divisible by $ax + b$, then the remainder will be zero, i.e., $p\left(-\frac{b}{a}\right) = 0$.

On putting $3x + 1 = 0$, we get $x = -\frac{1}{3}$.

Using the remainder theorem, we can find the value of m as follows:

$$p\left(-\frac{1}{3}\right)=0$$

$$\Rightarrow m\left(-\frac{1}{3}\right)^3+17\left(-\frac{1}{3}\right)^2-31\left(-\frac{1}{3}\right)-2m=0$$

$$\Rightarrow m\left(-\frac{1}{27}\right)+17\times\frac{1}{9}+\frac{31}{3}-2m=0$$

$$\Rightarrow -\frac{m}{27}-2m+\frac{17}{9}+\frac{31}{3}=0$$

$$\Rightarrow \frac{-m-54m}{27}+\frac{17+93}{9}=0$$

$$\Rightarrow -\frac{55m}{27}+\frac{110}{9}=0$$

$$\Rightarrow -\frac{55m}{27}=-\frac{110}{9}$$

$$\Rightarrow m=-\frac{110}{9}\times\left(-\frac{27}{55}\right)$$

⇒ $m = 6$

Thus, when $m = 6$, $mx^3 + 17x^2 - 31x - 2m$ is completely divisible by $3x + 1$.

Hard

Example 1:

Find the value of k for which $p(x) = 4kx^3 - 13x - 3k + 2$

i) is exactly divisible by $2x - 1$.

ii) leaves 3 as the remainder when divided by $2x + 3$.

Solution:

i)We have $p(x) = 4kx^3 - 13x - 3k + 2$

As per the remainder theorem, if $p(x)$ is divided by $ax - b$, then the remainder will be $p\left(\frac{b}{a}\right)$.

On putting $2x - 1 = 0$, we get $x = 1/2$.

Now, if $4kx^3 - 13x - 3k + 2$ is exactly divisible by $2x - 1$, then the remainder will be zero, i.e., $p\left(\frac{1}{2}\right) = 0$.

Using the remainder theorem, we can find the value of k as follows:

$$p\left(\frac{1}{2}\right) = 0$$
$$\Rightarrow 4k\left(\frac{1}{2}\right)^3 - 13\left(\frac{1}{2}\right) - 3k + 2 = 0$$
$$\Rightarrow 4k \times \frac{1}{8} - \frac{13}{2} - 3k + 2 = 0$$
$$\Rightarrow \frac{k}{2} - \frac{13}{2} - 3k + 2 = 0$$
$$\Rightarrow -\frac{5k}{2} - \frac{9}{2} = 0$$
$$\Rightarrow -\frac{5k}{2} = \frac{9}{2}$$

$$\Rightarrow k = \frac{-9}{5}$$

Thus, when $k = \frac{-9}{5}$, $4kx^3 - 13x - 3k + 2$ is exactly divisible by $2x - 1$.

ii)As per the remainder theorem, if $p(x)$ is divided by $ax + b$, then the remainder will be $p\left(-\frac{b}{a}\right)$.

On putting $2x + 3 = 0$, we get $x = -\frac{3}{2}$.

It is given that the division of $4kx^3 - 13x - 3k + 2$ by $2x + 3$ yields the remainder 3, i.e., $p\left(-\frac{3}{2}\right) = 3$.

Using the remainder theorem, we can find the value of k as follows:

$$p\left(-\frac{3}{2}\right)=3$$

$$\Rightarrow 4k\left(-\frac{3}{2}\right)^3-13\left(-\frac{3}{2}\right)-3k+2=3$$

$$\Rightarrow 4k\times\left(-\frac{27}{8}\right)+\frac{39}{2}-3k+2=3$$

$$\Rightarrow \frac{-27k}{2}+\frac{39}{2}-3k+2=3$$

$$\Rightarrow -\frac{33}{2}k+\frac{43}{2}=3$$

$$\Rightarrow -\frac{33}{2}k=3-\frac{43}{2}$$

$$\Rightarrow -\frac{33}{2}k=-\frac{37}{2}$$

$$\Rightarrow k=\frac{37}{33}$$

Thus, when k = 37/33, the division of $4kx^3 - 13x - 3k + 2$ by $2x + 3$ leaves 3 as the remainder.

Example 2:

Find the values of a and b for which $p(x) = x^3 + ax^2 + bx - 20$ leaves 0 and −2 as the remainders when divided by $x - 5$ and $x - 3$ respectively.

Solution:

We have $p(x) = x^3 + ax^2 + bx - 20$

As per the remainder theorem, if $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

On putting $x - 5 = 0$, we get $x = 5$.

On putting $x - 3 = 0$, we get $x = 3$.

Now, if the division of $x^3 + ax^2 + bx - 20$ by $x - 5$ leaves 0 as the remainder, then $p(5) = 0$.

$\Rightarrow 5^3 + a \times 5^2 + b \times 5 - 20 = 0$
$\Rightarrow 125 + 25a + 5b - 20 = 0$
$\Rightarrow 25a + 5b + 105 = 0$
$\Rightarrow 5a + b = -21 \qquad ...(1)$

Also, if the division of $x^3 + ax^2 + bx - 20$ by $x - 3$ leaves −2 as the remainder, then $p(3) = -2$.

$\Rightarrow 3^3 + a \times 3^2 + b \times 3 - 20 = -2$
$\Rightarrow 27 + 9a + 3b = -2 + 20$
$\Rightarrow 9a + 3b = 18 - 27$
$\Rightarrow 9a + 3b = -9$
$\Rightarrow 3a + b = -3 \qquad ...(2)$

On solving equations 1 and 2, we get:

$5a + (-3 - 3a) = -21$ ($\because b = -3 - 3a$)

$\Rightarrow 5a - 3a - 3 = -21$

$\Rightarrow 2a = -21 + 3$

$\Rightarrow 2a = -18$

$\Rightarrow a = -9$

Now, $b = -3 - 3a$

$\Rightarrow b = -3 - 3 \times (-9)$

$\Rightarrow b = -3 + 27$

$\Rightarrow b = 24$

Thus, when $a = -9$ and $b = 24$, the divisions of $x^3 + ax^2 + bx - 20$ by $x - 5$ and $x - 3$ leave 0 and −2 respectively as the remainders.

Factor Theorem and Its Applications

Factor Theorem

We know the relation between a number and its factor. If we divide 91 by 7, then we get 13 as the quotient and zero as the remainder. In this case, we say that 7 is a factor of

91 as the remainder is zero. Now, if we divide 107 by 9, then we get 11 as the quotient and 8 as the remainder. In this case, we say that 9 is not a factor of 107 as the remainder is not zero.

Thus, the relation between a number and its factor is given as follows:

If a number is completely divisible by another number, i.e., the remainder is zero, then the second number is a factor of the first number.

Similarly, a polynomial $p(x)$ is said to be completely divisible by a polynomial $q(x)$ if we get zero as the remainder on dividing $p(x)$ by $q(x)$. In this case, we say that $q(x)$ is a factor of $p(x)$.

We have studied the remainder **theorem** that helps us to find the remainder. Similarly, we have a **factor theorem** that helps us to determine whether or not a polynomial is a factor of another polynomial, without actually performing the division.

In this lesson, we will study the factor theorem and solve some problems based on it.

Understanding the Factor Theorem

We can easily determine whether a polynomial $q(x)$ is a factor of a polynomial $p(x)$ without performing the division. This can be done by using the factor theorem, which can be stated as follows:

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number c,

i) if $p(c) = 0$, then $x - c$ will be a factor of $p(x)$ and

ii) if $x - c$ is a factor of $p(x)$, then $p(c)$ will be equal to zero.

Consider the polynomial, $p(x) = x^2 - 3x + 2$.

On putting $x = 2$ in $p(x)$, we get:

$p(2) = 2^2 - 3 \times 2 + 2$

$= 4 - 6 + 2$

$= 0$

Thus, we can say that $x - 2$ is a factor of $p(x)$, where 2 is a real number.

Proof of the Factor Theorem

Statement

For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number c,

i) if $p(c) = 0$, then $x - c$ will be a factor of $p(x)$ and

ii) if $x - c$ is a factor of $p(x)$, then $p(c)$ will be equal to zero.

Proof

Let $p(x)$ be a polynomial of a degree greater than or equal to 1 and c be any real number such that $p(c) = 0$. Let quotient $q(x)$ be obtained when $p(x)$ is divided by $x - c$.

i) <u>$p(c) = 0$</u>

By the remainder theorem, the remainder obtained is $p(c)$.

$\Rightarrow p(x) = (x - c)\, q(x) + p(c)$

$\Rightarrow p(x) = (x - c)\, q(x)$ [$\because p(c) = 0$]

$\Rightarrow x - c$ is a factor of $p(x)$.

ii) $x - c$ is a factor of $p(x)$

$\Rightarrow$ When divided by $x - c$, $p(x)$ leaves zero as the remainder.

However, by the remainder theorem, the remainder obtained is $p(c)$.

$\Rightarrow p(c) = 0$

Notes

1) $x + c$ will be a factor of $p(x)$ if $p(-c) = 0$

2) $cx - d$ will be a factor of $p(x)$ if $p\left(\frac{d}{c}\right) = 0$

3) $cx + d$ will be a factor of $p(x)$ if $p\left(-\frac{d}{c}\right) = 0$

4) $(x - c)\,(x - d)$ will be a factor of $p(x)$ if $p(c) = 0$ and $p(d) = 0$

Example Based on the Theorem

Example Based on the Theorem

Solved Examples

Easy

Example 1:

Check whether or not $x - 1$ is a factor of $x^3 - 2x^2 - x + 2$.

Solution:

Let $p(x) = x^3 - 2x^2 - x + 2$

According to the factor theorem, $x - 1$ will be a factor of $p(x)$ if $p(1) = 0$.

$p(1) = 1^3 - 2 \times 1^2 - 1 + 2$

$= 1 - 2 - 1 + 2$

$= 0$

Thus, $x - 1$ is a factor of $x^3 - 2x^2 - x + 2$.

Example 2:

Using the factor theorem, show that $2x + 1$ is a factor of $2x^3 + 3x^2 - 11x - 6$.

Solution:

Let $p(x) = 2x^3 + 3x^2 - 11x - 6$

According to the factor theorem, $2x + 1$ will be a factor of $p(x)$ if $p\left(-\frac{1}{2}\right) = 0$.

$$p\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2-11\left(-\frac{1}{2}\right)-6$$
$$=-\frac{1}{4}+\frac{3}{4}+\frac{11}{2}-6$$
$$=\frac{-1+3+22-24}{4}$$
$$=0$$

Thus, $2x + 1$ is a factor of $2x^3 + 3x^2 - 11x - 6$.

Medium

Example 1:

For what value of m is $x - 3$ a factor of $3x^2 - 3x + m$?

Solution:

Let $p(x) = 3x^2 - 3x + m$

According to the factor theorem, $x - 3$ will be a factor of $p(x)$ if $p(3) = 0$.

$p(3) = 3 \times 3^2 - 3 \times 3 + m$

So, $3 \times 3^2 - 3 \times 3 + m = 0$

$\Rightarrow 3 \times 9 - 9 + m = 0$

$\Rightarrow 27 - 9 + m = 0$

$\Rightarrow 18 + m = 0$

$\Rightarrow m = -18$

Thus, $x - 3$ is a factor of $3x^2 - 3x + m$ when $m = -18$.

Example 2:

Check whether or not $2x^2 - 11x + 25$ is exactly divisible by $2x - 3$.

Solution:

Let $p(x) = 2x^2 - 11x + 25$ and $q(x) = 2x - 3$

We know that $p(x)$ will be exactly divisible by $q(x)$ if $q(x)$ is a factor of $p(x)$.

On putting $2x - 3 = 0$, we get $x = 3/2$.

On using the factor theorem, we get:

$$p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 11\left(\frac{3}{2}\right) + 25$$
$$= \frac{9}{2} - \frac{33}{2} + 25$$
$$= 13$$
$$\neq 0$$

Thus, $q(x)$ is not a factor of $p(x)$.

Hence, $2x^2 - 11x + 25$ is not exactly divisible by $2x - 3$.

Example 3:

Using the factor theorem, determine whether or not $g(x)$ is a factor of $f(x)$, where

$f(x) = 7x^2 - 2\sqrt{8}\, x - 6$ and $g(x) = x - \sqrt{2}$.

Solution:

It is given that $f(x) = 7x^2 - 2\sqrt{8}\, x - 6$ and $g(x) = x - \sqrt{2}$

According to the factor theorem, $g(x)$ will be a factor of $f(x)$ if $f\left(\sqrt{2}\right) = 0$.

$f\left(\sqrt{2}\right) = 7(\sqrt{2})^2 - 2\sqrt{8} \times \sqrt{2} - 6$

$= 7 \times 2 - 2\sqrt{16} - 6$

$= 14 - 8 - 6$

$= 0$

Therefore, $g(x)$ is a factor of $f(x)$.

Hard

Example 1:

Using the factor theorem, show that $a-b$, $b-c$ and $c-a$ are factors of
$a\,(b^2-c^2) + b\,(c^2-a^2) + c\,(a^2-b^2)$.

Solution:

We have the given expression as $a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$.

As per the factor theorem, $x - k$ will be a factor of a polynomial $p(x)$ if $p(x) = 0$ when $x = k$.

Let us consider $p(a) = a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$ to be a polynomial in variable 'a'. Take b and c as constants for the time being.

Now, as per the factor theorem, $a - b$ will be a factor of $p(a)$ if $p(a) = 0$ when $a = b$.

On putting $a = b$ in $p(a)$, we get:

$b\,(b^2 - c^2) + b\,(c^2 - b^2) + c\,(b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3 + c \times 0$

$= 0$

Thus, $a - b$ is a factor of $a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$.

Now, suppose $p(b) = a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$ is a polynomial in variable 'b' and a and c are constants. Then, $b - c$ will be a factor of $p(b)$ if $p(b) = 0$ when $b = c$.

On substituting $b = c$ in $p(b)$, we find that the result is zero.

Similarly, we can take $p(c) = a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$ to be a polynomial in variable 'c' and a and b as constants. Then, $c - a$ will be a factor of $p(c)$ if $p(c) = 0$ when $c = a$. On substituting $c = a$ in $p(c)$, we find that the result is zero.

Hence, $b - c$ and $c - a$ are also factors of $a\,(b^2 - c^2) + b\,(c^2 - a^2) + c\,(a^2 - b^2)$.

Factorisation of Quadratic Polynomials Using Factor Theorem and Splitting Middle Term

Factorisation of Quadratic Polynomials

We know that $7 \times 6 = 42$. Here, 7 and 6 are factors of 42. Now, consider the linear polynomials $x - 2$ and $x + 1$. On multiplying the two, we get: $x\,(x + 1) - 2\,(x + 1)$
$= x^2 + x - 2x - 2 = x^2 - x - 2$, which is a quadratic polynomial. So, $x - 2$ and $x + 1$ are

factors of the quadratic polynomial

$x^2 - x - 2$. A quadratic polynomial can have a maximum of two factors.

In the above example, we found the quadratic polynomial from its two factors. We can also find the factors from the quadratic polynomial. This process of decomposing a polynomial into a product of its factors (which when multiplied give the original expression) is called **factorisation**.

There are two ways of finding the factors of quadratic polynomials viz., by applying the factor theorem and by splitting the middle term. We will discuss these methods of factorisation in this lesson and also solve some examples based on them.

Factorisation of Quadratic Polynomials Using the Factor Theorem

The factor theorem states that: **For a polynomial $p(x)$ of a degree greater than or equal to 1 and for any real number a, if $p(a) = 0$, then $x - a$ will be a factor of $p(x)$.**

Consider the quadratic polynomial, $p(x) = x^2 - 5x + 6$. To find its factors, we need to ascertain the value of x for which the value of the polynomial comes out to be zero. For this, we first determine the factors of the constant term in the polynomial, and then check the value of the polynomial at these points.

In the given polynomial, the constant term is 6 and its factors are ±1, ±2, ±3 and ±6.

Let us now check the value of the polynomial for each of these factors of 6.

$p(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Hence, $x - 3$ is also a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum two factors which are already obtained as: $(x - 2)$ and $(x - 3)$.

Thus, the given polynomial = $p(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$

Solved Examples

Easy

Example 1:

Factorise $x^2 - 7x + 10$ using the factor theorem.

Solution:

Let $p(x) = x^2 - 7x + 10$

The constant term is 10 and its factors are ±1, ±2, ±5 and ±10.

Let us check the value of the polynomial for each of these factors of 10.

$p(1) = 1^2 - 7 \times 1 + 10 = 1 - 7 + 10 = 4 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 7 \times 2 + 10 = 4 - 14 + 10 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(5) = 5^2 - 7 \times 5 + 10 = 25 - 35 + 10 = 0$

Hence, $x - 5$ is a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x - 2$ and $x - 5$.

Thus, we can write the given polynomial as:

$p(x) = x^2 - 7x + 10 = (x - 2)(x - 5)$

Hard

Example 1:

Factorise $x^4y^2 - 5x^2y^2 + 6y^2$.

Solution:

$x^4y^2 - 5x^2y^2 + 6y^2 = y^2 (x^4 - 5x^2 + 6)$

Let $x^2 = a$

$\Rightarrow (x^2)^2 = a^2$

$\Rightarrow x^4 = a^2$

$\therefore x^4y^2 - 5x^2y^2 + 6y^2 = y^2 (a^2 - 5a + 6)$

$= y^2 \times f(a)$, where $f(a) = a^2 - 5a + 6$

Here, $f(a)$ is a quadratic polynomial and the factors of the constant term '6' are ±1, ±2, ±3 and ±6.

$f(1) = 1^2 - 5 \times 1 + 6 = 1 - 5 + 6 = 2 \neq 0$

Thus, $a - 1$ is not a factor of $f(a)$.

$f(2) = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

Thus, $a - 2$ is a factor of $f(a)$.

$f(3) = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

Thus, $a - 3$ is a factor of $f(a)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $a - 2$ and $a - 3$.

Thus, we can write the given polynomial as:

$f(a) = a^2 - 5a + 6 = (a - 2) (a - 3)$

Hence, $x^4y^2 - 5x^2y^2 + 6y^2 = y^2 (a - 2) (a - 3)$

$= y^2 (x^2 - 2) (x^2 - 3)$

Example 2:

Factorise $4x (y^2 + x - 1 + 3/x) + y^2 (y^2 - 2) - 20$.

Solution:

$4x (y^2 + x - 1 + 3/x) + y^2 (y^2 - 2) - 20$

$= 4xy^2 + 4x^2 - 4x + 12 + (y^2)^2 - 2y^2 - 20$

$= (2x)^2 + (y^2)^2 + 2 \times 2x \times y^2 - 4x - 2y^2 + 12 - 20$

$= (2x + y^2)^2 - 2\,(2x + y^2) - 8$

$= a^2 - 2a - 8$

$= f(a)$, where $a = 2x + y^2$

Here, $f(a)$ is a quadratic polynomial and the factors of the constant term '8' are ±1, ±2, ±4 and ±8.

$f(1) = 1^2 - 2 \times 1 - 8 = 1 - 2 - 8 = -9 \neq 0$

Thus, $a - 1$ is not a factor of $f(a)$.

$f(-1) = (-1)^2 - 2 \times (-1) - 8 = 1 + 2 - 8 = -5 \neq 0$

Thus, $a + 1$ is not a factor of $f(a)$.

$f(2) = 2^2 - 2 \times 2 - 8 = 4 - 4 - 8 = -8 \neq 0$

Thus, $a - 2$ is not a factor of $f(a)$.

$f(-2) = (-2)^2 - 2 \times (-2) - 8 = 4 + 4 - 8 = 0$

Thus, $a + 2$ is a factor of $f(a)$.

$f(4) = 4^2 - 2 \times 4 - 8 = 16 - 8 - 8 = 0$

Thus, $a - 4$ is a factor of $f(a)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $a + 2$ and $a - 4$.

Thus, we can write the given polynomial as:

$f(a) = a^2 - 2a - 8 = (a + 2)\,(a - 4)$

Hence, $4x\,(y^2 + x - 1 + 3/x) + y^2\,(y^2 - 2) - 20 = (2x + y^2 + 2)\,(2x + y^2 - 4)$

Solved Examples

Easy

Example 1:

Factorise $12x^2 - \sqrt{2}\,x - 12$ by splitting the middle term.

Solution:

The given polynomial is $12x^2 - \sqrt{2}x - 12$.

Here, $ac = 12 \times (-12) = -144$. The middle term is $-\sqrt{2}$.

Therefore, we have to split $-\sqrt{2}$ into two numbers such that their product is −144 and their sum is $-\sqrt{2}$.

These numbers are $-9\sqrt{2}$ and $8\sqrt{2}$ ($\because -9\sqrt{2} + 8\sqrt{2} = -\sqrt{2}$ and $-9\sqrt{2} \times 8\sqrt{2} = -144$).

Thus, we have:

$12x^2 - \sqrt{2}x - 12 = 12x^2 - 9\sqrt{2}x + 8\sqrt{2}x - 12$

$= 3\sqrt{2}x(2\sqrt{2}x - 3) + 4(2\sqrt{2}x - 3)$

$= (2\sqrt{2}x - 3)(3\sqrt{2}x + 4)$

Example 2:

Factorise $2x^2 - 11x + 15$ by splitting the middle term.

Solution:

The given polynomial is $2x^2 - 11x + 15$.

Here, $ac = 2 \times 15 = 30$. The middle term is −11. Therefore, we have to split −11 into two numbers such that their product is 30 and their sum is −11. These numbers are −5 and −6 [$\because (-5) + (-6) = -11$ and $(-5) \times (-6) = 30$].

Thus, we have:

$2x^2 - 11x + 15 = 2x^2 - 5x - 6x + 15$

$= x(2x - 5) - 3(2x - 5)$

$= (2x - 5)(x - 3)$

Medium

Example 1:

Factorise $(3y - 1)^2 - 6y + 2$.

Solution:

$(3y - 1)^2 - 6y + 2 = 9y^2 + 1 - 6y - 6y + 2$

$= 9y^2 - 12y + 3$

$= 3\ (3y^2 - 4y + 1)$

Here, $ac = 1 \times 3 = 3$. The middle term is −4. Therefore, we have to split −4 into two numbers such that their product is 3 and their sum is −4. These numbers are −1 and −3 [$\because$ (−3) + (−1) = −4 and (−3) × (−1) = 3].

Thus, we have:

$3\ (3y^2 - 4y + 1) = 3\ (3y^2 - 3y - y + 1)$

$= 3\ [3y\ (y - 1) - 1\ (y - 1)]$

$= 3\ (y - 1)\ (3y - 1)$

Example 2:

Find the dimensions of a rectangle whose area is given by the polynomial $20p^2 + 69p + 54$.

Solution:

We know that area of a rectangle = Length × Breadth

Area of the rectangle is given by the polynomial $20p^2 + 69p + 54$. So, its factors will be the required dimensions of the rectangle.

In the given polynomial, $ac = 20 \times 54 = 1080$. The middle term is 69. Therefore, we have to split 69 into two numbers such that their product is 1080 and their sum is 69. These numbers are 45 and 24 ($\because$ 45 + 24 = 69 and 45 × 24 = 1080).

Thus, we have:

$20p^2 + 69p + 54 = 20p^2 + 45p + 24p + 54$

$= 5p\ (4p + 9) + 6\ (4p + 9)$

= (4*p* + 9) (5*p* + 6)

Hence, the dimensions of the rectangle are 5*p* + 6 and 4*p* + 9.

Hard

Example 1:

Factorise $2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}+\frac{9}{19}\right)$.

Solution:

$$2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}+\frac{9}{19}\right)$$
$$=2\left(3x+\frac{4}{5x}\right)^2+19\left(3x+\frac{4}{5x}\right)+9$$
$$=2\left(3x+\frac{4}{5x}\right)^2+18\left(3x+\frac{4}{5x}\right)+\left(3x+\frac{4}{5x}\right)+9$$
$$=2\left(3x+\frac{4}{5x}\right)\left[\left(3x+\frac{4}{5x}\right)+9\right]+1\left[\left(3x+\frac{4}{5x}\right)+9\right]$$
$$=\left[\left(3x+\frac{4}{5x}\right)+9\right]\left[2\left(3x+\frac{4}{5x}\right)+1\right]$$
$$=\left(3x+\frac{4}{5x}+9\right)\left(6x+\frac{8}{5x}+1\right)$$

Factorisation of Cubic Polynomial Using Factor Theorem

Factorization of Cubic Polynomials

A cubic polynomial can be written as $p(x) = ax^3 + bx^2 + cx + d$, where *a*, *b*, *c* and *d* are real numbers. We cannot factorize a cubic polynomial in the manner in which we factorize a quadratic polynomial. We use a different approach for this purpose.

A cubic polynomial can have a maximum of three linear factors. By knowing one of these factors, we can reduce it to a quadratic polynomial. Thus, to factorize a cubic polynomial, we first find a factor by the hit and trial method or by using the factor theorem, and then reduce the cubic polynomial into a quadratic polynomial.

The resultant quadratic polynomial is solved by splitting its middle term or by using the factor theorem.

In this lesson, we will learn how to factorize a cubic polynomial and solve some examples related to the same.

Know More

Hit and trial method

Hit and trial method is used to find the factors or roots of a polynomial of degree more than two.

In this method, we put some value in the given polynomial to see if it satisfies the polynomial. If it does, then it is the zero of that polynomial. Using this method, we can reduce a polynomial of degree, say n, to a polynomial of degree $n - 1$.

Solved Examples

Easy

Example 1:

Factorize $x^3 - 3x^2 - x + 3$.

Solution:

Let $p(x) = x^3 - 3x^2 - x + 3$

The constant term is 3.

The factors of 3 are ±1 and ±3.

Let us take $x = 1$ and find the value of $p(x)$.

$p(1) = 1^3 - 3 \times 1^2 - 1 + 3$

$= 1 - 3 - 1 + 3$

$= 0$

Thus, $x - 1$ is a factor of $p(x)$, using factor theorem.

Now, we have to group the terms of $p(x)$ such that we can take $x - 1$ as common.

Thus, we have:

$p(x) = x^3 - 3x^2 - x + 3$

$= x^3 - x^2 - 2x^2 + 2x - 3x + 3$

$= x^2 (x - 1) - 2x (x - 1) - 3 (x - 1)$

$= (x - 1) (x^2 - 2x - 3)$ … (1)

Next, we factorize $x^2 - 2x - 3$ by splitting its middle term.

The middle term is −2. We have to find two numbers such that their product is −3 and their sum is 2. These two numbers are 3 and −1.

Thus, we have:

$x^2 - 2x - 3 = x^2 - (3 - 1)x - 3$

$= x^2 - 3x + x - 3$

$= x (x - 3) + 1 (x - 3)$

$= (x - 3) (x + 1)$

On substituting in equation 1, we get:

$p(x) = (x - 1) (x - 3) (x + 1)$

Example 2:

If $x + 3$ is a factor of the polynomial $f(x) = x^3 - 7x + 6$, then factorize $f(x)$.

Solution:

We have $x + 3$ as a factor of the polynomial $f(x) = x^3 + 0x^2 - 7x + 6$.

Let us divide $f(x)$ by $x + 3$.

$$
\begin{array}{r l}
 & x^2 - 3x + 2 \\
x + 3 \big) & x^3 + 0x^2 - 7x + 6 \\
 & x^3 + 3x^2 \\
 & - \quad - \\
\hline
 & -3x^2 - 7x \\
 & -3x^2 - 9x \\
 & + \quad + \\
\hline
 & 2x + 6 \\
 & 2x + 6 \\
 & - \quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

$\therefore f(x) = x^3 - 7x + 6 = (x + 3)(x^2 - 3x + 2)$

$= (x + 3)(x^2 - x - 2x + 2)$

$= (x + 3)[x(x - 1) - 2(x - 1)]$

$= (x + 3)(x - 1)(x - 2)$

Medium

Example 1:

Factorize $2x^3 - 7x^2 + 7x - 2$.

Solution:

Let $p(x) = 2x^3 - 7x^2 + 7x - 2$

Let us take $x = 1$ and find the value of $p(x)$.

$p(1) = 2 \times 1^3 - 7 \times 1^2 + 7 \times 1 - 2$

$= 2 - 7 + 7 - 2$

$= 0$

Thus, $x - 1$ is a factor of $p(x)$.

Now, we have to group the terms of $p(x)$ such that we can take $x - 1$ as common.

Thus, we have:

$p(x) = 2x^3 - 2x^2 - 5x^2 + 5x + 2x - 2$

$= 2x^2 (x - 1) - 5x (x - 1) + 2 (x - 1)$

$=(x - 1) (2x^2 - 5x + 2)$ … (1)

Next, we factorize $2x^2 - 5x + 2$ by splitting its middle term. The middle term is −5. We have to find two numbers such that their product is 4 and their sum is 5. These two numbers are 4 and 1.

Thus, we have:

$2x^2 - 5x + 2 = 2x^2 - (4 + 1)x + 2$

$= 2x^2 - 4x - x + 2$

$= 2x (x - 2) - 1 (x - 2)$

$= (2x - 1) (x - 2)$

On substituting in equation 1, we get:

$p(x) = (x - 1) (2x - 1) (x - 2)$

Example 2:

Factorize $x^3 - 23x^2 + 142x - 120$.

Solution:

Let $p(x) = x^3 - 23x^2 + 142x - 120$
Let us take $x = 1$ and find the value of $p(x)$.

$p(1)= 1^3 - 23 \times 1^2 + 142 \times 1 - 120$

$= 1 - 23 + 142 - 120$

$= 0$

Thus, $x - 1$ is a factor of $p(x)$.

Now, we have to group the terms of $p(x)$ such that we can take $x - 1$ as common.

Thus, we have:

$p(x)= x^3 − 23x^2 + 142x − 120$

$= x^3 − x^2 − 22x^2 + 22x + 120x − 120$
$= x^2 (x −1) − 22x (x − 1) + 120 (x − 1)$

$= (x − 1) (x^2 − 22x + 120)$ … (1)

Next, we factorize $x^2 − 22x + 120$ by splitting its middle term. The middle term is −22. We have to find two numbers such that their product is 120 and their sum is 22. These two numbers are 12 and 10.

Thus, we have:

$x^2 − 22x + 120= x^2 − 12x − 10x + 120$

$= x (x − 12) − 10 (x − 12)$

$= (x − 12) (x − 10)$

On substituting in equation (1), we get:

$x^3 − 23x^2 − 142x − 120 = (x − 1) (x − 12) (x − 10)$

Hard

Example 1:

Factorize the cubic polynomial $p(x) = 6x^3 + 5x^2 − 12x + 4$.

Solution:

We have $p(x) = 6x^3 + 5x^2 − 12x + 4$

Let us take $x = −2$ and then find the value of $p(x)$.

$p(−2)= 6 × (−2)^3 + 5 × (−2)^2 − 12 × (−2) + 4$

$= −48 + 20 + 24 + 4$

$= 0$

Thus, $x + 2$ is a factor of $p(x)$.

Now, let us divide $p(x)$ by $x + 2$.

$$
\begin{array}{r|l}
 & 6x^2 - 7x + 2 \\
\hline
x + 2 & 6x^3 + 5x^2 - 12x + 4 \\
 & 6x^3 + 12x^2 \\
 & -\quad - \\
\hline
 & -7x^2 - 12x \\
 & -7x^2 - 14x \\
 & +\quad + \\
\hline
 & 2x + 4 \\
 & 2x + 4 \\
 & -\quad - \\
\hline
 & 0 \\
\hline
\end{array}
$$

$\therefore 6x^3 + 5x^2 - 12x + 4 = (x + 2)(6x^2 - 7x + 2)$

$= (x + 2)(6x^2 - 4x - 3x + 2)$

$= (x + 2)[2x(3x - 2) - 1(3x - 2)]$

$= (x + 2)(3x - 2)(2x - 1)$