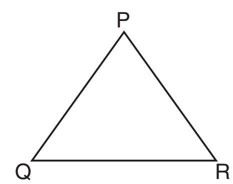
6 Plane Geometry –Triangles

KEY FACTS

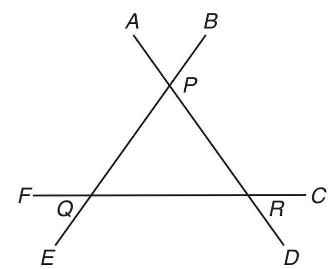
I. Definitions

A triangle is a three sided closed figure formed by three non-collinear points.

The three points P, Q and R in the given figure are called the **vertices,** line segments joining the three vertices, *i.e.*, PQ, QR and PR are called the **sides** and $\angle P$, $\angle Q$ and $\angle R$ are the **interior angles** of the triangle.

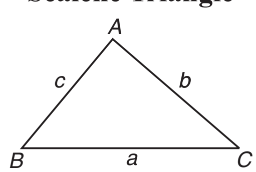
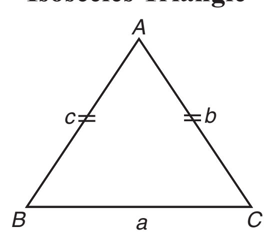
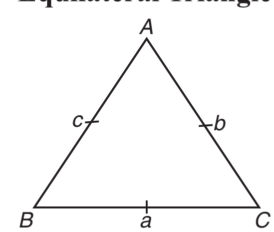


If the sides of a triangle are produced as shown in the given diagram, then the angles $\angle PRC$, $\angle QRD$, $\angle PFQ$, $\angle RQE$, $\angle QPA$ and $\angle RPB$ are the exterior angles of ΔABC.

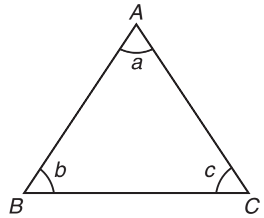
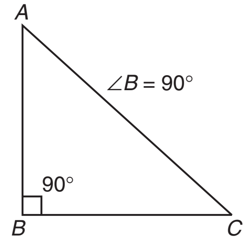
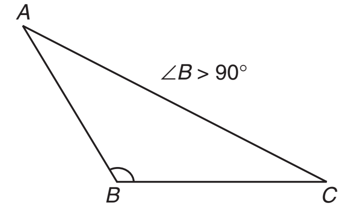


II. Types of Triangles:

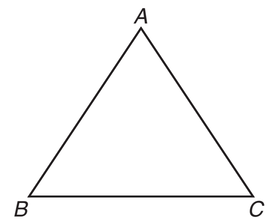
a. By sides:

Scalene Triangle	Isosceles Triangle	Equilateral Triangle
 $a \neq b \neq c$ (All the sides are unequal)	 (At least two sides are equal. Here, $AB = AC$) ***Angles opposite equal sides are also equal,*** *i.e.*, $\angle C = \angle B$.	 $a = b = c$ (All sides are equal) ***All angles are equal to 60º***

b. By angles:

Acute Angled Triangle	Right Angled Triangle	Obtuse Angled Triangle
 All angles are acute, *i.e.*, $\angle A < \angle 90º$, $\angle B < 90º$, $\angle C < 90º$	 One of the angles is a right angle. The other two are ***complementary*** to each other	 One of the angles is an obtuse angle.

III. Some Important Properties of Triangles:

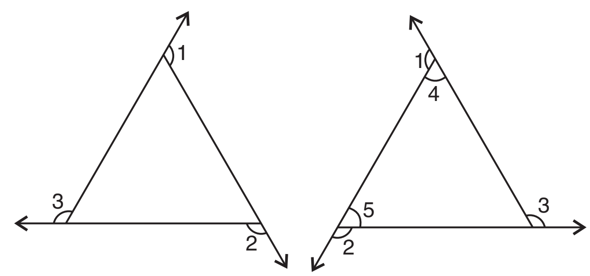


a. ***The sum of the three interior angles of a triangle is always 180º, i.e.,*** **$\angle BAC + \angle ABC + \angle BCA = 180º$.**

b. (*i*) ***If the sides of a triangle are produced in order then, the sum of the three (ordered) exterior angles of a triangle is 360º, i.e.,*** **in both the figures, $\angle 1 + \angle 2 + \angle 3 = 360°$**

(*ii*) ***The measure of an exterior angle is equal to the sum of the measures of the interior opposite angles, i.e.,*** **in figure (ii) $\angle 3 = \angle 4 + \angle 5$.**

(*iii*) ***The measure of an exterior angle is greater than the measure of each of the interior opposite angles, i.e.,*** **in figure (ii) $\angle 3 > \angle 4$ and $\angle 3 > \angle 5$.**



(i) (ii)

(*iv*) ***The sum of the measure of exterior angle at a vertex and its adjacent interior angle is 180º.***

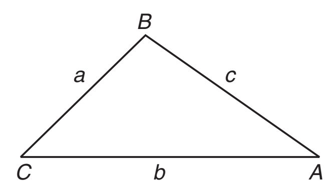
IV. Triangle Inequalities:

(*i*) ***Sum of any two sides of a triangle is always greater than the third side.***

(*ii*) ***The difference of any two sides is always less than the third side.***

(*iii*) ***If two sides of a triangle are not equal, then the angle opposite to the greater side is greater and vice versa.***

(*iv*) Let a, b, c be the three sides of a triangle ΔABC where $AB = c$ is the longest side (say). Then,

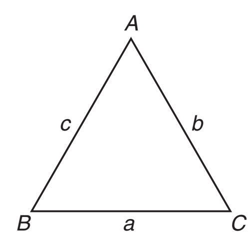


- if $c^2 < a^2 + b^2$, then the triangle is acute angled.
- if $c^2 = a^2 + b^2$, then the triangle is right angled.
- if $c^2 > a^2 + b^2$, then the triangle is obtuse angled.

V. Sine Rule:

In a ΔABC, if a, b, c be the three sides opposite to the angles A, B and C respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



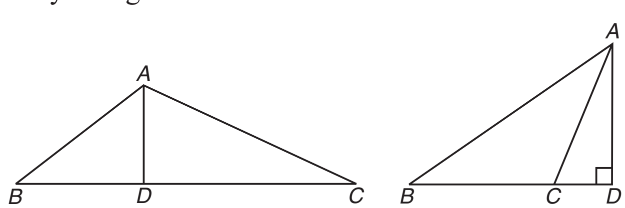
VI. Cosine Rule:

In a ΔABC, if a, b, c be the sides opposite to the angles A, B and C respectively, then

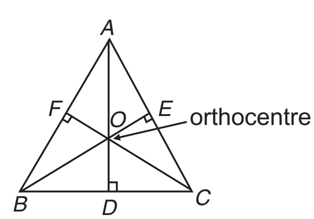
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

VII. Some Important Definitions:

(*i*) **Altitude** ***of a triangle is the perpendicular drawn from a vertex to the opposite side*** (produced if necessary). Every triangle has three altitudes.

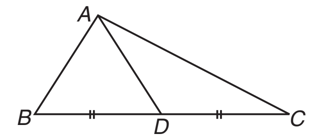


AD is the altitude on side BC.



Orthocentre ***is the point of intersection of the three altitudes of a triangle***. Hence, O is the orthocentre of ΔABC, where $\angle BOC = 180º - \angle A$, $\angle AOB = 180º - \angle C$, $\angle COA = 180º - \angle B$.

(*ii*) **Median** ***is the straight line segment joining the mid-point of any side to the opposite vertex.*** Every triangle has three medians and a median bisects the area of a Δ.

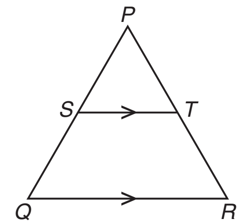


VIII. Important Theorems on Triangles

1. **Basic Proportionality Theorem (BPT):** ***Any line parallel to one side of a triangles divides the other two sides proportionally.***

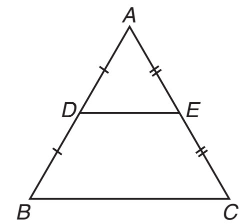
   Thus, if ST is drawn parallel to side QR of ΔPQR, then

$$\frac{PS}{SQ} = \frac{PT}{TR} \quad \text{or} \quad \frac{PS}{PQ} = \frac{PT}{PR}$$



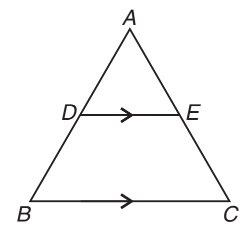
2. **Mid-point Theorem:** ***The line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.***

   Thus, if D and E are the midpoints of sides AB and AC respectively of ΔABC, then $DE \parallel BC$ and $DE = \frac{1}{2}BC$.



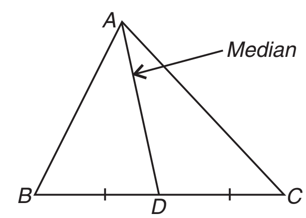
   Converse of Mid-point Theorem: ***The straight line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side.***

   Thus, a line drawn through D, the mid-point of side AB of ΔABC, parallel to BC bisects AC, *i.e.*, E is the mid-points of AC, *i.e.*, $AE = EC$.



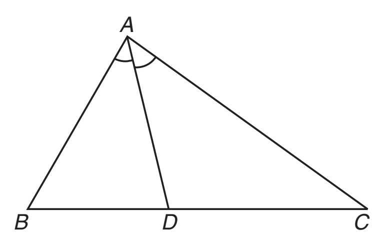
3. **Apollonius Theorem:** ***In a triangle, the sum of the squares of any two sides of a triangle is equal to twice the sum of the square of the median to the third side and square of half the third side,***

   i.e., $AB^2 + AC^2 = 2\left(AD^2 + \left(\frac{BC}{2}\right)^2\right) = 2(AD^2 + BD^2) = 2(AD^2 + CD^2)$.

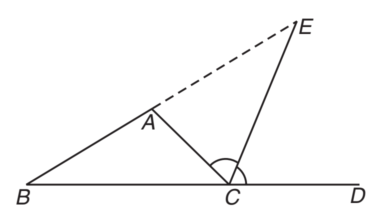


4. **Interior Angle Bisector Theorem:** ***In a triangle, the angle bisector of an angle divides the opposite side to the angle in the ratio of the remaining two sides.***

   Thus, if AD is the internal bisector of $\angle A$ of ΔABC, then $\frac{BD}{CD} = \frac{AB}{AC}$ and $BD \times AC = DC \times AB = AD^2$



5. **External Angle Bisector Theorem:** In a triangle, the angle bisector of any exterior angle of a triangle divides the side opposite to the external angle in the ratio of the remaining two sides, *i.e.*, If CE is the bisector of external angle ACD, then $\frac{BE}{AE} = \frac{BC}{AC}$.

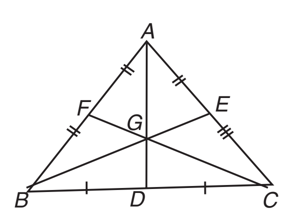


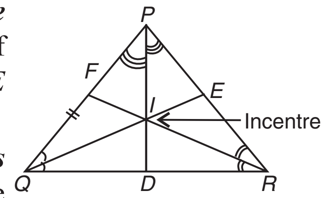
   ∴ Area (ΔABD) = Area (ΔACD) = $\frac{1}{2}$ Area (ΔABC).

   Centroid ***is point of intersection or point of concurrence of the three medians of a triangle***. Also it divides each median in the ratio 2:1 (vertex : base)

   G is the centroid of ΔABC, *i.e.*, the point of concurrence of medians AD, BE and CF. Also,

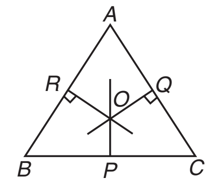
$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$$





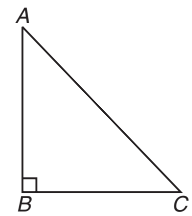
6. Angle Bisector: ***A line segment from a vertex of triangle which bisects the angle at the vertex is called the angle bisector***. PD, QE and RF are angle bisectors of $\angle P$, $\angle Q$ and $\angle R$ respectively of ΔPQR such that $\angle QPD = \angle RPD$, $\angle PQE = \angle RQE$ and $\angle PRF = \angle QRF$.

Incentre ***is the point of intersection of the angle bisectors of a triangle and it is equidistant from all the three sides of the triangle***, *i.e.*, perpendicular distance between the side and incentre is equal for all the three sides.



7. Perpendicular Bisector: ***A line segment bisecting a side of a triangle at right angles*** *is* ***called a perpendicular bisector***. The point of concurrence of the perpendicular bisectors is called the **circumcentre**. Here O is the circumcentre of ΔABC. The circumcentre is equidistant from the vertices of a triangle, *i.e.*, $OA = OB = OC$.

> **Note:** The orthocentre, centroid, incentre and circumcentre coincide in case of an equilateral triangle.



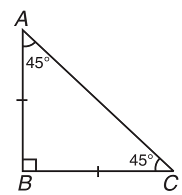
8. Pythagoras' Theorem: ***The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.***

Here ΔABC is right angled at C. So AC is the hypotenuse. Hence, according to Pythagoras, Theorem $AC^2 = AB^2 + BC^2$.

Converse of Pythagoreas' Theorem: ***If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the triangle is right angled.***

If a, b, c, be the sides opposite to $\angle A$, $\angle B$ and $\angle C$ respectively of a ΔABC and $c^2 = a^2 + b^2$, then ΔABC is right angled at C.

Such numbers or triplets a, b, c which satisfy the condition $a^2 + b^2 = c^2$ are called Pythagorean triplets. Some examples of Pythagorean triplets are: (3, 4, 5) ; (5, 12, 13), (7, 24, 25), etc.

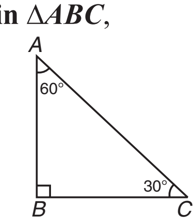


9. 45º – 45º – 90º triangle: ***If the angles of a triangle are 45º, 45º and 90º, then the hypotenuse (the longest side) is $\sqrt{2}$ times any smaller side, i.e.,***
In ΔABC, $\angle B = 90°$, $\angle A = \angle C = 45°$, then

$\boldsymbol{AB = BC}$ **and** $\boldsymbol{AC = \sqrt{2}\,AB}$ **or** $\boldsymbol{\sqrt{2}\,BC}$.

The converse of the above theorem: If the ratio of the sides of a triangle is $1 : 1 : \sqrt{2}$, then it is a 45º – 45º – 90º triangle.

10. 30º – 60º – 90º triangle: ***If the angles of a triangle are 30º, 60º and 90º, then the side opposite to the 30º angle is half of the hypotenuse and side opposite to 60º is $\frac{\sqrt{3}}{2}$ times the hypotenuse, i.e.,*** **in ΔABC**, where $\angle B = 90°$ and $\angle C = 30°$, $\angle A = 60°$, then

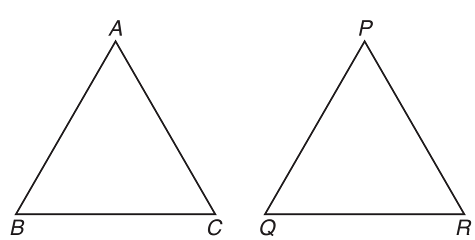


$\boldsymbol{AB = \frac{1}{2}\,AC}$ **and** $\boldsymbol{BC = \frac{\sqrt{3}}{2}\,AC}$, then

i.e., the ratio of the sides is $1 : \sqrt{3} : 2$. The converse of the above also holds true.

IX. Congruency of Triangles: **Two triangles are congruent to each other**

(*i*) ***if each of the three sides and three angles of one triangle are equal to the respective sides and angles of the other.***

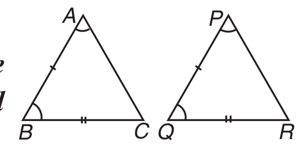


$\Delta ABC \cong \Delta PQR$ if

$AB = PQ$		$\angle A = \angle P$
$AC = PR$	and	$\angle B = \angle Q$
$BC = QR$		$\angle C = \angle R$

Tests of Congruency:

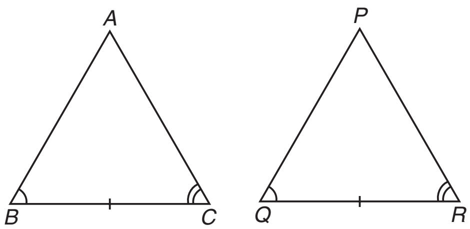
(*i*) **SAS Axiom (Side-angle-side):** ***If the two sides and the included angle of one triangle are respectively equal to the two sides and the included angle of the other, the triangles are congruent.***



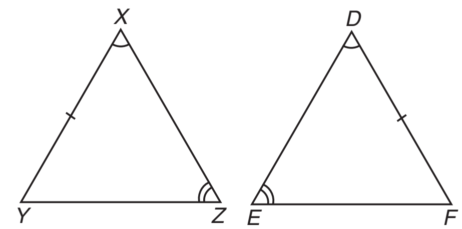
$AB = PQ$
$AC = PR$
$BC = QR$

Note: The two equal sides must be opposite to angles which are known to be equal.

(*ii*) **ASA or AAS Axiom (Two angles, corresponding side):** ***If two angles and one side of a triangle are respectively equal to two angles and the corresponding side of the other triangles, the triangles are congruent. The side may be in included side.***

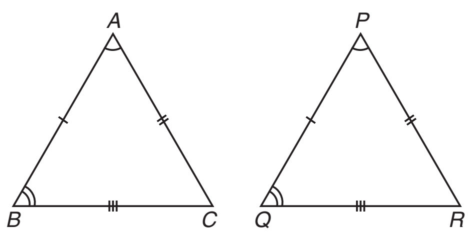


$\angle B = \angle Q$
$\angle C = \angle R$
$BC = QR$
$\Delta ABC \cong \Delta PQR$ (ASA)



$\angle X = \angle D$
$\angle Z = \angle E \quad \therefore \Delta XYZ \cong \Delta DEF$ (AAS)
$XY = DF$

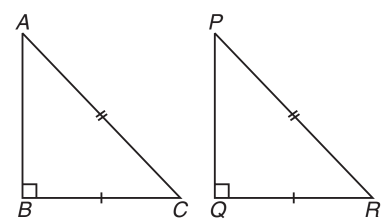
(*iii*) **SSS Axiom (Three sides):** ***If three sides of one triangle are respectively equal to the corresponding three sides of the other triangle, the triangles are congruent :***



$$\left.\begin{array}{l} AB = PQ \\ AC = PR \\ BC = QR \end{array}\right\} \Rightarrow \quad \Delta ABC \cong \Delta PQR \ \text{(SSS)}$$

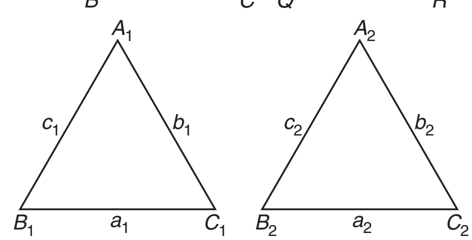
(*iv*) **RHS Axiom (Right angle-Hypotenuse-side):** ***If the hypotenuse and one side of a right angled triangle are respectively equal to the hypotenuse and corresponding side of the other right angled triangle, the two triangles are congruent.***

$\therefore \quad AC = PR$, $BC = QR$ and $\angle B = \angle Q = 90^\circ \Rightarrow \Delta ABC \cong PQR \quad (RHS)$



X. Similarity of Triangles:

Two triangles are said to be similar, if their corresponding angles are equal, and their corresponding sides are proportional.



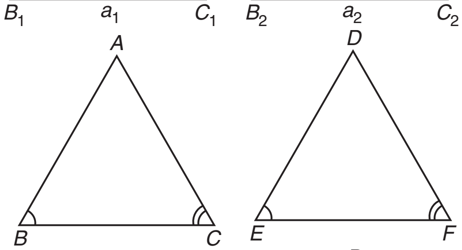
Thus, $\Delta A_1B_1C_1$ is similar to $\Delta A_2B_2C_2$ or $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$ if

(*i*) $\angle A_1 = \angle A_2$; $\angle B_1 = \angle B_2$; $\angle C_1 = \angle C_2$

(*ii*) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = k$

Tests for Similarity:

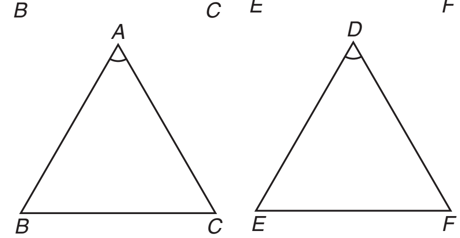
(*i*) **A-A axiom of similarily:** ***If two angles of one triangle are equal to the corresponding two angles of the other triangle, then the triangles are said to be similar.***



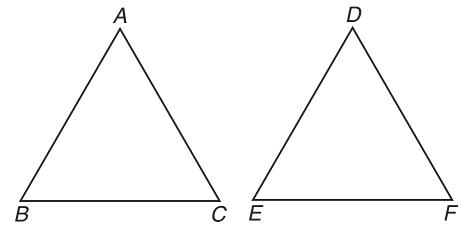
Note : If two pairs of corresponding angles in two triangles are equal, then the third pair will obviously be equal.

$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE \Rightarrow \Delta ABC \sim \Delta DEF$. (*AA similarity*)

(*ii*) **S-A-S axiom of similarity:** ***If two triangles have a pair of corresponding angles equal and the sides including them proportional, then the triangles are similar.***



In Δs ABC and DEF, $\angle A = \angle D$, $\dfrac{AB}{DE} = \dfrac{AC}{DF} = k$, then $\Delta ABC \sim \Delta DEF$.

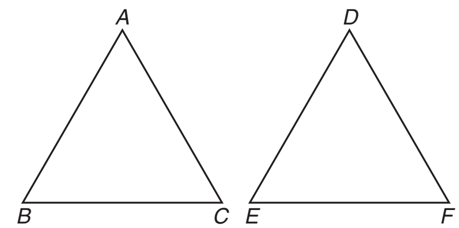


(*iii*) S-S-S axiom of similarity: ***If two triangles have their pairs of corresponding sides proportional, then the triangles are similar.***

If $\dfrac{BC}{EF} = \dfrac{AB}{DE} = \dfrac{AC}{DF}$, then, $\Delta ABC \sim \Delta DEF$

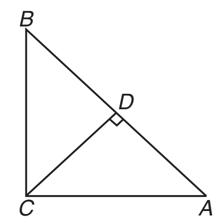
XI. Properties of Similar Triangles:

For a pair of similar triangles,



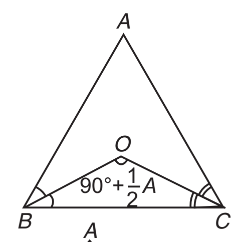
1. **Ratio of sides** ***= Ratio of Altitudes***
   = Ratio of Medians
   = Ratio of angle bisectors
   = Ratio of in-radii
   = Ratio of circums-radii.
2. **Ratio of areas =** ***Ratio of squares of corresponding sides.***

$$\frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2} = \frac{BC^2}{EF^2}$$

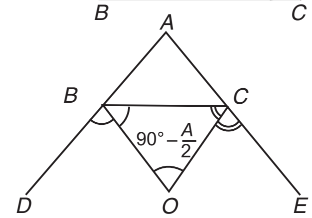


3. **In a right angled triangle*****, the triangles on each side of the altitude drawn from the vertex to the right angle to the hypotenuse are similar to the original triangle and to each other too*****.**
   i.e.**, $\Delta BCA \sim \Delta BDC \sim \Delta CDA$.**

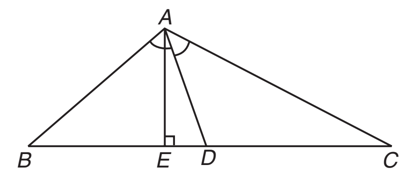
XII. Some Useful Results for a Triangle:



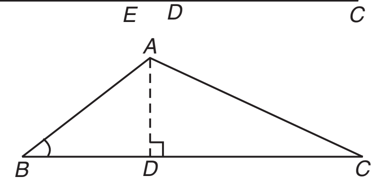
1. In a ΔABC, if the bisectors of $\angle ABC$ and $\angle ACB$ meet at O, then $\boldsymbol{\angle BOC = 90° + \frac{1}{2}\angle A}$.



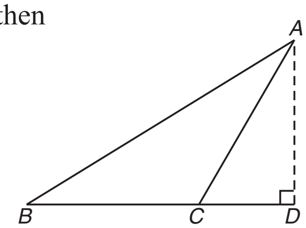
2. In a ΔABC, if the sides AB and AC are produced to D and E respectively and the bisectors of $\angle DBC$ and $\angle ECB$ intersect at O, then $\boldsymbol{\angle BOC = 90° - \frac{\angle A}{2}}$.



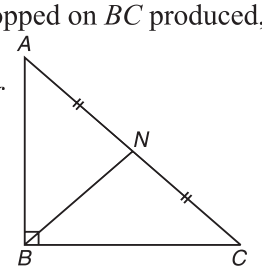
3. In a ΔABC, if AD is the angle bisector of $\angle BAC$ and $AE \perp BC$, then $\boldsymbol{\angle DAE = \frac{1}{2}\angle ABC - \angle ACB}$.



4. In an acute angled triangle, ABC, AD is the perpendicular dropped on BC, then $\boldsymbol{AC^2 = AB^2 + BC^2 - 2BD.DC}$.
5. In an obtuse angled triangle, if AD is the perpendicular dropped on BC produced, then $\boldsymbol{AC^2 = AB^2 + BC^2 + 2BD.DC}$.

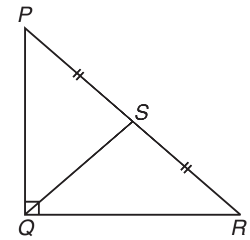


6. **In a right angled Δ,** ***the median to the hypotenuse is half the hypotenuse.***
   BN is the median from *B* on *AC* such that $AN = NC$.
   Then, $\boldsymbol{BN = \frac{1}{2}AC = AN = NC}$.



7. If in a right angled ΔPQR, $\angle Q = 90^\circ$, PR is the hypotenuse, and a perpendicular QS is dropped on the hypotenuse from the right angle vertex Q, then

(*i*) $\boldsymbol{QS = \dfrac{PQ \times QR}{PR}}$ (*ii*) $\boldsymbol{\dfrac{1}{QS^2} = \dfrac{1}{PQ^2} + \dfrac{1}{QR^2}}$



XIII. Area Formulae for a Triangle:

1. General Formula: Area of a triangle = $\boldsymbol{\dfrac{1}{2} \times \text{base} \times \text{height}}$

2. • **Area of a scalene triangle =** $\boldsymbol{\sqrt{s(s-a)(s-b)(s-c)}}$, where a, b, c are the sides of the triangle and s is the semi-perimeter of the triangle, *i.e.*, $s = \dfrac{a+b+c}{2}$.

• Also, **Area of a** $\boldsymbol{\Delta = r \times s = \dfrac{abc}{4R}}$, where $r \to$ in radius, $R \to$ circumradius.

3. Area of a right angled triangle = $\boldsymbol{\dfrac{1}{2} \times \text{base} \times \text{height.}}$

For a right angled triangle,

(*i*) **Inradius =** $\boldsymbol{\dfrac{AB + BC - AC}{2}}$, where ΔABC is rt $\angle d$ at $\angle B$

(*ii*) **Inradius =** $\boldsymbol{\dfrac{\text{Area}}{\text{Semi-perimeter}}}$

(*iii*) **Circum radius =** $\boldsymbol{\dfrac{\text{Hypotenuse}}{2}}$

4. Area of an isosceles triangle = $\boldsymbol{\dfrac{b}{4}\sqrt{4a^2 - b^2}}$, where a is the length of one of the equal sides, b is the length of third side.

5. For an equilateral triangle,

Area = $\boldsymbol{\dfrac{\sqrt{3}}{4}}$ **(side)**^{2}, **Inradius =** $\boldsymbol{\dfrac{\text{Side}}{2\sqrt{3}}}$, **Circumradius =** $\boldsymbol{\dfrac{\text{Side}}{\sqrt{3}}}$

$\therefore$ **Circumradius = 2 × Inradius.**

> **Note :** (*i*) *For the given perimeter of a triangle*, **the area of an equilateral triangle is maximum.**
> (*ii*) *For the given area of a triangle*, **the perimeter of an equilateral triangle is minimum.**

6. Two triangles on equal (or same bases and lying between same parallel lines have equal area.)

SOLVED EXAMPLES

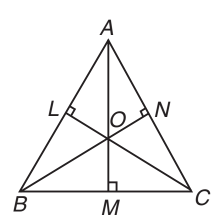
Ex. 1. Let *O* be any point inside a triangle *ABC*. Let *L*, *M* and *N* be the points on *AB*, *BC* and *CA* respectively, where perpendicular from *O* meet these lines. Show that :

$\boldsymbol{AL^2 + BM^2 + CN^2 = AN^2 + CM^2 + BL^2}$

Sol. Join O to A, B and C

In right Δs OAL, OBM and OCN.

$$\left.\begin{aligned} OL^2 + AL^2 &= OA^2 \quad \ldots(i) \\ OM^2 + BM^2 &= OB^2 \quad \ldots(ii) \\ ON^2 + CN^2 &= OC^2 \quad \ldots(iii) \end{aligned}\right\} \left(\begin{array}{c}\text{Pythagoras'} \\ \text{Theoram}\end{array}\right)$$



$\therefore$ Adding (*i*), (*ii*) and (*iii*), we get

$OL^2 + AL^2 + OM^2 + BM^2 + ON^2 + CN^2 = OA^2 + OB^2 + OC^2$

$AL^2 + BM^2 + CN^2 = (OA^2 + OB^2 + OC^2) - (OL^2 + OM^2 + ON^2)$ ...(1)

Similarly in right Δs OAN, OBL and OMC,

$ON^2 + AN^2 = OA^2$ ...(*iv*)

$OL^2 + BL^2 = OB^2$ ...(*v*)

$OM^2 + CM^2 = OC^2$ ...(*vi*)

$\therefore$ Adding (*iv*), (*v*) and (*vi*)

$ON^2 + AN^2 + OL^2 + BL^2 + OM^2 + CM^2 = OA^2 + OB^2 + OC^2$

$\Rightarrow$ $AN^2 + BL^2 + CM^2 = (OA^2 + OB^2 + OC^2) - (ON^2 + OL^2 + OM^2)$ ...(2)

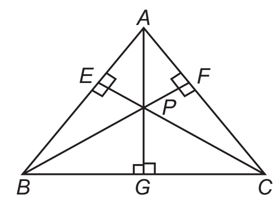
From (1) and (2)

$AL^2 + BM^2 + CN^2 = AN^2 + BL^2 + CM^2$

Ex. 2. A point is selected at random inside an equilateral triangle. From this point a perpendicular is dropped to each side. Prove that the sum of these perpendiculars is independent of the location of the given point.

Sol. Let P be any point in the equilateral ΔABC with each side = S units.

Let PE, PF and PG be the lengths of the perpendiculars from P on AB, AC and BC respectively. Then.



Area of ΔABC = Area (ΔPAB) + Area (ΔPBC) + Area (ΔPAC)

$$= \frac{1}{2} \times AB \times PE + \frac{1}{2} \times AC \times PF + \frac{1}{2} \times BC \times PG$$

$$= \frac{1}{2} \times S \times (PE + PF + PG)$$

where $AB = BC = CA = S$

Also, let h be the height be ΔABC, then

Area of $\Delta ABC = \frac{1}{2} \times S \times h$

$\therefore$ $\frac{1}{2} \times S \times h = \frac{1}{2} \times S \times (PE + PF + PG)$

$\Rightarrow$ $h = (PE + PF + PG)$

$\Rightarrow$ Sum of the perpendiculars = height of equilateral Δ = a constant

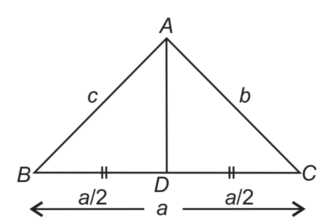
$\Rightarrow$ Sum of perpendiculars is independent of location of P.

Ex. 3. In a ΔABC, the lengths of sides BC, CA and AB are a, b and c respectively. Median AD drawn from A is perpendicular to BC. Express b in terms of a and c.

Sol. By Appolonius theorem, in ΔABC,

$AB^2 + AC^2 = 2\,(AD^2 + BD^2)$

$c^2 + b^2 = 2\left(c^2 - \frac{a^2}{4} + \frac{a^2}{4}\right)$ ($\because$ In rt. ΔABD, $AD = \sqrt{AB^2 - BD^2}$)



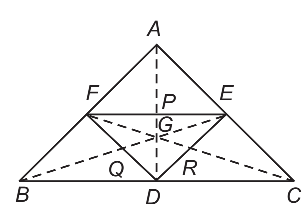
$\Rightarrow$ $b^2 + c^2 = a^2 - 2c^2$

$\Rightarrow$ $b^2 = a^2 - 3c^2$

$\Rightarrow$ $b = \sqrt{a^2 - 3c^2}$

Ex. 4. If the medians AD, BE and CF of ΔABC meet at G, prove that G is the centroid of ΔDEF also.

Sol. Since D and E are the mid-points of sides BC and AC of ΔABC, therefore,



$DE \parallel BA$ and $DE = \frac{1}{2}BA$ (By mid-point theorem)

$\Rightarrow$ $DE \parallel FA$ and $DE = FA$ ($\because$ $FA = \frac{1}{2}$ BA)

Also, $DF \parallel AC$ and $DF = \frac{1}{2}AC$ (By mid-point theorem)

$\Rightarrow$ $DF \parallel AE$ and $DF = AE$.

$\therefore$ $DEAF$ is a parallelogram, whose diagonals AD and FE intersect at P.

Since the diagonals of a parallelogram bisect each other, therefore, $AP = PD$ and $FP = PE$

$\Rightarrow$ P is the mid-point of FE

$\Rightarrow$ DP is the median of ΔDEF.

Similarly it can be shown that, $FDEC$ is a parallelogram and hence R is the mid-point of ED and hence FR is the median of ΔDEF.

$\therefore$ Medians DP and FR intersect at point G, where G is the centroid of ΔDEF.

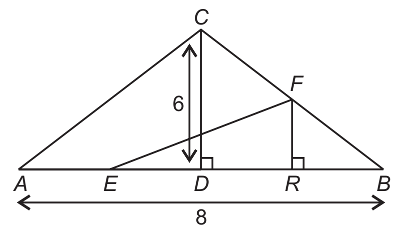
Ex. 5. *ABC* is an acute angled triangle. *CD* is the altitude through *C*. If *AB* = 8 units, *CD* = 6 units, find the distance between the mid-points of *AD* and *BC*.

Sol. Let E be the mid-point of AD and F the mid-point of BC.

Draw $FR \perp AB$.

$\because$ $CD \perp AB \Rightarrow FR \parallel CD$.

Also, F being the mid-point of BC and $FR \parallel CD \Rightarrow R$ is the mid-point of BD (By converse of mid-point theorem).



Now by basic proportionality theorem, $FR = \frac{1}{2}\ CD = FR = 3$ units.

Also, $ER = ED + DR = \frac{1}{2}(AD + DB) = \frac{1}{2} \times AB\ = 4$ units.

$\therefore$ In ΔFER, $EF = \sqrt{FR^2 + ER^2} = \sqrt{9+16} = \sqrt{25} = \mathbf{5}$ **units.**

Ex. 6. In the given figure, $\angle MON = \angle MPO = \angle NQO = 90^\circ$ and *OQ* is the bisector of $\angle MON$ and *QN* = 10, *OR* = 40/7. Find *OP*. (*CDS 2012*)

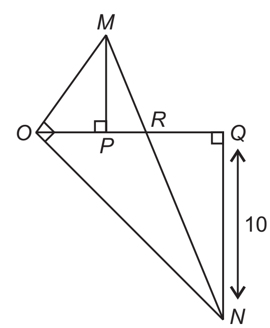
Sol. In ΔOMP, $\angle MOP = 45^\circ$ (OQ bisects $\angle MON$)

$\therefore$ $\angle OMP = 45^\circ$.

$\Rightarrow$ $OP = PM = x$ (say) (sides opp. equal $\angle s$ are equal)

Also in OQN, $\angle QON = 45^\circ$ (OQ bisects $\angle MON$)

$\Rightarrow$ $\angle QNO = 45^\circ \Rightarrow OQ = ON = 10$ (sides opp. equal $\angle s$ are equal)



$\therefore$ $QR = OQ - OR = 10 - \frac{40}{7} = \frac{30}{7}$.

$\Delta PMR \sim \Delta QNS$, since $\angle MPR = \angle NQR = 90^\circ$ and $\angle MRP = \angle QRP$ (vert. opp. $\angle s$)

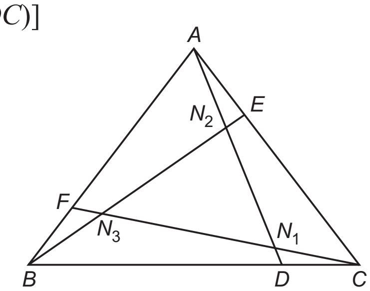
$$\Rightarrow \frac{PM}{PR} = \frac{QN}{QR} \Rightarrow \frac{x}{\frac{40}{7} - x} = \frac{10}{\frac{30}{7}} = \frac{7}{3}$$

$$\Rightarrow \frac{7x}{40 - 7x} = \frac{7}{3} \Rightarrow 21x = 280 - 49x \Rightarrow x = \mathbf{4.}$$

Ex. 7. In the figure *CD*, *AE* and *BF* are one-third of their respective sides. It is given that $AN_2 : N_2N_1 : N_1D$ = 3 : 3 : 1 and similarly for the lines *BE* and *CF*. Show that the area of $\Delta N_1 N_2 N_3$ is $\frac{1}{7}$ Area (ΔABC).

Sol. Ar $(\Delta N_1 N_2 N_3)$ = Area (ΔABC) – [(Area (ΔCBF) + Area (ΔABE) + Area (ΔADC)]

+ (Area (ΔN_1CD) + Area (N_2AE) + Area (ΔN_3FB))



$\because$ $AE = \frac{1}{3}AC$ and height of ΔABE and ΔABC is same, keeping AC as the base,

Area of $\Delta ABE = \frac{1}{3}$ Area (ΔABC) (Area $= \frac{1}{2} \times b \times h$)

Similarly, Area (ΔCBF) = Area (ΔADC) = $\frac{1}{3}$ Area (ΔABC)

In ΔCBF, $FN_3 : N_3N_1 : N_1C = 3 : 3 : 1 \Rightarrow$ Area of $\Delta N_1CD = \frac{1}{7}$ Area (ΔBFC)

$\Rightarrow$ Area (ΔN_1CD) = $\frac{1}{7} \times \frac{1}{3}$ Area (ΔABC) = $\frac{1}{21}$ Area (ΔABC)

Similarly, Area (ΔN_2AE) = Area ($\Delta N_3\,FB$) = $\frac{1}{21}$ Area (ΔABC)

$\therefore$ Area ($\Delta N_1N_2N_3$) = Area (ΔABC) – $3 \cdot \frac{1}{3}$ Area (ΔABC) + $3 \cdot \frac{1}{21}$ Area (ΔABC) = $\mathbf{\frac{1}{7}}$ **Area (ΔABC)**

Ex. 8. In the diagram AB and AC are the equal sides of an isosceles triangle ABC, in which is inscribed equilateral triangle DEF. Designate angle BFD by a, angle ADE by b and angle FEC by c. Then show that $a = \frac{b+c}{2}$.

Sol. For ΔBDF, ext $\angle ADF = \angle B + a$ …(i)

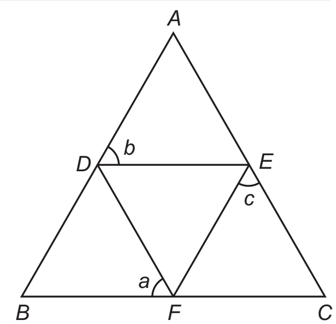
$\Rightarrow$ $b + 60^\circ = \angle B + a$ …(ii)

Similarly $a + 60^\circ = c + \angle C$

$\therefore$ Eq (i) – Eq (ii) $\Rightarrow b - a = a - c + \angle B - \angle C$

$\because$ $AB = AC \Rightarrow \angle B = \angle C$ (Isosceles Δ property)

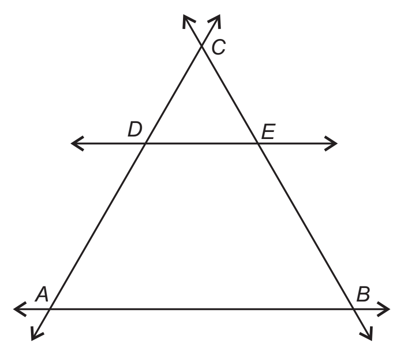
$\therefore$ $b - a = a - c \Rightarrow b + c = 2a \Rightarrow a = \mathbf{\frac{b+c}{2}}$.



Ex. 9. In the given figure, line DE is parallel to line AB. $CD = 3$ while $DA = 6$. Which of the following must be true?

I. $\Delta CDE \sim \Delta CAB$ **II. $\frac{\text{Area of } \Delta CDE}{\text{Area of } \Delta CAB} = \left(\frac{CD}{CA}\right)^2$** **III. If $AB = 4$, then $DE = 2$**

Sol. I. Since $DE \parallel AB$, $\angle CDE = \angle CAB$
$\angle CED = \angle CBA$
$\therefore$ $\Delta CDE \sim \Delta CAB$ (AA similarly)
} Corresponding angles



II. Since ratio of the areas of similar triangles is the square of the ratio of the corresponding sides.

$\therefore$ $\frac{\text{Area } (\Delta CDE)}{\text{Area } (\Delta CAB)} = \left(\frac{CD}{CA}\right)^2$

III. $CA = CD + DA = 3 + 6 = 9$.

$\therefore$ $\Delta CDE \sim \Delta CAB \Rightarrow \frac{CD}{CA} = \frac{DE}{AB}$

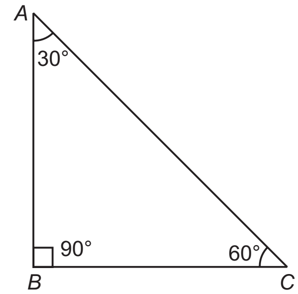
$\Rightarrow$ $\frac{3}{9} = \frac{DE}{4} \Rightarrow DE = \frac{12}{9} = \frac{4}{3}$

$\therefore$ If $AB = 4$, then $DE = \frac{4}{3}$

$\therefore$ I and II are true and III is false.

Ex. 10. If the angles of a triangle are in the ratio 1 : 2 : 3, then find the ratio of the corresponding opposite sides.

Sol. If the angles are in the ratio 1 : 2 : 3, then the angles of the triangle are 30º, 60º, 90º. Therefore, the triangle is a right angled triangle.



The side ratios opposite to the angles 30º, 60º and 90º is $BC : AB : AC$, *i.e.*, AC sin 30º; AC sin 60º : AC,

i.e., $-\frac{1}{2}\times AC : \frac{\sqrt{3}}{2}\times AC : AC$, *i.e.*, $\mathbf{1 : \sqrt{3} : 2.}$

Ex. 11. The angles of a triangle are in the ratio 8 : 3 : 1. What is the ratio of the longest side of the triangle to the next longest side?

Sol. The angles are $\frac{8}{12}\times 180^\circ, \frac{3}{12}\times 180^\circ, \frac{1}{12}\times 180^\circ$ *i.e.*, 120º, 45º and 15º.

Also we know that the longest side is opposite the greatest angle and so on.

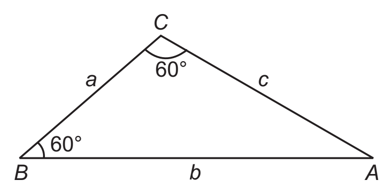
$\therefore$ Let the longest side opposite the greatest angle 120º be x and let the next longest side opposite angle 45º be y. Then, by the sine rule

$$\frac{\sin 120^\circ}{x} = \frac{\sin 45^\circ}{y}$$

$$\Rightarrow \quad \frac{x}{y} = \frac{\sin 120^\circ}{\sin 45^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{\mathbf{6}}}{\mathbf{2}}.$$

Ex. 12. In Δ*ABC*, *a* = 2*x*, *b* = 3*x* + 2, *c* = $\sqrt{12}$ and ∠*c* = 60º. Find *x*.

Sol. Here we use the law of cosines. So $c^2 = a^2 + b^2 - 2ab\cos C$



$\Rightarrow \quad 12 = (2x)^2 + (3x + 2)^2 - 2\,(2x)\,(3x + 2)\cos 60^\circ$

$\Rightarrow \quad 12 = 4x^2 + 9x^2 + 12x + 4 - (12x^2 + 8x) \times \frac{1}{2}$

$\Rightarrow \quad 7x^2 + 8x - 8 = 0$

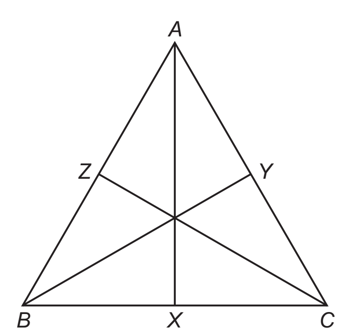
$$x = \frac{-8 \pm \sqrt{64+224}}{14} = \frac{-8 \pm \sqrt{288}}{14} \quad \left(\text{Recall that roots of a quadratic eq. } ax^2 + bx + c \text{ are } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right).$$

Since the side of a triangle must be positive, therefore,

$$x = \frac{-8+\sqrt{288}}{14} \simeq \mathbf{0.64.}$$

Ex. 13. The bisectors of the angles of a triangle *ABC* meet *BC*, *CA* and *AB* at *X*, *Y* and *Z* respectively. Prove that $\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB} = 1.$

Sol. We make use of the bisector theorem here, *i.e.*, the bisector (internal or external) of an angle of triangle divides the opposite side in the ratio of the sides containing the angle.



AX bisects $\angle A \Rightarrow \frac{BX}{XC} = \frac{AB}{AC}$ ...(*i*)

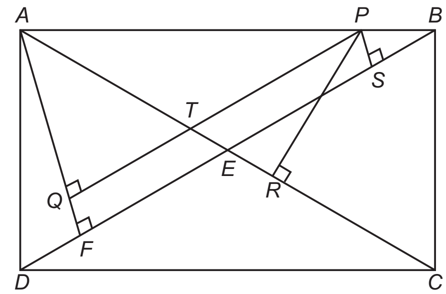
BY bisects $\angle B \Rightarrow \frac{CY}{YA} = \frac{BC}{BA}$ ...(*ii*)

CZ bisects $\angle C \Rightarrow \frac{AZ}{ZB} = \frac{CA}{CB}$ ...(*iii*)

$\therefore$ Multiplying (i), (ii) and (iii), we get

$$\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB} = \frac{AB}{AC}.\frac{BC}{BA}.\frac{CA}{CB} = \mathbf{1}.$$

Ex. 14. As given in the diagram alongside, P is any point on AB, $PS \perp BD$ and $PR \perp AC$. $AF \perp BD$ and $PQ \perp AF$. Then show that $PR + PS$ is equal to AF.



Sol. $\Delta PTR \sim \Delta ATQ$ (AA similarity)

($\because \angle ATQ = \angle PTR$ (vert. opp. $\angle s$) and $\angle PRT = \angle AQT = 90^\circ$)

$\therefore \quad \dfrac{PR}{AQ} = \dfrac{PT}{AT}$ …(i)

$\therefore \quad AF \perp BD$ and $PQ \perp AF \Rightarrow PQ \parallel BD \Rightarrow \angle TPA = \angle PBS$ …(ii)

Also, diagonals of a rectangle are equal and bisect each other

$\Rightarrow \quad EA = EB \Rightarrow \angle EAB = \angle EBA \Rightarrow \angle TAP = \angle PBS$ …(iii)

$\therefore$ From (i) and (ii), $\angle TPA = \angle TAP \Rightarrow PT = AT$ (Isosceles triangle property)

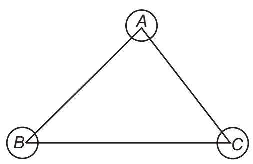
$\Rightarrow \quad PR = AQ$

Also, $PQ \parallel BD$ and $PS \perp BD$ and $QF \perp BD \Rightarrow PS = QF$

$\therefore \quad PR + PS = AQ + QF = AF$.

PRACTICE SHEET

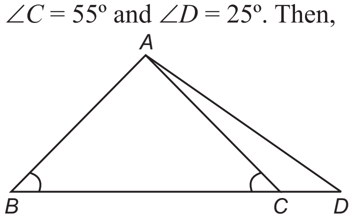
1. In the given figure, what is the sum of the angles formed around A, B, C except at angles of ΔABC.



(a) 360º (b) 720º
(c) 900º (d) 1000º

(CDS 2010)

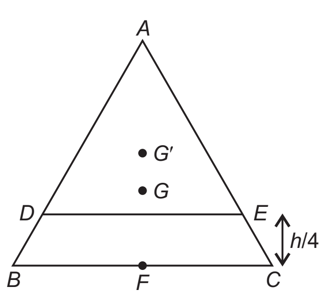
2. In the given figure, $\angle B = \angle C = 55^\circ$ and $\angle D = 25^\circ$. Then,



(a) $BC < CA < CD$
(b) $BC > CA > CD$
(c) $BC < CA, CA > CD$
(d) $BC > CA, CA < CD$

3. In-centre of a triangle lies in the interior of:
(a) An isosceles triangle only
(b) Any triangle
(c) An equilateral triangle only
(d) A right triangle only

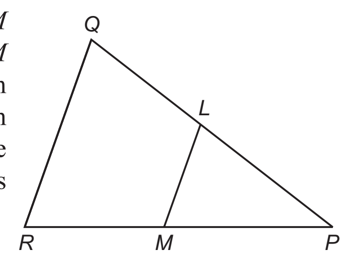
4. G is the centroid of ΔABC with height h units. If a line DE parallel to BC cuts ΔABC at a height $h/4$ from BC, find the distance GG' in terms of AG if G' is the centroid of ΔADE.



(a) $\dfrac{1}{2}AG$ (b) $\dfrac{3}{4}AG$
(c) $\dfrac{1}{4}AG$ (d) $\dfrac{2}{3}AG$

5. In an equilateral triangle if a, b and c denote the lengths of the perpendicular from A, B and C respectively on the opposite sides, then
(a) $a > b > c$ (b) $a > b < c$
(c) $a < b < c$ (d) $a = b = c$

6. In the given figure, LM is parallel to QR. If LM divides the ΔPQR such that the area of trapezium $LMRQ$ is two times the area of ΔPLM, then what is $\dfrac{PL}{LQ}$ equal to?



(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{3}$

(CDS 2011)

7. If the medians of two equilateral triangles are in the ratio 3 : 2, then what is the ratio of their sides?
(a) 1 : 1 (b) 2 : 3 (c) 3 : 2 (d) $\sqrt{3} : \sqrt{2}$

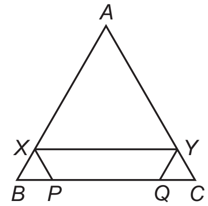
(CDS 2009)

8. If A is the area of the right angled triangle and b is one of the sides containing the right angle, then what is the length of the altitude on the hypotenuse?

(a) $\dfrac{2Ab}{\sqrt{b^4 + 4A^2}}$ (b) $\dfrac{2A^2b}{\sqrt{b^4 + 4A^2}}$

(c) $\dfrac{2Ab^2}{\sqrt{b^4 + 4A^2}}$ (d) $\dfrac{2A^2b^2}{\sqrt{b^4 + A^2}}$ ***(CDS 2008)***

9. In the given figure, ABC is an equilateral triangle of side length 30 cm. XY is parallel to BC, XP is parallel to AC and YQ is parallel to AB. If $(XY + XP + YQ)$ is 40 cm, then what is PQ equal to?



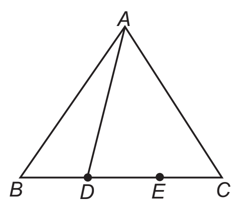
(a) 5 cm (b) 12 cm
(c) 15 cm (d) None of these **(CDS 2010)**

10. If AD is the median of ΔABC, then
(a) $AB^2 + AC^2 = 2AD^2 + 2BD^2$
(b) $AB^2 + AC^2 = 2AD^2 + BD^2$
(c) $AB^2 + AC^2 = AD^2 + BD^2$
(d) $AB^2 + AC^2 = AD^2 + 2BD^2$

11. Let ABC be a triangle of area 16 cm^2. XY is drawn parallel to BC dividing AB in the ratio 3 : 5. If BY is joined, then the area of triangle BXY is
(a) 3.5 cm^2 (b) 3.7 cm^2 (c) 3.75 cm^2 (d) 4.0 cm^2

12. From a point O in the interior of a ΔABC if perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB respectively, then which of the following statements is true?
(a) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$
(b) $AB^2 + BC^2 = AC^2$
(c) $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2$
(d) $AF^2 + BD^2 + CE^2 = OD^2 + OE^2 + OF^2$

13. In an equilateral triangle ABC, the side BC is trisected at D. Then AD^2 is equal to



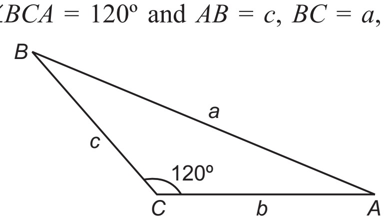
(a) $\frac{9}{7}AB^2$ (b) $\frac{7}{9}AB^2$
(c) $\frac{3}{4}AB^2$ (d) $\frac{4}{5}AB^2$

14. In a ΔABC, $AB = 10$ cm, $BC = 12$ cm and $AC = 14$ cm. Find the length of median AD. If G is the centroid, find the length of GA.
(a) $\frac{5}{3}\sqrt{7}, \frac{5}{9}\sqrt{7}$ (b) $5\sqrt{7}, 4\sqrt{7}$
(c) $\frac{10}{\sqrt{3}}, \frac{8}{3}\sqrt{7}$ (d) $4\sqrt{7}, \frac{8}{3}\sqrt{7}$

15. If a, b, c are the sides of a triangle and $a^2 + b^2 + c^2 = bc + ca + ab$, then the triangle is:
(a) equilateral (b) isosceles
(c) right-angled (d) obtuse angled
(CAT 2000)

16. H is the orthocentre of ΔABC whose altitudes are AD, BE and CF. Then the orthocentre of ΔHBC is
(a) F (b) E (c) A (d) D

17. In the given figure, $\angle BCA = 120°$ and $AB = c$, $BC = a$, $AC = b$. Then



(a) $c^2 = a^2 + b^2 + ba$
(b) $c^2 = a^2 + b^2 - ba$
(c) $c^2 = a^2 + b^2 - 2ba$
(d) $c^2 = a^2 + b^2 + 2ab$

18. In the ΔABC, $AB = 2$ cm, $BC = 3$ cm and $AC = 4$ cm. D is the middle-point of AC. If a square is constructed on the side BD, what is the area of the square?
(a) 4.5 cm^2 (b) 2.5 cm^2 (c) 6.35 cm^2 (d) None of these
(CDS 2009)

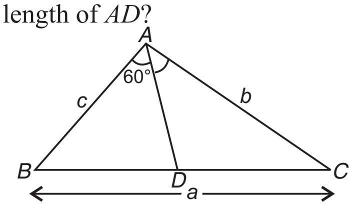
19. In the given figure, (not drawn to scale), P is a point on AB such that $AP : PB = 4 : 3$. PQ is parallel to AC and QD is parallel to CP. In ΔARC, $\angle ARC = 90°$ and in ΔPQS, $\angle PSQ = 90°$. The length of $QS = 6$ cm. What is the ratio of $AP : PD$?
(a) 10 : 3 (b) 2 : 1 (c) 7 : 3 (d) 8 : 3
(CAT 2003)

20. In ΔLMN, LO is the median. Also LO is the bisector of $\angle MLN$. If $LO = 3$ cm, and $LM = 5$ cm, then find the area of ΔLMN.
(a) 12 cm^2 (b) 10 cm^2 (c) 4 cm^2 (d) 6 cm^2
(CAT 2009)

21. A point within an equilateral triangle whose perimeter is 30 m is 2 m from one side and 3 m from another side. Find its distance from third side.
(a) $\sqrt{5} - 3$ (b) $5\sqrt{3} - 5$ (c) $5\sqrt{5} - 3$ (d) $5\sqrt{3} - 3$

22. A city has a park shaped as a right angled triangle. The length of the longest side of this park is 80 m. The Mayor of the city wants to construct three paths from the corner point opposite to the longest side such that these paths divide the longest side into four equal segments. Determine the sum of the squares of the lengths of the three paths.
(a) 4000 m (b) 4800 m (c) 5600 m (d) 6400 m
(XAT 2012)

23. In a triangle ABC, AD is the angle bisector of $\angle BAC$ and $\angle BAD = 60°$. What is the length of AD?

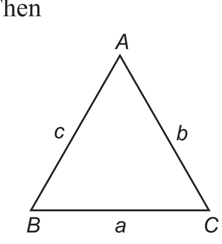


(a) $\frac{b+c}{bc}$ (b) $\frac{bc}{b+c}$
(c) $\sqrt{b^2+c^2}$ (d) $\frac{(b+c)^2}{bc}$

24. In a ΔABC, the internal bisector of angle A meets BC at D. If $AB = 4$, $AC = 3$ and $\angle A = 60°$, then the length of AD is
(a) $2\sqrt{3}$ (b) $\frac{12\sqrt{3}}{7}$ (c) $\frac{15\sqrt{3}}{8}$ (d) $\frac{6\sqrt{3}}{7}$
(CAT 2002)

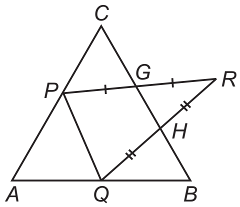
25. Suppose the medians PP' and QQ' of ΔPQR intersect at right angles. If $QR = 3$ and $PR = 4$, then the length of side PQ is
(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) $\sqrt{5}$ (d) $\sqrt{6}$

26. In the given triangle ABC, the length of sides AB and AC is same ($i.e.$, $b = c$) and $60° < A < 90°$. Then



(a) $b < a < b\sqrt{3}$
(b) $c < a < c\sqrt{2}$
(c) $b < a < 2b$
(d) $\frac{c}{3} < a < 3c$

27. In the given figure, P and Q are the mid-points of AC and AB. Also, $PG = GR$ and $HQ = HR$. What is the ratio of the Area of ΔPQR : Area of ΔABC



(a) 1 : 2 (b) 2 : 3
(c) 3 : 4 (d) 3 : 5

28. The lengths of the sides a, b, c of a ΔABC are connected by the relation $a^2 + b^2 = 5c^2$. The angle between medians drawn to the sides 'a' and 'b' is

(a) 60º (b) 45º (c) 90º (d) None of these

29. ABC is a triangle with $\angle BAC = 60º$. A point P lies on one-third of the way from B to C and AP bisects $\angle BAC$. $\angle APC$ equals

(a) 90º (b) 45º (c) 60º (d) 120º

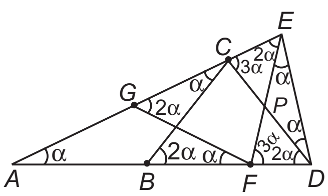
(XAT 2007)

30. A rectangle inscribed in a triangle has its base coinciding with the base $\boldsymbol{b}$ of the triangle. If the altitude of the triangle is $\boldsymbol{h}$, and the altitude $\boldsymbol{x}$ of the rectangle is half the base of the rectangle, then

(a) $x = \frac{1}{2}h$ (b) $x = \frac{bh}{h+b}$

(c) $x = \frac{bh}{2h+b}$ (d) $x = \sqrt{\frac{hb}{2}}$ ***(FMS 2011)***

31. In the given figure $AB = BC = CD = DE = EF = FG = GA$. Then $\angle DAE$ is approximately:



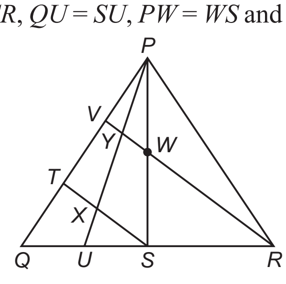
(a) 15º (b) 20º
(c) 30º (d) 25º

(CAT 2000)

32. If ABC is a triangle in which $\angle B = 2\ \angle C$. D is a point on side BC such that AD bisects $\angle BAC$ and $AD = CD$, then $\angle BAC =$

(a) 62º (b) 72º (c) 76º (d) 84º

33. In the figure shown here, $QS = SR$, $QU = SU$, $PW = WS$ and $ST \parallel RV$. What is the value of $\frac{\text{Area of } \Delta PSX}{\text{Area of } \Delta PQR}$?



(a) $\frac{1}{5}$ (b) $\frac{1}{3}$
(c) $\frac{1}{6}$ (d) $\frac{1}{7}$

34. In a ΔABC, AD, BE and CF are the medians drawn from the vertices A, B and C respectively. Then study the following statements and choose the correct option.

I. $3\ (AB + BC + AC) > 2\ (AD + BE + CF)$
II. $3\ (AB + BC + AC) < 2\ (AD + BE + CF)$
III. $3\ (AB + BC + AC) < 4\ (AD + BE + CF)$
IV. $3\ (AB + BC + AC) > 4\ (AD + BE + CF)$

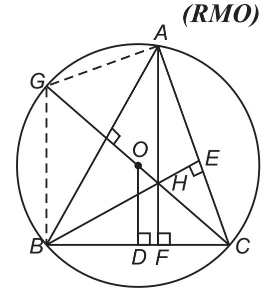
(a) I and IV are true (b) I and III are true
(c) II and IV are true (d) II and III are true

35. In a ΔABC, $AB = AC$. P and Q are points on AC and AB respectively such that $CB = BP = PQ = QA$. Then $\angle AQP =$

(a) $\frac{2\pi}{7}$ (b) 3π (c) $\frac{5\pi}{7}$ (d) $\frac{4\pi}{7}$

(RMO)

36. In a triangle, the ratio of the distance between a vertex and the orthocentre and the distance of the circumcentre from the side opposite the vertex is



(a) 3 : 1 (b) 4 : 1
(c) 2 : 1 (d) $\sqrt{2} : 1$

37. In a ΔABC, angle A is twice angle B. Then,

(a) $a^2 = b\ (a + c)$ (b) $a^2 = \sqrt{bc}$
(c) $a^2 = b\ (b + c)$ (d) $a^2 = b + c$

38. Let ABC be a triangle. Let D, E, F be points respectively on segments BC, CA, AB such that AD, BE and CF concur at point K. Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$, then

(a) $\angle ABE = \angle CAD$ (b) $\angle ABE = \angle AFC$
(c) $\angle ABE = \angle FKB$ (d) $\angle ABE = \angle BCF$

(RMO 2011)

39. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. $\angle BAC$ equals.

(a) 45º (b) 90º (c) 60º (d) 75º

(RMO 2009)

40. In a ΔABC, let D be the mid-point of BC. If $\angle ADB = 45º$ and $\angle ACD = 30º$. then $\angle BAD$ equals.

(a) 45º (b) 60º (c) 30º (d) 15º

(RMO 2005)

ANSWERS

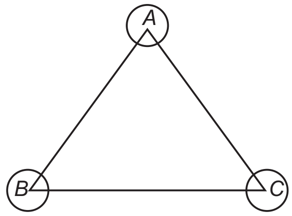
1. (c)	**2.** (d)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (a)
21. (b)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (b)	**27.** (a)	**28.** (c)	**29.** (d)	**30.** (c)
31. (d)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (b)	**40.** (c)

HINTS AND SOLUTIONS

1. $\angle A = 360^\circ$ – Ext. $\angle A$

$\angle B = 360^\circ$ – Ext. $\angle B$

$\angle C = 360^\circ$ – Ext. $\angle C$



We know,

$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow$ 360° – Ext. $\angle A$ + 360° – Ext. $\angle B$ + 360° – Ext. $\angle C = 180^\circ$

$\Rightarrow$ Ext. $\angle A$ + Ext. $\angle B$ + Ext. $\angle C = 1080^\circ - 180^\circ$ **= 900º.**

2. $\angle B = \angle C = 55^\circ \Rightarrow AC = AB$

$\Rightarrow \angle BAC = 180^\circ - (\angle B + \angle C) = 180^\circ - 110^\circ = 70^\circ$

Now in ΔACD, $\angle ACD = 180^\circ - 55^\circ = 125^\circ$

$\therefore\ \angle CAD = 180^\circ - (125^\circ + 25^\circ) = 30^\circ$

$\Rightarrow \angle CAD > \angle CDA \Rightarrow CD > AC$

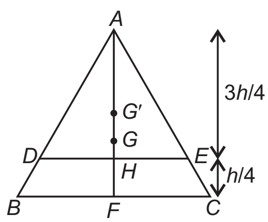
($\because$ In a given triangle, the greater angle has greater side opposite to it)

Also $\angle BAC > \angle ABC \Rightarrow BC > AC$

$\therefore$ ***BC* > *CA* and *CA* < *CD*.**

4. AF is the median to BC from A in ΔABC. G is the centroid of ΔABC

$\Rightarrow AG = \frac{2}{3}AF \ \Rightarrow AF = \frac{3}{2}AG.$



Now AH is the median to DE from A in ΔADE. G' is the centroid of ΔADE

$\Rightarrow AG' = \frac{2}{3}AH$

Since $DE \parallel BC$, $AH = \frac{3}{4}AF$

(By basic proportionality theorem)

$$\therefore\ AG' = \frac{2}{3}\times\frac{3}{4}AF = \frac{1}{2}AF = \frac{1}{2}\times\frac{3}{2}AG = \frac{3}{4}AG$$

$$\therefore\ GG' = AG - AG' = AG - \frac{3}{4}AG = \mathbf{\frac{1}{4}}\boldsymbol{AG.}$$

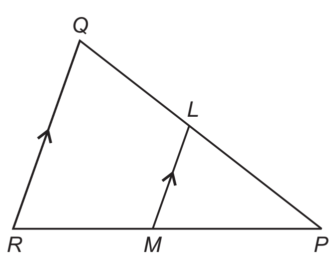
6. $\because\ LM \parallel QR$

$\left.\begin{aligned}\angle PLM = \angle PQR\\ \angle PML = \angle PRQ\end{aligned}\right\}$ *Corresponding* $\angle s$

$\Rightarrow \Delta PQR \sim \Delta PLM$

$$\therefore\ \frac{\text{Area of } \Delta PLM}{\text{Area of } \Delta PQR} = \frac{PL^2}{PQ^2} \qquad \ldots(i)$$

Area of ΔPQR = Area of ΔPLM + Area of trap. $LMRQ$



Given, Area of trapezium $LMRQ$ = 2 Area of ΔPLM

$\therefore$ Area of ΔPQR = Area of ΔPLM + 2 Area of ΔPLM

= 3 Area of ΔPLM

$\therefore$ From (i), we have

$$\frac{\text{Area of } \Delta PLM}{3\times \text{Area of } \Delta PLM} = \frac{PL^2}{PQ^2}$$

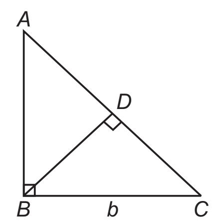
$$\Rightarrow \frac{PL^2}{PQ^2} = \frac{1}{3} \Rightarrow \frac{PL}{PQ} = \mathbf{\frac{1}{\sqrt{3}}}.$$

7. Median of an equilateral triangle = $\frac{\sqrt{3}}{2}\times$side

Let the sides of the two equilateral triangles be a_1 and a_2 respected. Then,

$$\frac{\frac{\sqrt{3}}{2}a_1}{\frac{\sqrt{3}}{2}a_2} = \frac{3}{2} \ \Rightarrow \frac{a_1}{a_2} = \mathbf{\frac{3}{2}}.$$

8. Let ABC be the given right angled triangle, right angled at B. Let $BC = b$.



Then, Area of $\Delta ABC = \frac{1}{2}\times BC\times AB$

$$\Rightarrow\ A = \frac{1}{2}\times b\times AB \Rightarrow AB = \frac{2A}{b}$$

In ΔABC, $AC^2 = AB^2 + BC^2$

$$AC^2 = \frac{4A^2}{b^2} + b^2$$

$$\Rightarrow\ AC = \sqrt{\frac{4A^2}{b^2} + b^2}$$

Again in ΔABC,

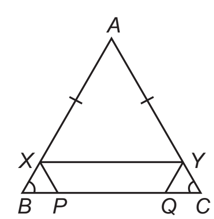
$$A = \frac{1}{2}\times AC\times BD$$

$$\Rightarrow\ A = \frac{1}{2}\times\sqrt{\frac{4A^2 + b^4}{b^2}}\times BD$$

$$\Rightarrow\ BD = \mathbf{\frac{2Ab}{\sqrt{4A^2 + b^4}}}.$$

9. $AB \parallel YQ \ \Rightarrow\ \angle XBP = \angle YQC$ (corresponding angles) and

$XP \parallel AC \ \Rightarrow\ \angle XPB = \angle YCQ$



ΔABC being an equilateral triangle,

$\angle B = \angle C = 60^\circ$

$\Rightarrow \angle XPB = \angle YQC = 60°$

$\Rightarrow \angle XPB = \angle QYC = 60^\circ$

$\Rightarrow \Delta XBP$ and ΔYQC are equilateral triangles.

Now, $XY \parallel BC \Rightarrow \frac{AX}{AB} = \frac{XY}{BC} \ \Rightarrow AX = XY \quad (\because\ AB = BC)$

Also, $XY + XP + YQ = 40 \Rightarrow AX + XB + YQ = 40$

($\because\ AX = XY, XP = XB$)

$\Rightarrow AB + YQ = 40 \Rightarrow YQ = 40 - AB = 40 - 30 = 10$

$\therefore\ XP = YQ = 10$ cm.

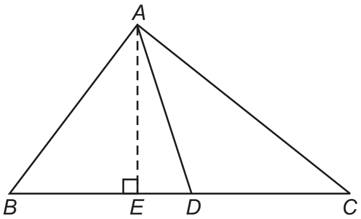
$\Rightarrow BP = QC = 10$ cm (ΔXBP and ΔYQC are equilateral)

$\Rightarrow PQ = BC - (BP + QC) = 30 - 10 - 10 =$ **10 cm.**

10. Let AD be the median of ΔABC

$\therefore BD = DC$

Let AE be the perpendicular from A on BC. Then,



In rt. ΔABE,

$AB^2 = BE^2 + AE^2$

$= (BD - ED)^2 + AE^2$

$= BD^2 + \underbrace{ED^2 + AE^2} - 2BD.ED$

$= BD^2 + AD^2 - 2BD.ED$ …(i)

In rt. ΔACE,

$AC^2 = AE^2 + EC^2$

$= AE^2 + (ED + DC)^2$

$= \underbrace{AE^2 + ED^2} + DC^2 + 2ED.DC$

$= AD^2 + DC^2 + 2ED.DC$

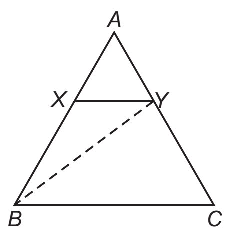
$= AD^2 + BD^2 + 2ED.BD$ $(\because BD = DC)$…(ii)

Adding (i) and (ii), we get

$\boldsymbol{AB^2 + AC^2 = 2AD^2 + 2BD^2}$**.**

11. Let the areas of ΔAXY and ΔBXY be A_1 and A_2 respectively. The heights of the ΔAXY and ΔBXY are equal, so

$$\frac{A_1}{A_2} = \frac{\frac{1}{2} \times AX \times \text{height}}{\frac{1}{2} \times BX \times \text{height}}$$



$$\Rightarrow \frac{A_1}{A_2} = \frac{AX}{BX} = \frac{3}{5} \quad \ldots(i)$$

Also, $\Delta AXY \sim \Delta ABC$

$$\Rightarrow \frac{\text{Area of } \Delta AXY}{\text{Area of } \Delta ABC} = \left(\frac{AX}{AB}\right)^2$$

$$\Rightarrow \frac{A_1}{16} = \left(\frac{3x}{3x+5x}\right)^2 \Rightarrow \frac{A_1}{16} = \frac{9}{64}$$

$$\Rightarrow A_1 = \frac{9}{64} \times 16 = \frac{9}{4} \quad \ldots(ii)$$

$\therefore$ From (i) and (ii) $A_2 = \frac{5}{3} \times A_1 = \frac{5}{3} \times \frac{9}{4} = \frac{15}{4}$ $\mathbf{= 3.75\ cm^2.}$

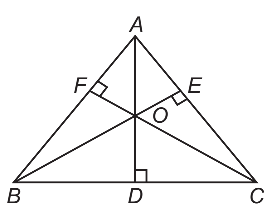
12. Join OA, OB and OC.

By Pythagoras, theorem,

In ΔAOF, $AF^2 = AO^2 - OF^2$ …(i)

In ΔBOD, $BD^2 = BO^2 - OD^2$ …(ii)

In ΔCOE, $CE^2 = CO^2 - OE^2$ …(iii)



Adding (i), (ii) and (iii), we get

$AF^2 + BD^2 + CE^2 = AO^2 - OF^2 + BO^2 - OD^2 + CO^2 - OE^2$

$= \underbrace{AO^2 - OE^2} + \underbrace{BO^2 - OF^2} + \underbrace{CO^2 - OD^2}$

$\boldsymbol{= AE^2 + BF^2 + CD^2}$**.**

13. ABC being an equilateral triangle, $AB = BC = AC$.

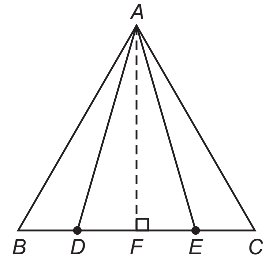
Also BC being trisected at D

$$\Rightarrow \quad BD = \frac{1}{3}BC \quad \ldots(i)$$

Let AF be drawn perpendicular to $BC \Rightarrow BF = FC$.

Also E is given as the other point of trisection, so $BD = DE = EC$

Also, $BD = 2DF$ …(ii)



Now $AB^2 = BF^2 + AF^2$ and

$AD^2 = DF^2 + AF^2$

Now $AB^2 = BF^2 + AF^2$

$$\Rightarrow AB^2 = \left(\frac{1}{2}BC\right)^2 + AF^2$$

$$\Rightarrow AB^2 = \frac{1}{4}BC^2 + AF^2 \quad (\because BC = AB)$$

$$\Rightarrow AB^2 = \frac{1}{4}AB^2 + AF^2 \Rightarrow AF^2 = \frac{3}{4}AB^2 \quad \ldots(iii)$$

Also, $AD^2 = DF^2 + AF^2$

$$= \left(\frac{1}{2} \times \frac{1}{3}BC\right)^2 + AF^2 \quad \text{(From } (i) \text{ and } (ii))$$

$$= \left(\frac{1}{6}BC\right)^2 + AF^2$$

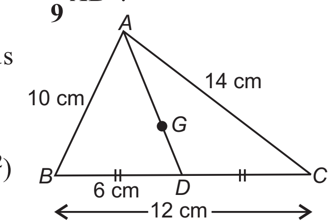
$$= \frac{1}{36}BC^2 + AF^2$$

$$= \frac{1}{36}AB^2 + \frac{3}{4}AB^2$$

($\because BC = AB$ and putting the value from (iii))

$$= \frac{28}{36}AB^2 \Rightarrow AD^2 = \boldsymbol{\frac{7}{9}AB^2}.$$

14. Applying the Apollosnius theorem,



$AB^2 + AC^2 = 2(BD^2 + AD^2)$

$\Rightarrow 100 + 196 = 2\,(36 + AD^2)$

$\Rightarrow 2AD^2 = 296 - 72 = 224$

$\Rightarrow AD^2 = 112 \Rightarrow AD = \mathbf{4\sqrt{7}}$

As G is the centroid, so $\frac{AG}{GD} = \frac{2}{1}$

$$\Rightarrow AG = \frac{2}{3}AD = \frac{2}{3} \times 4\sqrt{7} = \boldsymbol{\frac{8}{3}\sqrt{7}}.$$

15. $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

Sum of perfect squares = 0 $\Rightarrow$ Each term of the sum is zero

$\Rightarrow (a - b) = 0 = (b - c) = (c - a)$

$\Rightarrow a = b = c$

$\Rightarrow$ The triangle is equilateral.

16. H is the orthocentre of ΔABC

$\Rightarrow AD \perp BC$, $BE \perp CA$, $CF \perp AB$

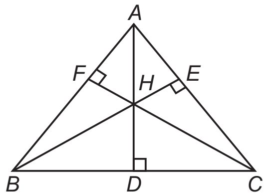
In ΔBHC,

$AD \perp BC \Rightarrow HD \perp BC$

$CF \perp AB \Rightarrow BF \perp HC$ (Produced)

$\Rightarrow$ The altitudes HD and BF of ΔHBC intersect in A.

$\Rightarrow A$ is the orthocentre of ΔHBC.



17. By the cosine rule, we have,

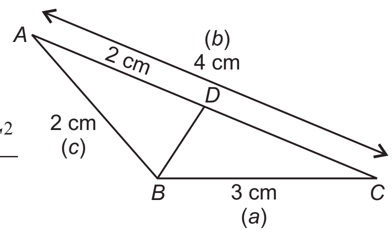
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \cos 120^\circ = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow -\frac{1}{2} = \frac{a^2+b^2-c^2}{2ab}$$

$\Rightarrow \boldsymbol{c^2 = a^2 + b^2 + ab.}$

18. In ΔABC,



$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{AC^2+AB^2-BC^2}{2\times AC\times AB}$$

$$= \frac{4^2+2^2-3^2}{2\times4\times2}$$

$$= \frac{16+4-9}{16} = \frac{11}{16}$$

In ΔBAD,

$$\cos A = \frac{AD^2+AB^2-BD^2}{2\times AD\times AB}$$

$$\Rightarrow \frac{11}{16} = \frac{4+4-BD^2}{2\times2\times2} = \frac{8-BD^2}{8}$$

$\Rightarrow 11 = 16 - 2BD^2 \Rightarrow 2BD^2 = 5 \Rightarrow BD^2 = \mathbf{2.5}$

$\therefore$ Area of square on $BD = (BD)^2 = 2.5$ cm^2.

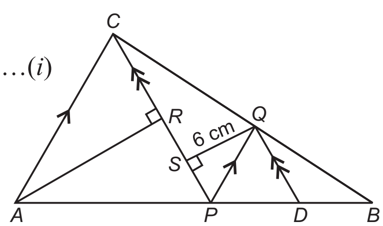
19. $PQ \parallel AC$

$$\Rightarrow \frac{CQ}{QB} = \frac{AP}{PB} = \frac{4}{3} \quad \ldots(i)$$

Also, $QD \parallel CP$

$$\Rightarrow \frac{PD}{DB} = \frac{CQ}{QB} = \frac{4}{3}$$

(From (i))



$$\frac{PD}{DB} = \frac{4}{3} \Rightarrow \frac{PD}{PB} = \frac{PD}{PD+DB} = \frac{4}{4+3} = \frac{4}{7} \quad \ldots(ii)$$

$\therefore$ From (i) and (ii)

$$\frac{AP}{PB} = \frac{4}{3} \text{ and } \frac{PB}{PD} = \frac{7}{4}$$

$$\therefore \frac{AP}{PB}\times\frac{PB}{PD} = \frac{4}{3}\times\frac{7}{4} \Rightarrow \frac{AP}{PD} = \frac{7}{3} \Rightarrow AP : PD = \mathbf{7 : 3.}$$

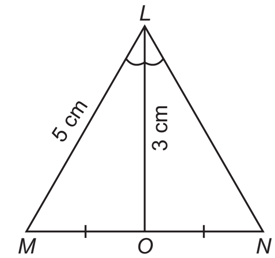
20. LO being the median on MN,

$MO = ON$

Also, LO being the internal bisector of $\angle MLN$,

$$\frac{LM}{LN} = \frac{MO}{ON} = 1$$

(By bisector theorem)



$\Rightarrow LM = LN$

$\Rightarrow \Delta LMN$ is isosceles triangle.

Now $\Delta LOM \cong \Delta LON$ (By SSS)

$\Rightarrow \angle LOM = \angle LON = 90^\circ$ (cpct)

$\therefore$ In ΔLOM, $MO = \sqrt{LM^2 - LO^2}$

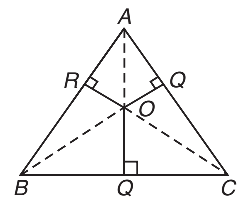
$= \sqrt{25-9} = \sqrt{16} = 4$ cm .

Area of $\Delta LMN = \frac{1}{2}\times MN\times LO$

$= \frac{1}{2}\times8\times3$ cm^2 $= \mathbf{12\ cm^2}$.

21. Given, ABC is an equilateral triangle such that $AB = BC = CA = 10$ m

If O is any point in the ΔABC, then



Area of ΔABC

= Area (ΔOAB) + Area (ΔOAC) + Area (ΔOBC)

$$= \frac{1}{2}\times AB\times OR + \frac{1}{2}\times AC\times OP + \frac{1}{2}\times AC\times OQ$$

$$= \frac{1}{2}\times AB\times(OR+OP+OQ) \qquad (\because AB = BC = CA)$$

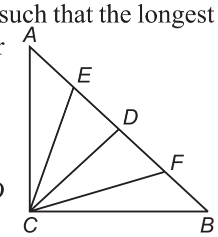
$$= \frac{1}{2}\times10\times(2+3+OQ)$$

$\because$ Area of an equilateral $\Delta = \frac{\sqrt{3}}{4}(\text{side})^2$

$$\therefore \frac{\sqrt{3}}{4}\times(10)^2 = \frac{1}{2}\times10\times(5+OQ)$$

$\Rightarrow 5\sqrt{3} = 5 + OQ \Rightarrow OQ = \mathbf{5\sqrt{3} - 5.}$

22. CE, CD and CF are the required paths such that the longest side of the park AB is divided into four equal segments.



$AE = ED = DF = FB = 20$ m.

Let $AC = b$, $BC = a$

Then, by Apollonius theorem in ΔACD

$AC^2 + CD^2 = 2\,(CE^2 + 20^2)$

$$\Rightarrow \frac{1}{2}\,(b^2 + CD^2) = CE^2 + 20^2 \quad \ldots(i)$$

Similarly in ΔCDB, $(CB^2 + CD^2) = 2\,(CF^2 + 20^2)$

$$\Rightarrow CF^2 + 20^2 = \frac{1}{2}(a^2 + CD^2) \quad \ldots(ii)$$

Adding (i) and (ii),

$$CE^2 + CF^2 + 2\times20^2 = \frac{1}{2}(a^2+b^2+2\times CD^2)$$

$$\Rightarrow CE^2 + CF^2 = \frac{1}{2}(80^2 + 2\times40^2) - 2\times20^2$$

$(\because AC^2 + CB^2 = AB^2 \Rightarrow a^2 + b^2 = 80^2)$

($\because$ $CD = 40$, $\because$ line joining the vertex at the right $\angle$ to the mid-point of the hypotenuse is half the hypotenuse)

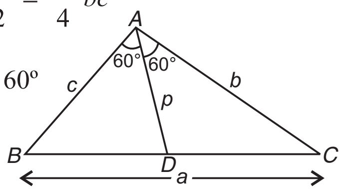
$$\Rightarrow CE^2 + CF^2 = \frac{1}{2}(6400 + 3200) - 800$$
$$= \mathbf{4000.}$$

Now $CE^2 + CF^2 + CD^2 = 4000 + 40^2 = \mathbf{5600.}$

23. Let $AD = p$

Area of $\Delta ABC = \frac{1}{2}bc\ \sin\angle BAC = \frac{1}{2}bc\ \sin 120^\circ$

$$= \frac{1}{2}bc \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}bc$$



Area of $\Delta BAD = \frac{1}{2}cp\ \sin 60^\circ$

$$= \frac{\sqrt{3}}{4}cp$$

Area of $\Delta CAD = \frac{1}{2}bp\ \sin 60^\circ = \frac{\sqrt{3}}{4}bp$

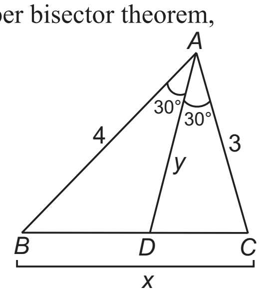
Now Area (ΔABC) = Area (ΔBAD) + Area (ΔCAD)

$$\Rightarrow \frac{\sqrt{3}}{4}bc = \frac{\sqrt{3}}{4}cp + \frac{\sqrt{3}}{4}bp \Rightarrow bc = p(b+c) \Rightarrow p = \mathbf{\frac{bc}{b+c}.}$$

24. Let $BC = x$ and $AD = y$, then as per bisector theorem,

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{4}{3}$$

$\therefore\ BD = \frac{4x}{7}$ and $DC = \frac{3x}{7}$



Now in ΔABD using cosine rule,

$$\cos 30^\circ = \frac{AB^2 + AD^2 - BD^2}{2 \times AB \times AD}$$

$$= \frac{16 + y^2 - \frac{16x^2}{49}}{2 \times 4 \times y} \Rightarrow \frac{\sqrt{3}}{2} = \frac{16 + y^2 - \frac{16x^2}{49}}{2 \times 4 \times y}$$

$$\Rightarrow 4\sqrt{3}y = 16 + y^2 - \frac{16x^2}{49} \quad \ldots(i)$$

In ΔACD using the cosine rule,

$$\cos 30^\circ = \frac{AC^2 + AD^2 - CD^2}{2 \times AC \times AD}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{9 + y^2 - \frac{9x^2}{49}}{2 \times 3 \times y}$$

$$\Rightarrow 3\sqrt{3}y = 9 + y^2 - \frac{9x^2}{49} \quad \ldots(ii)$$

Also in ΔABC, $\cos 60^\circ = \frac{AB^2 + AC^2 - BC^2}{2 \times AB \times AC}$

$$\Rightarrow \frac{1}{2} = \frac{16 + 9 - BC^2}{2 \times 4 \times 3} \Rightarrow 12 = 25 - BC^2$$

$\Rightarrow BC^2 = 13 \Rightarrow x^2 = 13$

$\therefore$ Subtracting eqn (ii) from (i), we get

$$\sqrt{3}y = 7 - \frac{7x^2}{49} = 7 - \frac{7 \times 13}{49} = 7 - \frac{13}{7} = \frac{49 - 13}{7} = \frac{36}{7}$$

$$\Rightarrow y = \frac{36}{7 \times \sqrt{3}} = \mathbf{\frac{12\sqrt{3}}{7}}.$$

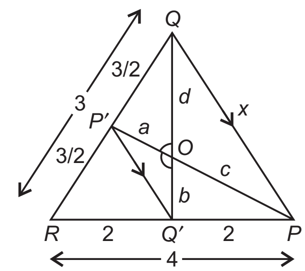
25. Join $P'Q'$.

P', Q' being the mid-points of QR and PR respectively, we have $P'Q' \parallel PQ$ and $P'Q' = \frac{1}{2}PQ$

(By the mid-point theoram)

Let $OP' = a$, $OQ' = b$, $OP = c$, $OQ = d$ and $PQ = x$

Then, $P'Q' = \frac{1}{2}x$.



$\therefore$ In rt. Δ $OP'Q'$,

$$a^2 + b^2 = \frac{x^2}{4} \quad \ldots(i)$$

In rt. Δ $OP'Q$,

$$a^2 + d^2 = \frac{9}{4} \quad \ldots(ii)$$

In rt. ΔOQP,

$$c^2 + d^2 = x^2 \quad \ldots(iii)$$

In rt. Δ $OQ'P$,

$$b^2 + c^2 = 4 \quad \ldots(iv)$$

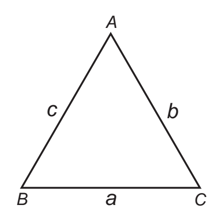
$\therefore$ Eq (i) – Eq (ii) + Eq (iii) – Eq (iv)

$$\Rightarrow a^2 + b^2 - (a^2 + d^2) + (c^2 + d^2) - (b^2 + c^2) = \frac{x^2}{4} - \frac{9}{4} + x^2 - 4$$

$$\Rightarrow 0 = \frac{5x^2}{4} - \frac{25}{4} \Rightarrow \frac{5x^2}{4} = \frac{25}{4} \Rightarrow x^2 = 5 \Rightarrow \mathbf{x = \sqrt{5}.}$$

26. When $\angle A = 60^\circ$ and $b = c$, then ΔABC is equilateral

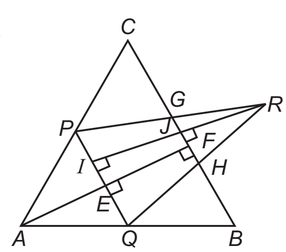
$\Rightarrow a = b = c$



when $A = 90^\circ$, then ΔABC is an isosceles right angled triangle and $a = \sqrt{2}b$ or $a = \sqrt{2}c$.

$\therefore$ $60^\circ < A < 90^\circ$ = $\mathbf{c < a < \sqrt{2}c.}$

27. P and Q being the mid-points of AC and AB respectively, $PQ \parallel BC$ and $PQ = \frac{1}{2}BC$



Let $AF \perp BC$ be drawn such that it intersects PQ and BC in E and F respectively.

$$PQ \parallel BC \Rightarrow \frac{AE}{EF} = \frac{AP}{PC} = 1 \Rightarrow AE = EF$$

Also let $RI \perp PQ$ be drawn such that it intersect BC and PQ in J and I respectively. G and H being the mid-points of sides PR and RQ of ΔPQR, $GH \parallel PQ$ and $GH = \frac{1}{2}PQ$

(By midpoint Theorem)

Also, $GH \parallel PQ \Rightarrow \dfrac{RJ}{JI} = \dfrac{RG}{GP} = 1 \Rightarrow RJ = JI$

(By Basic Proportionality Theorem)

But $EF = JI$

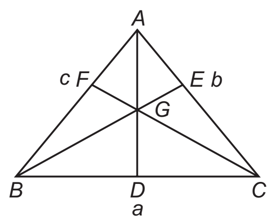
$\therefore\ AE = EF = RJ = JI$

$\therefore\ AF = AE + EF = RJ + JI = RI = h$ (say)

Then, $\dfrac{\text{Area }(\Delta PQR)}{\text{Area }(\Delta ABC)} = \dfrac{\frac{1}{2} \times PQ \times h}{\frac{1}{2} \times BC \times h} = \dfrac{PQ}{BC} = \mathbf{\dfrac{1}{2}}.$

28. AD being the median to BC,

$AB^2 + AC^2 = 2\,(BD^2 + AD^2)$ (Apollonius Th.)



$\Rightarrow c^2 + b^2 = 2\left(\dfrac{a^2}{4} + AD^2\right)$

$\Rightarrow 2c^2 + 2b^2 = 4\left(\dfrac{a^2}{4} + AD^2\right)$

$\Rightarrow 4AD^2 = 2c^2 + 2b^2 - a^2$

$\Rightarrow AD = \dfrac{1}{2}\sqrt{2c^2 + 2b^2 - a^2}$

Now G divides AD in ratio 2 : 1 $\quad \therefore\ AG = \dfrac{2}{3}AD$

$$\Rightarrow \quad AG^2 = \frac{4}{9}AD^2 = \frac{4}{9} \times \frac{1}{4}(2c^2 + 2b^2 - a^2)$$

$$= \frac{1}{9}(2c^2 + 2b^2 - a^2)$$

Similarly, $\quad GB^2 = \dfrac{1}{9}(2c^2 + 2a^2 - b^2)$

$$AG^2 + GB^2 = \frac{1}{9}\ [2c^2 + 2b^2 - a^2 + 2c^2 + 2a^2 - b^2]$$

$$= \frac{1}{9}\ [a^2 + b^2 + 4c^2]$$

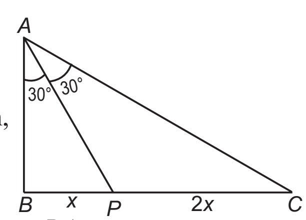
$$= \frac{1}{9}[5c^2 + 4c^2] = \frac{9c^2}{9} = c^2 = BC^2$$

$\Rightarrow AG \perp GB \Rightarrow$ Angle between the medians drawn to sides $\boldsymbol{a}$ and $\boldsymbol{b}$ is **90º**.

29. Let $BP = x$. Then $PC = 2x$

$\because\ AP$ bisects $\angle BAC$,

By the angle bisector theorem,



$\dfrac{AB}{AC} = \dfrac{BP}{PC} = \dfrac{1}{2}$

Using the sine formula, $\dfrac{AC}{\sin B} = \dfrac{BA}{\sin C}$

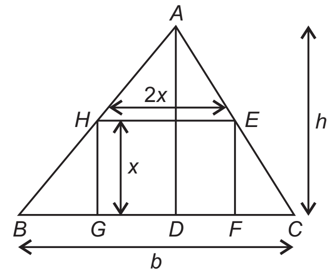
$\Rightarrow \dfrac{\sin C}{\sin B} = \dfrac{BA}{AC} = \dfrac{1}{2}$

$\Rightarrow \dfrac{\sin C}{\sin B} = \dfrac{\frac{1}{2}}{1} \Rightarrow \sin C = \dfrac{1}{2}$ and $\sin B = 1$

$\Rightarrow \angle C = 30°$ and $\angle B = 90°$

$\therefore$ In ΔAPC, $APC = 180º - (30º + 30º) = \mathbf{120º}$.

30. Let AD be the height of the given triangle ABC, where $AD = h$ and $BC = b$ Also let height HG of rectangle $EFGH$ equal x so that $HE = GF = 2x$



Now $\Delta BGH \sim \Delta BDA$

$\Rightarrow \dfrac{BG}{BD} = \dfrac{HG}{AD}$

$\Rightarrow \dfrac{BG}{BD} = \dfrac{x}{h} \Rightarrow BG = \dfrac{x}{h}BD \qquad \ldots(i)$

Also, $\Delta CFE \sim \Delta CDA$

$\Rightarrow \dfrac{CF}{CD} = \dfrac{EF}{AD} = \dfrac{x}{h}$

$\Rightarrow CF = \dfrac{x}{h}CD \qquad \ldots(ii)$

$BG + CF = BC - GF = b - 2x \qquad \ldots(iii)$

$\therefore$ From (i), (ii) and (iii)

$b - 2x = \dfrac{x}{h}(BD + CD) \quad \Rightarrow \quad b - 2x = \dfrac{x}{h}BC = \dfrac{x}{h}b$

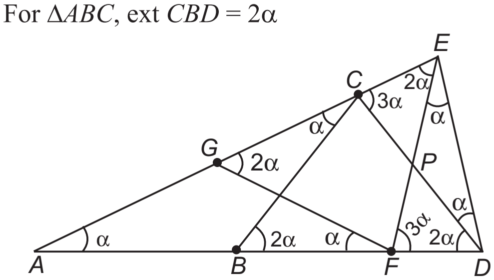
$\Rightarrow bh - 2xh = xb \Rightarrow bh = xb + 2xh = x(b + 2h)$

$\Rightarrow x = \boldsymbol{\dfrac{bh}{b + 2h}}.$

31. Let $\angle EAD = \alpha$. Then,

In ΔABC, $AB = BC \Rightarrow \angle BCA = \alpha$] sides opp. equal
In ΔAGF, $AG = GF \Rightarrow \angle AFG = \alpha$] angles are equal

$\therefore$ For ΔABC, ext $CBD = 2\alpha$



In ΔCBD, $CB = CD \Rightarrow \angle CDB = 2\alpha$

For ΔAFG, ext $\angle FGC = 2\alpha$

$\because$ In ΔGFE, $GF = EF \Rightarrow \angle FEG = \angle FGE = 2\alpha$

For ΔEAF, ext. $\angle EFD = 3\alpha$

$\because\ EF = ED \ \therefore\ \angle EDF = \angle EFD = 3\alpha \Rightarrow \angle EDP = \alpha$

For ΔCAD, ext. $\angle DCE = 3\alpha$

In ΔECD, $DC = ED \Rightarrow \angle DEC = \angle DCE = 3\alpha$

$\Rightarrow \angle FED = \angle DEC - \angle FEC = 3\alpha - 2\alpha = \alpha$.

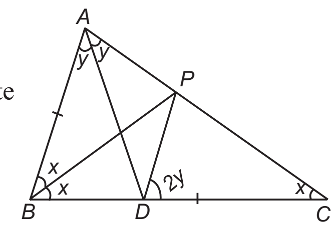
$\therefore$ In ΔEFD, $\alpha + 3\alpha + 3\alpha = 180º$

$\Rightarrow 7\alpha = 180º \Rightarrow \alpha = \dfrac{180º}{7} = 26º$ or approximately 25º.

32. In ΔABC, let BP bisect $\angle ABC$

Let $\angle C = x \Rightarrow \angle B = 2x$

$\therefore\ \angle PBC = \angle ABP = x$



In ΔPBC,

$\angle PBC = \angle PCB = x$

$\Rightarrow PC = PB$ (sides opposite equal angles are equal)

Now in ΔAPB and ΔBPC

$AB = CD$ (Given)

$PB = PC$ (Proved above)

$\angle ABP = \angle DCP = x$

$\therefore \; \Delta APB \cong \Delta DPC$

$\Rightarrow \angle BAP = \angle PDC = 2y$ and $AP = DP$

$\therefore$ In ΔAPD, $AP = DP \Rightarrow \angle PDA = \angle PAD = y$

$\therefore \; \angle DPA = 180^\circ - 2y$ …(i)

Also from ΔDPC, $\angle DPC = 180^\circ - (x + 2y)$ …(ii)

$\therefore$ From (i) and (ii), $\angle DPA + \angle DPC = 180^\circ$

$\Rightarrow 180^\circ - 2y + 180^\circ - (x + 2y) = 180^\circ$

$\Rightarrow x + 4y = 180^\circ$ …(1)

Also in ΔABC, $\angle A + \angle B + \angle C = 180^\circ$

$2y + 2x + x = 180^\circ$

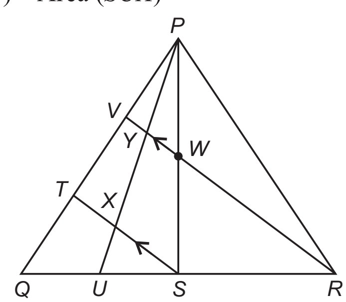
$3x + 2y = 180^\circ$ …(2)

$\therefore$ (2) – 3 × (1) $\Rightarrow 3x + 2y - (3x + 12y) = 180^\circ - 3 \times 180^\circ$

$\Rightarrow -10y = -360^\circ \Rightarrow y = 36^\circ$

$\therefore \; \angle BAC = 2y = 2 \times 36^\circ = \mathbf{72^\circ}$.

33. Area (ΔPSX) = Area (PUS) – Area (SUX)



In ΔPXS, $WY \parallel SX$

$$\Rightarrow \frac{PY}{YX} = \frac{PW}{WS} = 1$$

(Given, $PW = WS$)

$\Rightarrow PY = YX$

In ΔRUY,

$$SX \parallel RY = \frac{UX}{XY} = \frac{US}{SR} = \frac{1}{2}$$

($\because$ $QS = SR$ and $QV = US$)

$$= UX = \frac{1}{2}(XY)$$

$$\therefore \text{ In } \Delta PUS,\; UX = \frac{1}{2}XY = \frac{1}{2}\left(\frac{1}{2}PX\right) = \frac{1}{4}PX$$

$$PU = UX + PX = \frac{1}{4}PX + PX = \frac{5}{4}PX$$

$$\therefore \; \frac{\text{Area of } \Delta SUX}{\text{Area of } \Delta PUS} = \frac{1}{5}$$

$$\text{Now Area of } \Delta PUS = \frac{1}{4}(\text{Area of } \Delta PQR)$$

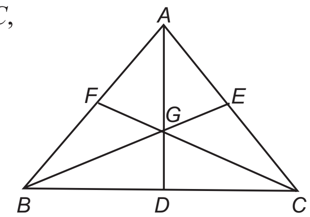
$$\text{Area of } \Delta SUX = \frac{1}{4} \times \frac{1}{5} \times \text{Area of } \Delta PQR$$

$$= \frac{1}{20}\text{Area of } \Delta PQR$$

$$\therefore \frac{\text{Area of } \Delta PSX}{\text{Area of } \Delta PQR} = \frac{\frac{1}{4}\text{Area of } \Delta PQR - \frac{1}{20}\text{Area of } \Delta PQR}{\text{Area of } \Delta PQR}$$

$$= \frac{5-1}{20} = \mathbf{\frac{1}{5}}.$$

34. G being the centroid of ΔABC,



$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$$

In ΔABD,

$AB + BD > AD$

$$\Rightarrow AB + \frac{BC}{2} > AD \qquad \ldots(i)$$

In ΔBEC,

$$BC + CE > BE \Rightarrow BC + \frac{AC}{2} > BE \qquad \ldots(ii)$$

In ΔAFC,

$$AC + AF > CF \Rightarrow AC + \frac{AB}{2} > CF \qquad \ldots(iii)$$

Adding (i), (ii) and (iii), we get

$$AB + BC + AC + \frac{BC}{2} + \frac{AC}{2} + \frac{AB}{2} > AD + BE + CF$$

$$\Rightarrow \frac{2AB + 2BC + 2AC + BC + AC + AB}{2} > AD + BE + CF$$

$\Rightarrow 3(AB + BC + AC) > 2\,(AD + BE + CF) \Rightarrow$ ***I* is true**

Also, in $\Delta\, BGC$,

$BG + GC > BC$

$$\left(\because \frac{BG}{GE} = \frac{2}{1} \text{ and } \frac{CG}{GF} = \frac{2}{1} \Rightarrow BG = \frac{2}{3}BE \text{ and } CG = \frac{2}{3}CF\right)$$

$$\Rightarrow \frac{2}{3}BE + \frac{2}{3}CF > BC \Rightarrow 2BE + 2CF > 3BC \qquad \ldots(iv)$$

Similarly in ΔBGA,

$BG + GA > AB$

$$\Rightarrow \frac{2}{3}BE + \frac{2}{3}AD > AB \Rightarrow 2BE + 2AD > 3AB \qquad \ldots(v)$$

In ΔCGA,

$CG + GA > AC$

$$\Rightarrow \frac{2}{3}CF + \frac{2}{3}AD > AC$$

$\Rightarrow 2CF + 2AD > 3AC$ …(vi)

Adding (iii), (iv) and (v), we get

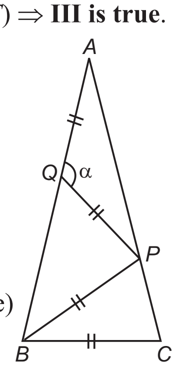
$2BE + 2CF + 2BE + 2AD + 2CF + 2AD$

$> 3BC + 3AB + 3AC$

$\Rightarrow 4(AD + BE + CF) > 3(AB + BC + AC)$

$\Rightarrow 3(AB + BC + AC) < 4(AD + BE + CF) \Rightarrow$ **III is true**.

35. Let $\angle AQP = \alpha$



In ΔAQP, $AQ = QP$

$$\Rightarrow \angle QAP = \angle QPA = \frac{1}{2}(\pi - \alpha)$$

$$= \frac{\pi}{2} - \frac{\alpha}{2}$$

$\angle PQB = \pi - \alpha$

(AQB being a straight line)

In ΔPQB,

$PQ = PB \Rightarrow \angle PBQ = \angle PQB = \pi - \alpha$

$\therefore \; \angle QPB = \pi - 2\,(\pi - \alpha) = 2\alpha - \pi$

In ΔABC,

$\angle BAC + \angle ABC + \angle ACB = \pi$

$\Rightarrow \angle QPA + \angle ACB + \angle ACB = \pi$

$(\because \angle QAP = \angle BAC, AB = AC \Rightarrow \angle ACB = \angle ABC)$

$\Rightarrow \dfrac{\pi}{2} - \dfrac{\alpha}{2} + 2\angle ACB = \pi$

$\Rightarrow 2\angle ACB = \pi - \dfrac{\pi}{2} + \dfrac{\alpha}{2} = \dfrac{\pi}{2} + \dfrac{\alpha}{2} \Rightarrow \angle ACB = \dfrac{\pi}{4} + \dfrac{\alpha}{4}$

$\therefore$ In ΔBPC, $BP = BC$

$\Rightarrow \angle BPC = \angle BCP = \angle ACB = \dfrac{\pi}{4} + \dfrac{\alpha}{4}$

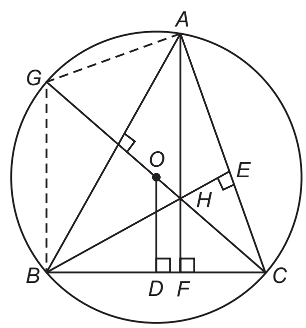
Now APC being a straight line,

$\angle APQ + \angle BPQ + \angle BPC = \pi$

$\Rightarrow \dfrac{\pi}{2} - \dfrac{\alpha}{2} + 2\alpha - \pi + \dfrac{\pi}{4} + \dfrac{\alpha}{4} = \pi$

$\Rightarrow \dfrac{7\alpha}{4} = \dfrac{5\pi}{4} \Rightarrow \alpha = \boldsymbol{\dfrac{5\pi}{7}}$.

36. Let ABC be the given triangle whose circumcentre is O. Produce CO to meet the circle at $G \Rightarrow COG$ is the diameter of the circle.



$\Rightarrow \angle GAC = \angle GBC = 90^\circ$ (Angles in a semicircle)

Let AF and BE be the perpendiculars from vertex A and B respectively on sides BC and AC.

$\therefore$ H is the orthocentre of ΔABC. We need to find the ratio $AH : OD$, where OD is the perpendicular distance of the circumcentre O from side BC.

$OD \perp BC \Rightarrow BD = DC$ (Perpendicular from the centre of the circle bisects the chord)

Also, $GB \perp BC$ and $OD \perp BC \Rightarrow OD \parallel GB$.

$\therefore$ In ΔBGC, by the midpoint theorem, O and D being the mid-points of sides GC and BC, $OD \parallel GB$ and $OD = \dfrac{1}{2}GB$

From (i)

Now GA and BE are both perpendiculars to AC

$\Rightarrow GA \parallel BE \Rightarrow GA \parallel BH$

Also, $GB \parallel AF \Rightarrow GB \parallel AH$

$\Rightarrow GAHB$ is a parallelogram

$\Rightarrow GB = AH = 2OD$

$\therefore \dfrac{AH}{OD} = \dfrac{2OD}{OD} = \boldsymbol{\dfrac{2}{1}}$.

37. In Δs ABC and DAC

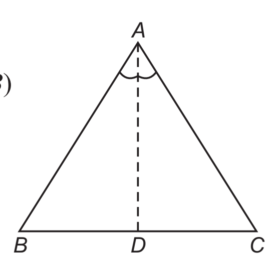
$\angle ABC = \angle DAC$

$(\because AD$ bisects $\angle A$ and $\angle A = 2\angle B)$

Also, $\angle ACB = \angle DCA$

$\Rightarrow \Delta ABC \sim DAC$ (AA similarity)

$\dfrac{AC}{DC} = \dfrac{BC}{AC} \Rightarrow AC^2 = BC.DC$



$\Rightarrow DC = \dfrac{AC^2}{BC} = \dfrac{b^2}{a}$ …(i)

(In a ΔABC, $AB = c$, $BC = a$, $AC = b$)

Also, AD being the angle bisector of $\angle A$,

$\dfrac{BD}{CD} = \dfrac{AB}{AC} = \dfrac{c}{b}$

$\therefore \dfrac{BD}{CD} + 1 = \dfrac{c}{b} + 1$

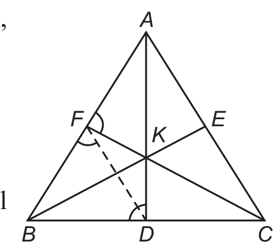
$\Rightarrow \dfrac{BD + CD}{CD} = \dfrac{c + b}{b}$

$\Rightarrow \dfrac{BC}{CD} = \dfrac{C + b}{b} \Rightarrow \dfrac{a}{CD} = \dfrac{c + b}{b} \Rightarrow CD = \dfrac{ab}{b + c}$ …(ii)

$\therefore$ From (i) and (ii), $\dfrac{b^2}{a} = \dfrac{ab}{b + c} \Rightarrow \boldsymbol{a^2 = b(b + c)}$.

38. Since $\dfrac{BD}{DC} = \dfrac{BF}{FA}$

By basic proportionality theorem, we have $FD \parallel AC$.



Also, $\angle BDK = \angle ADB = \angle AFC$

$= 180^\circ - \angle BFK$

$\therefore \angle BDK + \angle BFK = 180^\circ$

$\Rightarrow BDKF$ is a cyclic quadrilateral

$\Rightarrow \angle FBK = \angle FDK$

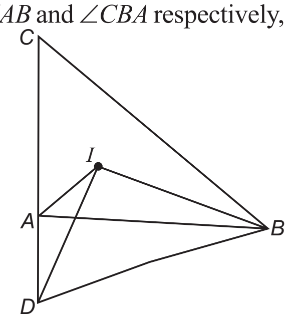
(Angles in the same segment)

$\therefore \angle \boldsymbol{ABE} = \angle FBK = \angle FDK = \angle FDA = \angle \boldsymbol{DAC}$

$(\because FD \parallel AC$, alt. $\angle s$ are equal)

39. AI and BI are the bisectors of $\angle CAB$ and $\angle CBA$ respectively,

$\therefore$ In ΔABC,



$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow \angle A + \angle B = 180^\circ - \angle C$

$\dfrac{\angle A}{2} + \dfrac{\angle B}{2} = 90^\circ - \dfrac{\angle C}{2}$

Also, in ΔAIB,

$\angle AIB = 180^\circ - \left(\dfrac{\angle A}{2} + \dfrac{\angle B}{2}\right)$

$= 180^\circ - \left(90^\circ - \dfrac{\angle C}{2}\right) = 90^\circ + \dfrac{\angle C}{2}$

Extend CA to D such that $AD = AI$.

Then, $BC = AB + AI \Rightarrow BC = CA + AD = CD$

(By hypothesis, $AB = AC$, $AD = AI$)

$\Rightarrow \angle CDB = \angle CBD = 90^\circ - \dfrac{\angle C}{2}$

(Since in ΔCDB, $CD = CB$ and $\angle C + \angle D + \angle B = 180°$)

Thus, $\angle AIB + \angle ADB = 90^\circ + \dfrac{\angle C}{2} + 90^\circ - \dfrac{\angle C}{2} = 180^\circ$

$\Rightarrow AI\,BD$ is a cyclic quadrilateral

Also $\angle ADI = \angle ABI = \dfrac{\angle B}{2}$ (angles in the same segment)

$\therefore$ In ΔDAI, $\angle DAI$, $= 180^\circ - 2(\angle ADI) = 180^\circ - \angle B$

$(\because AD = AI \Rightarrow \angle ADI = \angle AID)$

Thus, $\angle CAI = B \Rightarrow A = 2B$ (CAD is a straight angle)

Since, $AC = AB \Rightarrow \angle B = \angle C$

$\therefore$ In ΔABC, $\angle A + \angle B + \angle C = 180^\circ$

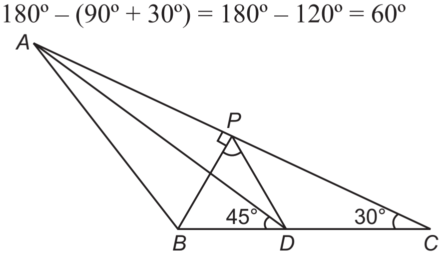
$\Rightarrow 4\angle B = 180^\circ \Rightarrow \angle B = 45^\circ \Rightarrow \angle A =$ **90º.**

40. Draw $BP \perp AC$ and join P to D.

In ΔBPC,

$\angle PBC = 180^\circ - (\angle BPC + \angle BCP)$

$= 180^\circ - (90^\circ + 30^\circ) = 180^\circ - 120^\circ = 60^\circ$



In ΔBPC,

$\sin 30^\circ = \dfrac{BP}{BC} \Rightarrow \dfrac{BP}{BC} = \dfrac{1}{2} \Rightarrow BP = \dfrac{1}{2}BC = BD$

Now in ΔBPD, $BP = BD \Rightarrow \angle BDP = \angle PBD = 60^\circ$

$\Rightarrow \angle BPD = 60^\circ \Rightarrow \Delta BPD$ is equilateral

$\therefore$ $PB = PD$ and $\angle ADP = 60^\circ - 45^\circ = 15^\circ$

In ΔADC, ext. $\angle ADB = \angle ACD + \angle DAC$

$45^\circ = 30^\circ + \angle DAC \Rightarrow \angle DAC = 15^\circ$

$\therefore$ In ΔAPD, $\angle ADP = \angle PAD \Rightarrow PD = PA$

We have $PD = PA = PB$

$\Rightarrow$ P is the circumcentre of ΔADB as circumcentre is equidistant from the vertices of a Δ.

$\Rightarrow BAD = \dfrac{1}{2}\angle BPD$

($\because$ Angle subtended by an are at the centre of a circle is half the angle subtended by the same are at any other point on the remaining part of the circumference.)

$\Rightarrow BAD = \dfrac{1}{2} \times 60^\circ =$ **30°.**