CHAPTER – 7

INTEGRALS

Exercise – 7.6

Question 1: Integrate the functions.

$x \sin x$

Answer:

Let I = $x \sin x$

Now, integrating by parts, we get,

$I = x \int \sin x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin x \, dx \right\} dx$

= $x(-\cos x) - \int 1.(-\cos x) \, dx$

= $-x \cos x + \sin x + C$

Question 2: Integrate the functions.

$x \sin 3x$

Answer:

Let I = $x \sin 3x$

Now, integrating by parts, we get,

$I = x \int \sin 3x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x \, dx \right\} dx$

= $x \left(\frac{-\cos 3x}{3} \right) - \int 1. \left(\frac{-\cos 3x}{3} \right) dx$

= $\frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C$

Question 3: Integrate the functions.

$x^2 e^x$

Answer:

Let I = $x^2 e^x$

Now, integrating by parts, we get,

$I = x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$

= $x^2 e^x - \int 2x . e^x \, dx$

= $x^2 e^x - 2 \int x . e^x \, dx$

Again integrating by parts, we get,

= $x^2 e^x - 2 \left[x \int e^x \, dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx \right]$

= $x^2 e^x - 2[x e^x - \int e^x \, dx]$

= $x^2 e^x - 2[x e^x - e^x]$

= $x^2 e^x - 2x e^x + e^x + C$

= $e^x (x^2 - 2x + 2) + C$

Question 4: Integrate the functions.

$x \log x$

Answer:

Let I = $x \log x$

Now, integrating by parts, we get,

Taking, Logarithmic function as first function and algebraic function as second function,

$I = \log x \int x\, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x\, dx \right\} dx$

$= \log x \left(\frac{x^2}{2} \right) - \int \frac{1}{2} . \frac{x^2}{2} dx$

$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$

$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

Question 5: Integrate the functions.

$x \log 2x$

Answer:

Let I = xlog2x

Now, integrating by parts, we get,

$I = \log 2x \int x\, dx - \int \left\{ \left(\frac{d}{dx} 2\log x \right) \int x\, dx \right\} dx$

$= \log 2x \left(\frac{x^2}{2} \right) - \int \frac{2}{2x} . \frac{x^2}{2} dx$

$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$

$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$

Question 6: Integrate the functions.

$x^2 \log x$

Answer:

Let I = $x^2 \log x$

Now, integrating by parts, we get,

$I = \log x \int x^2\, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2\, dx \right\} dx$

$= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} . \frac{x^3}{3} dx$

$= \frac{x^3 \log x}{2} - \int \frac{x^2}{2} dx$

$= \frac{x^3 \log x}{2} - \frac{x^3}{4} + C$

Question 7: Integrate the functions.

$x \sin^{-1} x$

Answer:

Let I = $x \sin^{-1} x$

Now, integrating by parts, we get,

$I = \sin^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x\, dx \right\} dx$

$= \sin^{-1} x . \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2} dx$

$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$

$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx$

$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \right\} dx$

$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C$

$= \frac{1}{4} (2x^2 - 1) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + \mathrm{C}$

Question 8: Integrate the functions.

$x \tan^{-1} x$

Answer:

Let I = $x \tan^{-1} x$

Now, integrating by parts, we get,

$$I = \tan^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x \, dx \right\} dx$$

$$= \tan^{-1} x . \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} + \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} + \frac{1}{2} \int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} + \frac{1}{2} \int \left\{ 1 - \frac{1}{1+x^2} \right\} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} + \frac{1}{2} (x - \tan^{-1} x) + C$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

Question 9: Integrate the functions.

$x \cos^{-1} x$

Answer:

Let I = $x \cos^{-1} x$

Now, integrating by parts, we get,

$$I = \cos^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x \, dx \right\} dx$$

$= \cos^{-1} x . \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2} dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{-1-x^2-1}{\sqrt{1-x^2}} dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x$ ...(1)

Now, $I_1 = \int \sqrt{1-x^2}\, dx$

$I_1 = x\sqrt{1-x^2} - \int \frac{d}{dx} \sqrt{1-x^2} \int x\, dx$

$I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} x . dx$

$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} x . dx$

$= x\sqrt{1-x^2} - \int \frac{1+x^2-1}{\sqrt{1-x^2}} x . dx$

$= x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2} dx + \int \frac{dx}{\sqrt{1-x^2}} \right\}$

$I_1 = x\sqrt{1-x^2} - \{ I_1 + \cos^{-1} x \}$

$I_2 = \frac{x}{2} \sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x$

Now, substituting in (1), we get,

$I = \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \left(\frac{x}{2} \sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x \right) - \frac{1}{2} \cos^{-1} x$

$= \frac{(2x^2-1)}{4} \cos^{-1} x - \frac{x}{4} \sqrt{1-x^2} + C$

Question 10: Integrate the functions.

$(\sin^{-1} x)^2$

Let I =$(\sin^{-1} x)^2$

Now, integrating by parts, we get,

$I = (\sin^{-1} x) \int x dx - \int \left\{\left(\frac{d}{dx}(\sin^{-1} x)^2\right). \int x\, dx\right\} dx$

$= (\sin^{-1} x)^2 . x - \int \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} . x dx$

$= x(\sin^{-1} x)^2 + \int \sin^{-1} x \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$

$= x(\sin^{-1} x)^2 + \left[\sin^{-1} x \int \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx - \int \left\{\left(\frac{d}{dx}(\sin^{-1} x)^2\right) \int \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx\right\} dx\right]$

$= x(\sin^{-1} x)^2 + \left[\sin^{-1} x . 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} . 2\left(\sqrt{1-x^2}\right) dx\right]$

$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - \int 2\, dx$

$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C$

Question 11: Integrate the functions.

$\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$

Answer:

Let I = $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$

$I = -\frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} . \cos^{-1} x\, dx$

Now, integrating by parts, we get,

$I = -\frac{1}{2} \int \cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{\left(\frac{d}{dx} \cos^{-1} x\right) \int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx$

$= -\frac{1}{2}\left[\cos^{-1} x . 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} . 2\sqrt{1-x^2} dx\right]$

$= -\frac{1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + \int 2\,dx\right]$

$= \frac{1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + 2x\right]$ + C

$= -\left[\sqrt{1-x^2}\cos^{-1}x + x\right]$ + C

Question 12: Integrate the functions.

$x\sec^2 x$

Answer:

Let I = $x\sec^2 x$

Now, integrating by parts, we get,

$I = x\int \sec^2 x\,dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sec^2 x\,dx\right\}dx$

$= x\tan x - \int 1\tan x\,dx$

$= x\tan x + \log|\cos x| + C$

Question 13: Integrate the functions.

$\tan^{-1}x$

Answer:

Let I = $\tan^{-1}x$

So, now integrating by parts, we get,

$I = \tan^{-1}x\int 1.dx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int 1.dx\right\}dx$

$= \tan^{-1}x.x - \int\frac{1}{1+x^2}x.dx$

$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$

$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + C$

$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$

Question 14: Integrate the functions.

$x(\log x)^2$

Answer:

Let I = $x(\log x)^2$

Integrating by parts, we get,

$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int x dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x dx\right\} dx\right]$

$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2} - \log x - \int \frac{1}{x}.\frac{x^3}{2} dx\right]$

$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x. dx$

$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$

Question 15: Integrate the functions.

$(x^2+1)\log x$

Answer:

Let I = $\int(x^2+1)\log x dx \ = \int x^2 \log x dx + \int \log x\ dx$

Now, Let I = $I_1 + I_2$ … (1)

Where, $I_1 = \int x^2 \log x dx$ and $I_2 = \int \log x dx$

So, now

$I_1 = \int x^2 \log x dx$

Integrating by parts, we get,

$I_1 = \log x - \int x^2 dx - \int \int \left\{\left(\frac{d}{dx}\log x\right)\int x dx\right\} dx$

$= \log x . \frac{x^3}{3} - \int \frac{1}{x} . \frac{x^3}{3} dx$

$= \frac{x^3}{3}\log x - \frac{1}{3}(\int x^2 . dx)$

$= \frac{x^3}{3}\log x \ \frac{x^3}{9} + C_1$ …(2)

Now, $I_2 = \int \log x dx$

Integrating by parts, we get,

$I_2 = \log x \int 1 . dx - \int \left\{\left(\frac{d}{dx}\log x\right)\int 1 . dx\right\}$

$= \log x . x - \int \frac{1}{x} . x dx$

$= x \log x - \int 1 . dx$

$= x \log x + C_2$ …(3)

Now putting the value of I_1 and I_2, in (1), we get,

I = $\frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$

$= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2)$

$= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$

Question 16: Integrate the functions.

$e^x(\sin x + \cos x)$

Answer:

$I = \int e^x(\sin x + \cos x)dx$

Now,

Let $\sin x = f(x) \Rightarrow f'(x) = \cos x$

We know that,

$\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

Thus,

$\int e^x(\sin x + \cos x)dx = e^x \sin x + C$

Question 17: Integrate the functions.

$\frac{xe^x}{(1+x)^2}$

Answer:

$I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{\frac{x}{(1+x)^2}\right\} dx$

$= \int e^x \left\{\frac{1+x-1}{(1+x)^2}\right\} dx$

$\int e^x \left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\} dx$

Now,

Let $\frac{1}{1+x} = f(x) \Rightarrow f'(x) = \frac{1}{(1+x)^2}$

We know that,

$\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

Thus,

$\int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + \mathrm{C}$

Question 18: Integrate the functions.

$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

Answer:

$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

$= e^x\left(\frac{\sin^2\frac{x}{2}\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$

$=\frac{e^x\left(\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$

$=\frac{1}{2}e^x.\left(\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$

$=\frac{1}{2}e^x\left(\tan\frac{x}{2}+1\right)^2$

$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$

$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$

$\Rightarrow e^x\left[\frac{1+\sin x}{1+\cos x}\right]=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$

Now,

Let $\tan\frac{x}{2}=f(x)\Rightarrow f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

We know that,

$\int e^x\{f(x)+f'(x)\}dx=e^xf(x)+C$

Thus,

$\int e^x \left[\frac{1+\sin x}{1+\cos x}\right] dx = e^x \tan\frac{x}{2} + C$

Question 19: Integrate the functions.

$e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$

Answer:

Let I = $\int e^x \left[\frac{1}{x} - \frac{1}{x^2}\right] dx$

Now, let

$\frac{1}{x} = f(x) \Rightarrow f'(x) = \frac{1}{x^2}$

We know that,

$\int e^x\{\, f(x) + f'(x)\}dx = e^x f(x) + C$

Thus,

$I = \frac{e^x}{x}$ + C

Question 20: Integrate the functions.

$\frac{(x-3)e^x}{(x-1)^3}$

Answer:

$\int e^x \left[\frac{(x-3)}{(x-1)^3}\right] dx = \int e^x \left\{\frac{x-1-2}{(x-1)^3}\right\} dx$

$= \int e^x \left\{\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right\}$

Now, let f(x) = $\frac{1}{(x-1)^2}$

$\Rightarrow f'(x) = \frac{2}{(x-1)^3}$

We know that,

$$\int e^x\{ f(x) + f'(x)\}dx = e^x f(x) + C$$

Thus,

$$\int e^x \left\{\frac{(x-3)}{(x-1)^3}\right\} dx = \frac{e^x}{(x-1)^2} + C$$

Question 21: Integrate the functions.

$e^{2x} \sin x$

Answer:

Let I = $e^{2x} \sin x$

Integrating by parts, we get,

$$I = \sin x \int e^{2x} dx - \int \left\{\left(\frac{d}{dx} \sin x\right) \int e^{2x} dx\right\} dx$$

$$= \sin x. \frac{e^{2x}}{2} - \int \cos x. \frac{e^{2x}}{2} dx$$

Again, integrating by parts, we get,

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos \int e^{2x}\, dx - \int \left\{\left(\frac{d}{dx} \sin x\right) \int e^{2x} dx\right\} dx\right]$$

$$= \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos x. \frac{e^{2x}}{2} - \int (-\sin x). \frac{e^{2x}}{2} dx\right]$$

$$= \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{2} - \frac{1}{4} I$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{2}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{2}$$

$$\Rightarrow \frac{4}{5}\left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{2}\right]$$

Question 22: Integrate the functions.

$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Answer:

Let x = tan θ

⇒ dx = $\sec^2\theta\, d\theta$

Thus,

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \int 2\theta d\theta = 2\int \theta \sec^2\theta d\theta$

Integrating by parts, we get

$2\left[\theta. \int \theta \sec^2\theta d\theta - \int \left\{\left(\frac{d}{d\theta}\right)\int \sec^2\theta d\theta\right\} d\theta\right]$

$= 2[\theta. \tan\theta - \int \tan\theta\, d\theta]$

= 2[θ tan θ + log |cos θ | + C

$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right| + C\right]$

$= 2x\tan^{-1}x + 2\log(1+x^2)^{\frac{1}{2}} + C$

$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log(1+x)^2\right] + C$

$= 2x\tan^{-1}x + \log(1+x^2) + C$

Question 23: choose the correct answer:

$\int x^2 e^{x^3}\, dx\ equals$

A. $\frac{1}{3}e^{x^3} + C$

B. $\frac{1}{3}e^{x^2} + C$

C. $\frac{1}{2}e^{x^3}$ + C

D. $\frac{1}{2}e^{x^2} + C$

Answer:

Let I = $\int x^2 e^{x^3} dx$

Also, let $x^3 = t$

$\Rightarrow 3x^2dx = dt$

Thus,

$\Rightarrow$ I = $\frac{1}{3}\int e^t dt$

$\Rightarrow \frac{1}{3}(e^t) + C$

$\Rightarrow \frac{1}{3}\left(e^{x^3}\right) + C$

Question 24: choose the correct answer:

$\int e^x \sec x\,(1 + \tan x)dx$ equals

A. $e^x \cos x + C$

B. $e^x \sec x + C$

C. $e^x \sin x + C$

D. $e^x \tan x + C$

Answer:

$\int e^x \sec x\,(1 + \tan x)dx$

Let I = $\int e^x \sec x\,(1 + \tan x)dx = \int e^x\,(\sec x + \sec x \tan x)dx$

Also, let sec x = f(x)

$\Rightarrow$ sec x tan x = f'(x)

We know that, $\int e^x\{ f(x) + f'(x)\}dx = e^x f(x) + C$

Thus, I = $e^x \sec x$ + C