10 Trigonometry

KEY FACTS

I. The common systems of measuring angles are:

(*a*) Sexagesimal System: Here, one complete revolution is divided into 360 equal parts, each called a **degree**, a degree is divided into sixty equal parts, each called a **second**. Thus,

One complete revolution = 360°

$1° = 60'$ (minutes)

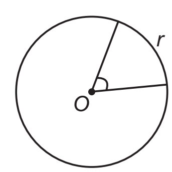
$1' = 60''$ (seconds)

(*b*) Circular System: The unit of measuring an angle in this system is **radian**. A **radian** is the measure of the central angle subtended by an arc equal in length to the radius of the circle. **1 radian** is written as $\mathbf{1^c}$.

II. π radians = 180 degrees

(*a*) To convert an angle in radians to its equivalent in degrees, multiply the number of radians by $\dfrac{180°}{\pi}$.

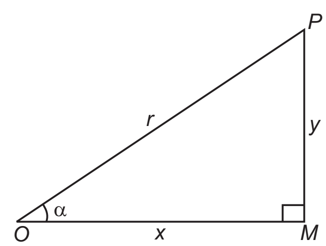
$\therefore \dfrac{\pi}{3}$ radians $= \dfrac{\pi}{3} \times \dfrac{180°}{\pi} = 60°$



(*b*) To convert an angle in degrees to its equivalent in radians, multiply the number of degrees by $\dfrac{\pi}{180°}$.

$\therefore 45° = \dfrac{\pi}{180°} \times 45° = \dfrac{\pi}{4}$ rad.

III. Trigonometrical functions (or Ratios)



In a right angled triangle *OMP*, where $\angle OMP = 90°$, $\angle POM = \alpha$, base $OM = x$, perpendicular $PM = y$ and hypotenuse $OP = r$,

(*a*) $\sin \alpha = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{y}{r}$

(*b*) $\cos \alpha = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{x}{r}$

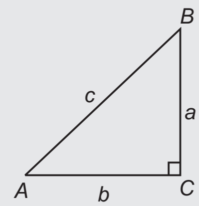
(*c*) $\tan \alpha = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{y}{x}$

(*d*) $\text{cosec } \alpha = \dfrac{1}{\sin \alpha} = \dfrac{\text{Hypotenuse}}{\text{Prependicular}} = \dfrac{r}{y}$

(*e*) $\sec \alpha = \dfrac{1}{\cos \alpha} = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{r}{y}$

(*f*) $\cot \alpha = \dfrac{1}{\tan \alpha} = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{x}{y}$

Note:



1. In the right $\angle d\ \Delta ABC$, it can be easily seen that,

(*a*) $\sin A = \dfrac{a}{c} = \cos B$ (*b*) $\cos A = \dfrac{b}{c} = \sin B$

(*c*) $\tan A = \dfrac{a}{b} = \cot B$ (*d*) $\tan B = \dfrac{b}{a} = \cot A$

2. $(\sin\theta)^2 = \sin^2\theta$, $(\cos\theta)^3 = \cos^3\theta$ read as sine square θ and cos cube θ respectively.

IV. Fundamental relations between trigonometric functions

1. Reciprocal relations

(*i*) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} \Rightarrow \operatorname{cosec}\theta \times \sin\theta = 1$ (*ii*) $\sec\theta = \dfrac{1}{\cos\theta} \Rightarrow \sec\theta \times \cos\theta = 1$

(*iii*) $\cot\theta = \dfrac{1}{\tan\theta} \Rightarrow \cot\theta \times \tan\theta = 1$

2. Quotient relations

(*i*) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (*ii*) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

3. Square relations (Pythagorean Identities)

(*i*) $\sin^2\theta + \cos^2\theta = 1$

(*ii*) $\sec^2\theta = 1 + \tan^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta - 1 = \tan^2\theta$

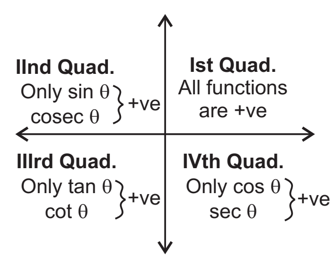
(*iii*) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$

V. Signs of trigonometric functions

The sign of a particular *t*-function is any quadrant can be remembered by the phrase. "All – sin – tan – cos" (**A**dd **S**ugar **T**o **C**offee).

↓ ↓ ↓ ↓

Ist quad· 2nd quad· 3rd quad· 4th quad



The *t*-function stated in the given phrase corresponding to the particular quadrant along with its reciprocal is positive in that quadrant and the rest are negative.

VI. Values of trigonometric ratios of some standard angles

θ / *t*-function	**0°**	**30° or $\frac{\pi}{3}$**	**45° or $\frac{\pi}{4}$**	**60° or $\frac{\pi}{6}$**	**90° or $\frac{\pi}{2}$**	**180° or π**
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0
cosec θ	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	undefined
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined	–1
cot θ	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	undefined

VII. Trigonometric functions of angles of any magnitude

The following formulae are helpful in reducing functions of any larger angle to function of smaller angles.

	– θ	**90° – θ**	**90° + θ**	**180° – θ**	**180° + θ**	**270° – θ**	**270° + θ**	**360° – θ**	**360° + θ**
sin	– sin	cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ
cos	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ	cos θ	cos θ
tan	– tan θ	cot θ	– cot θ	– tan θ	tan θ	cot θ	– cot θ	– tan θ	tan θ
cosec	– cosec θ	sec θ	sec θ	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ
sec	sec θ	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ	sec θ	sec θ
cot	– cot θ	tan θ	– tan θ	– cot θ	cot θ	tan θ	– tan θ	cot θ	cot θ

	2*n* π – θ or *n* . 360° – θ	**2*n* π + θ or *n* . 360° + θ**
sin	–sin θ	sin θ
cos	cos θ	cos θ
tan	–tan θ	tan θ

VIII. Some important formulae

(*a*) Sum Formulae

(*i*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(*ii*) $\cos(A+B) = \cos A \cos B + \sin A \sin B$

(*iii*) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(*iv*) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(*v*) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(*b*) Difference formulae

(*i*) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(*ii*) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(*iii*) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(*iv*) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(*c*) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

(*d*) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

$\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

(*e*) Product formulae

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

(*f*) Double angle formulae

- $\sin 2A = 2 \sin A \cos A$

  $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A$

  $\cos 2A = 1 - 2\sin^2 A$

  $\cos 2A = 2\cos^2 A - 1$

  $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $2\sin^2 A = 1 - \cos 2A$

  $\Rightarrow \sin^2 A = \dfrac{1-\cos 2A}{2}$
- $2\cos^2 A = 1 + \cos 2A$

  $\Rightarrow \cos^2 A = \dfrac{1+\cos 2A}{2}$
- $\tan^2 A = \dfrac{\sin^2 A}{\cos^2 A} = \dfrac{1-\cos 2A}{1+\cos 2A}$

(*g*) Half angle formulae

- $\sin A = 2\sin A/2 \cos A/2$

  $\sin A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$
- $\cos A = \cos^2 A/2 - \sin^2 A/2$

  $\cos A = 1 - 2\sin^2 A/2$

  $\cos A = 2\cos^2 A/2 - 1$

  $\cos A = \dfrac{1-\tan^2 A/2}{1+\tan^2 A/2}$
- $\tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$
- $\sin A/2 = \pm\sqrt{\dfrac{1-\cos A}{2}}$
- $\cos A/2 = \pm\sqrt{\dfrac{1-\cos A}{2}}$
- $\tan A/2 = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

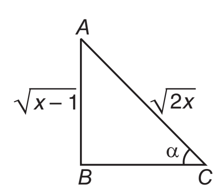
(*h*) Triple-angle formulae

- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$
- $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

SOLVED EXAMPLES

Ex. 1. If α is an acute angle and $\sin \alpha = \sqrt{\dfrac{x-1}{2x}}$, then what is tan α equal to? ***(CDS 2010)***

Sol. In ΔABC where $\angle B = 90°$, $\sin\alpha = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{\sqrt{x-1}}{\sqrt{2x}} = \dfrac{AB}{AC}$



$\therefore\ BC^2 = AC^2 - AB^2 = 2x - (x-1) = x + 1$

$\Rightarrow BC = \sqrt{x+1}$

$\therefore\ \tan\alpha = \dfrac{\text{Prependicular}}{\text{Base}} = \dfrac{AB}{BC} = \dfrac{\sqrt{x-1}}{\sqrt{x+1}}.$

Ex. 2. What is the value of the expression

$$3\tan^2\frac{\pi}{6} + \frac{4}{3}\cos^2\frac{\pi}{6} - \frac{1}{2}\cot^3\frac{\pi}{4} - \frac{2}{3}\sin^2\frac{\pi}{3} + \frac{1}{8}\sec^4\frac{\pi}{3}\ ?$$

Sol. $3\tan^2\frac{\pi}{6}+\frac{4}{3}\cos^2\frac{\pi}{6}-\frac{1}{2}\cot^3\frac{\pi}{4}-\frac{2}{3}\sin^2\frac{\pi}{3}+\frac{1}{8}\sec^4\frac{\pi}{3}$

$$= 3\tan^2 30° + \frac{4}{3}\cos^2 30° - \frac{1}{2}\cot^3 45° - \frac{2}{3}\sin^2 60° + \frac{1}{8}\sec^4 60°$$

$$= 3\times\left(\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\times\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}\times(1)^3-\frac{2}{3}\times\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{8}\times(2)^4$$

$$= 3\times\frac{1}{3}+\frac{4}{3}\times\frac{3}{4}-\frac{1}{2}-\frac{2}{3}\times\frac{3}{4}+\frac{1}{8}\times 16 = 1+1-\frac{1}{2}-\frac{1}{2}+2=\mathbf{3.}$$

Ex. 3. If $A + B = 90°$, then what is the value of $\sqrt{\sin A\sec B-\sin A\cos B}$?

Sol. Given $A + B = 90°$

$$\therefore\ \sqrt{\sin A\sec B-\sin A\cos B}=\sqrt{\sin A\sec(90°-A)-\sin A\cos(90°-A)}$$

$$=\sqrt{\sin A.\text{cosec } A-\sin A.\sin A}=\sqrt{1-\sin^2 A}=\sqrt{\cos^2 A}=\mathbf{\cos A.}$$

Ex. 4. If $\sin(A+B+C)=1$, $\tan(A-B)=\frac{1}{\sqrt{3}}$ and $\sec(A+C)=2$, then find the values of the angles A, B and C in degrees.

Sol. $\sin(A+B+C)=1 \Rightarrow \sin(A+B+C)=\sin 90° \Rightarrow A+B+C=90°$ …(*i*)

$\tan(A-B)=\frac{1}{\sqrt{3}} \Rightarrow \tan(A-B)=\tan 30° \Rightarrow A-B=30°$ …(*ii*)

$\sec(A+C)=2 \Rightarrow \sec(A+C)=\sec 60° \Rightarrow A+C=60°$ …(*iii*)

Eq (*i*) – Eq (*iii*) $\Rightarrow B=30°$

$\therefore$ From eqn (*ii*), $A-30°=30° \Rightarrow A=60°$

$\therefore$ From eqn (*i*), $60°+30°+C=90° \Rightarrow C=0° \Rightarrow A=\mathbf{60°}, B=\mathbf{30°}, C=\mathbf{0°}.$

Ex. 5. If $\sec\theta=\sqrt{2}$ and $\frac{3\pi}{2}<\theta<2\pi$, find the value of $\frac{1+\tan\theta+\text{cosec}\,\theta}{1+\cot\theta-\text{cosec}\,\theta}$.

Sol. $\sec\theta=\sqrt{2} \Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$

$$\therefore\ \sin\theta=\pm\sqrt{1-\cos^2\theta}=\pm\sqrt{1-\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}$$

Since θ lies in the fourth quadrant, so $\sin\theta$ is –ve and $\cos\theta$ is +ve.

$\therefore\ \sin\theta=-\frac{1}{\sqrt{2}}$, $\text{cosec}\,\theta=-\sqrt{2}$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=-\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{1}=-1 \Rightarrow \cot\theta=-1$$

$$\therefore\ \frac{1+\tan\theta+\text{cosec}\,\theta}{1+\cot\theta-\text{cosec}\,\theta}=\frac{1-1-\sqrt{2}}{1-1+\sqrt{2}}=\mathbf{-1.}$$

Ex. 6. What is the value of cot (– 870º)?

Sol. $\cot(-870°) = -\cot 870°$ $(\because \cot(-\theta)=-\cot\theta)$

$$= -\cot(720°+150°) = -\cot(2\times 360°+150°)$$

$= -\cot 150° = -\cot(90° + 60°)$ $(\because \cot(n.360° + \theta) = \cot\theta, n \in N)$

$= -(-\tan 60°)$ $(\because \cot(90° + \theta) = -\tan\theta)$

$= \tan 60° = \sqrt{3}$ **.**

Ex. 7. What is the value of cos 480º sin 150º + sin 600º cos 390º? **(*Kerala PET 2006*)**

Sol. $\cos 480° \sin 150° + \sin 600° \cos 390°$

$= \cos(360° + 120°) \sin(180° - 30°) + \sin(2 \times 360° - 120°) \cos(360° + 30°)$

$= \cos 120° \sin 30° - \sin(120°) \cos 30°$

$= -\cos 60° \sin 30° - \sin(180° - 60°) \cos 30°$

$= -\cos 60° \sin 30° - \sin 60° \cos 30°$

$= -(\sin 30° \cos 60° + \sin 60° \cos 30°)$

$= -\sin(30° + 60°)$ $(\because \sin(A + B) = \sin A \cos B + \cos A \sin B)$

$= -\sin 90° = \mathbf{-1.}$

$$\left[\begin{array}{l}\because \cos(360° + \theta) = \cos\theta \\ \sin(n.360° - \theta) = -\sin\theta, n \in N \\ \sin(180° - \theta) = \sin\theta \\ \sin(180° + \theta) = -\sin\theta \\ \cos 120° = \cos(180° - 60°) = -\cos 60°\end{array}\right]$$

Ex. 8. If $x = y\cos\left(\frac{2\pi}{3}\right) = z\cos\left(\frac{4\pi}{3}\right)$, then what is $xy + yz + zx$ equal to? ***(NDA/NA 2011)***

Sol. $x = y\cos 120° = z\cos 240°$

$\Rightarrow x = y\cos(180° - 60°) = z\cos(180° + 60°)$

$\Rightarrow x = -y\cos 60° = -z\cos 60°$ $(\because \cos(180° - \theta) = -\cos\theta = \cos(180° + \theta))$

$\Rightarrow x = -\frac{1}{2}y = -\frac{z}{2} \Rightarrow 2x = -y = -z$

$\Rightarrow \frac{x}{\frac{1}{2}} = \frac{y}{(-1)} = \frac{z}{(-1)} = k \Rightarrow x = \frac{k}{2}, y = -k, z = -k$

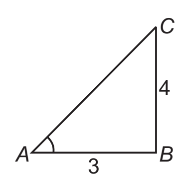
$\therefore xy + yz + zx = \left(\frac{k}{2}\right)(-k) + (-k)(-k) + (-k)\left(\frac{k}{2}\right) = \frac{-k^2}{2} + k^2 - \frac{k^2}{2} = \mathbf{0.}$

Ex. 9. If A lies in the third quadrant and $3\tan A - 4 = 0$, then what is the value of $5\sin 2A + 3\sin A + 4\cos A$?

Sol. $3\tan A - 4 = 0$

$\Rightarrow 3\tan A = 4 \Rightarrow \tan A = \frac{4}{3}$

$\Rightarrow \tan A = \frac{4}{3} \Rightarrow \frac{BC}{AB} = \frac{4}{3} \Rightarrow AC = \sqrt{AB^2 + BC^2} = \sqrt{9+16} = \sqrt{25} = \mathbf{5.}$



$\therefore$ A being in the third quadrant, $\sin A$ and $\cos A$ are negative

So, $\sin A = -\frac{4}{5}$ and $\cos A = -\frac{3}{5}$.

$\therefore$ Given expression $5\sin 2A + 3\sin A + 4\cos A$

$= 5 \times 2\sin A\cos A + 3\sin A + 4\cos A$

$= 10 \times \left(\frac{-4}{5}\right) \times \left(\frac{-3}{5}\right) + 3 \times \left(\frac{-4}{5}\right) + 4 \times \left(\frac{-3}{5}\right)$

$= \frac{24}{5} - \frac{12}{5} - \frac{12}{5} = \mathbf{0.}$

Ex. 10. Show that $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\}=1$.

Sol. $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\}$

$= \cos(270°+x)\cos(360°+x)\{\cot(270°-x)+\cot(360°+x)\}$

$\left[\begin{array}{l}\because \cos(360°+\theta)=\cos\theta \\ \cot(360°+\theta)=\cot\theta \\ \cos(270°+\theta)=+\sin\theta \\ \cot(270°-\theta)=\tan\theta\end{array}\right]$

$= \sin x\cos x\{\tan x+\cot x\}$

$= \sin x\cos x\left[\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right] = \sin x\cos x\left[\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right] = \mathbf{1.}$

Ex. 11. What is the value of $\frac{\cos 15°+\cos 45°}{\cos^3 15°+\cos^3 45°}$? ***(NDA/NA 2013)***

Sol. $\frac{\cos 15°+\cos 45°}{\cos^3 15°+\cos^3 45°} = \frac{\cos 15°+\cos 45°}{(\cos 15°+\cos 45°)(\cos^2 15°+\cos^2 45°-\cos 45°\cos 15°)}$

($\because a^3+b^3=(a+b)(a^2-ab+b^2)$)

$= \frac{1}{(\cos^2 15°+\cos^2 45°-\cos 45°\cos 15°)}$ ...(*i*)

$\left[\begin{array}{l}\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30° \\ \qquad =\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}\end{array}\right]$

$\therefore \frac{\cos 15°+\cos 45°}{\cos^3 15°+\cos^3 45°} = \frac{1}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2-\frac{1}{\sqrt{2}}\times\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}$ (From (*i*))

$= \frac{1}{\frac{3+1+2\sqrt{3}}{8}+\frac{1}{2}-\left(\frac{\sqrt{3}+1}{4}\right)} = \frac{1}{\frac{4+4+2\sqrt{3}-2\sqrt{3}-2}{8}} = \frac{8}{6} = \mathbf{\frac{4}{3}.}$

Ex. 12. What is the value of $\sqrt{3}$ cosec 20° – sec 20° ?

Sol. $\sqrt{3}$ cosec 20° – sec 20°

$= \frac{\sqrt{3}}{\sin 20°}-\frac{1}{\cos 20°}=\frac{\sqrt{3}\cos 20°-\sin 20°}{\sin 20°\cos 20°}$

$= \frac{4}{2\sin 20°\cos 20°}\left(\frac{\sqrt{3}}{2}\cos 20°-\frac{1}{2}\sin 20°\right)$ (Multiplying numerator and denominator by 4)

$= \frac{4}{\sin 40°}(\sin 60°\cos 20°-\cos 60°\sin 20°)$ ($\because \sin 2\theta = 2\sin\theta\cos\theta$)

$= \frac{4}{\sin 40°}(\sin(60°-20°))$ ($\because \sin(A-B)=\sin A\cos B-\cos A\sin B$)

$= \frac{4}{\sin 40°}.\sin 40° = \mathbf{4.}$

Ex. 13. Given, 3 sin θ + 4 cos θ = 5, then what is 3 cos θ – 4 sin θ equal to?

Sol. $3 \sin\theta + 4\cos\theta = 5$

$\Rightarrow (3\sin\theta + 4\cos\theta)^2 = 25 \Rightarrow 9\sin^2\theta + 16\cos^2\theta + 24\sin\theta\cos\theta = 25$

$\Rightarrow 9(1-\cos^2\theta) + 16(1-\sin^2\theta) + 24\sin\theta\cos\theta = 25$ $\quad (\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow 9 - 9\cos^2\theta + 16 - 16\sin^2\theta + 24\sin\theta\cos\theta = 25$

$\Rightarrow 9\cos^2\theta + 16\sin^2\theta - 24\sin\theta\cos\theta = 0$

$\Rightarrow (3\cos\theta - 4\sin\theta)^2 = 0 \Rightarrow 3\cos\theta - 4\sin\theta = \mathbf{0.}$

Ex. 14. If tan *x* = b/a, then what is the value of *a* cos 2*x* + *b* sin 2*x*? **(*UPSEAT 2007, AMU 2002*)**

Sol. Given $\tan x = b/a$

$$a\cos 2x + b\sin 2x = a\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) + b\left(\frac{2\tan x}{1+\tan^2 x}\right) = \frac{a(1-\tan^2 x) + b(2\tan x)}{(1+\tan^2 x)}$$

$$= \frac{a(1-b^2/a^2) + b(2.b/a)}{1+b^2/a^2} = \frac{a\left(\frac{a^2-b^2}{a^2}\right) + \frac{2b^2}{a}}{\frac{a^2+b^2}{a^2}}$$

$$= \frac{\frac{a^2-b^2}{a} + \frac{2b^2}{a}}{\frac{a^2+b^2}{a^2}} = \frac{\frac{a^2-b^2+2b^2}{a}}{\frac{a^2+b^2}{a^2}} = \frac{a^2+b^2}{a} \cdot \frac{a^2}{a^2+b^2} = \boldsymbol{a.}$$

Ex. 15. If θ and φ are angles in the first quadrant such that tan θ = $\mathbf{\frac{1}{7}}$ and sin φ = $\mathbf{\frac{1}{\sqrt{10}}}$, then show that θ + 2φ = 45°.

Sol. Since θ and φ lie in the Ist quadrant, sin θ, sin φ; cos θ, cos φ and tan θ, tan φ are all positive.

$$\Rightarrow \sin\phi = \frac{1}{\sqrt{10}} \Rightarrow \cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3} \quad \therefore \quad \tan 2\phi = \frac{2\tan\phi}{1-\tan^2\phi} = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$$

Also, given $\tan\theta = \frac{1}{2}$

$$\therefore \tan(\theta + 2\phi) = \frac{\tan\theta + \tan 2\phi}{1-\tan\theta\tan 2\phi} \qquad \left(\because \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}\right)$$

$$= \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}} = \frac{\frac{4+21}{28}}{1-\frac{3}{28}} = \frac{\frac{25}{28}}{\frac{25}{28}} = 1$$

$\Rightarrow \tan(\theta + 2\phi) = \tan 45° \Rightarrow \theta + 2\phi = \mathbf{45°.}$

Ex. 16. If $A + B + C = \frac{\pi}{2}$, then find sin 2*A* + sin 2*B* + sin 2*C*.

Sol. $\sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + 2 \sin C \cos C$

$= 2 \sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + 2 \sin C \cos C$ $\quad \left(\text{Using } \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}\right)$

$= 2 \sin (A + B) \cos (A - B) + 2 \sin C \cos C$ $\quad \left(\text{Given } A + B + C = \frac{\pi}{2} \Rightarrow A + B = \frac{\pi}{2} - C\right)$

$= 2 \sin\left(\frac{\pi}{2} - C\right) \cos (A - B) + 2 \sin C \cos C$

$= 2 \cos C \cos (A - B) + 2 \sin C \cos C$ $\quad$ (Using $\sin (90° - \theta) = \cos \theta$)

$= 2 \cos C (\cos (A - B) + \sin C)$

$= 2 \cos C \left(\cos (A - B) + \sin \left\{\frac{\pi}{2} - (A + B)\right\}\right)$

$= 2 \cos C (\cos (A - B) + \cos (A + B))$

$= 2 \cos C \left(2 \cos\left(\frac{A-B+A+B}{2}\right)\cos\left(\frac{A-B-A-B}{2}\right)\right)$ $\quad \left(\text{Using } \cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}\right)$

$= 2 \cos C (2 \cos A \,.\, \cos (-B)) = \mathbf{4 \cos A \cos B \cos C}$ $\quad$ (Using $\cos (-\theta) = \cos \theta$)

Ex. 17. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then show that tan α = tan β = 2tan γ. ***(IIT 2003)***

Sol. $\alpha + \beta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} - \beta \Rightarrow \tan \alpha = \tan\left(\frac{\pi}{2} - \beta\right)$

$\Rightarrow \tan \alpha = \cot \beta \Rightarrow \tan \alpha \tan \beta = 1.$ …(i)

Now, $\beta + \gamma = \alpha \Rightarrow \gamma = \alpha - \beta$

$\Rightarrow \tan \gamma = \tan (\alpha - \beta) \Rightarrow \tan \gamma = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$\Rightarrow \tan \gamma = \frac{\tan \alpha - \tan \beta}{1+1} = \frac{\tan \alpha - \tan \beta}{2}$ $\quad$ ($\because \tan \alpha \tan \beta = 1$)

$\Rightarrow 2 \tan \gamma = \tan \alpha - \tan \beta \Rightarrow$ $\boldsymbol{\tan \alpha = \tan \beta + 2 \tan \gamma}$.

Ex. 18. If $\cos 2B = \frac{\cos (A + C)}{\cos (A - C)}$, then show that tan *A*, tan *B* and tan *C* are in *G.P.*

Sol. $\cos 2B = \frac{\cos (A + C)}{\cos (A - C)}$

$\Rightarrow \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{\cos A \cos C - \sin A \sin C}{\cos A \cos C + \sin A \sin C}$

$\Rightarrow \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{1 - \tan A \tan C}{1 + \tan A \tan C}$ $\quad$ (On dividing the numerator and denominator of RHS by cos *A* cos *C*)

$\Rightarrow 1 + \tan A \tan C - \tan^2 B - \tan A \tan^2 B \tan C = 1 + \tan^2 B - \tan A \tan C - \tan A \tan^2 B \tan C$

$\Rightarrow 2 \tan A \tan C = 2 \tan^2 B \Rightarrow \tan A \,.\, \tan C = \tan^2 B \Rightarrow$ tan *A*, tan *B*, tan *C* are in *G.P.*

Ex. 19. If $\cos x = \dfrac{2\cos y - 1}{2 - \cos y}$, where $x \in (0, \pi)$, then show that $\tan \dfrac{x}{2} \cot \dfrac{y}{2} = \sqrt{3}$. ***(Manipal Engineering 2010)***

Sol. $\cos x = \dfrac{2\cos y - 1}{2 - \cos y}$

$$\Rightarrow \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{2\left[\dfrac{1-\tan^2 y/2}{1+\tan^2 y/2}\right]-1}{2-\left[\dfrac{1-\tan^2 y/2}{1+\tan^2 y/2}\right]} \quad \Rightarrow \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{2(1-\tan^2 y/2)-(1+\tan^2 y/2)}{2(1+\tan^2 y/2)-(1-\tan^2 y/2)}$$

$$\Rightarrow \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{1-3\tan^2 y/2}{1+3\tan^2 y/2}$$

$$\Rightarrow 1 + 3\tan^2 y/2 - \tan^2 x/2 - 3\tan^2 x/2 \tan^2 y/2 = 1 - 3\tan^2 y/2 + \tan^2 x/2 - 3\tan^2 x/2 \tan^2 y/2$$

$$\Rightarrow 6\tan^2 y/2 = 2\tan^2 x/2 \quad \Rightarrow \tan^2 x/2 \,.\, \frac{1}{\tan^2 y/2} = 3 \quad \Rightarrow \mathbf{\tan x/2 \,.\, \cot y/2 = \sqrt{3}}.$$

Ex. 20. If A, B and C are the angles of a triangle such that sec $(A - B)$, sec A and sec $(A + B)$ are in arithmetic progression then show that $2\sec^2 A = \sec^2 \dfrac{B}{2}$. ***(UPSEEE 2011)***

Sol. Since sec $(A - B)$, sec A, sec $(A + B)$ are in A.P., therefore

$$\sec A = \frac{\sec(A-B)+\sec(A+B)}{2} = \frac{1}{2}\left[\frac{1}{\cos(A-B)} + \frac{1}{\cos(A+B)}\right]$$

$$= \frac{\cos(A+B)+\cos(A-B)}{2\cos(A-B)\cos(A+B)} = \frac{2\cos A\cos B}{2[(\cos A\cos B+\sin A\sin B)(\cos A\cos B-\sin A\sin B)]}$$

$$= \frac{\cos A\cos B}{(\cos^2 A\cos^2 B-\sin^2 A\sin^2 B)} = \frac{\cos A\cos B}{(\cos^2 A\cos^2 B)-(1-\cos^2 A)(1-\cos^2 B)}$$

$$\Rightarrow \sec A = \frac{\cos A\cos B}{[\cancel{\cos^2 A\cos^2 B} - 1 + \cos^2 A + \cos^2 B - \cancel{\cos^2 A\cos^2 B}]}$$

$\Rightarrow \cos^2 A + \cos^2 B - 1 = \cos^2 A \cos B \Rightarrow \cos^2 A\,(1 - \cos B) = 1 - \cos^2 B$

$\Rightarrow \cos^2 A = 1 + \cos B \Rightarrow \cos^2 A = 2\cos^2 \dfrac{B}{2} \Rightarrow \sec^2 A = \dfrac{1}{2}\sec^2 \dfrac{B}{2} \Rightarrow \mathbf{2\sec^2 A = \sec^2 \dfrac{B}{2}}$.

Ex. 21. If $\tan\theta + \sec\theta = 4$, then what is the value of $\sin\theta$? ***(NDA/NA 2012)***

Sol. Given, $\tan\theta + \sec\theta = 4$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = 4 \quad \Rightarrow \frac{1+\sin\theta}{\cos\theta} = 4$$

$$\Rightarrow \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\theta/2\cos\theta/2}{(\cos^2\theta/2 - \sin^2\theta/2)} = 4$$

(Using the formulas $\sin^2\theta + \cos^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2 - \sin^2\theta$)

$$\Rightarrow \frac{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)} = 4$$

Alternatively,

Given, $\tan\theta + \sec\theta = 4$ ...(*i*)

$\sec^2\theta = 1 + \tan^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta = 1$ ...(*ii*)

eq. (*ii*) ÷ eq. (*i*)

$\Rightarrow \sec\theta - \tan\theta = \dfrac{1}{4}$ ...(*iii*)

eq. (*i*) + eq. (*iii*)

$\Rightarrow 2\sec\theta = \dfrac{17}{4} \Rightarrow \sec\theta = \dfrac{17}{8}$

$\Rightarrow \cos\theta = \dfrac{8}{17}$

Now, use $\sin\theta = \sqrt{1-\cos^2\theta}$

$$\Rightarrow \frac{(\sin\theta/2+\cos\theta/2)}{\cos\theta/2-\sin\theta/2}=4 \qquad \Rightarrow \frac{1+\tan\theta/2}{1-\tan\theta/2}=4$$

$$\Rightarrow 1+\tan\theta/2 = 4-4\tan\theta/2 \quad \Rightarrow 5\tan\theta/2 = 3 \Rightarrow \tan\theta/2 = 3/5$$

$$\therefore \sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}=\frac{2\times 3/5}{1+9/25}=\frac{6/5}{34/25}=\frac{30}{34}=\mathbf{\frac{15}{17}.}$$

Ex. 22. Find the value of tan $\left(7\frac{1^{o}}{2}\right)$.

Sol. $$\tan 7\frac{1}{2}^{\circ} = \frac{\sin 7\frac{1}{2}^{\circ}}{\cos 7\frac{1}{2}^{\circ}}=\frac{\sin^2 7\frac{1}{2}^{\circ}}{\sin 7\frac{1}{2}^{\circ}\cos 7\frac{1}{2}^{\circ}}=\frac{2\sin^2 7\frac{1}{2}^{\circ}}{2\sin 7\frac{1}{2}^{\circ}\cos 7\frac{1}{2}^{\circ}}$$

$$=\frac{1-\cos 15^\circ}{\sin 15^\circ}$$ (Using $1-\cos 2\theta = 2\sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$)

$$=\frac{1-\cos(45^\circ-30^\circ)}{\sin(45^\circ-30^\circ)}=\frac{1-(\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ)}{\sin 45^\circ\cos 30^\circ-\sin 30^\circ\sin 45^\circ}$$

$$=\frac{1-\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}.\frac{1}{2}\right)}{\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{1}{\sqrt{2}}}=\frac{1-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{3-1}$$

$$=\frac{2\sqrt{6}-2\sqrt{3}-4-2\sqrt{2}}{2}=\mathbf{\sqrt{6}-\sqrt{3}-\sqrt{2}-2.}$$

Ex. 23. If tan $\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=p\sec 2\theta$, then find the value of p.

Sol. $$\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=p\sec 2\theta$$

$$\Rightarrow \frac{\tan\pi/4+\tan\theta}{1-\tan\pi/4.\tan\theta}+\frac{\tan\pi/4-\tan\theta}{1+\tan\pi/4.\tan\theta}=p\sec 2\theta$$ $\left(\because \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\right)$

$$\Rightarrow \frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}=p\sec\theta$$ $\left(\because \tan\frac{\pi}{4}=\tan 45^o=1\right)$

$$\Rightarrow \frac{(1+\tan\theta)^2+(1-\tan\theta)^2}{1-\tan^2\theta}=p\sec^2\theta$$

$$\Rightarrow \frac{1+2\tan\theta+\tan^2\theta+1-\ 2\tan\theta+\tan\theta}{1-\tan^2\theta}=p\sec 2\theta$$

$$\Rightarrow \frac{2(1+\tan^2\theta)}{1-\tan^2\theta}=p\sec 2\theta \quad \Rightarrow \frac{2}{\cos 2\theta}=p\sec 2\theta$$ $\left(\text{Using }\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$\Rightarrow 2\sec 2\theta = p\sec 2\theta \Rightarrow \mathbf{p=2.}$$

Ex. 24. Prove that $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$.

Sol. LHS $=\dfrac{(\sin 7A+\sin A)+(\sin 5A+\sin 3A)}{(\cos 7A+\cos A)+(\cos 5A+\cos 3A)}$

$$=\frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$\left(\text{Using } \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right);\ \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right)$

$$=\frac{2\sin 4A\cos 3A+2\sin 4A\cos A}{2\cos 4A\cos 3A+2\cos 4A\cos A}=\frac{2\sin 4A(\cos 3A+\cos A)}{2\cos 4A(\cos 3A+\cos A)}=\mathbf{\tan 4A}.$$

Ex. 25. If $\cos\alpha+\cos\beta+\cos\gamma=0$, then prove that $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=12\cos\alpha\cos\beta\cos\gamma$.

Sol. $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=(4\cos^3\alpha-3\cos^3\alpha)+(4\cos^3\beta-3\cos\beta)+(4\cos^3\gamma-3\cos\gamma)$

$=4(\cos^3\alpha+\cos^3\beta+\cos^3\gamma)-3(\cos\alpha+\cos\beta+\cos\gamma)$

$=4(\cos^3\alpha+\cos^3\beta+\cos^3\gamma)-3\times 0=4(\cos^3\alpha+\cos^3\beta+\cos^3\gamma)$

$=4\times 3\cos\alpha\cos\beta\cos\gamma$ $\quad(\because a+b+c=0\Rightarrow a^3+b^3+c^3=3abc)$

$=\mathbf{12\cos\alpha\cos\beta\cos\gamma}$.

Ex. 26. If $A=\dfrac{41\pi}{12}$, then what is the value of $\dfrac{1-3\tan^2 A}{3\tan A-\tan^3 A}$.

Sol. $\because \tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

$$\therefore \frac{1-3\tan^2 A}{3\tan A-\tan^3 A}=\frac{1}{\tan 3A}=\frac{1}{\tan 3\times\frac{41\pi}{12}}=\frac{1}{\tan\frac{41\pi}{4}}$$

$$=\frac{1}{\tan\left(10\pi+\frac{\pi}{4}\right)}=\frac{1}{\tan\frac{\pi}{4}}=\mathbf{1}.$$ [Using $\tan(2n\pi+\theta)=\tan\theta$]

Ex. 27. If $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of the equation $x^2-px+q=0$, then prove that $p^2=q(q+2)$.

(Kerala PET 2007)

Sol. Sum of roots $=\sec\alpha+\operatorname{cosec}\alpha=p$

$\Rightarrow \dfrac{1}{\cos\alpha}+\dfrac{1}{\sin\alpha}=p\Rightarrow\dfrac{\sin\alpha+\cos\alpha}{\sin\alpha\cos\alpha}=p$ ...(*i*)

Product of roots $=\sec\alpha\,.\operatorname{cosec}\alpha=q\ \Rightarrow\ \dfrac{1}{\sin\alpha\cos\alpha}=q$ ...(*ii*)

Now $p^2=\left(\dfrac{\sin\alpha+\cos\alpha}{\sin\alpha\cos\alpha}\right)^2=\dfrac{\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha}{\sin\alpha^2\cos^2\alpha}$

$\Rightarrow p^2=\dfrac{1+2\sin\alpha\cos\alpha}{\sin^2\alpha\cos^2\alpha}=\dfrac{1}{\sin^2\alpha\cos^2\alpha}+\dfrac{2}{\sin\alpha\cos\alpha}=q^2+2q=\mathbf{q(q+2)}.$

Ex. 28. If θ and φ are acute angles such that sin θ = $\frac{1}{2}$ and cos φ = $\frac{1}{3}$, than θ + φ lies in

(*a*) $\left]\frac{\pi}{3},\frac{\pi}{2}\right[$ (*b*) $\left]\frac{2\pi}{3},\frac{5\pi}{3}\right[$ (*c*) $\left]\frac{\pi}{2},\frac{2\pi}{3}\right[$ (*d*) $\left]\frac{5\pi}{6},\pi\right[$ ***(IIT 2004)***

Sol. $\sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$ (∵ θ and φ are acute angles lying in the first quadrant) …(*i*)

Now $\cos\phi = \frac{1}{3} \Rightarrow 0 < \cos\phi < \frac{1}{2} \Rightarrow \cos\frac{\pi}{2} < \cos\phi < \cos\frac{\pi}{3} \Rightarrow \frac{\pi}{2} < \phi < \frac{\pi}{3}$ …(*ii*)

∴ From (*i*) and (*ii*) $\frac{\pi}{2}+\frac{\pi}{6} < \theta+\phi < \frac{\pi}{3}+\frac{\pi}{6}$

$\Rightarrow \frac{\pi}{2} < \theta+\phi < \frac{2\pi}{3} \Rightarrow$ θ + φ lies in the open interval $\left]\frac{\pi}{2},\frac{2\pi}{3}\right[$.

Hence (*c*) is the correct option.

Ex. 29. The angle *A* lies in the third quadrant and it satisfies the equation 4 ($\sin^2 x + \cos x$) = 1. What is the measure of angle *A*? ***(NDA/NA 2010)***

Sol. $4\sin^2 x + 4\cos x = 1$

$\Rightarrow 4\sin^2 x + 4\cos x - 1 = 0 \quad \Rightarrow \quad 4(1-\cos^2 x) + 4\cos x - 1 = 0$

$\Rightarrow -4\cos^2 x + 4\cos x + 3 = 0 \quad \Rightarrow \quad 4\cos^2 x - 4\cos x - 3 = 0$

$\Rightarrow 4\cos^2 x - 6\cos x + 2\cos x - 3 = 0 \quad \Rightarrow \quad 2\cos x(2\cos x - 3) + 1(2\cos x - 3) = 0$

$\Rightarrow (2\cos x + 1)(2\cos x - 3) = 0$

$\Rightarrow \cos x = -\frac{1}{2}$ and $\cos x = \frac{3}{2}$ (not possible)

Now $\cos x = -\frac{1}{2}$

Since *A* lies in the third quadrant and $\cos A = -\frac{1}{2}$, therefore,

$\cos A = \cos(180° + 60°) = \cos 240° \Rightarrow A =$ **240°.**

Ex. 30. Find the value of $\cos^2 76 + \cos^2 16 - \cos 76° \cos 16°$. ***(EAMCET)***

Sol. $\cos^2 76° + \cos^2 16° - \cos^2 76° \cos 16°$

$= \frac{1+\cos(76°\times 2)}{2} + \frac{1+\cos(16°\times 2)}{2} - \frac{1}{2}[\cos(76° - 16°) + \cos(76° - 16°)]$

[∵ $2\cos^2\theta = 1 + \cos 2\theta$, $\cos(A + B) + (A - B) = 2\cos A\cos B$]

$= \frac{1}{2}[1 + \cos 152° + 1 + \cos 32° - \cos 92° + \cos 60°]$

$= \frac{1}{2}\left[\left(2 - \frac{1}{2}\right) + \cos 152° + \cos 32° - \cos 92°\right]\left[\because \cos 60° = \frac{1}{2}\right]$

$= \frac{1}{2}\left[\frac{3}{2} + 2\cos\left(\frac{152° + 32°}{2}\right)\cos\left(\frac{152° - 32°}{2}\right) - \cos 92°\right]$

$= \frac{1}{2}\left[\frac{3}{2} + 2\cos 92°\cos 60° - \cos 92°\right]\left(\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right)$

$= \frac{1}{2}\left[\frac{3}{2} + \cos 92° - \cos 92°\right] = \mathbf{\frac{3}{4}}.$ $\left(\because \cos 60° = \frac{1}{2}\right)$

PRACTICE SHEET

1. If $\cot\theta = \frac{2xy}{x^2-y^2}$, then what is $\cos\theta$ equal to?

(*a*) $\frac{x^2-y^2}{x^2+y^2}$ (*b*) $\frac{x^2+y^2}{x^2-y^2}$

(*c*) $\frac{2xy}{x^2+y^2}$ (*d*) $\frac{2xy}{\sqrt{x^2+y^2}}$ **(*CDS 2009*)**

2. If $\sec\theta = \frac{13}{5}$, then what is the value of $\frac{2\sin\theta-3\cos\theta}{4\sin\theta-9\cos\theta}$?

(*a*) 1 (*b*) 3 (*c*) 2 (*d*) 4

(*CDS 2007*)

3. If cosec $A = 2$, then the value of $\frac{1}{\tan A}+\frac{\sin A}{1+\cos A}$ is

(*a*) $\sqrt{2}-1$ (*b*) $\sqrt{3}+2$

(*c*) 0 (*d*) 2

4. The value of $4\cot^2 45° - \sec^2 60° + \sin^2 60° + \cos^2 90°$ is

(*a*) $\frac{1}{2}$ (*b*) $\frac{3}{4}$ (*c*) 1 (*d*) $\frac{11}{16}$

5. Evaluate $\frac{1}{4}\left(\cot^4 30° - \text{cosec}^4 60°\right)$

$$+\frac{3}{2}\left(\sec^2 45° - \tan^2 30°\right) - 5\cos^2 60°.$$

(*a*) $\frac{55}{18}$ (*b*) $\frac{11}{6}$ (*c*) $\frac{7}{4}$ (*d*) 0

6. If $2x^2\cos 60° - 4\cot^2 45° - 2\tan 60° = 0$, what is the value of x?

(*a*) 2 (*b*) 3 (*c*) $\sqrt{3}-1$ (*d*) $\sqrt{3}+1$

(*CDS 2007*)

7. The cotangent of the angles $\pi/3$, $\pi/4$ and $\pi/6$ are in

(*a*) A.P (*b*) G.P. (*c*) H.P.

(*d*) None of these **(*AMU*)**

8. The value of cos1°, cos 2°... cos100° is

(*a*) –1 (*b*) 0 (*c*) 1

(*d*) None of these **(*AIEEE 2002*)**

9. If $\pi = 22/7$, then a unit radian is approximately equal to

(*a*) 57° 16′ 22″ (*b*) 57° 15′ 22″

(*c*) 57° 16′ 20″ (*d*) 57° 15′ 20″ **(*CDS 2007*)**

10. What is the value of

$$\frac{5\sin 75°\sin 77° + 2\cos 13°\cos 15°}{\cos 15°\sin 77°} - \frac{7\sin 81°}{\cos 90}?$$

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) 2

(*CDS 2009*)

11. What is the value of

$\sin^2 15° + \sin^2 20° + \sin^2 25° + ... + \sin^2 75°$?

(*a*) $\tan^2 15° + \tan^2 20° + \tan^2 25° + ... + \tan^2 75°$

(*b*) $\cos^2 15° + \cos^2 20° + \cos^2 25° + ... + \cos^2 75°$

(*c*) $\cot^2 15° + \cot^2 20° + \cot^2 25° + ... + \cot^2 75°$

(*d*) $\sec^2 15° + \sec^2 20° + \sec^2 25° + ... + \sec^2 75°$.

12. If $\tan (x^2 - 8x + 60)° = \cot (6x - 5)°$, what is one of the values of x?

(*a*) 7 (*b*) 8 (*c*) 9 (*d*) 10.

(*CDS 2005*)

13. What is the value of $\sec (90 - \theta)°. \sin\theta \sec 45°$?

(*a*) 1 (*b*) $\frac{\sqrt{3}}{2}$ (*c*) $\sqrt{2}$ (*d*) $\sqrt{3}$

(*CDS 2012*)

14. If $x + y = 90°$, then what is the value of $\left(1+\frac{\tan x}{\tan y}\right)\sin^2 y$?

(*a*) 0 (*b*) 1/2 (*c*) 1 (*d*) 2

(*CDS 2006*)

15. If $\frac{\tan 26° + \tan 19°}{x(1-\tan 26° \tan 19°)} = \cos 60°$, then the value of x is

(*a*) 1 (*b*) $\sqrt{2}$ (*c*) 2 (*d*) $\sqrt{3}$

16. The value of cot 105° is

(*a*) $\sqrt{3}-2$ (*b*) $2-\sqrt{3}$

(*c*) $\sqrt{2}+3$ (*d*) $\sqrt{3}+2$ **(*Odisha JEE*)**

17. sin 120° cos 150° – cos 240° sin 330° is equal to

(*a*) $-\left(\frac{\sqrt{3}+1}{4}\right)$ (*b*) –1

(*c*) 1 (*d*) $\frac{2}{3}$ **(*EAMCET 2006*)**

18. If A, B, C, D are the successive angles of a cyclic quadrilateral, then what is $\cos A + \cos B + \cos C + \cos D$ equal to:

(*a*) 4 (*b*) 2 (*c*) 1 (*d*) 0

(*CDS 2011*)

19. What is the value of tan (–1575°)?

(*a*) 1 (*b*) $\frac{1}{2}$ (*c*) 0 (*d*) –1

(*NDA/NA 2009*)

20. The value of $\frac{\sin 300° \tan 330° \sec 420°}{\tan 135° \sin 210° \sec 315°}$ is equal to

(*a*) $\frac{1}{\sqrt{3}}$ (*b*) $\frac{1}{\sqrt{2}}$ (*c*) $\sqrt{2}$ (*d*) $\sqrt{3}$

21. cos 1° + cos 2° + cos 3° + ... + cos180° is equal to

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) 2

(*KCET 2003*)

22. What is $\dfrac{\sin^6\theta - \cos^6\theta}{\sin^2\theta - \cos^2\theta}$ equal to?

(*a*) $\sin^4\theta - \cos^4\theta$ (*b*) $1 - \sin^2\theta\cos^2\theta$
(*c*) $1 + \sin^2\theta\cos^2\theta$ (*d*) $1 - 3\sin^2\theta\cos^2\theta$

(*CDS 2011*)

23. If $\dfrac{\cos x}{1+\text{cosec } x} + \dfrac{\cos x}{\text{cosec } x - 1} = 2,$ which one of the following is one of the values of x ?

(*a*) $\pi/2$ (*b*) $\pi/3$ (*c*) $\pi/4$ (*d*) $\pi/6$

(*CDS 2009*)

24. If $x = a(1 + \cos\theta\cos\phi)$, $y = b(1 + \cos\theta\sin\phi)$ and $z = c(1 + \sin\theta)$, then which of the following is correct?

(*a*) $\left(\dfrac{x-a}{a}\right)^2 + \left(\dfrac{y-b}{b}\right)^2 + \left(\dfrac{z-c}{c}\right)^2 = 1$

(*b*) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(*c*) $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$

(*d*) $\dfrac{(x-a)^2}{a} + \dfrac{(y-b)^2}{b} + \dfrac{(z-c)^2}{c} = 1$ **(*CDS 2005*)**

25. If $\sin^4 x + \sin^2 x = 1$ then what is 1 are the value of $\cot^4 x + \cot^2 x$?

(*a*) $\cos^2 x$ (*b*) $\sin^2 x$ (*c*) $\tan^2 x$ (*d*) 1

(*CDS 2006*)

26. If $a\sin\theta + b\cos\theta = c$, what is/are the values of $(a\cos\theta - b\sin\theta)$?

(*a*) $c - a + b$ (*b*) $c - b + a$
(*c*) $\pm\sqrt{a^2 + b^2 - c^2}$ (*d*) $\pm\sqrt{c^2 + b^2 - a^2}$

(*CDS 2005*)

27. What is the value of $\sin A\cos A\tan A + \cos A\sin A\cot A$?

(*a*) $\sin^2 A + \cos A$ (*b*) $\sin^2 A + \tan^2 A$
(*c*) $\sin^2 A + \cot^2 A$ (*d*) $\text{cosec}^2 A - \cot^2 A$

28. $\cos\alpha\sin(\beta - \gamma) + \cos\beta\sin(\gamma - \alpha) + \cos\gamma\sin(\alpha - \beta)$ is equal to

(*a*) 0 (*b*) $1/2$
(*c*) 1 (*d*) $4\cos\alpha\cos\beta\cos\gamma$

(*EAMCET*)

29. If $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in geometric progression, then $\cot^6\theta - \cot^2\theta$ is equal to

(*a*) $\dfrac{1}{2}$ (*b*) 1 (*c*) 2 (*d*) 3

30. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos\left(\theta - \dfrac{\pi}{4}\right)$ is

(*a*) $\dfrac{1}{2\sqrt{2}}$ (*b*) $\dfrac{1}{\sqrt{2}}$ (*c*) $\dfrac{1}{2}$ (*d*) $\dfrac{1}{4}$

(*EAMCET 2013*)

31. If $3\sin x + 4\cos x = 5$, then $6\tan\dfrac{x}{2} - 9\tan^2\dfrac{x}{2}$ is equal to

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 4

(*EAMCET 2012*)

32. $\cos^2\alpha + \cos^2(\alpha - 120°) + \cos^2(\alpha + 120°)$ is equal to

(*a*) 0 (*b*) $\dfrac{1}{2}$ (*c*) 1 (*d*) $\dfrac{3}{2}$

(*MPPET*)

33. The expression $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A}$ can be written as

(*a*) $\sin A\cos A + 1$ (*b*) $\sec A\,\text{cosec } A + 1$
(*c*) $\tan A + \cot A$ (*d*) $\sec A + \text{cosec } A$

(*IIT 2013*)

34. The value of $\tan 5\theta$ is

(*a*) $\dfrac{\tan^5\theta + 10\tan^3\theta - 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}$

(*b*) $\dfrac{5\tan\theta + 10\tan^3\theta - \tan^5\theta}{1 + 10\tan^2\theta - 5\tan^4\theta}$

(*c*) $\dfrac{\tan^5\theta - 10\tan^3\theta - 5\tan\theta}{5\tan^4\theta + 10\tan^2\theta + 1}$

(*d*) $\dfrac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

35. If α and β are such that $\tan\alpha = 2\tan\beta$, then what is $\sin(\alpha + \beta)$ equal to?

(*a*) 1 (*b*) $2\sin(\alpha - \beta)$
(*c*) $\sin(\alpha - \beta)$ (*d*) $3\sin(\alpha - \beta)$

(*NDA/NA 2007*)

36. If $\sin x + \text{cosec } x = 2$, then $\sin^n x + \text{cosec}^n x$ is equal to

(*a*) 2 (*b*) 2^n (*c*) 2^{n-1} (*d*) 2^{n-2}

(*AMU 2008*)

37. The value of $\cos\dfrac{2\pi}{15}.\cos\dfrac{4\pi}{15}.\cos\dfrac{8\pi}{15}.\dfrac{16\pi}{15}$ is equal to:

(*a*) $\dfrac{1}{16}$ (*b*) $\dfrac{1}{32}$ (*c*) $\dfrac{1}{64}$ (*d*) $\dfrac{1}{8}$

(*Manipal Engineering 2010*)

38. If $\alpha + \beta + \gamma = \pi$, then the value of $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$ is equal to

(*a*) $2\sin\alpha$ (*b*) $2\sin\alpha\cos\beta\sin\gamma$
(*c*) $2\sin\alpha\sin\beta\cos\gamma$ (*d*) $2\sin\alpha\sin\beta\sin\gamma$

(*Manipal Engineering 2012*)

39. What is the value of cos15°?

(*a*) $\dfrac{1}{2}\left(\sqrt{2-\sqrt{3}}\right)$ (*b*) $\dfrac{1}{2}\left(\sqrt{2+\sqrt{3}}\right)$
(*c*) $\sqrt{2}+\sqrt{3}$ (*d*) $\sqrt{2}-\sqrt{3}$

(*NDA/NA 2008*)

40. What is $\sqrt{2+\sqrt{2+\sqrt{2+\cos 4A}}}$ equal to?

(*a*) cos *A* (*b*) cos (2*A*) (*c*) $2\cos A/2$ (*d*) $\sqrt{2\cos A}$

(*NDA/NA 2008*)

41. If $\cos x \neq 1$, then what is $\frac{\sin x}{1+\cos x}$ equal to?

(*a*) $-\cot\frac{x}{2}$ (*b*) $\cot\frac{x}{2}$ (*c*) $\tan\frac{x}{2}$ (*d*) $-\tan\frac{x}{2}$

(*NDA/NA 2010*)

42. The value of $\frac{\sin 70° + \cos 40°}{\cos 70° + \sin 40°}$ is equal to

(*a*) $\frac{1}{2}$ (*b*) 0 (*c*) $\sqrt{3}$ (*d*) $\frac{\sqrt{3}}{2}$

43. If $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3-\cos 2\beta}$, then tan α is equal to

(*a*) $\frac{\tan\beta}{\sqrt{2}}$ (*b*) tan β (*c*) $\frac{\tan^2\beta}{\sqrt{2}}$ (*d*) $\sqrt{2}\tan\beta$

(*Odisha JEE 2006*)

44. $\frac{\sin 5\theta}{\sin\theta}$ is equal to

(*a*) $16\cos^4\theta - 12\cos^2\theta - 1$

(*b*) $16\cos^4\theta - 12\cos^2\theta + 1$

(*c*) $16\cos^4\theta + 12\cos^2\theta - 1$

(*d*) $16\cos^4\theta + 12\cos^2\theta + 1$ **(*EAMCET 2001*)**

45. If $\sin A + \sin B = a$, $\cos A - \cos B = b$, then the value of $\cos(A-B)$ is

(*a*) $\frac{2ab}{a^2+b^2}$ (*b*) $\frac{2ab}{a^2-b^2}$

(*c*) $\frac{a^2-b^2}{a^2+b^2}$ (*d*) $\frac{a^2+b^2}{a^2-b^2}$

46. The value of sin16° + cos16° is

(*a*) $\frac{1}{\sqrt{3}}\left(\sqrt{2}\cos 1° + \sin 1°\right)$ (*b*) $\frac{1}{\sqrt{2}}\left(\cos 1° + \sqrt{3}\sin 1°\right)$

(*c*) $\frac{1}{\sqrt{3}}\left(\cos 1° + \sqrt{2}\sin 1°\right)$ (*d*) $\frac{1}{\sqrt{2}}\left(\sqrt{3}\cos 1° + \sin 1°\right)$

47. $\left[1+\cos\frac{\pi}{8}\right]\left[1+\cos\frac{3\pi}{8}\right]\left[1+\cos\frac{5\pi}{8}\right]\left[1+\cos\frac{7\pi}{8}\right]$ is equal to

(*a*) $\frac{1}{8}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{1+\sqrt{2}}{2\sqrt{2}}$ (*d*) $\cos\frac{\pi}{8}$

(*AIEEE 2002, DCE 2003*)

48. Evaluate:

$\cos 2(\theta+\phi) + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\sin^2\phi$

(*a*) sin 2θ (*b*) cos 2θ (*c*) sin 3θ (*d*) cos 3θ

(*Odisha CET 2004*)

49. If tan *A* – tan *B* = *x* and cot *B* – cot *A* = *y*, then what is cot (*A* – *B*) equal to?

(*a*) $\frac{1}{y}-\frac{1}{x}$ (*b*) $\frac{1}{x}-\frac{1}{y}$ (*c*) $\frac{1}{x}+\frac{1}{y}$ (*d*) $-\frac{1}{x}-\frac{1}{y}$

(*NDA/NA 2011*)

50. tan α + 2 tan 2α + 4 tan 4α + 8cot 8α is equal to

(*a*) 0 (*b*) 2 tan α (*c*) cot α (*d*) tan 16α

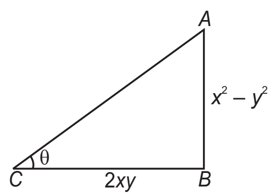
(*IIT*)

ANSWERS

1. (*c*)	**2.** (*b*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a*)	**6.** (*d*)	**7.** (*b*)	**8.** (*b*)	**9.** (*a*)
10. (*b*)	**11.** (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*c*)	**16.** (*a*)	**17.** (*b*)	**18.** (*d*)
19. (*a*)	**20.** (*c*)	**21.** (*a*)	**22.** (*b*)	**23.** (*c*)	**24.** (*a*)	**25.** (*d*)	**26.** (*c*)	**27.** (*d*)
28. (*a*)	**29.** (*b*)	**30.** (*a*)	**31.** (*b*)	**32.** (*d*)	**33.** (*b*)	**34.** (*d*)	**35.** (*d*)	**36.** (*a*)
37. (*a*)	**38.** (*c*)	**39.** (*b*)	**40.** (*c*)	**41.** (*c*)	**42.** (*c*)	**43.** (*d*)	**44.** (*b*)	**45.** (*c*)
46. (*d*)	**47.** (*a*)	**48.** (*b*)	**49.** (*c*)	**50.** (*c*)				

HINTS AND SOLUTIONS

1. $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{BC}{AB}$ (where $\angle ACB = \theta$)



$\therefore\ BC = (2xy)\,k$

$AB = (x^2 - y^2)\,k$

$\Rightarrow AC^2 = AB^2 + BC^2$

$= k^2\,((x^2+y^2)^2 + (2xy)^2)$

$= k^2\,[x^4 + y^4 - 2x^2y^2 + 4x^2y^2]$

$= k^2\,[x^4 + y^4 + 2x^2y^2] = k^2\,(x^2+y^2)^2$

$\Rightarrow AC = k\,(x^2+y^2)$

$\therefore\ \cos\theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{BC}{AC} = \frac{\mathbf{2xy}}{\mathbf{x^2+y^2}}$.

2. Given, $\sec\theta = \frac{13}{5} \Rightarrow \sec^2\theta = \frac{169}{25}$

$\Rightarrow\ 1 + \tan^2\theta = \frac{169}{25}$ $(\because \sec^2\theta - \tan^2\theta = 1)$

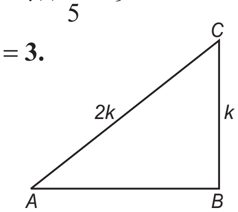
$\Rightarrow\ \tan^2\theta = \frac{169}{25} - 1 = \frac{144}{25} \Rightarrow \tan\theta = \frac{12}{5}$

$$\therefore \quad \frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} = \frac{\frac{2\sin\theta}{\cos\theta} - 3}{\frac{4\sin\theta}{\cos\theta} - 9}$$

(On dividing each term of numerator and denominator by cos θ)

$$= \frac{2\tan\theta - 3}{4\tan\theta - 9} = \frac{2 \times \frac{12}{5} - 3}{4 \times \frac{12}{5} - 9}$$

$$= \frac{24 - 15}{48 - 45} = \frac{9}{3} = \mathbf{3.}$$



3. $\text{cosec } A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{2}{1}$

$\Rightarrow AC = 2k, BC = k$

$\Rightarrow AB^2 = \sqrt{AC^2 - BC^2} = \sqrt{4k^2 - k^2} = \sqrt{3k^2} = k\sqrt{3}$

$\therefore \tan A = \frac{\text{Perp.}}{\text{Base}} = \frac{BC}{AB} = \frac{k}{k\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\sin A = \frac{1}{\text{cosec } A} = \frac{1}{2}$

$\cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{k\sqrt{3}}{2k} = \frac{\sqrt{3}}{2}$

$$\therefore \quad \frac{1}{\tan A} + \frac{\sin A}{1 + \cos A} = \frac{1}{1/\sqrt{3}} + \frac{1/2}{1 + \sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{1} + \frac{1/2}{\frac{2+\sqrt{3}}{2}} = \frac{\sqrt{3}}{1} + \frac{1}{2+\sqrt{3}}$$

$$= \frac{2\sqrt{3} + 3 + 1}{2 + \sqrt{3}} = \frac{2\sqrt{3} + 4}{2 + \sqrt{3}} = \frac{2\left(\sqrt{3} + 2\right)}{2 + \sqrt{3}} = \mathbf{2.}$$

4. $4 \cot^2 45° - \sec^2 60° + \sin^2 60° + \cos^2 90°$

$= 4 (\cot 45°)^2 - (\sec 60°)^2 + (\sin 60°)^2 + (\cos 90°)^2$

$= 4 \times 1^2 - (2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (0)^2$

$= 4 - 4 + \frac{3}{4} + 0 = \mathbf{\frac{3}{4}.}$

5. $\frac{1}{4}(\cot^4 30° - \text{cosec}^4 60°)$

$$+\frac{3}{2}(\sec^2 45° - \tan^2 30°) - 5\cos^2 60°$$

$$= \frac{1}{4}\left[(\cot 30°)^4 - (\text{cosec} 60°)^4\right]$$

$$+\frac{3}{2}\left[(\sec 45°)^2 - (\tan 30°)^2\right] - 5(\cos 60°)^2$$

$$= \frac{1}{4}\left[\left(\sqrt{3}\right)^4 - \left(\frac{2}{\sqrt{3}}\right)^4\right] + \frac{3}{2}\left[\left(\sqrt{2}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] - 5\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}\left(9 - \frac{16}{9}\right) + \frac{3}{2}\left(2 - \frac{1}{3}\right) - 5 \times \frac{1}{4}$$

$$= \frac{1}{4}\left[\frac{81 - 16}{9}\right] + \frac{3}{2}\left[\frac{6-1}{3}\right] - \frac{5}{4}$$

$$= \frac{1}{4} \times \frac{65}{9} + \frac{3}{2} \times \frac{5}{3} - \frac{5}{4} = \frac{65}{36} + \frac{5}{2} - \frac{5}{4}$$

$$= \frac{65 + 90 - 45}{36} = \frac{110}{36} = \mathbf{\frac{55}{18}.}$$

6. $2x^2 \cos 60° - 4 \cot^2 45° - 2 \tan 60° = 0$

$\Rightarrow 2x^2 \times \frac{1}{2} - 4 \times (1)^2 - 2 \times \sqrt{3} = 0$

$\Rightarrow x^2 - 4 - 2\sqrt{3} = 0 \Rightarrow x^2 = 4 + 2\sqrt{3}$

$\Rightarrow x^2 = 3 + 1 + 2\sqrt{3} \Rightarrow x^2 = (\sqrt{3})^2 + (1)^2 + 2\sqrt{3}$

$\Rightarrow x^2 = \left(\sqrt{3} + 1\right)^2 \Rightarrow x = \mathbf{\sqrt{3} + 1.}$

7. $\cot\frac{\pi}{3} = \cot 60° = \frac{1}{\sqrt{3}}, + \cot\frac{\pi}{4} = \cot 45° = 1,$

$\cot\frac{\pi}{6} = \cot 30° = \sqrt{3}$

We can see that

$$\left(\cot\frac{\pi}{3}\right).\left(\cot\left(\frac{\pi}{6}\right)\right) = \frac{1}{\sqrt{3}}.\sqrt{3} = 1 = \left(\cot\frac{\pi}{4}\right)^2$$

$\therefore$ cot π/3, cot π/4, cot π/6 are in G.P

8. $\because \cos 90° = 0$

$\therefore$ cos 1°. cot 2°. .... cos 90° . cos 91° ..... . cos 100° = **0.**

9. π radians = 180°

$$\Rightarrow 1 \text{ radian} = \frac{180}{\pi} \text{ degree} = \frac{180}{22} \times 7 \text{ degree}$$

$$= \frac{630}{11} \text{ degree} = 57\frac{3}{11} \text{ degree}$$

$$= 57° + \frac{3}{11} \times 60 \text{ min} = 57° + \frac{180}{11} \text{ min}$$

$$= 57° + 16\frac{4}{11} \text{ min}$$

$$= 57° + 16' + \frac{4}{11} \times 60 \text{ s}$$

$= 57° + 16' + 21.8''$

$= \mathbf{57° \; 16' \; 22''}$ (approx).

10. $$\frac{5 \sin 75° \sin 77° + 2 \cos 13° \cos 15°}{\cos 15° \sin 77°} - \frac{7 \sin 81°}{\cos 9°}$$

$$= \frac{5 \sin(90° - 15°) \sin 77° + 2 \cos(90° - 77°) \cos 15°}{\cos 15° \sin 77°}$$

$$- \frac{7 \sin(90° - 9°)}{\cos 9°}$$

$= \dfrac{5\cos 15°\sin 77° + 2\sin 77°\cos 15°}{\cos 15°\sin 77°} - \dfrac{7\sin 9°}{\cos 9°}$

$= \dfrac{7\cos 15°\sin 77°}{\cos 15°\sin 77°} - \dfrac{7\cos 9°}{\cos 9°} = 7 - 7 = \mathbf{0.}$

11. $\sin^2 15° + \sin^2 20° + \sin^2 25° + \ldots\ldots + \sin^2 75°$

$= \sin^2(90° - 75°) + \sin^2(90° - 70°) + \ldots + \sin^2(90° - 15°)$

$= \cos^2 75° + \cos^2 70° + \ldots\ldots + \cos^2 \mathbf{15°}$

12. $\tan(x^2 - 8x + 60)° = \cot(6x - 5)°$

$\Rightarrow \tan(x^2 - 8x + 60)° = \tan[90 - (6x - 5)]°$

$\Rightarrow (x^2 - 8x + 60)° = 90° - (6x - 5)°$

$\Rightarrow x^2 - 8x + 60 = 90 - 6x + 5 \Rightarrow x^2 - 2x - 35 = 0$

$\Rightarrow (x - 7)(x + 5) = 0 \Rightarrow x = 7, -5 \Rightarrow x = \mathbf{7.}$

13. $\sec(90 - \theta)° \sin\theta \sec 45°$

$= \operatorname{cosec}\theta \sin\theta.\ (\sqrt{2}) = \dfrac{1}{\sin\theta}.\sin\theta.\sqrt{2} = \mathbf{\sqrt{2}.}$

14. $\left(1 + \dfrac{\tan x}{\tan y}\right).\sin^2 y$

$= \left(1 + \dfrac{\tan(90° - y)}{\tan y}\right).\sin^2 y \quad (\because x + y = 90°)$

$= (1 + \cot y.\cot y).\sin^2 y$

$= (1 + \cot^2 y).\sin^2 y = \operatorname{cosec}^2 y.\sin^2 y = \mathbf{1.}$

15. $\dfrac{\tan 26° + \tan 19°}{x(1 - \tan 26°\tan 19°)} = \cos 60°$

$= \dfrac{\tan 26° + \tan 19°}{1 - \tan 26°\tan 19°} = x\cos 60°$

$= \tan(26° + 19°) = x \times \dfrac{1}{2}$

$\left[\because \tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}\right]$

$= \tan 45° = x/2 \Rightarrow x/2 = 1 \Rightarrow x = \mathbf{2.}$

16. $\cot 105° = \cot(90° + 15°) = -\tan 15°$

$= -\tan(45° - 30°) = -\left[\dfrac{\tan 45° - \tan 30°}{1 + \tan 45°\tan 30°}\right]$

$\left(\because \tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}\right)$

$= -\left[\dfrac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}}\right] = -\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$= -\dfrac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = -\dfrac{4 - 2\sqrt{3}}{2} = \mathbf{\sqrt{3} - 2.}$

17. $\sin 120°\cos 150° - \cos 240°\sin 330°$

$= \sin(180° - 60°)\cos(180° - 30°) - \cos(180° + 60°)\sin(360° - 30°)$

$= (\sin 60°).(-\cos 30°) - (-\cos 60°)(-\sin 30°)$

$= \dfrac{\sqrt{3}}{2} \times -\dfrac{\sqrt{3}}{2} - \left(\dfrac{-1}{2}\right)\left(\dfrac{-1}{2}\right) = -\dfrac{3}{4} - \dfrac{1}{4} = -\dfrac{4}{4} = \mathbf{-1.}$

18. In a cyclic quadrilateral, the sum of the opposite angles is 180°.

$\Rightarrow A + C = 180°$ and $B + D = 180°$

$\therefore \cos A + \cos B + \cos C + \cos D$

$= \cos A + \cos B + \cos(180° - C) + \cos(180° - B)$

$= \cos A + \cos B - \cos A - \cos B$

$= \mathbf{0.}$

19. $\tan(-1575°)$

$= -\tan(1575°) \quad (\because \tan(-\theta) = -\tan\theta)$

$= -\tan(4 \times 360° + 135°)$

$= -\tan(135°) \quad (\because \tan(n.360° + \theta) = \tan\theta)$

$= -\tan(90° + 45°) \quad (\because \tan(90° + \theta) = -\cot\theta)$

$= -(-\cot 45°) = \cot 45° = \mathbf{1.}$

20. $\dfrac{\sin 300°\tan 330°\sec 420°}{\tan 135°\sin 210°\sec 315°}$

$= \dfrac{\sin(360° - 60°)\tan(360° - 30°)\sec(360° + 60°)}{\tan(180° - 45°)\sin(180° + 30°)\sec(360° - 45°)}$

$= \dfrac{(-\sin 60°)(-\tan 30°)(\sec 60°)}{(-\tan 45°)(-\sin 30°)(\sec 45°)}$

$= \dfrac{\sin 60° \times \tan 30° \times \sec 60°}{\tan 45° \times \sin 30° \times \sec 45°}$

$= \dfrac{\sqrt{3}/2 \times 1/\sqrt{3} \times 2}{1 \times 1/2 \times \sqrt{2}} = \mathbf{\sqrt{2}.}$

21. $\cos 1° + \cos 2° + \cos 3° + \ldots\ldots + \cos 180°$

$= (\cos 1° + \cos 179°) + (\cos 2° + \cos 178°) + \ldots\ldots + (\cos 89° + \cos 91°) + \cos 90° + \cos 180°$

$= [\cos 1° + \cos(180° - 1°)] + [\cos 2° + \cos(180° - 2°)] + \ldots\ldots + [\cos 89° + \cos(180° - 91°)] + \cos 90° + \cos 180°$

$= (\cos 1° - \cos 1°) + (\cos 2° - \cos 2°) + \ldots\ldots + (\cos 89° - \cos + 89°) + 0 + (-1) = \mathbf{-1.}$

$(\because \cos(180° - \theta) = -\cos\theta)$

22. $\dfrac{\sin^6\theta - \cos^6\theta}{\sin^2\theta - \cos^2\theta} = \dfrac{(\sin^2\theta)^3 - (\cos^2\theta)^3}{\sin^2\theta - \cos^2\theta}$

$= \dfrac{(\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta)}{(\sin^2\theta - \cos^2\theta)}$

(Using $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$)

$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta$

$= (\sin^2\theta + \cos^2\theta)^2 - \sin^2\theta\cos^2\theta$

$= \mathbf{1 - \sin^2\theta\cos^2\theta} \quad (\because \sin^2\theta + \cos^2\theta = 1)$

23. $\dfrac{\cos x}{1+\operatorname{cosec} x}+\dfrac{\cos x}{\operatorname{cosec} x-1}=2$

$\Rightarrow \dfrac{\cos x\,(\operatorname{cosec} x-1)+\cos x\,(1+\operatorname{cosec} x)}{\operatorname{cosec}^2 x-1}=2$

$\Rightarrow \dfrac{2\cos x \operatorname{cosec} x}{\cot^2 x}=2$

$\Rightarrow \dfrac{\cos x}{\sin x}\times\dfrac{\sin^2 x}{\cos^2 x}=1$

$\Rightarrow \tan x = 1 \Rightarrow \tan x = \tan\dfrac{\pi}{4} \Rightarrow x = \boldsymbol{\dfrac{\pi}{4}}.$

24. Given, $x = a\,(1+\cos\theta\cos\phi)$

$\Rightarrow \dfrac{x}{a}=1+\cos\theta\cos\phi \quad \Rightarrow \dfrac{x}{a}-1=\cos\theta\cos\phi$

$\Rightarrow \dfrac{x-a}{a}=\cos\theta\cos\phi$ ...(*i*)

Similarly, $y = b\,(1+\cos\theta\sin\phi)$

$\Rightarrow \dfrac{y-b}{b}=\cos\theta\sin\phi$ ...(*ii*)

$z = c\,(1+\sin\theta) \Rightarrow \dfrac{z-c}{c}=\sin\theta$ ...(*iii*)

Squaring eqns (*i*), (*ii*) and (*iii*) and adding, we get

$\left(\dfrac{x-a}{a}\right)^2+\left(\dfrac{y-b}{b}\right)^2+\left(\dfrac{z-c}{c}\right)^2$

$=\cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta$

$=\cos^2\theta\,(\cos^2\phi+\sin^2\phi)+\sin^2\theta$

$=\cos^2\theta+\sin^2\theta=\mathbf{1}.$

25. $\sin^4 x+\sin^2 x=1$

$\Rightarrow \sin^4 x = 1-\sin^2 x$

$\Rightarrow \sin^4 x=\cos^2 x$ ...(*i*)

$\therefore \cot^4 x+\cot^2 x$

$=\cot^2 x\,(1+\cot^2 x)=\cot^2 x\,.\operatorname{cosec}^2 x$

$=\dfrac{\cos^2 x}{\sin^2 x\,.\sin^2 x}=\dfrac{\cos^2 x}{\sin^4 x}$

$=\dfrac{\cos^2 x}{\cos^2 x}=\mathbf{1}.$ [Using (*i*)]

26. Given, $a\sin\theta-b\cos\theta=c$

$(a\cos\theta-b\sin\theta)^2$

$=a^2\cos^2\theta-2ab\cos\theta\sin\theta+b^2\sin^2\theta$

$=a^2(1-\sin^2\theta)-2\,(a\sin\theta)\,(b\cos\theta)+b^2(1-\cos^2\theta)$

$=a^2-a^2\sin^2\theta-2\,(a\sin\theta)\,(b\cos\theta)+b^2-b^2\cos^2\theta$

$=a^2+b^2-[a^2\sin^2\theta+2\,(a\sin\theta)\,(b\cos\theta)+b^2\cos^2\theta]$

$=a^2+b^2-(a\sin\theta+b\cos\theta)^2=a^2+b^2-c^2$

$\Rightarrow (a\cos\theta-b\sin\theta)=\pm\boldsymbol{\sqrt{a^2+b^2-c^2}}.$

27. $\sin A\cos A\tan A+\cos A\sin A\cot A$

$=\sin A\cos A\,.\dfrac{\sin A}{\cos A}+\cos A\sin A\dfrac{\cos A}{\sin A}$

$=\sin^2 A+\cos^2 A=1=\boldsymbol{\operatorname{cosec}^2 A-\cot^2 A}.$

28. $\cos\alpha\sin(\beta-\gamma)+\cos\beta\sin(\gamma-\alpha)+\cos\gamma\sin(\alpha-\beta)$

$=\cos\alpha\,[\sin\beta\cos\gamma-\cos\beta\sin\gamma]$

$+\cos\beta\,[\sin\gamma\cos\alpha-\cos\alpha\sin\beta]$

$+\cos\gamma\,[\sin\alpha\cos\beta-\cos\alpha\sin\beta]$

$=\cos\alpha\sin\beta\cos\gamma-\cos\alpha\cos\beta\sin\gamma$

$+\cos\beta\sin\gamma\cos\alpha-\cos\beta\cos\alpha\sin\gamma$

$+\cos\gamma\sin\alpha\cos\beta-\cos\gamma\cos\alpha\sin\beta$

$=\mathbf{0}.$

29. $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P.

$\Rightarrow \sin\theta\times\tan\theta=\cos^2\theta \Rightarrow \sin\theta\,.\dfrac{\sin\theta}{\cos\theta}=\cos^2\theta$

$\Rightarrow \sin^2\theta=\cos^3\theta$ …(*i*)

Now, $\cot^6\theta-\cot^2\theta=\dfrac{\cos^6\theta}{\sin^6\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{(\cos^3\theta)^2}{\sin^6\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{\sin^4\theta}{\sin^6\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$ (By (*i*))

$=\dfrac{1}{\sin^2\theta}-\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{1-\cos^2\theta}{\sin^2\theta}=\dfrac{\sin^2\theta}{\sin^2\theta}=\mathbf{1}.$

30. $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$

$\Rightarrow \tan(\pi\cos\theta)=\tan(\pi/2-\pi\sin\theta)$

$\Rightarrow \pi\cos\theta=\dfrac{\pi}{2}-\pi\sin\theta$

$\Rightarrow \pi(\cos\theta-\sin\theta)=\dfrac{\pi}{2} \quad \Rightarrow \cos\theta+\sin\theta=\dfrac{1}{2}$

$\Rightarrow \dfrac{1}{\sqrt{2}}\cos\theta+\dfrac{1}{\sqrt{2}}\sin\theta=\dfrac{1}{2\sqrt{2}}$

(Multiplying both sides by $\dfrac{1}{\sqrt{2}}$)

$\Rightarrow \cos\dfrac{\pi}{4}\cos\theta+\sin\dfrac{\pi}{4}\sin\theta=\dfrac{1}{2\sqrt{2}}$

$\Rightarrow \cos(\pi/4-\theta)=\boldsymbol{\dfrac{1}{2\sqrt{2}}}$

$(\because \cos(A-B)=\cos A\cos B+\sin A\sin B)$

31. $3\sin x+4\cos x=5$

$\Rightarrow 3\left(\dfrac{2\tan x/2}{1+\tan^2 x/2}\right)+4\left(\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)=5$

$\left(\because \sin x=\dfrac{2\tan x/2}{1+\tan^2 x/2},\ \cos x=\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)$

$\Rightarrow \dfrac{6\tan x/2+4-4\tan^2 x/2}{1+\tan^2 x/2}=5$

$\Rightarrow 6\tan x/2+4-4\tan^2 x/2=5+5\tan^2 x/2$

$\Rightarrow 6\tan x/2-9\tan^2 x/2=\mathbf{1}.$

32. $\cos^2\alpha + \cos^2(\alpha - 120°) + \cos^2(\alpha + 120°)$

$= \cos^2\alpha + \{\cos\alpha\ \cos 120° + \sin\alpha\ \sin 120°\}^2 + \{\cos\alpha\ \cos 120° - \sin\alpha\ \sin 120°\}^2$

$(\because \cos(A+B) = \cos A\cos B - \sin A\ \sin B)$

$= \cos^2\alpha + \left\{-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right\}^2 + \left\{-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right\}^2$

$\left(\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}\right)$

$= \cos^2\alpha + \frac{1}{4}\cos^2\alpha + \frac{3}{4}\sin^2\alpha - \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha + \frac{1}{4}\cos^2\alpha + \frac{3}{4}\sin^2\alpha + \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha$

$= \cos^2\alpha + \frac{1}{4}\cos^2\alpha + \frac{1}{4}\cos^2\alpha + \frac{3}{4}\sin^2\alpha + \frac{3}{4}\sin^2\alpha$

$= \frac{3}{2}\cos^2\alpha + \frac{3}{2}\sin^2\alpha = \frac{3}{2}(\cos^2\alpha + \sin^2\alpha) = \mathbf{\frac{3}{2}.}$

33. $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A}$

$= \frac{\sin A}{\cos A} \times \frac{1}{1-\frac{\cos A}{\sin A}} + \frac{\cos A}{\sin A} \times \frac{1}{1-\frac{\sin A}{\cos A}}$

$= \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$

$= \frac{\sin^2 A}{\cos A(\sin A - \cos A)} - \frac{\cos^2 A}{\sin A(\sin A - \cos A)}$

$= \frac{1}{(\sin A - \cos A)}\left[\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right]$

$= \frac{(\sin A - \cos A)(\sin^2 A + \sin A\cos A + \cos^2 A)}{(\sin A - \cos A)\cos A\sin A}$

$= \frac{1 + \sin A\cos A}{\cos A\sin A}$ $(\because \sin^2 A + \cos^2 A = 1)$

$= \mathbf{\sec A\ \mathrm{cosec}\ A + 1.}$

34. $\tan 5\theta = \tan(2\theta + 3\theta) = \frac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta\tan 3\theta}$

$\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}\right]$

$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}}{1 - \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\cdot\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)}$

$= \frac{2\tan\theta(1-3\tan^2\theta) + (3\tan\theta - \tan^3\theta)(1-\tan^2\theta)}{(1-\tan^2\theta)(1-3\tan^2\theta) - (2\tan\theta)(3\tan\theta - \tan^3\theta)}$

$= \frac{2\tan\theta - 6\tan^3\theta + 3\tan\theta - \tan^3\theta - 3\tan^3\theta + \tan^5\theta}{1 - \tan^2\theta - 3\tan^2\theta + 3\tan^4\theta - 6\tan^2\theta + 2\tan^4\theta}$

$= \mathbf{\frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}}.$

35. Given, $\tan\alpha = 2\tan\beta \Rightarrow \frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{\sin\beta}{\cos\beta}} = 2$

$\Rightarrow \frac{\sin\alpha\ \cos\beta}{\cos\alpha\ \sin\beta} = \frac{2}{1}$

Using componendo and dividendo, we get

$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{2+1}{2-1}$

$\Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{3}{1} \Rightarrow \sin(\alpha+\beta) = \mathbf{3\sin(\alpha - \beta).}$

36. $\sin x + \mathrm{cosec}\ x = 2$ ...(*i*)

Squaring both the sides, we have

$\sin^2 x + \mathrm{cosec}^2 x + 2 = 4$ ...(*ii*)

$\Rightarrow \sin^2 x + \mathrm{cosec}^2 x = 2$, *i.e.,* for $n = 2$, $\sin^n x + \cos^n x = 2$

Cubing both the sides of (*i*), we have

$\sin^3 x + \mathrm{cosec}^3 x + 3\sin x\ \mathrm{cosec}\ x(\sin x + \mathrm{cosec}\ x) = 8$

$\Rightarrow \sin^3 x + \mathrm{cosec}^3 + 3 \times 2 = 8$

$\Rightarrow \sin^3 x + \mathrm{cosec}^3 = 8 - 6 = 2,$

i.e., for $n = 3$, $\sin^n x + \cos^n x = 2$

Squaring both the sides of (*ii*), we have

$\sin^4 x + \mathrm{cosec}^4 x + 2 = 4$

$\Rightarrow \sin^4 x + \mathrm{cosec}^4 x = 2$, *i.e.,* for $n = 4$, $\sin^n x + \cos^n x = 2$

Proceeding in the same way, we see that

$\sin^n x + \mathrm{cosec}^n x = 2$ for all $n \in N$.

37. $\cos\frac{2\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15}$

$= \frac{1}{2\sin\frac{2\pi}{15}}.\left(2\sin\frac{2\pi}{15}.\cos\frac{2\pi}{15}\right)\cos\frac{4\pi}{15}.\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15}$

$\left(\text{On multiplying and dividing the expression by } 2\sin\frac{2\pi}{15}\right)$

$= \frac{1}{2\sin\frac{2\pi}{15}}.\sin\frac{4\pi}{15}.\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15}$

$(\because \sin 2A = 2\sin A\cos A)$

$= \frac{1}{4\sin\frac{2\pi}{15}}.\left(2\sin\frac{4\pi}{15}.\cos\frac{4\pi}{15}\right)\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15}$

$= \frac{1}{4\sin\frac{2\pi}{15}}.\sin\frac{8\pi}{15}.\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15}$

$$= \frac{1}{8\sin\frac{2\pi}{15}}\cdot\left(2\sin\frac{8\pi}{15}.\cos\frac{8\pi}{15}\right).\cos\frac{16\pi}{15}$$

$$= \frac{1}{8\sin\frac{2\pi}{15}}.\sin\frac{16\pi}{15}.\cos\frac{16\pi}{15}$$

$$= \frac{1}{16\sin\frac{2\pi}{15}}.\sin\frac{32\pi}{15}$$

$$= \frac{1}{16\sin\frac{2\pi}{15}}.\left(\sin 2\pi + \frac{2\pi}{15}\right) \text{(Note the step)}$$

$$= \frac{1}{16\sin\frac{2\pi}{15}}.\sin\frac{2\pi}{15} \quad (\because \sin(360° + \theta) = \sin\theta)$$

$$= \mathbf{\frac{1}{16}}.$$

38. $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(\beta-\gamma)$

[Using $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$]

$= \sin^2\alpha + \sin(\pi-\alpha)\sin(\beta-\gamma)$ $(\because \alpha+\beta+\gamma=\pi)$

$= \sin^2\alpha + \sin\alpha\sin(\beta-\gamma)$ $(\because \sin(180° - \theta) = \sin\theta)$

$= \sin\alpha\,[\sin\alpha + \sin(\beta-\gamma)]$

$= \sin\alpha\,[\sin(\pi-(\beta+\gamma)) + \sin(\beta-\gamma)]$

$= \sin\alpha\,[\sin(\beta+\gamma) + \sin(\beta-\gamma)]$

$= \sin\alpha\,[\sin\beta\cos\gamma + \cos\beta\sin\gamma + \sin\beta\cos\gamma - \cos\beta\sin\gamma]$

$(\because \sin(A+B) = \sin A\cos B + \cos A\sin B)$

$= \mathbf{2\sin\alpha\sin\beta\cos\gamma}.$

39. We know that $\cos 2\theta = 2\cos^2\theta - 1$

Putting $\theta = 15°$ in the above expression,

$\cos 30° = 2\cos^2 15° - 1$

$$\Rightarrow \frac{\sqrt{3}}{2} + 1 = 2\cos^2 15°$$

$$\Rightarrow \cos^2 15° = \frac{\sqrt{3}+2}{4} \Rightarrow \cos 15° = \mathbf{\frac{1}{2}\sqrt{\sqrt{3}+2}}.$$

40. $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4A}}}$

$$= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 4A)}}}$$

$$= \sqrt{2+\sqrt{2+\sqrt{4\cos^2 2A}}} \quad \left(\begin{array}{l}\because \cos 2A = 2\cos^2 A - 1\\ \text{or } 1+\cos 2A = 2\cos^2 A\end{array}\right)$$

$$= \sqrt{2+\sqrt{2+2\cos 2A}}$$

$$= \sqrt{2+\sqrt{2(1+\cos 2A)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 A)}} = \sqrt{2+2\cos A}$$

$$= \sqrt{2(1+\cos A)} = \sqrt{2\times 2\cos^2\frac{A}{2}}$$

$$= \sqrt{4\cos^2 A/2} = \mathbf{2\cos A/2}.$$

41. $$\frac{\sin x}{1+\cos x} = \frac{2\sin x/2\cos x/2}{1+2\cos^2 x/2 - 1}$$

$(\because \sin 2\theta = 2\sin\theta\cos\theta \text{ and } \cos 2\theta = 2\cos^2 - 1)$

$$= \frac{2\sin x/2\cos x/2}{2\cos^2 x/2} = \frac{\sin x/2}{\cos x/2} = \mathbf{\tan x/2}.$$

42. $$\frac{\sin 70° + \cos 40°}{\cos 70° + \sin 40°} = \frac{\sin 70° + \cos(90° - 50°)}{\cos 70° + \sin(90° - 50°)}$$

$$= \frac{\sin 70° + \sin 50°}{\cos 70° + \cos 50°} \quad \left[\begin{array}{l}\because \sin(90° - \theta) = \cos\theta\\ \cos(90° - \theta) = \sin\theta\end{array}\right]$$

$$= \frac{2\sin\left(\frac{70°+50°}{2}\right)\cos\left(\frac{70°-50°}{2}\right)}{2\cos\left(\frac{70°+50°}{2}\right)\cos\left(\frac{70°-50°}{2}\right)}$$

$$\left[\begin{array}{l}\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\ \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\end{array}\right]$$

$$= \frac{2\sin 60°\cos 10°}{2\cos 60°\cos 10°} = \tan 60° = \mathbf{\sqrt{3}}.$$

43. $$\tan^2\alpha = \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{2\sin^2\alpha}{2\cos^2\alpha} = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

$(\because 1 - \cos 2\theta = 2\sin^2\theta,\ 1 + \cos 2\theta = 2\cos^2\theta)$

$$= \frac{1 - \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}}{1 + \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}} = \frac{3 - \cos 2\beta - 3\cos 2\beta + 1}{3 - \cos 2\beta + 3\cos 2\beta - 1}$$

$$= \frac{4 - 4\cos 2\beta}{2 + 2\cos 2\beta} = \frac{4(1-\cos 2\beta)}{2(1+\cos 2\beta)}$$

$$= 2\left(\frac{\sin^2\beta)}{2\cos^2\beta)}\right) = 2\tan^2\beta$$

$\therefore$ $\tan^2\alpha = 2\tan^2\beta \Rightarrow \tan\alpha = \mathbf{\sqrt{2}\tan\beta}.$

44. $$\frac{\sin 5\theta}{\sin\theta} = \frac{\sin(2\theta + 3\theta)}{\sin\theta}$$

$$= \frac{1}{\sin\theta}\{\sin 2\theta\cos 3\theta + \cos 2\theta\sin 3\theta\}$$

$$= \frac{1}{\sin\theta}\Big\{2\sin\theta\cos\theta\left(4\cos^3\theta - 3\cos\theta\right) + \left(2\cos^2\theta - 1\right)\left(3\sin\theta - 4\sin^3\theta\right)\Big\}$$

$(\because \cos 3A = 4\cos^3 A - 3\cos A,\ \sin 3A = 3\sin A - 4\sin^3 A)$

$= \{2\cos\theta\,(4\cos^3\theta - 3\cos\theta) + (2\cos^2\theta - 1)(3 - 4\sin^2\theta)\}.$

(Note the step)

$= 8\cos^4\theta - 6\cos^2\theta + 6\cos^2\theta - 3 - 8\cos^2\theta\sin^2\theta + 4\sin^2\theta$

$= 8\cos^4\theta - 3 - 8\cos^2\theta\,(1 - \cos^2\theta) + 4\,(1 - \cos^2\theta)$

$(\because \sin^2\theta + \cos^2\theta = 1)$

$= 8\cos^4\theta - 3 - 8\cos^2\theta + 8\cos^4\theta + 4 - 4\cos^2\theta$

$= \mathbf{16\cos^4\theta - 12\cos^2\theta + 1}.$

45. $\sin A + \sin B = a$

$\Rightarrow 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = a$ ...(*i*)

$\cos A - \cos B = b$

$\Rightarrow 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right) = b$

$\Rightarrow 2\sin\left(\frac{A+B}{2}\right)\sin\left[-\left(\frac{A-B}{2}\right)\right] = b$

$\Rightarrow -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) = b$ ...(*ii*)

($\because \sin(-\theta) = -\sin\theta$)

Dividing eqn (*ii*) by eqn (*i*), we get

$$\frac{-\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} = \frac{b}{a}$$

$\Rightarrow -\tan\left(\frac{A-B}{2}\right) = b/a \Rightarrow \tan\left(\frac{A-B}{2}\right) = -b/a$

$$\therefore \cos(A-B) = \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}$$

$\left(\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \frac{1-b^2/a^2}{1+b^2/a^2} = \boldsymbol{\frac{a^2-b^2}{a^2+b^2}}.$

46. $\sin 16° + \cos 16° = \sin(15° + 1°) + \cos(15° + 1°)$

$= (\sin 15° \cos 1° + \cos 15° \sin 1°)$

$+ (\cos 15° \cos 1° - \sin 15° \sin 1°)$

$= \sin 1° (\cos 15° - \sin 15°) + \cos 1° (\sin 15° + \cos 15°)$

Now, $\cos 15° = \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

and $\sin 15° = \sin(45° - 30°)$

$= \sin 45° \cos 30° - \cos 45° \sin 30°$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\therefore \sin 16° + \cos 16°$

$= \sin 1°\left(\frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos 1°\left(\frac{\sqrt{3}-1}{2\sqrt{2}} + \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$= \sin 1°\left(\frac{2}{2\sqrt{2}}\right) + \cos 1°\left(\frac{2\sqrt{3}}{2\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}\sin 1° + \frac{\sqrt{3}}{\sqrt{2}}\cos 1° = \boldsymbol{\frac{1}{\sqrt{2}}\left(\sin 1° + \sqrt{3}\cos 1°\right)}.$

47. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\left(\pi-\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{8}\right)\right)$

$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

($\because \cos(\pi - \theta) = -\cos\theta$)

$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) = \sin^2\frac{\pi}{8}\ \sin^2\frac{3\pi}{8}$

$= \sin^2\frac{\pi}{8}\ \cos^2\frac{\pi}{8}$ $\left(\because \sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = \cos\frac{\pi}{8}\right)$

$= \frac{1}{4}\left(4\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right) = \frac{1}{4}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2$

$= \frac{1}{4}\left(\sin\frac{\pi}{4}\right)^2 \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2 = \boldsymbol{\frac{1}{8}}.$

(Using, $2\sin\theta\cos\theta = \sin 2\theta$)

48. $\cos 2(\theta+\phi) + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\sin^2\phi$

$= \{\cos 2\theta\cos 2\phi - \sin 2\theta \sin 2\phi\}$

$+ 4\{(\cos\theta\cos\phi - \sin\theta\sin\phi)\sin\theta\sin\phi\} + 2\sin^2\phi$

$= \{(1-2\sin^2\theta)(1-2\sin^2\phi) - 2\sin\theta\cos\theta \, . \, 2\sin\phi\cos\phi\}$

$+ [4\cos\theta\cos\phi\sin\theta\sin\phi - 4\sin^2\theta\sin^2\phi] + 2\sin^2\phi$

$= 1 - 2\sin^2\theta - 2\sin^2\phi + 4\sin^2\theta\sin^2\phi$

$- 4\sin\theta\sin\phi\cos\theta\cos\phi + 4\cos\theta\cos\phi\sin\theta\sin\phi$

$- 4\sin^2\theta\sin^2\phi + 2\sin^2\phi$

$= 1 - 2\sin^2\theta = \boldsymbol{\cos 2\theta}.$

49. $\tan A - \tan B = x$

$\Rightarrow \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} = x$

$\Rightarrow \frac{\sin A\cos B - \sin B\cos A}{\cos A\cos B} = x$

$\Rightarrow \frac{\sin(A-B)}{\cos A\cos B} = x$

$\Rightarrow \frac{1}{x} = \frac{\cos A\cos B}{\sin(A-B)}$ …(*i*)

$\cot B - \cot A = y$

$\Rightarrow \frac{\cos B}{\sin B} - \frac{\cos A}{\sin A} = y$

$\Rightarrow \frac{\cos B\sin A - \cos A\sin B}{\sin B\sin A} = y$

$\Rightarrow \frac{\sin(A-B)}{\sin A\sin B} = y \Rightarrow \frac{1}{y} = \frac{\sin A\sin B}{\sin(A-B)}$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$\frac{1}{x} + \frac{1}{y} = \frac{\cos A\cos B + \sin A\sin B}{\sin(A-B)}$

$\Rightarrow \frac{\cos(A-B)}{\sin(A-B)} = \boldsymbol{\cot(A-B)}.$

50. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{8}{\tan 8\alpha}$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{8}{\dfrac{2\tan 4\alpha}{1-\tan^2 4\alpha}}$

$\left(\because \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}\right)$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{4(1-\tan^2 4\alpha)}{\tan 4\alpha}$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \dfrac{4}{\tan 4\alpha} - 4\tan 4\alpha$

$= \tan\alpha + 2\tan 2\alpha + \dfrac{4}{\tan 4\alpha}$

$= \tan\alpha + 2\tan 2\alpha + \dfrac{4}{\dfrac{2\tan 2\alpha}{1-\tan^2 2\alpha}}$

$= \tan\alpha + 2\tan 2\alpha + \dfrac{2(1-\tan^2 2\alpha)}{\tan 2\alpha}$

$= \tan\alpha + \dfrac{2}{\tan 2\alpha} = \tan\alpha + \dfrac{2}{\dfrac{2\tan\alpha}{1-\tan^2\alpha}}$

$= \tan\alpha + \dfrac{1-\tan^2\alpha}{\tan\alpha} = \dfrac{1}{\tan\alpha} =$ **$\cot\alpha$.**

SELF ASSESSMENT SHEET

1. If q cosec $\theta = p$ and θ is acute, then what is the value of $\sqrt{p^2 - q^2}\,\tan\theta$?

(*a*) p (*b*) q (*c*) pq (*d*) $\sqrt{p^2+q^2}$

2. If $0 < x < 45°$ and $45° < y < 90°$, then which one of the following is correct?

(*a*) $\sin x = \sin y$ (*b*) $\sin x < \sin y$
(*c*) $\sin x > \sin y$ (*d*) $\sin x \le \sin y$

3. If $\tan^2 y\ \ \text{cosec}^2 x - 1 = \tan^2 y$, then which one of the following is correct?

(*a*) $x - y = 0$ (*b*) $x = 2y$
(*c*) $y = 2x$ (*d*) $x - y = 1°$

4. If α and β the complementary angles, then what is $\sqrt{\text{cosec}\,\alpha\,.\,\text{cosec}\,\beta}\left(\dfrac{\sin\alpha}{\sin\beta}+\dfrac{\cos\alpha}{\cos\beta}\right)^{-1/2}$ equal to

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) None of there

(*CDS 2011*)

5. What is $\left(\dfrac{\sec 18°}{\sec 144°}+\dfrac{\text{cosec}\ 18°}{\text{cosec}\ 144°}\right)$ equal to?

(*a*) sec 18° (*b*) cosec 18°
(*c*) – sec 18° (*d*) – cosec 18°

6. If an angle α is divided into two parts A and B such that $A - B = x$ and $\tan A : \tan B = 2:1$, then what is $\sin x$ equal to?

(*a*) $3\sin\alpha$ (*b*) $\dfrac{2\sin\alpha}{3}$
(*c*) $\dfrac{\sin\alpha}{3}$ (*d*) $2\sin\alpha$ **(*NDA/NA 2011*)**

7. If $\cos\theta = \dfrac{8}{17}$ and θ lies in the first quadrant, then the value of $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$ is

(*a*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}-1}{2}-\dfrac{1}{\sqrt{2}}\right)$ (*b*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}+1}{2}-\dfrac{1}{\sqrt{2}}\right)$

(*c*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}-1}{2}+\dfrac{1}{\sqrt{2}}\right)$ (*d*) $\dfrac{23}{17}\left(\dfrac{\sqrt{3}+1}{2}+\dfrac{1}{\sqrt{2}}\right)$

(*DCE 2008*)

8. Let A, B, C be the angles of a plain triangle $\tan(A/2) = \dfrac{1}{3}, \tan\left(\dfrac{B}{2}\right) = \dfrac{2}{3}$. Then $\tan(C/2)$ is equal to

(*a*) $\dfrac{2}{9}$ (*b*) $\dfrac{1}{3}$ (*c*) $\dfrac{7}{9}$ (*d*) $\dfrac{2}{3}$

9. If $\dfrac{1+\cos A}{1-\cos A} = \dfrac{m^2}{n^2}$, then $\tan A$ is equal to

(*a*) $\pm\dfrac{2mn}{m^2-n^2}$ (*b*) $\pm\dfrac{2mn}{m^2+n^2}$

(*c*) $\dfrac{m^2+n^2}{m^2-n^2}$ (*d*) $\dfrac{m^2-n^2}{m^2+n^2}$

(*J&K CET 2007*)

10. If α, β, γ, are in A.P, then $\dfrac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha}$. is equal to

(*a*) $\sin\alpha$ (*b*) $\cos\alpha$ (*c*) $\tan\beta$ (*d*) $\cot\beta$

ANSWERS

1. (*b*) **2.** (*b*) **3.** (*a*) **4.** (*b*) **5.** (*a*) **6.** (*c*) **7.** (*c*) **8.** (*c*) **9.** (*a*) **10.** (*d*)

HINTS AND SOLUTIONS

1. q cosec $\theta = p$

$\therefore \sqrt{p^2 - q^2}\ \tan\theta$

$= \sqrt{q^2 \operatorname{cosec}^2\theta - q^2}\,.\tan\theta = \sqrt{q^2(\operatorname{cosec}^2\theta - 1)}\,.\tan\theta$

$= \sqrt{q^2\cot^2\theta}\,.\tan\theta = q\cot\theta\,.\tan\theta = \boldsymbol{q.}$

2. Since sin θ in cosec from θ = 0° to θ = 90°, *i.e.*, where 0° < θ < 90° then 0 < sin θ < 1. ∴ **sin *x* < sin *y*.**

3. $\tan^2 y\ \operatorname{cosec}^2 x - 1 = \tan^2 y$

$\Rightarrow \tan^2 y\ \operatorname{cosec}^2 x - \tan^2 y = 1$

$\Rightarrow \tan^2 y\ (\operatorname{cosec}^2 x - 1) = 1 \Rightarrow \tan^2 y\,.\cot^2 x = 1$

$\Rightarrow \cot^2 x = \cot^2 y \Rightarrow x = y \Rightarrow \boldsymbol{x - y = 0.}$

4. Given α + β = 90°

$\therefore \sqrt{\operatorname{cosec}\alpha\,.\operatorname{cosec}\beta}\left(\frac{\sin\alpha}{\sin\beta} + \frac{\cos\alpha}{\cos\beta}\right)^{-1/2}$

$= \frac{1}{(\sin\alpha\sin\beta)^{1/2}}\left(\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\beta\cos\beta}\right)^{-1/2}$

$= \frac{1}{(\sin\alpha\sin\beta)^{1/2}}\left(\frac{\sin(\alpha+\beta)}{\sin\beta\cos\beta}\right)^{-1/2}$

$= \frac{1}{(\sin\alpha\sin\beta)^{1/2}}\left(\frac{\sin 90°}{\sin\beta\cos(90° - \alpha)}\right)^{-1/2}$

$= \frac{1}{(\sin\alpha\sin\beta)^{\frac{1}{2}}}\left(\frac{1}{\sin\beta\sin\alpha}\right)^{-\frac{1}{2}}$

$= \frac{1}{(\sin\alpha\sin\beta)^{\frac{1}{2}}} \times (\sin\alpha\sin\beta)^{\frac{1}{2}} = \mathbf{1.}$

5. $\frac{\sec 18°}{\sec 144°} + \frac{\operatorname{cosec} 18°}{\operatorname{cosec} 144°}$

$= \frac{\sec 18°}{\sec(180° - 36°)} + \frac{\operatorname{cosec} 18°}{\operatorname{cosec}(180° - 36°)}$

$= -\frac{\sec 18°}{\sec 36°} + \frac{\operatorname{cosec} 18°}{\operatorname{cosec} 36°}$ $\left(\begin{array}{l}\because \sec(180° - \theta) = -\sec\theta \\ \operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta\end{array}\right)$

$= \frac{\sin 36°}{\sin 18°} - \frac{\cos 36°}{\cos 18°} = \frac{\sin 36°\cos 18° - \cos 36°\sin 18°}{\sin 18°\cos 18°}$

(∵ sin (*A* – *B*) = sin *A* cos *B* – cos *A* sin *B*)

$= \frac{\sin(36° - 18°)}{\sin 18°\cos 18°} = \frac{\sin 18°}{\sin 18°\cos 18°} = \frac{1}{\cos 18°} = \mathbf{\sec 18°.}$

6. $\alpha = A + B$

$x = A - B$

$\Rightarrow A = \frac{\alpha + x}{2}$ and $B = \frac{\alpha - x}{2}$

Given, $\frac{\tan A}{\tan B} = \frac{2}{1} \Rightarrow \frac{\tan\left(\frac{\alpha + x}{2}\right)}{\tan\left(\frac{\alpha - x}{2}\right)} = \frac{2}{1}$

$\Rightarrow \frac{\sin\left(\frac{\alpha + x}{2}\right).\cos\left(\frac{\alpha - x}{2}\right)}{\cos\left(\frac{\alpha + x}{2}\right).\sin\left(\frac{\alpha - x}{2}\right)} = \frac{2}{1}$

$\Rightarrow \frac{2\sin\left(\frac{\alpha + x}{2}\right)\cos\left(\frac{\alpha - x}{2}\right)}{2\cos\left(\frac{\alpha + x}{2}\right)\sin\left(\frac{\alpha - x}{2}\right)} = \frac{2}{1}$

$\Rightarrow \frac{\sin\alpha + \sin x}{\sin\alpha - \sin x} = \frac{2}{1}$

$\left(\begin{array}{l}\because 2\sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right) = \sin C + \sin D \\ \text{and } 2\cos\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right) = \sin C - \sin D\end{array}\right)$

$\Rightarrow \sin\alpha + \sin x = 2\sin\alpha - 2\sin x$

$\Rightarrow 3\sin x = \sin\alpha \Rightarrow \mathbf{\sin x = \frac{\sin\alpha}{3}.}$

7. cos (30° + θ) + cos (45° – θ) + cos (120° – θ)

= cos 30° cos θ – sin 30° sin θ + cos 45° cos θ + sin 45° sin θ + cos 120° cos θ + sin 120° sin θ

$\left(\cos\theta = \frac{8}{17} \Rightarrow \sin\theta = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{\frac{289 - 64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}\right)$

$= \frac{\sqrt{3}}{2} \times \frac{8}{17} - \frac{1}{2} \times \frac{15}{17} + \frac{1}{\sqrt{2}} \times \frac{8}{17} + \frac{1}{\sqrt{2}} \times \frac{15}{17} + \left(\frac{-1}{2}\right) \times \frac{8}{17} + \frac{\sqrt{3}}{2} \times \frac{15}{17}$

$\left[\begin{array}{l}\because \cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2} \\ \sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}\end{array}\right]$

$= \frac{8}{17}\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \frac{15}{17}\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} - \frac{1}{2}\right)$

$= \frac{8}{17}\left(\frac{\sqrt{3} - 1}{2} + \frac{1}{\sqrt{2}}\right) + \frac{15}{17}\left(\frac{\sqrt{3} - 1}{2} + \frac{1}{\sqrt{2}}\right)$

$= \mathbf{\frac{23}{17}\left(\frac{\sqrt{3} - 1}{2} + \frac{1}{\sqrt{2}}\right).}$

8. $A + B + C = 180°$ (Angle sum property of a triangle)

$\Rightarrow\ A + B = 180° - C$

$\Rightarrow\ \dfrac{A+B}{2} = 90° - \dfrac{C}{2}$

$\Rightarrow\ \tan\left(\dfrac{A}{2} + \dfrac{B}{2}\right) = \tan\left(90° - \dfrac{C}{2}\right)$

$\Rightarrow\ \dfrac{\tan A/2 + \tan B/2}{1 - \tan A/2 \tan B/2} = \cot C/2$

$\Rightarrow\ \dfrac{\frac{1}{3} + \frac{2}{3}}{1 - \frac{1}{3} \times \frac{2}{3}} = \cot C/2\ \Rightarrow\ \dfrac{1}{1 - \frac{2}{9}} = \cot C/2$

$\Rightarrow\ \dfrac{9}{7} = \cot C/2\ \Rightarrow\ \mathbf{\tan C/2 = 7/9}$**.**

9. $\dfrac{1 + \cos A}{1 - \cos A} = \dfrac{m^2}{n^2}$

$\Rightarrow\ \dfrac{2\cos^2 A/2}{2\sin^2 A/2} = \dfrac{m^2}{n^2}$

($\because$ $1 + \cos 2A = 2\cos^2 A$, $1 - \cos^2 A = 2\sin^2 A$)

$\Rightarrow\ \tan^2 A/2 = \dfrac{n^2}{m^2} \Rightarrow \tan A/2 = \pm\dfrac{n}{m}$

$\therefore\ \tan A = \dfrac{2\tan A/2}{1 - \tan^2 A/2} = \pm\dfrac{\frac{2n}{m}}{1 - \frac{n^2}{m^2}}$

$= \pm\dfrac{\frac{2n}{m}}{\frac{m^2 - n^2}{m^2}} = \pm\dfrac{\mathbf{2nm}}{\mathbf{m^2 - n^2}}$**.**

10. Since α, β and γ are in A.P.

$\alpha + \gamma = 2\beta$

$\Rightarrow \beta = \dfrac{\alpha + \gamma}{2}$

$\therefore\ \dfrac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha} = \dfrac{2\cos\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}{2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)}$

$= \cot\left(\dfrac{\alpha+\gamma}{2}\right) = \mathbf{\cot\beta}$**.**