CHAPTER 5

Mathematical Induction and Binomial Theorem

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. The larger of $99^{50} + 100^{50}$ and 101^{50} is .... *(1982 - 2 Marks)*
2. The sum of the coefficients of the plynomial $(1 + x - 3x^2)^{2163}$ is .... *(1982 - 2 Marks)*
3. If $(1 + ax)^n = 1 + 8x + 24x^2 + ...$ then $a =$ ...... and $n =$ .... *(1983 - 2 Marks)*
4. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1 + x)^n$ are in A.P., then the value of n is .... *(1994 - 2 Marks)*
5. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is .... *(1997 - 2 Marks)*

C MCQs with One Correct Answer

1. Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)$th and $(r + 2)$th terms in the binomial expansion of $(1+x)^{2n}$ are equal . Then *(1983 - 1 Mark)*
(a) $n = 2r$ (c) $n = 2r + 1$
(c) $n = 3r$ (d) none of these
2. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is *(1983 - 1 Mark)*
(a) $\frac{405}{256}$ (b) $\frac{504}{259}$
(c) $\frac{450}{263}$ (d) none of these
3. The expression $\left(x + (x^3 - 1)^{\frac{1}{2}}\right)^5 + \left(x - (x^3 - 1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree *(1992 - 2 Marks)*
(a) 5 (b) 6 (c) 7 (d) 8
4. If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively, then m is *(1999 - 2 Marks)*
(a) 6 (b) 9 (c) 12 (d) 24
5. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ *(2000S)*
(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$ (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$
6. In the binomial expansion of $(a - b)^n$, $n \ge 5$, the sum of the 5th and 6th terms is zero. Then a/b equals *(2001S)*
(a) $(n - 5)/6$ (b) $(n - 4)/5$
(c) $5/(n - 4)$ (d) $6/(n - 5)$
7. The sum $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if $p < q$) is maximum when m is *(2002S)*
(a) 5 (b) 10 (c) 15 (d) 20
8. Coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is *(2003S)*
(a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$
9. If ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$, then $k \in$ *(2004S)*
(a) $(-\infty, -2]$ (b) $[2, \infty)$ (c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$
10. The value of
$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} ..... + \binom{30}{20}\binom{30}{30}$$ is where $\binom{n}{r} = {}^nC_r$ *(2005S)*
(a) $\binom{30}{10}$ (b) $\binom{30}{15}$ (c) $\binom{60}{30}$ (d) $\binom{31}{10}$
11. For r = 0, 1, …, 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to *(2010)*
(a) $B_{10} - C_{10}$ (b) $A_{10}(B^2_{10}C_{10}A_{10})$
(c) 0 (d) $C_{10} - B_{10}$
12. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is *(JEE Adv. 2014)*
(a) 1051 (b) 1106 (c) 1113 (d) 1120

D MCQs with One or More than One Correct

1. If C_r stands for ${}^nC_r$, then the sum of the series
$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^n (n + 1)C_n^2],$$
where n is an even positive integer, is equal to *(1986 - 2 Marks)*

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(a) 0 (b) $(-1)^{n/2}(n+1)$

(c) $(-1)^{n/2}(n+2)$ (d) $(-1)^n n$

(e) none of these.

2. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals *(1998 - 2 Marks)*

(a) $(n-1)a_n$ (b) na_n

(c) $\frac{1}{2}na_n$ (d) None of the above

E Subjective Problems

1. Given that *(1979)*
$C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots\ldots\ldots + 2n\, C_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$
where $C_r = \frac{(2n)!}{r!(2n-r)!}$ $r = 0, 1, 2, \ldots\ldots\ldots\ldots, 2n$
Prove that
$C_1^2 - 2C_2^2 + 3C_3^2 - \ldots\ldots\ldots\ldots - 2nC_{2n}^2 = (-1)^n n\, C_n$.

2. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n. *(1982 - 5 Marks)*

3. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time, represented by $\sum\limits_{0 \le i < j \le n}\sum C_iC_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$
(1983 - 3 Marks)

4. Use mathematical Induction to prove : If n is any odd positive integer, then $n(n^2 - 1)$ is divisible by 24. *(1983 - 2 Marks)*

5. If p be a natural number then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n. *(1984 - 4 Marks)*

6. Given $s_n = 1 + q + q^2 + \ldots\ldots + q^n$;
$$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots\ldots + \left(\frac{q+1}{2}\right)^n, q \ne 1 \text{ Prove that}$$
$${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots\ldots + {}^{n+1}C_n s_n = 2^n S_n$$
(1984 - 4 Marks)

7. Use method of mathematical induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n > 0$ *(1985 - 5 Marks)*

8. Prove by mathematical induction that – *(1987 - 3 Marks)*
$$\frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{1/2}} \text{ for all positive Integers n.}$$

9. Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+4}$. *(1988 - 5 Marks)*

10. Using mathematical induction, prove that *(1989 - 3 Marks)*
$${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \ldots\ldots\ldots\ldots\ {}^mC_k\,{}^nC_0 = {}^{(m+n)}C_k,$$
where m, n, k are positive integers, and ${}^pC_q = 0$ for $p < q$.

11. Prove that *(1989 - 5 Marks)*
$C_0 - 2^2C_1 + 3^2C_2 - \ldots\ldots\ldots\ldots + (-1)^n(n+1)^2C_n = 0$,
$n > 2$, where $C_r = {}^nC_r$.

12. Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer n. *(1990 - 2 Marks)*

13. Using induction or otherwise, prove that for any non-negative integers m, n, r and k, *(1991 - 4 Marks)*
$$\sum_{m=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

14. If $\sum\limits_{r=0}^{2n} a_r(x-2)^r = \sum\limits_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$ *(1992 - 6 Marks)*

15. Let $p \ge 3$ be an integer and α, β be the roots of $x^2 - (p+1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$.
(i) is an integer and (ii) is not divisible by p
(1992 - 6 Marks)

16. Using mathematical induction, prove that
$\tan^{-1}(1/3) + \tan^{-1}(1/7) + \ldots\ldots \tan^{-1}\{1/(n^2+n+1)\}$
$= \tan^{-1}\{n/(n+2)\}$ *(1993 - 5 Marks)*

17. Prove that $\sum\limits_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. *(1993 - 5 Marks)*

18. If x is not an integral multiple of 2π use mathematical induction to prove that : *(1994 - 4 Marks)*
$$\cos x + \cos 2x + \ldots\ldots\ldots + \cos nx = \cos\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}$$

19. Let n be a positive integer and *(1994 - 5 Marks)*
$(1+x+x^2)^n = a_0 + a_1x + \ldots\ldots\ldots\ldots + a_{2n}x^{2n}$
Show that $a_0^2 - a_1^2 + a_2^2 \ldots\ldots\ldots\ldots + a_{2n}^2 = a_n$

20. Using mathematical induction prove that for every integer $n \ge 1$, $(3^{2n}-1)$ is divisible by 2^{n+2} but not by 2^{n+3}. *(1996 - 3 Marks)*

21. Let $0 < A_i < \pi$ for $i = 1, 2 \ldots, n$. Use mathematical induction to prove that
$$\sin A_1 + \sin A_2 \ldots + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + \ldots\ldots + A_n}{n}\right)$$
where ≥ 1 is a natural number.
{You may use the fact that
$p\sin x + (1-p)\sin y \le \sin[px + (1-p)y]$,
where $0 \le p \le 1$ and $0 \le x, y \le \pi$.} *(1997 - 5 Marks)*

22. Let p be a prime and m a positive integer. By mathematical induction on m, or otherwise, prove that whenever r is an integer such that p does not divide r, p divides ${}^{mp}C_r$. *(1998 - 8 Marks)*
[Hint: You may use the fact that $(1+x)^{(m+1)p} = (1+x)^p(1+x)^{mp}$]

23. Let n be any positive integer. Prove that **(1999 - 10 Marks)**

$$\sum_{k=0}^{m} \frac{\binom{2n-k}{k}}{\binom{2n-k}{n}} \cdot \frac{(2n-4k+1)}{(2n-2k+1)} 2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} 2^{n-2m}$$

for each non-be gatuve integer $m \le n$. $\left(\text{Here} \binom{p}{q} = {}^pC_q\right)$.

24. For any positive integer m, n (with $n \ge m$), let $\binom{n}{m} = {}^nC_m$.

Prove that $\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + ... + \binom{m}{m} = \binom{n+1}{m+2}$.

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + ... + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}.$$

(2000 - 6 Marks)

25. For every positive integer n, prove that $\sqrt{(4n+1)} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$. Hence or otherwise, prove that $[\sqrt{n} + \sqrt{(n+1)}] = [\sqrt{4n+1}]$, where $[x]$ denotes the greatest integer not exceeding x. **(2000 - 6 Marks)**

26. Let a, b, c be positive real numbers such that $b^2 - 4ac > 0$ and let $\alpha_1 = c$. Prove by induction that

$$\alpha_{n+1} = \frac{a\alpha_n^2}{(b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_n))} \text{ is well – defined and}$$

$\alpha_{n+1} < \frac{\alpha_n}{2}$ for all $n = 1, 2, ...$ (Here, 'well – defined' means that the denominator in the expression for α_{n+1} is not zero.) **(2001 - 5 Marks)**

27. Use mathematical induction to show that $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$ for all $n = 1, 2, ...$ **(2002 - 5 Marks)**

28. Prove that **(2003 - 2 Marks)**

$$2^k \binom{n}{0}\binom{n}{k} - 2^{k-1\binom{n}{2}}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n-2}{k-2} - .....(-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}.$$

29. A coin has probability p of showing head when tossed. It is tossed n times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1 = 1$, $p_2 = 1 - p^2$ and $p_n = (1-p).\ p_{n-1} + p(1-p)\,p_{n-2}$ for all $n \ge 3$.

Prove by induction on n, that $p_n = A\alpha^n + B\beta^n$ for all $n \ge 1$, where α and β are the roots of quadratic equation

$x^2 - (1-p)\,x - p\,(1-p) = 0$ and $A = \frac{p^2 + \beta - 1}{\alpha\beta - \alpha^2}$, $B = \frac{p^2 + \alpha - 1}{\alpha\beta - \beta^2}$.

(2000 - 5 Marks)

I Integer Value Correct Type

1. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = **(JEE Adv. 2013)**

2. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is **(JEE Adv. 2016)**

Section-B JEE Main / AIEEE

1. The coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ are **[2002]**

(a) equal
(b) equal with opposite signs
(c) reciprocals of each other
(d) none of these

2. If the sum of the coefficients in the expansion of $(a + b)^n$ is 4096, then the greatest coeficient in the expansion is **[2002]**

(a) 1594 (b) 792 (c) 924 (d) 2924

3. The positive integer just greater than $(1 + 0.0001)^{10000}$ is **[2002]**

(a) 4 (b) 5 (c) 2 (d) 3

4. r and n are positive integers $r > 1$, $n > 2$ and coefficient of $(r+2)^{\text{th}}$ term and $3r^{\text{th}}$ term in the expansion of $(1 + x)^{2n}$ are equal, then n equals **[2002]**

(a) $3r$ (b) $3r+1$ (c) $2r$ (d) $2r+1$

5. If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + ...}}}$ haing n radical signs then by methods of mathematical induction which is true **[2002[**

(a) $a_n > 7\ \forall\ n \ge 1$ (b) $a_n < 7\ \forall\ n \ge 1$
(c) $a_n < 4\ \forall\ n \ge 1$ (d) $a_n < 3\ \forall\ n \ge 1$

6. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is **[2003]**

(a) 6th term (b) 7th term (c) 5th term (d) 8th term.

7. The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is **[2003]**

(a) 35 (b) 32 (c) 33 (d) 34

8. Let $S(K) = 1 + 3 + 5... + (2K - 1) = 3 + K^2$. Then which of the following is true **[2004]**

(a) Principle of mathematical induction can be used to prove the formula
(b) $S(K) \Rightarrow S(K+1)$
(c) $S(K) \not\Rightarrow S(K+1)$
(d) $S(1)$ is correct

9. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same if α equals **[2004]**

(a) $\frac{3}{5}$ (b) $\frac{10}{3}$ (c) $\frac{-3}{10}$ (d) $\frac{-5}{3}$

10. The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is

(a) $(-1)^{n-1}n$ (b) $(-1)^n(1-n)$ **[2004]**

(c) $(-1)^{n-1}(n-1)^2$ (d) $(n-1)$

11. The value of ${}^{50}C_4+\sum_{r=1}^{6}{}^{56-r}C_3$ is **[2005[**

(a) ${}^{55}C_4$ (b) ${}^{55}C_3$ (c) ${}^{56}C_3$ (d) ${}^{56}C_4$

12. If $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then which one of the following holds for all $n\geq 1$, by the principle of mathematical induction **[2005]**

(a) $A^n=nA-(n-1)I$ (b) $A^n=2^{n-1}A-(n-1)I$

(c) $A^n=nA+(n-1)I$ (d) $A^n=2^{n-1}A+(n-1)I$

13. If the coefficient of x^7 in $\left[ax^2+\left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax-\left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation **[2005]**

(a) $a-b=1$ (b) $a+b=1$

(c) $\frac{a}{b}=1$ (d) $ab=1$

14. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1+x)^{\frac{3}{2}}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{\frac{1}{2}}}$ may be approximated as

(a) $1-\frac{3}{8}x^2$ (b) $3x+\frac{3}{8}x^2$ **[2005]**

(c) $-\frac{3}{8}x^2$ (d) $\frac{x}{2}-\frac{3}{8}x^2$

15. If the expansion in powers of x of the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0+a_1x+a_2x^2+a_3x^3......$ then a_n is

(a) $\frac{b^n-a^n}{b-a}$ (b) $\frac{a^n-b^n}{b-a}$ **[2006]**

(c) $\frac{a^{n+1}-b^{n+1}}{b-a}$ (d) $\frac{b^{n+1}-a^{n+1}}{b-a}$

16. For natural numbers m, n if $(1-y)^m(1+y)^n$ $=1+a_1y+a_2y^2+.......$ and $a_1=a_2=10$, then (m,n) is

(a) $(20,45)$ (b) $(35,20)$ **[2006]**

(c) $(45,35)$ (d) $(35,45)$

17. In the binomial expansion of $(a-b)^n$, $n\geq 5$, the sum of 5th and 6th terms is zero, then a/b equals **[2007[**

(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$ (c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$.

18. The sum of the series **[2007]**

${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+.....-.....+{}^{20}C_{10}$ is

(a) 0 (b) ${}^{20}C_{10}$ (c) $-{}^{20}C_{10}$ (d) $\frac{1}{2}{}^{20}C_{10}$

19. **Statement -1 :** $\sum_{r=0}^{n}(r+1)\ {}^nC_r=(n+2)2^{n-1}$. **[2008]**

Statement-2 : $\sum_{r=0}^{n}(r+1)\ {}^nC_rx^r=(1+x)^n+nx(1+x)^{n-1}$.

(a) Statement -1 is false, Statement-2 is true

(b) Statement -1 is true, Statement-2 is true; Statement -2 is a correct explanation for Statement-1

(c) Statement -1 is true, Statement-2 is true; Statement -2 is not a correct explanation for Statement-1

(d) Statement -1 is true, Statement-2 is false

20. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9 is : **[2009]**

(a) 2 (b) 7 (c) 8 (d) 0

21. Let $S_1=\sum_{j=1}^{10}j(j-1)^{10}C_J$, $S_2=\sum_{j=1}^{10}j^{10}C_j$ and $S_3=\sum_{j=1}^{10}j^{2\ 10}C_j$.

Statement-1 : $S_3=55\times 2^9$.

Statement-2: $S_1=90\times 2^8$ and $S_2=10\times 2^8$. **[2010]**

(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.

(b) Statement -1 is true, Statement -2 is false.

(c) Statement -1 is false, Statement -2 is true .

(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

22. The coefficient of x^7 in the expansion of $(1-x-x^2+x^3)^6$ is **[2011]**

(a) –132 (b) –144 (c) 132 (d) 144

23. If n is a positive integer , then $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$ is : **[2012]**

(a) an irrational number

(b) an odd positive integer

(c) an even positive integer

(d) a rational number other than positive integers

24. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is **[JEE M 2013]**

(a) 4 (b) 120 (c) 210 (d) 310

25. If the coefficents of x^3 and x^4 in the expansion of $\left(1+ax+bx^2\right)(1-2x)^{18}$ in powers of x are both zero, then (a, b) is equal to: **[JEE M 2014]**

(a) $\left(14,\frac{272}{3}\right)$ (b) $\left(16,\frac{272}{3}\right)$ (c) $\left(16,\frac{251}{3}\right)$ (d) $\left(14,\frac{251}{3}\right)$

26. The sum of coefficients of integral power of x in the binomial expansion $\left(1-2\sqrt{x}\right)^{50}$ is : **[JEE M 2015]**

(a) $\frac{1}{2}\left(3^{50}-1\right)$ (b) $\frac{1}{2}\left(2^{50}+1\right)$

(c) $\frac{1}{2}\left(3^{50}+1\right)$ (d) $\frac{1}{2}\left(3^{50}\right)$

27. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x\neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is : **[JEE M 2016]**

(a) 243 (b) 729 (c) 64 (d) 2187

5 Mathematical Induction and Binomial Theorem

Section-A : JEE Advanced/ IIT-JEE

A	**1.**	$(101)^{50}$	**2.**	–1	**3.**	$a=2, n=4$	**4.**	7	**5.**	41		
C	**1.**	(a)	**2.**	(a)	**3.**	(c)	**4.**	(c)	**5.**	(d)	**6.**	(b)
	7.	(c)	**8.**	(d)	**9.**	(d)	**10.**	(a)	**11.**	(d)	**12.**	(c)
D	**1.**	(c)	**2.**	(c)								
I	**1.**	6	**2.**	5								

Section-B : JEE Main/ AIEEE

1.	(a)	**2.**	(c)	**3.**	(d)	**4.**	(c)	**5.**	(b)	**6.**	(d)
7.	(c)	**8.**	(b)	**9.**	(c)	**10.**	(b)	**11.**	(d)	**12.**	(a)
13.	(d)	**14.**	(c)	**15.**	(d)	**16.**	(d)	**17.**	(b)	**18.**	(d)
19.	(b)	**20.**	(a)	**21.**	(b)	**22.**	(b)	**23.**	(a)	**24.**	(c)
25.	(b)	**26.**	(c)	**27.**	(b)						

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Consider $(101)^{50}-\{(99)^{50}+(100)^{50}\}$

$=(100+1)^{50}-(100-1)^{50}-(100)^{50}$

$=(100)^{50}[(1+0.01)^{50}-(1-0.01)^{50}-1]$

$=(100)^{50}[2({}^{50}C_1(0.01)+{}^{50}C_3(0.01)^3+....)-1]$

$=(100)^{50}[2({}^{50}C_3(0.01)^3+....)]>0$

$\therefore\ (101)^{50}>(99)^{50}+(100)^{50}\quad \therefore\ (101)^{50}$ is greater.

2. If we put $x=1$ in the expansion of $(1+x-3x^2)^{2163}=A_0+A_1x+A_2x^2+...$ we will get the sum of coefficients of given polynomial, which clearly comes to be – 1.

3. $(1+ax)^n=1+8x+24x^2+...$

$\Rightarrow\ (1+ax)^n=1+nxa+\dfrac{n(n-1)}{2!}.a^2x^2+...$

$=1+8x+24x^2+...$

Comparing like powers of x we get

$nax=8x\ \Rightarrow\ na=8$ ....(1)

$\dfrac{n(n-1)a^2}{2}=24\ \Rightarrow\ n(n-1)a^2=48$ ....(2)

Solving (1) and (2), $n=4,\ a=2$

4. We know that for a +ve integer n

$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+......+{}^nC_nx^n$

ATQ coefficients of 2^{nd}, 3^{rd}, and 4^{th} terms are in A.P.

i.e. ${}^nC_1, {}^nC_2, {}^nC_3$ are in A.P.

$\Rightarrow\ 2.{}^nC_2={}^nC_1+{}^nC_3$

$\Rightarrow\ 2\times\dfrac{n(n-1)}{2}=n+\dfrac{n(n-1)(n-2)}{3!}$

$\Rightarrow\ n-1=1+\dfrac{n^2-3n+2}{6}\ \Rightarrow\ n^2-9n+14=0$

$\Rightarrow\ (n-7)(n-2)=0\Rightarrow\ n=7$ or 2

But for the existance of 4^{th} term, n = 7.

5. Let T_{r+1} be the general term in the expansion of $(\sqrt{2}+3^{1/5})^{10}$

$\therefore\ T_{r+1}={}^{10}C_r(\sqrt{2})^{10-r}.(3^{1/5})^r.(0\le r\le 10)$

$=\dfrac{10!}{r!(10-r)!}.2^{5-r/2}.3^{r/5}$

Let T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers.

$\Rightarrow\ 5-\dfrac{r}{2}$ and $\dfrac{r}{5}$ are integers.

$\Rightarrow$ r = 0 and r = 10 $\Rightarrow$ T_1 and T_{11} are rational terms.

$\Rightarrow$ Sum of T_1 and $T_{11}={}^{10}C_0 2^{5-0}.3^0+{}^{10}C_{10}2^{5-5}.3^2$

$=1.32.1+1.1.9=32+9=41$

C. MCQs with ONE Correct Answer

1. (a) Given that r and n are +ve integers such that r > 1, n > 2

Also in the expansion of $(1+x)^{2n}$

coeff. of $(3r)^{th}$ term = coeff. of $(r+2)^{th}$ term

$\Rightarrow\ {}^{2n}C_{3r-1}={}^{2n}C_{r+1}$

$\Rightarrow$ 3r – 1 = r + 1 or 3r – 1 + r + 1 = 2n

$\Rightarrow$ r = 1 or 2r = n

But r > 1 $\quad\therefore\quad$ n = 2r

2. (a) General term in the expansion $\left(\dfrac{x}{2}-\dfrac{3}{x^2}\right)^{10}$ is

$$T_{r+1}={}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{-3}{x^2}\right)^r={}^{10}C_r x^{10-3r}\frac{(-1)^r 3^r}{2^{10-r}}$$

For coeff of x^4, we should have

$10-3r=4 \Rightarrow r=2$

$\therefore$ Coeff of $x^4 = {}^{10}C_2 \dfrac{(-1)^2 3^2}{2^8} = \dfrac{405}{256}$

3. **(c)** The given expression is

$(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$

We know by binomial theorem, that

$(x+a)^n+(x-a)^n = 2\,[{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots\ldots]$

$\therefore$ The given expression is equal to

$2\,[{}^5C_0x^5 + {}^5C_2x^3(x^3-1) + {}^5C_4x\,(x^3-1)^2]$

Max. power of x involved here is 7, also only +ve integral powers of x are involved, therefore given expression is a polynomial of degree 7.

4. **(c)** We have $(1+x)^m\,(1-x)^n$

$\left[1+mx+\dfrac{m(m-1)}{2!}x^2+\ldots\right]\left[1-nx+\dfrac{n(n-1)}{2!}x^2-\ldots\right]$

$= 1+(m-n)x+\left[\dfrac{m(m-1)}{2}+\dfrac{n(n-1)}{2}-mn\right]x^2+\ldots$

Given, $m-n=3$ ....(1)

and $\dfrac{1}{2}m(m-1)+\dfrac{1}{2}n(n-1)-mn=-6$

$\Rightarrow m^2+n^2-2mn-(m+n)=-12$

$\Rightarrow (m-n)^2-(m+n)=-12$

$\Rightarrow m+n=9+12=21$ ....(2)

From (1) and (2), we get $m=12$

5. **(d)** $\dbinom{n}{r}+2\dbinom{n}{r-1}+\dbinom{n}{r-2}$

$=\left[\dbinom{n}{r}+\dbinom{n}{r-1}\right]+\left[\dbinom{n}{r-1}+\dbinom{n}{r-2}\right]$

NOTE THIS STEP : $\dbinom{n+1}{r}+\dbinom{n+1}{r-1}=\dbinom{n+2}{r}$

$[\because\ {}^nC_r+{}^nC_{r-1}={}^{n+1}C_r]$

6. **(b)** $(a-b)^n$, $n\ge 5$

In binomial expansion of above $T_5+T_6=0$

$\Rightarrow {}^nC_4\,a^{n-4}b^4+{}^nC_5\,a^{n-5}b^5=0$

$\Rightarrow \dfrac{{}^nC_4}{{}^nC_5}.\dfrac{a}{b}=1 \Rightarrow \dfrac{4+1}{n-4}.\dfrac{a}{b}=1 \Rightarrow \dfrac{a}{b}=\dfrac{n-4}{5}$

7. **(c)** $\sum\limits_{i=0}^{m}{}^{10}C_i\,{}^{20}C_{m-i} = {}^{10}C_0\,{}^{20}C_m+{}^{10}C_1\,{}^{20}C_{m-1} + {}^{10}C_2\,{}^{20}C_{m-2}+\ldots\ldots+{}^{10}C_m\,{}^{20}C_0$

= Coeff of x^m in the expansion of product $(1+x)^{10}(1+x)^{20}$

= Coeff of x^m in the expansion of $(1+x)^{30}$

$={}^{30}C_m$

To get max. value of given sum, ${}^{30}C_m$ should be max. which is so when $m=30/2=15$.

[Using the fact that $\max({}^nC_r)=\begin{cases}{}^nC_{n/2} & \text{if } n \text{ is even}\\ {}^nC_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\end{cases}$]

8. **(d)** $(1+t^2)^{12}\,(1+t^{12})\,(1+t^{24})$

$=(1+t^{12}+t^{24}+t^{36})\,(1+t^2)^{12}$

$\therefore$ Coeff. of $t^{24} = 1\times$ Coeff. of t^{24} in $(1+t^2)^{12}+1\times$ Coeff. of t^{12} in $(1+t^2)^{12}+1\times$ constant term in $(1+t^2)^{12}$

$={}^{12}C_{12}+{}^{12}C_6+{}^{12}C_0=1+{}^{12}C_6+1={}^{12}C_6+2$

9. **(d)** ${}^{n-1}C_r={}^nC_{r+1}(k^2-3) \Rightarrow k^2-3=\dfrac{{}^{n-1}C_r}{{}^nC_{r+1}}=\dfrac{r+1}{n}$

Since $0\le r\le n-1$

$\Rightarrow 1\le r+1\le n \Rightarrow \dfrac{1}{n}\le\dfrac{r+1}{n}\le 1 \Rightarrow \dfrac{1}{n}\le k^2-3\le 1$

$\Rightarrow 3+\dfrac{1}{n}\le k^2\le 4 \Rightarrow \sqrt{3+\dfrac{1}{n}}\le k\le 2$

as $n\to\infty \Rightarrow \sqrt{3}<k\le 2 \Rightarrow k\in(\sqrt{3},2]$

10. **(a)** To find

${}^{30}C_0\,{}^{30}C_{10}-{}^{30}C_1\,{}^{30}C_{11}+{}^{30}C_2\,{}^{30}C_{12}-\ldots.+{}^{30}C_{20}\,{}^{30}C_{30}$

We know that

$(1+x)^{30}={}^{30}C_0+{}^{30}C_1x+{}^{30}C_2x^2+\ldots.+{}^{30}C_{20}x^{20}+\ldots.{}^{30}C_{30}x^{30}$ ....(1)

$(x-1)^{30}={}^{30}C_0x^{30}-{}^{30}C_1x^{29}+\ldots.+{}^{30}C_{10}x^{20}-{}^{30}C_{11}x^{19}+{}^{30}C_{12}x^{18}+\ldots.{}^{30}C_{30}x^0$ ....(2)

Multiplying eq^n (1) and (2), we get

$(x^2-1)^{30}=(\ )\times(\ )$

Equating the coefficients of x^{20} on both sides, we get

${}^{30}C_{10}={}^{30}C_0\,{}^{30}C_{10}-{}^{30}C_1\,{}^{30}C_{11}+{}^{30}C_2\,{}^{30}C_{12}-\ldots.+{}^{30}C_{20}\,{}^{30}C_{30}$

$\therefore$ Req. value is ${}^{30}C_{10}$

11. **(d)** Clearly $A_r={}^{10}C_r,\ B_r={}^{20}C_r,\ C_r={}^{30}C_r$

Now $\sum\limits_{r=1}^{10}{}^{10}C_r\left({}^{20}C_{10}\,{}^{20}C_r-{}^{30}C_{10}\,{}^{10}C_r\right)$

$={}^{20}C_{10}\sum\limits_{r=1}^{10}{}^{10}C_r\,{}^{20}C_r-{}^{30}C_{10}\sum\limits_{r=1}^{10}{}^{10}C_r\times{}^{10}C_r$

$={}^{20}C_{10}\left({}^{10}C_1\,{}^{20}C_1+{}^{10}C_2\,{}^{20}C_2+\ldots+{}^{10}C_{10}\,{}^{20}C_{10}\right)$

$-{}^{30}C_{10}\left({}^{10}C_1\times{}^{10}C_1+{}^{10}C_2\times{}^{10}C_2+\ldots+{}^{10}C_{10}\,{}^{10}C_{10}\right)$....(1)

Now expanding $(1+x)^{10}$ and $(1+x)^{20}$ by binomial theorem and comparing the coefficients of x^{20} in their product, on both sides, we get

${}^{10}C_0\,{}^{20}C_0+{}^{10}C_1\,{}^{20}C_1+{}^{10}C_2\,{}^{20}C_2+\ldots+{}^{10}C_{10}\,{}^{20}C_{10}$

$=$ coeff of x^{20} in $(1+x)^{30}={}^{30}C_{20}={}^{30}C_{10}$

$\therefore {}^{10}C_1\,{}^{20}C_1+{}^{10}C_2\,{}^{20}C_2+\ldots+{}^{10}C_{10}\,{}^{20}C_{10}={}^{30}C_{10}-1$

Again expending $(1+x)^{10}$ and $(x+1)^{10}$ by binomial theorem and comparing the coefficients of x^{10} in their product on both sides, we get

$\therefore \left({}^{10}C_0\right)^2\left({}^{10}C_1\right)^2+\left({}^{10}C_2\right)^2+...+\left({}^{10}C_{10}\right)^2=$

coeff of x^{10} in $(1+x)^{20}={}^{20}C_{10}$

$\therefore \left({}^{10}C_1\right)^2+\left({}^{10}C_2\right)^2+...+\left({}^{10}C_{10}\right)^2={}^{20}C_{10}-1$

Substituting these values in equation (1), we get

$={}^{20}C_{10}\left({}^{30}C_{10}-1\right)-{}^{30}C_{10}\left({}^{20}C_{10}-1\right)$

$={}^{30}C_{10}-{}^{20}C_{10}=C_{10}-B_{10}$

12. (c) Coeff. of x^{11} in exp. of $\left(1+x^2\right)^4\left(1+x^3\right)^7\left(1+x^4\right)^{12}$

= (Coeff. of x^a) × (Coeff. of x^b) × (Coeff. of x^c)

Such that $a+b+c=11$

Here $a=2m, b=3n, c=4p$

$\therefore 2m+3n+4p=11$

Case I : $m=0, n=1, p=2$

Case II : $m=1, n=3, p=0$

Case III : $m=2, n=1, p=1$

Case IV : $m=4, n=1, p=0$

$\therefore$ Required coeff.

$={}^4C_0\times{}^7C_1\times{}^{12}C_2+{}^4C_1\times{}^7C_3\times{}^{12}C_0$

$+{}^4C_2\times{}^7C_1\times{}^{12}C_1+{}^4C_4\times{}^7C_1\times{}^{12}C_0$

$=462+140+504+7=1113$

D. MCQs with ONE or MORE THAN ONE Correct

1. (c) $\because$ n is even, let $n=2m$ then

$$\text{LHL}=S=\frac{2.m!m!}{(2m)!}[C_0^2-2C_1^2+3C_2^2.....$$

$$+(-1)^{2m}(2m+1)C_{2m}^2....]\qquad ....(1)$$

$$=\frac{2.m!.m!}{(2m)!}C_{2m}^2-2C_{2m-1}^2+3C_{2m-2}^2-.....$$

$$+(-1)^{2m}(2m+1)C_0^2\qquad [\text{Using } C_r=C_{n-r}]$$

$$\Rightarrow S=\frac{2.m!m!}{(2m)!}[(2m+1)\,C_0^2-2mC_1^2$$

$$+(2m-1)\,C_2^2......-2C_{2m-1}^2+C_{2m}^2]\qquad ....(2)$$

Adding (1) and (2):

$$2S=2\frac{m!m!}{(2m)!}[2m+2]\,[C_0^2-C_1^2+C_2^2+.....+C_{2m}^2]$$

Now keeping in mind that if n is even, then

$$C_0^2-C_1^2+C_2^2-......+C_n^2=(-1)^{n/2}\,{}^nC_{n/2}$$

$\therefore$ we get

$$S=\frac{m!m!}{(2m)!}(2m+2)\,[(-1)^m\,{}^{2m}C_m]=\left(2\frac{n}{2}+2\right)(-1)^{n/2}$$

$$=(-1)^{n/2}(n+2)$$

2. (c) Let $b=\sum_{r=0}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n-(n-r)}{{}^nC_r}$

$$=na_n-\sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}}\qquad [\because\ {}^nC_r={}^nC_{n-r}]$$

$$=na_n-b$$

$$\Rightarrow\ 2b=na_n\Rightarrow b=\frac{n}{2}a_n$$

E. Subjective Problems

1. Given that

$C_1+2C_2x+3C_3x^2+....+2nC_{2n}x^{2n-1}=2n\,(1+x)^{2n-1}$ ....(1)

where $C_r=\dfrac{2n!}{r!(2n-r)!}$

Integrating both sides with respect to x, under the limits 0 to x, we get

$[C_1x+C_2x^2+C_3x^3+....+C_{2n}x^{2n}]_0^x=[(1+x)^{2n}]_0^x$

$\Rightarrow\ C_1x+C_2x^2+C_3x^3+....+C_{2n}x^{2n}=(1+x)^{2n}-1$

$\Rightarrow\ C_0+C_1x+C_2x^2+C_3x^3+....+C_{2n}x^{2n}=(1+x)^{2n}$ ....(2)

Changing x by $-\dfrac{1}{x}$, we get

$$\Rightarrow\ C_0-\frac{C_1}{x}+\frac{C_2}{x^2}-\frac{C_3}{x^3}+....+(-1)^{2n}\frac{C_{2n}}{x^{2n}}=\left(1-\frac{1}{x}\right)^{2n}$$

$\Rightarrow\ C_0x^{2n}-C_1x^{2n-1}+C_2x^{2n-2}-C_3x^{2n-3}$

$+....+C_{2n}=(x-1)^{2n}$ ....(3)

Multiplying eqn. (1) and (3) and equating the coefficients of x^{2n-1} on both sides, we get

$-C_1^2+2C_2^2-3C_3^2+....+2n\,C_{2n}^2$

$=$ coeff.of x^{2n-1} in $2n(x-1)\,(x^2-1)^{2n-1}$

$=2n$ [coeff. of x^{2n-2} in $(x^2-1)^{2n-1}$

$-$ coeff. of x^{2n-1} in $(x^2-1)^{2n-1}$]

$=2n\,[{}^{2n-1}C_{n-1}(-1)^{n-1}-0]$

$=(-1)^{n-1}.\,2n\,{}^{2n-1}C_{n-1}$

$\Rightarrow\ C_1^2-2C_2^2+3C_3^2+....+2n\,C_{2n}{}^2$

$=(-1)^n.\,2n\,{}^{2n-1}C_{n-1}=(-1)^n n.\left(\frac{2n}{n}.{}^{2n-1}C_{n-1}\right)$

$=(-1)^n n.\,{}^{2n}C_n=(-1)^n n.C_n.\qquad (\because\ {}^{2n}C_n=C_n)$

Hence Proved.

2. $P(n): 7^{2n}+2^{3n-3}, 3^{n-1}$ is divisible by 25 $\forall n\in N$.

Let us prove it by Mathematical Induction :

$P(1): 7^2+2^0.3^0=49+1=50$ which is divisible by 25.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true that is $7^{2k}+2^{3k-3}, 3^{k-1}$ is divisible by 25.

$\Rightarrow\ 7^{2k}+2^{3k-3}.\,3^{k-1}=25m$ where $m\in Z$.

$\Rightarrow\ 2^{3k-3}.\ 3^{k-1}=25m-7^{2k}$ ....(1)

Consider $P(k+1)$:

$7^{2(k+1)}+2^{3(k+1)-3}.3^{k+1-1}=7^{2k}.7^2+2^{3k}.3^k$

$=49.\,7^{2k}+2^3.\,3.2^{3k-3}.\,3^{k-1}=49.\,7^{2k}+24\,(25m-7^{2k})$

(Using IH eq. (1))

$=49.\,7^{2k}+24\times 25m-24\times 7^{2k}$

$=25.\,7^{2k}+24\times 25m=25\,(7^{2k}+24\,m)$

$=25\times$ some integral value which is divisible by 25.

$\therefore$ $P(k+1)$ is also true.

Hence by the principle of mathematical induction

$P(n)$ is true $\forall\ n \in Z$.

3. $S = \sum\limits_{0 \le i < j \le n} \sum C_i C_j$

NOTE THIS STEP

$\Rightarrow\ S = C_0(C_1 + C_2 + C_3 + ... + C_n) + C_1(C_2 + C_3 + ... + C_n) + C_2(C_3 + C_4 + C_5 + ... C_n) + ... C_{n-1}(C_n)$

$\Rightarrow\ S = C_0(2^n - C_0) + C_1(2^n - C_0 - C_1) + C_2(2^n - C_0 - C_1 - C_2) + ... + C_{n-1}(2^n - C_0 - C_1 .... C_{n-1}) + C_n(2^n - C_0 - C_1 .... C_n)$

$\Rightarrow\ S = 2^n(C_0 + C_1 + C_2 + ... + C_{n-1} + C_n) - (C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) - S$

$\Rightarrow\ 2S = 2^n.2^n - \dfrac{2n!}{(n!)^2} = 2^{2n} - \dfrac{2n!}{(n!)^2}$

$\Rightarrow\ S = 2^{2n-1} - \dfrac{2n!}{2(n!)^2}$

4. $P(n) : n(n^2-1)$ is divisible by 24 for n odd +ve integer.

For $n = 2m - 1$, it can be restated as

$P(m) : (2m-1)(4m^2 - 4m) = 4m(m-1)(2m-1)$

is divisible by 24 $\forall\ m \in N$

$\Rightarrow\ P(m) : m(m-1)(2m-1)$ is divisible by 6 $\forall\ m \in N$.

Here $P(1) = 0$, divisible by 6.

$\therefore\ P(1)$ is true.

Let it be true for $m = k$, i.e.,

$k(k-1)(2k-1) = 6p$

$\Rightarrow\ 2k^3 - 3k^2 + k = 6p$ ....(1)

Consider $P(k+1) : k(k+1)(2k+1) = 2k^3 + 3k^2 + k$

$= 6p + 3k^2 + 3k^2$ (Using (1)

$= 6(p + k^2) \Rightarrow$ divisible by 6

$\therefore\ P(k+1)$ is also true.

Hence $P(m)$ is true $\forall\ m \in N$.

5. $P(n) : P^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$

For $n = 1, P(1) : p^2 + p + 1$ which is divisible by $p^2 + p + 1$.

$\therefore\ P(1)$ is true.

Let $P(k)$ be true, i.e.,

$p^{k+1} + (p+1)^{2k-1}$ is divisible by $p^2 + p + 1$

$\Rightarrow\ p^{k+1} + (p+1)^{2k-1} = (p^2 + p + 1)m$ ....(1)

Consider $P(k+1) : p^{k+2} + (p+1)^{2k+1}$

$= p.p^{k+1} + (p+1)^{2k-1}.(p+1)^2$

$= p[m(p^2+p+1) - (p+1)^{2k-1}] + (p+1)^{2k-1}(p+1)^2$

$= p(p^2+p+1)m - p(p+1)^{2k-1} + (p+1)^{2k-1}(p^2+2p+1)$

$= p(p^2+p+1)m + (p+1)^{2k-1}(p^2+p+1)$

$= (p^2+p+1)[mp + (p+1)^{2k-1}]$

$= (p^2+p+1)$ some integral value

$\therefore$ divisible by p^2+p+1 $\quad\therefore P(k+1)$ is also true.

Hence by principle of mathematical induction $P(n)$ is true $\forall\, n \in N$.

6. We have $s_n = \dfrac{1-q^{n+1}}{1-q}$ ....(1)

and $S_n = \dfrac{1-\left(\dfrac{q+1}{2}\right)^{n+1}}{1-\left(\dfrac{q+1}{2}\right)} = \dfrac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)}$ ....(2)

Now, ${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + ... + {}^{n+1}C_{n+1} s_n$

$= \dfrac{1}{1-q}[{}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3(1-q^3) + ... + ... + {}^{n+1}C_n(1-q^{n+1})]$ Using (1)

$= \dfrac{1}{1-q}\left[\left({}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right) - \left({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + ... + {}^{n+1}C_{n+1} q^{n+1}\right)\right.$

$= \dfrac{1}{1-q}\left[2^{n+1} - 1) - \left\{(1+q)^{n+1} - 1\right\}\right]$

$= \dfrac{2^{n+1} - (1+q)^{n+1}}{(1-q)} = 2^n S_n$ [Using eq. (2)]

7. Let $A_n = 2.7^n + 3.5^n - 5$

Then $A_1 = 2.7 + 3.5 - 5 = 14 + 15 - 5 = 24$.

Hence A_1 is divisible by 24.

Now assume that A_m is divisible by 24 so that we may write

$A_m = 2.7^m + 3.5^m - 5 = 24k,\ k \in N$ ....(1)

Then $A_{m+1} - A_m = 2(7^{m+1} - 7^m) + 3(5^{m+1} - 5^m) - 5 + 5$

$= 2.7^m(7-1) + 3.5^m(5-1) = 12.(7^m + 5^m)$

Since 7^m and 5^m are odd integers $\forall\ m \in N$, their sum must be an even integer, say $7^m + 5^m = 2p, p \in N$.

Hence $A_{m+1} - A_m = 12.2p = 24p$

or $A_{m+1} = A_m + 24p = 24k + 24p$ [by (1)]

Hence A_{m+1} is divisible by 24.

It follows by mathematical induction that A_n is divisible by 24 for all $n \in N$.

8. Let $P(n) : \dfrac{(2n)!}{2^{2n}(n!)^2} \le \dfrac{1}{(3n+1)^{1/2}}$

For $n = 1$, $P(1) : \dfrac{2!}{2^2(1!)^2} \le \dfrac{1}{(3+1)^{1/2}} \Rightarrow \dfrac{1}{4} \le \dfrac{1}{2}$

$\Rightarrow\ \dfrac{1}{2} \le \dfrac{1}{2}$ which is true for $n = 1$

Assume that P(k) is true, then

$P(k) : \dfrac{(2k)!}{2^{2k}(k!)^2} \le \dfrac{1}{(3k+1)^{1/2}}$ ....(1)

For $n = k+1$,

$\dfrac{[2(k+1)]!}{2^{2(k+1)}[(k+1)!]^2} = \dfrac{(2k+2)!}{2^{2k+2}[(k+1)!]^2}$

$= \dfrac{(2k+2)(2k+1)(2k)!}{4.2^{2k}(k+1)^2(k!)^2}$

$\le \dfrac{(2k+2)\,(2k+1)}{4(k+1)^2}.\dfrac{1}{(3k+1)^{1/2}}$

[Using Induction hypothesis (1)]

$= \dfrac{(2k+1)}{2.(k+1)\,(3k+1)^{1/2}}$

Thus, $\dfrac{[2(k+1)]!}{2^{2(k+1)}\,[(k+1)!]^2} \le \dfrac{(2k+1)}{2(k+1)\,(3k+1)^{1/2}}$ ....(2)

In order to prove $P(k+1)$, it is sufficient to prove that

$$\frac{(2k+1)}{2(k+1)(3k+1)^{1/2}} \le \frac{1}{(3k+4)^{1/2}} \quad ....(3)$$

Squaring eq. (3), we get

$$\frac{(2k+1)^2}{4(k+1)^2(3k+1)} \le \frac{1}{3k+4}$$

$\Rightarrow (2k+1)^2(3k+4) - 4(k+1)^2(3k+1) \le 0$

$\Rightarrow (4k^2+4k+1)(3k+4) - 4(k^2+2k+1)(3k+1) \le 0$

$\Rightarrow (12k^3+28k^2+19k+4) - (12k^3+28k^2+20k+4) \le 0$

$\Rightarrow -k \le 0$

which is true.

Hence from (2) and (3), we get

$$\frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2} \le \frac{1}{(3k+4)1/2}$$

Hence the above inequation is true for $n = k+1$ and by the principle of induction it is true for all $n \in N$.

9. We have $5\sqrt{5} - 11 = \frac{4}{5\sqrt{5}+11} < 1$

Therefore $0 < 5\sqrt{5} - 11 < 1$

This gives us $0 < (5\sqrt{5}-11)^{2n+1} < 1$ for every positive integer n.

Also $(5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$= 2[{}^{2n+1}C_1(5\sqrt{5})^{2n}.11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}.11^3 + ... + {}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2[{}^{2n+1}C_1(125)^n.11 + {}^{2n+1}C_3(125)^{n-1}.11^3 + ... + {}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2k \quad ....(1)$

where k is some positive integer.

Let $F = (5\sqrt{5}-11)^{2n+1}$

Then equation (1) becomes

$R - F = 2k$

$\Rightarrow [R] + R - [R] - F = 2k \Rightarrow [R] + f - F = 2k$

$\Rightarrow f - F = 2k - [R] \Rightarrow f - F$ is an integer.

But $0 \le f < 1$ and $0 < F < 1$ Therefore $-1 < f - F < 1$

Since $f - F$ is an integer, we must have $f - F = 0$

$\Rightarrow f = F.$

Now, $Rf = RF = (5\sqrt{5}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1}$

$= [(5\sqrt{5})^2 - 12]^{2n+1} = 4^{2n+1}$

10. Let the given statement be

$P(m,n): {}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + ... + {}^mC_k{}^nC_0 = {}^{m+n}C_k$

where $m, n, k \in N$ and ${}^pC_q = 0$ for $p < q$.

As k is a positive integer and ${}^pC_q = 0$ for $p < q$.

$\therefore$ k must be a positive integer less than or equal to the smaller of m and n,

We have $k = 1$, when $m = n = 1$

$\therefore$ $P(1, 1)$ is ${}^1C_0{}^1C_1 + {}^1C_1{}^1C_0 = {}^2C_1 \Rightarrow 1 + 1 = 2.$

Thus $P(1, 1)$ is true.

Now let us assume that $P(m, n)$ holds good for any fixed value of m and n i.e.

${}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + ... + {}^mC_k{}^nC_0 = {}^{m+n}C_k \quad ....(1)$

Then $P(m+1, n+1)$ will be

${}^{m+1}C_0{}^{n+1}C_k + {}^{m+1}C_1{}^{n+1}C_{k-1} + ... + {}^{m+1}C_k{}^{n+1}C_0$

$= {}^{m+n+2}C_k \quad ....(2)$

Consider LHS

$= {}^{m+1}C_0{}^{n+1}C_k + {}^{m+1}C_1{}^{n+1}C_{k-1} + ... + {}^{m+1}C_k{}^{n+1}C_0$

$= 1.({}^nC_{k-1} + {}^nC_k) + ({}^mC_0 + {}^mC_1)({}^nC_{k-2} + {}^nC_{k-1})$

$+ ({}^mC_1 + {}^mC_2)({}^nC_{k-3} + {}^nC_{k-2}) + ... + ({}^mC_{k-1} + {}^mC_k).1$

$= ({}^nC_{k-1} + {}^mC_1{}^nC_{k-2} + {}^mC_2{}^nC_{k-3} + ... + {}^mC_{k-1}{}^nC_0)$

$+ ({}^nC_k + {}^mC_1{}^nC_{k-1} + {}^mC_2{}^nC_{k-2} + ... + {}^mC_{k-1}{}^nC_1 + {}^mC_k)$

$+ ({}^mC_0{}^nC_{k-2} + {}^mC_1{}^nC_{k-3} + ... + {}^mC_{k-2}{}^nC_0)$

$+ ({}^mC_0{}^nC_{k-1} + {}^mC_1{}^nC_{k-2} + {}^mC_2{}^nC_{k-3} + ... + {}^mC_{k-2}{}^nC_1 + {}^mC_{k-1})$

$= {}^{m+n}C_{k-1} + {}^{m+n}C_k + {}^{m+n}C_{k-2} + {}^{m+n}C_{k-1}$ [Using (1)]

$= {}^{m+n+1}C_k + {}^{m+n+1}C_{k-1} = {}^{m+n+2}C_k$

Hence the theorem holds for the next integers $m+1$ and $n+1$. Then by mathematical induction the statement $P(m, n)$ holds for all positive integral values of m and n.

11. We know that

$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ... + (-1)^n C_nx^n$

Multiplying both sides by x, we get

$x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + ... + (-1)^n C_nx^{n+1}$

Differentiating both sides w.r. to x, we get

$(1-x)^n - nx(1-x)^{n-1}$

$= C_0 - 2C_1x + 3C_2x^2 - 4C_3x^3 + ... + (-1)^n(n+1)C_nx^n$

Again multiplying both sides by x, we get

$x(1-x)^n - nx^2(1-x)^{n-1}$

$= C_0x - 2C_1x^2 + 3C_2x^3 - 4C_3x^4 + ... + (-1)^n(n+1)C_nx^{n+1}$

Differentiating above with respect to x, we get

$(1-x)^n - nx(1-x)^{n-1} - 2nx(1-x)^{n-1} + nx^2(n-1)(1-x)^{n-2}$

$= C_0 - 2^2C_1x + 3^2C_2x^2 - 4^2C_3x^3 + ... + (-1)^n(n+1)^2C_nx^n$

Substituting $x = 1$, in above, we get

$0 = C_0 - 2^2C_1 + 3^2C_2 - 4^2C_3 + ... + (-1)^n(n+1)^2C_n$

Hence Proved.

12. We have

$P(n): \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer, $\forall n \in N$

$P(1): \frac{1}{7} + \frac{1}{5} + \frac{2}{2} - \frac{1}{105}$

$= \frac{15+21+70-1}{105} = \frac{105}{105} = 1$ an integer

$\therefore$ $P(1)$ is true

Let $P(k)$ be true i.e.

$$\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105} \text{ is an integer}$$

$$\Rightarrow \frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}=m,\text{(say)}$$

$m \in N$ ....(1)

Consider $P(k+1)$:

$$=\frac{(k+1)^7}{7}+\frac{(k+1)^5}{5}+\frac{2(k+1)^3}{3}-\frac{(k+1)}{105}$$

$$=\left(\frac{k^7+7k^6+21k^5+35k^4+35k^3+21k^2+7k+1}{7}\right)$$

$$+\left(\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}\right)$$

$$+2\left(\frac{k^3+3k^2+3k+1}{3}\right)-\left(\frac{k+1}{105}\right)$$

$$=\left(\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}\right)$$

$$+[k^6+3k^5+5k^4+5k^3+3k^2+k+k^4$$

$$+2k^3+2k^2+k+2k^2+2k]+\left(\frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}\right)$$

$= m +$ some integral value $+ 1$

$=$ some integral value

$\therefore$ $P(k+1)$ is also true.

Hence $P(n)$ is true $\forall\ n \in N$, (by the Principle of Mathematical Induction.)

13. Let $P(k)=\sum_{m=0}^{k}\frac{(n-m)(r+m)!}{m!}=\frac{(r+k+1)!}{k!}\left[\frac{n}{r+1}-\frac{k}{r+2}\right]$

For $k = 1$, we will have two terms, on LHS, in sigma for $m = 0$ and $m = 1$, so that

$$LHS=(n-0)\frac{r!}{0!}+(n-1)\frac{(r+1)!}{1!}$$

and $$RHS=\frac{(r+2)!}{1!}\left[\frac{n}{r+1}-\frac{1}{r+2}\right]$$

Hence LHS = RHS for $k = 1$.

Now let the formula holds for $k = s$, that is let

$$\sum_{m=0}^{s}\frac{(n-m)\,(r+m)!}{m!}=\frac{(r+s+1)!}{s!}\left(\frac{n}{r+1}-\frac{s}{r+2}\right) \quad ...(1)$$

Let us add next term corresponding to $m = s + 1$ i.e.

adding $\frac{(n-s-1)(r+s+1)!}{(s+1)!}$ to both sides, we get

$$\sum_{m=0}^{s+1}\frac{(n-m)\,(r+m)!}{m!}=\frac{(r+s+1)!}{s!}\left[\frac{n}{r+1}-\frac{s}{r+2}\right]$$

$$+\frac{(n-s-1)(r+s+1)!}{(s+1)!}$$

$$=\frac{(r+s+1)!}{(s+1)!}\left[\frac{(s+1)n}{r+1}-\frac{s(s+1)}{r+2}+n-s-1\right]$$

$$=\frac{(r+s+1)!}{(s+1)!}\left[n\left\{\frac{s+1}{r+1}+1\right\}-(s+1)\left\{\frac{s}{r+2}+1\right\}\right]$$

$$=\frac{(r+s+2)\,(r+s+1)!}{(s+1)!}\left[\frac{n}{r+1}-\frac{s+1}{r+2}\right]$$

Hence the formula holds for $k = s + 1$ and so by the induction principle, the formula holds for all natural numbers k.

14. Given that

$$\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r \quad ....(1)$$

and $a_k = 1, \forall\ k \ge n$

To prove $b_n={}^{2n+1}C_{n+1}$

In the given equation (1) let us put $x-3 = y$ so that $x - 2 = y + 1$ and we get

$$\sum_{r=0}^{2n}a_r(1+y)^r=\sum_{r=0}^{2n}b_r(y)^r$$

$$\Rightarrow a_0+a_1(1+y)+....a_{n-1}+(1+y)^{n-1}(1+y)^n+(1+y)^{n+1}+....(1+y)^{2n}$$

$$=\sum_{r=0}^{2n}b_r y^r \qquad [\text{Using } a_k=1, \forall\ k\ge n]$$

Equating the coefficients of y^n on both sides we get

NOTE THIS STEP:

$\Rightarrow {}^nC_n+{}^{n+1}C_n+{}^{n+2}C_n+....+{}^{2n}C_n=b_n$

$\Rightarrow ({}^{n+1}C_{n+1}+{}^{n+1}C_n)+{}^{n+2}C_n+....+{}^{2n}C_n=b_n$

[Using ${}^nC_n={}^{n+1}C_{n+1}=1$]

$\Rightarrow b_n={}^{n+2}C_{n+1}+{}^{n+2}C_n+....+{}^{2n}C_n$

[Using ${}^mC_r+{}^mC_{r-1}={}^{m+1}C_r$]

Combining the terms in similar way, we get

$\Rightarrow b_n={}^{2n}C_{n+1}+{}^{2n}C_n \Rightarrow b_n={}^{2n+1}C_{n+1}$

Hence Proved

15. Since α, β are the roots of $x^2-(p+1)x+1=0$

$\therefore$ $\alpha+\beta=p+1$; $\alpha\beta=1$

Here $p \ge 3$ and $p \in Z$

(i) To prove that $\alpha^n+\beta^n$ is an integer.

Let us consider the statement, "$\alpha^n+\beta^n$ is an integer."

Then for $n = 1$, $\alpha+\beta=p+1$ which is an integer, p being an integer.

$\therefore$ Statement is true for $n = 1$

Let the statement be true for $n \le k$, i.e., $\alpha^k+\beta^k$ is an integer

Then ,

$$\alpha^{k+1}+\beta^{k+1}=\alpha^k.\alpha+\beta^k.\beta$$

$$=\alpha(\alpha^k+\beta^k)+\beta(\alpha^k+\beta^k)-\alpha\beta^k-\alpha^k\beta$$

$$=(\alpha+\beta)(\alpha^k+\beta^k)-\alpha\beta(\alpha^{k-1}+\beta^{k-1})$$

$$=(\alpha+\beta)(\alpha^k+\beta^k)-(\alpha^{k-1}+\beta^{k-1}) \quad .....(1)$$

[as $\alpha\beta=1$]

= difference of two integers = some integral value

$\Rightarrow$ Statement is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction the given statement is true for $\forall\ n \in N$.

(ii) Let R_n be the remainder of $\alpha^n + \beta^n$ when divided by p where $0 \le R_n \le p-1$

Since $\alpha + \beta = p+1$ $\therefore$ $R_1 = 1$

Also $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (p+1)^2 - 2$

$= p^2 + 2p - 1 = p(p+1) + p - 1$

$\therefore$ $R_2 = p - 1$

Also from equation (1) of previous part (i), we have

$$\alpha^{n+1} + \beta^{n+1} = (p+1)(\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})$$

$$= p(\alpha^n + \beta^n) + (\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})$$

$\Rightarrow R_{n+1}$ is the remainder of $R_n - R_{n-1}$ when divided by p

$\therefore$ We observe that $R_2 - R_1 = p - 1 - 1$

$\therefore$ $R_3 = p - 2$

Similarly, R_4 is the remainder when $R_3 - R_2$ is divided by p where

$R_3 - R_2 = p - 2 - p + 1 = -1 = -p + (p-1)$ $\therefore$ $R_4 = p - 1$

$R_4 - R_3 = p - 1 - p + 1 = 1$ $\therefore$ $R_5 = 1$

$R_5 - R_4 = 1 - p + 1 = -p + 2$ $\therefore$ $R_6 = p - 2$

It is evident for above that the remainder is either 1 or $p-1$ or $p-2$.

Since $p \ge 3$, so none is divisible by p.

16. To prove

$$P(n):\ \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots\tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right)$$

For $n = 1$, LHS $= \tan^{-1}\frac{1}{3}$;

RHS $= \tan^{-1}\frac{1}{3}$ $\Rightarrow$ LHS = RHS.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true, i.e.

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots\tan^{-1}\left(\frac{1}{k^2+k+1}\right) = \tan^{-1}\left(\frac{k}{k+2}\right)$$

Consider $P(k+1)$

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \ldots\tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

$$+\tan^{-1}\left(\frac{1}{(k+1)^2+(k+1)+1}\right)$$

$$= \tan^{-1}\left[\frac{k+1}{(k+1)+2}\right]$$

$$\text{LHS} = \tan^{-1}\left[\frac{k}{(k+2}\right] + \tan^{-1}\left(\frac{1}{(k^2+3k+3}\right)$$

[Using equation (1)]

$$= \tan^{-1}\left[\frac{\frac{k}{(k+2} + \frac{1}{k^2+3k+3}}{1 - \left(\frac{k}{k+2}\right)\left(\frac{1}{k^2+3k+3}\right)}\right]$$

$$= \tan^{-1}\left[\frac{(k+1)(k^2+2k+2)}{(k+3)(k^2+2k+2)}\right] = \tan^{-1}\left(\frac{k+1}{k+3}\right) = RHS$$

$\therefore$ $P(k+1)$ is also true.

Hence by the principle of mathematical induction $P(n)$ is true for every natural number.

17. To evaluate $\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1}$ where $k = \frac{3n}{2}$ and n is +ve even interger.

Let $n = 2m$, where $m \in z^+$ $\therefore$ $k = \frac{3(2m)}{2} = 3m$

$$\therefore\ \sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1} = \sum_{r=1}^{3m}(-3)^{r-1}\ {}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\ {}^{6m}C_5 - \ldots\ldots\quad \ldots(1)$$

Now we know that

$(1+a)^{6m} - (1-a)^{6m}$

$= 2[{}^{6m}C_1 a + {}^{6m}C_3 a^3 + {}^{6m}C_5 a^5 + \ldots..]$ ....(2)

Keeping in mind the form of RHS in equation (1) and in equation (2)

We put $a = i\sqrt{3}$ in equation (2) to get

$$(1+i\sqrt{3})^{6m} - (1+i\sqrt{3})^{6m}$$

$$= 2[{}^{6m}C_1 i\sqrt{3} - {}^{6m}C_3 i3\sqrt{3} + {}^{6m}C_5 i3^2\sqrt{3}\ldots.]$$

$$\Rightarrow\ (1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m}$$

$$= 2\sqrt{3}i[{}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\ {}^{6m}C_5\ldots.]\ldots.(3)$$

But $1 + i\sqrt{3} = 2(\cos\pi/3 + i\sin\pi/3)$

$\therefore$ $(1+i\sqrt{3})^{6m} = 2^{6m}(\cos\pi/3 + i\sin\pi/3)^{6m}$

NOTE THIS STEP

$$= 2^{6m}\left(\cos\frac{6m\pi}{3} + i\sin\frac{6m\pi}{3}\right)\ \text{[Using D' Moivre's thm.]}$$

Similarly,

$$(1-i\sqrt{3})^{6m} = 2^{6m}\left(\cos\frac{6m\pi}{3} - i\sin\frac{6m\pi}{3}\right)$$

$$\therefore\ (1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m} = 2^{6m}.2\sin 2m\pi = 0$$

Substituting the above in equation (3) we get

$${}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\ {}^{6m}C_5 - \ldots.. = 0$$

$$\Rightarrow\ \sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1} = 0.$$

Hence Proved

18. Let $P(n): \cos x + \cos 2x + \ldots. + \cos nx$

$$= \cos\frac{n+1}{2}x\sin\frac{nx}{2}\ \text{cosec}\ \frac{x}{2}\quad \ldots.(1)$$

where x is not an integral multiple of 2π.

For $n = 1$ $P(1)$: L.H.S. $= \cos x$

$$R.H.S. = \cos\frac{1+1}{2}x\sin\frac{x}{2}\ \text{cosec}\ \frac{x}{2} = \cos x$$

L.H.S. = R.H.S.

$\Rightarrow$ $P(1)$ is true.

Let $P(k)$ be true i.e.

$\cos x + \cos 2x + ... + \cos kx$

$= \cos\frac{k+1}{2}x \sin\frac{kx}{2} \text{ cosec } \frac{x}{2}$ ....(2)

Consider $P(k+1)$:

$\cos x + \cos 2x + ... + \cos kx + \cos (k+1)x$

$= \cos\left(\frac{k+2}{2}\right) x \sin\frac{(k+1)x}{2} \text{ cosec } \frac{x}{2}$

L.H.S. $[\cos x + \cos 2x + ... + \cos kx + \cos (k+1)x$

$= \cos\left(\frac{k+1}{2}\right) x \sin \text{cosec} \frac{kx}{2}\ \frac{x}{2} + \cos(k+1)x$

[Using (2)]

$= \left[\cos\left(\frac{k+1}{2}\right) x \sin\frac{kx}{2} + \cos(k+1)x \sin\frac{x}{2}\right] \text{cosec } \frac{x}{2}$

$= \frac{1}{2}\left[2\cos\frac{(k+1)x}{2} \sin\frac{kx}{2} + 2\cos(k+1)x \sin\frac{x}{2}\right] \text{cosec } \frac{x}{2}$

$= \frac{1}{2}[\sin\left(\frac{2k+1}{2}\right) x - \sin\frac{x}{2}$

$+ \sin\left(xk + \frac{3x}{2}\right) - \sin\left(xk + \frac{x}{2}\right)] \text{cosec } \frac{x}{2}$

$= \frac{1}{2}\left[\sin\left(xk + \frac{3x}{2}\right) - \sin\frac{x}{2}\right] \text{cosec } \frac{x}{2}$

$= \frac{1}{2}\left[2\cos\frac{(k+2)x}{2} \sin\frac{(k+1)}{2}\right] \text{cosec } \frac{x}{2}$

$= \cos\frac{(k+2)x}{2} \sin\frac{(k+1)x}{2} \text{cosec } \frac{x}{2} = R.H.S.$

$\therefore$ $P(k+1)$ is also true.

Hence by the principle of mathematical induction

$P(n)$ is true $\forall n \in N$.

19. Given that,

$(1 + x + x^2)^n = a_0 + a_1 x + ... + a_{2n} x^{2n}$ ....(1)

where n is a +ve integer.

Replacing x by $-\frac{1}{x}$ in eq^n (1), we get

$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + ... + \frac{a_{2n}}{x^{2n}}$ ....(2)

Multiplying eq.'s (1) and (2):

$\frac{(1 + x + x^2)^n (x^2 - x + 1)^n}{x^{2n}}$

$= (a_0 + a_1 x + ... + a_{2n} x^{2n}) \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^{2n}}\right)$

Equating the constant terms on both sides we get

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 =$ constant term in the expansion of $\frac{[(1 + x + x^2)(1 - x + x^2)]^n}{x^{2n}}$

= Coeff. of x^{2n} in the expansion of $(1 + x^2 + x^4)^n$

But replacing x by x^2 in eq's (1), we have

$(1 + x^2 + x^4)^n = a_0 + a_1 x^2 + ... + a_{2n}(x^2)^{2n}$

$\therefore$ Coeff of $x^{2n} = a_n$

Hence we obtain, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 = a_n$

20. For $n = 1$, $3^{2^n} - 1 = 3^{2^1} - 1 = 9 - 1 = 8$ which is divisible by $2^{n+2} = 2^3 = 8$ but is not divisible by $2^{n+3} = 2^4 = 16$

Therefore, the result is true for $n = 1$.

Assume that the result is true for $n = k$. That is, assume that $3^{2^k} - 1$ is divisible by 2^{k+2} but is not divisible by 2^{k+3},

Since $3^{2^k} - 1$ is divisble by 2^{k+2} but not by 2^{k+3}, we can write $3^{2^k} - 1 = (m) 2^{k+2}$

where m must be an odd positive integer, for otherwise $3^{2k} - 1$ will become divisible by 2^{k+3}.

For $n = k + 1$, we have $3^{2^{k+1}} - 1 = 3^{2^k . 2} - 1 = \left(3^{2^k}\right)^2 - 1$

$= (m.2^{k+2} + 1)^2 - 1$ [Using (1)]

$= m^2.(2^{k+2})^2 + 2m.2^{k+2} + 1 - 1$

$= m^2.2^{2k+4} + m.2^{k+3} = 2^{k+3}(m^2.2^{k+1} + m.)$

$\Rightarrow$ $3^{2^{k+1}} - 1$ is divisible by 2^{k+3}.

But $3^{2^{k+1}} - 1$ is not divisible by 2^{k+4} for otherwise we must have 2 divides $m^2.\ 2^{k+1} + m$. But this is not possible as m is odd. Thus, the result is true for $n = k + 1$.

21. For $n = 1$, the inequalitity becomes

$\sin A_1 \le \sin A_1$, which is clearly true.

Assume that the inequality holds for $n = k$ where k is some positive integer. That is, assume that

$\sin A_1 + \sin A_2 + ... + \sin A_k \le k \sin\left(\frac{A_1 + A_2 + ... + A_k}{k}\right)$ ....(1)

for same positive integer k.

We shall now show that the result holds for $n = k + 1$ that is, we show that

$\sin A_1 + \sin A_2 + ... + \sin A_k + \sin A_{k+1}$

$\le (k+1) \sin\left(\frac{A_1 + A_2 + ... + A_{k+1}}{k+1}\right)$ ....(2)

L.H.S. of (2)

$= \sin A_1 + \sin A_2 + ... + \sin A_k + \sin A_{k+1}$

$\le k \sin\left(\frac{A_1 + A_2 + ... + A_k}{k}\right) + \sin A_{k+1}$

[Induction assumption]

$= (k+1)\left[\frac{k}{k+1} \sin\alpha + \frac{1}{k+1} \sin A_{k+1}\right]$;

where $\alpha + \frac{A_1 + A_2 + ...A_k}{k}$

$\therefore$ L.H.S. of (2) $\le (k+1)\left[\left(1 - \frac{k}{k+1}\right) \sin\alpha + \frac{1}{k+1} \sin A_{k+1}\right]$

$\le (k+1) \sin\left\{\left(1 - \frac{k}{k+1}\right)\alpha + \frac{1}{k+1} A_{k+1}\right\}$

[Using the fact $p \sin x + (1-p) \sin y \le \sin[px + (1-p)y]$ for $0 \le p \le 1, 0 \le x, y \le \pi$]

$$= (k+1)\sin\left\{\frac{k}{k+1}\left(\frac{A_1 + A_2 + ...A_k}{k}\right) + \frac{1}{k+1}A_{k+1}\right\}$$

$$= (k+1)\sin\left(\frac{A_1 + A_2 + ...A_{k+1}}{k+1}\right)$$

Thus, the inequality holds for $n = k + 1$. Hence, by the principle of mathematical induction the inequality holds for all $n \in N$.

22. We know that ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

$\therefore\quad {}^{mp}C_r = \frac{mp}{r}\,{}^{mp-1}C_{r-1} = \left[\frac{m.{}^{mp-1}C_{r-1}}{r}\right]p$

Now, L.H.S is an integer

$\Rightarrow$ RHS must be an integer

But p and r are coprime (given)

$\therefore$ r must divide $m.\,{}^{mp-1}C_{r-1}$

or $\frac{m.\,{}^{mp-1}C_{r-1}}{r}$ is an integer.

$\Rightarrow \frac{{}^{mp}C_r}{p}$ is an integer or ${}^{mp}C_r$ is divisible by p.

23. Let $P(m) = \sum_{k=0}^{m} \frac{\binom{2n-k}{k}^{(2n-4k+1)}}{\binom{2n-k}{n}^{(2n-2k+1)}} 2^{n-2k}$

$$= \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}.2^{n-2m} \qquad ....(1)$$

For $m=0$, LHS $= \frac{\binom{2n}{0}}{\binom{2n}{n}}.\frac{2n+1}{2n+1}\,.\,2^n = \frac{1}{\binom{2n}{n}}2^n$,

R.H.S. $= \frac{\binom{n}{0}}{\binom{2n}{n}}\,.\,2^n = \frac{1}{\binom{2n}{n}}2^n = L.H.S$

$[\because m=0 \Rightarrow k=0]$

$\therefore$ $P(0)$ holds true. Now assuming $P(m)$

L.H.S. of $P(m+1) =$ L.H.S. of

$$P(m) + \frac{\binom{2n-m-1}{m+1}}{\binom{2n-m-1}{n}}\cdot\frac{(2n-4m-3)}{(2n-2m-1)}.2^{n-2m-2}$$

$$= \frac{n!(n-m)!}{m!(2n-2m)!}.2^{n-2m}$$

$$+ \frac{n!(n-m-1)!(2n-4m-3)}{(m+1)!(2n-2m-2)!(2n-2m-1)}.2^{n-2m-2}$$

$$= \frac{n!(n-m-1)!2^{n-2m-2}}{(m+1)!(2n-2m-1)!} \times \left\{\frac{(n-m).4\,(m+1)}{(2n-2m)} + (2n-4m-3)\right\}$$

$$= \frac{n!(n-m-1)!2^{n-2m-2}(2n-2m-1)}{(m+1)!(2n-2m-1)!}$$

$$= \frac{n!(n-m-1)!2^{n-2m-2}}{(m+1)!(2n-2m-2)!} = \frac{\binom{n}{m+1}}{\binom{2n-2m-2}{n-m-1}}.2^{n-2m-2}$$

= R.H.S. of $P(m+1)$.

Hence by mathematical induction, result follows for all $0 \le m \le n$.

24. Given that for positive integers m and n such that $n \ge m$, then to prove that

$${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m = {}^{n+1}C_{m+1}$$

L.H.S. ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^{n-1}C_m + {}^nC_m$

[writing L.H.S. in reverse order]

$= ({}^{m+1}C_{m+1} + {}^{m+1}C_m) + {}^{m+2}C_m + ... + {}^{n-1}C_m + {}^nC_m$

$[\because\ {}^mC_m = {}^{m+1}C_{m+1}]$

$= ({}^{m+2}C_{m+1} + {}^{m+2}C_m) + {}^{m+3}C_m + ... + {}^nC_m$

$[\because\ {}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}]$

$= {}^{m+3}C_{m+1} + {}^{m+3}C_m + ... + {}^nC_m$

Combining in the same way we get

$= {}^nC_{m+1} + {}^nC_m = {}^{n+1}C_{m+1} =$ R.H.S.

Again we have to prove

${}^nC_m + 2\,{}^{n-1}C_m + 3\,{}^{n-2}C_m + ... + (n-m+1)\,{}^mC_m$

$= {}^{n+2}C_{m+2}$

$= [{}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m] + [{}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m] + [{}^{n-2}C_m + ... + {}^mC_m] + ... + [{}^mC_m]$

$[n-m+1$ bracketed terms$]$

$= {}^{n+1}C_{m+1} + {}^nC_{m+1}\ {}^{n-1}C_{m+1}\ .... + {}^{m+1}C_{m+1}$

[using previous result.]

$= {}^{n+2}C_{m+2}$

[Replacing n by $n+1$ and m by $m+1$ in the previous result.]

= R.H.S.

25. For $n > 0, \sqrt{4n+1} > 0, \sqrt{n} + \sqrt{n+1} > 0$ *and* $\sqrt{4n+2} > 0$

Now, $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$ to be proved.

I. To prove $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$

Squaring both sides in $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$

$\Rightarrow\ 4n+1 < n+n+1+2\sqrt{n(n+1)}$

$\Rightarrow\ 2n < 2\sqrt{n(n+1)} \Rightarrow n < \sqrt{n(n+1)}$ which is true.

II. To prove $\sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$

Squaring both sides,

$n+n+1+2\sqrt{n(n+1)} < 4n+2$

$\Rightarrow\ 2\sqrt{n(n+1)} < 2n+1$ Squaring again

$4\,[n\,(n+1)] < 4n^2 + 1 + 4n$ or $0 < 1$ which is true

Hence $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$

Further to prove $[\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+1}]$, we have to prove that there is no positive integer which lies between $\sqrt{4n+1}$ and $\sqrt{4n+2}$ or $[\sqrt{4n+1}]=[\sqrt{4n+2}]$. Using Mathematical induction.

We have to check $[\sqrt{4n+1}]=[\sqrt{4n+2}]$ for $n = 1$

$[\sqrt{5}]=[\sqrt{6}] \Rightarrow 2 = 2$, which is true

Assume for $n = k$ (arbitrary)

i.e., $[\sqrt{4k+1}]=[\sqrt{4k+2}]$ To prove for $n = k+1$

To check $[\sqrt{4k+5}]=[\sqrt{4k+6}]$ since $k \geq 0$

Here $4k + 5$ is an odd number and $4k + 6$ is even number. Their greatest integer will be different iff $4k + 6$ is a perfect square that is $4k + 6 = r^2$

$\Rightarrow \quad k=\dfrac{r^2}{4}-\dfrac{6}{4}, \dfrac{6}{4}$ is not integer. But k has to be integer.

So 4k + 6 cannot be perfect square.

$\Rightarrow \quad [\sqrt{4k+5}]=[\sqrt{4k+6}]$

By Sandwich theorem

$\Rightarrow \quad [\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+1}]$

26. We have a, b, c the +ve real number s.t. $b^2 - 4ac > 0$; $\alpha_1 = c$.

$$P(n) : \alpha_{n+1} = \frac{a\alpha_n^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ...\alpha_n)}$$

is well defined and $\alpha_{n+1} < \dfrac{\alpha_n}{2}, \forall\, n = 1, 2, ...$

For $n = 1$, $\alpha_2 = \dfrac{a\alpha_1^2}{b^2 - 2a\alpha_1} = \dfrac{ac^2}{b^2 - 2ac}$

Now, $b^2 - 4ac > 0 \Rightarrow b^2 - 2ac > 2ac > 0$

$\therefore \quad \alpha_2$ is well defined (as denomination is not zero)

Also $\begin{bmatrix} \because & b^2 - 2ac > 2ac \\ \Rightarrow & \dfrac{1}{b^2 - 2ac} < \dfrac{1}{2ac} \end{bmatrix} \Rightarrow \dfrac{\alpha_2}{c} < \dfrac{1}{2} \Rightarrow \dfrac{\alpha_2}{\alpha_1} < \dfrac{1}{2}$

$\therefore \quad P(n)$ is true for $n = 1$.

Let the statement be true for $1 \leq n \leq k$ i.e.,

$$\alpha_{k+1} = \frac{a\alpha_k^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ...\alpha_k)} \text{ is well defined}$$

and $\alpha_{k+1} < \dfrac{\alpha_k}{2}$

Now, we will prove that $P(k + 1)$ is also true

i.e., $\alpha_{k+2} = \dfrac{a\alpha_{k+1}^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k + \alpha_{k+1})}$ is

well defined and $\alpha_{k+2} < \dfrac{\alpha_{k+1}}{2}$.

We have

$\alpha_1 = c, \alpha_2 < \dfrac{c}{2}, \alpha_3 < \dfrac{\alpha_2}{2} < \dfrac{c}{2^2}, \alpha_4 < \dfrac{\alpha_3}{2} < \dfrac{c}{2^3}, ...$ (by IH)

Now, $(\alpha_{1+} + \alpha_2 + ...\alpha_k + \alpha_{k+1} < c + \dfrac{c}{2} + \dfrac{c}{2^k} + ... + \dfrac{c}{2^k}$

$$= \frac{c\left(1 - \dfrac{1}{2^{k+1}}\right)}{1 - 1/2} = 2c\left(1 - \frac{1}{2^{k+1}}\right) < 2c$$

$\therefore \quad \alpha_1 + \alpha_2 + ... + \alpha_{k+1} < 2c$

$\Rightarrow \quad -2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1}) > -4ac$

$\Rightarrow \quad b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1}) > b^2 - 4ac > 0$

$\therefore \quad \alpha_{k+2}$ is well defined. Again by IH we have

$\alpha_{k+1} < \dfrac{\alpha_k}{2} \Rightarrow 2\alpha_{k+1} < \alpha_k$

$\Rightarrow \quad 4\alpha_{k+1}^2 < \alpha_k^2$ [$As\ by\ def. \alpha_{k+1}, \alpha_k\ are + ve$]

$\Rightarrow \quad 4\alpha_{k+1} < \dfrac{\alpha_k^2}{\alpha_{k+1}}$

$\Rightarrow \quad 4\alpha_{k+1} < \dfrac{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k)}{a}$

$\Rightarrow \quad 4a\alpha_{k+1} < b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k)$

$\Rightarrow \quad 2a\alpha_{k+1} < b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k + \alpha_{k+1})$

$\Rightarrow \quad \dfrac{a\alpha_{k+1}^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1})} < \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{a\alpha_{k+1}}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1})} < \dfrac{\alpha_{k+1}}{2}$

$\Rightarrow \quad \alpha_{k+2} < \dfrac{\alpha_{k+1}}{2}$

$\therefore \quad P(k+1)$ is also true.

Thus by the Principle of Mathematical Induction the Statement $P(n)$ is true $\forall\, n \in N$.

27. Let $P(n) : (25)^{n+1} - 24n + 5735$

For n = 1.

P(1) : $625 - 24 + 5735 = 6336 = (24)^2 \times (11)$, which is divisible by 24^2. Hence P(1) is true

Let $P(k)$ be true, where $k \geq 1$

$\Rightarrow \quad (25)^{k+1} - 24k + 5735 = (24)^2 \lambda$ where $\lambda \in N$

For $n = k + 1, P(k + 1) : (25)^{k+2} - 24(k + 1) + 5735$

$= 25[(25)^{k+1} - 24k + 5735]$

$\quad + 25.24.k - (25)(5735) + 5735 - 24(k + 1)$

$= 25(24)^2 \lambda + (24)^2 k - 5735 \times 24 - 24$

$= 25(24)^2 \lambda + (24)^2 k - (24)(5736)$

$= 25(24)^2 \lambda + (24)^2 k - (24)^2(239)$,

$= (24)^2[25\lambda + k - 239]$ which is divisible by $(24)^2$.

Hence, by the method of mathematical induction result is true $\forall\, n \in N$.

28. To prove that

$$2^k\, {}^nC_0\, {}^nC_k - 2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1} + 2^{k-2}\, {}^nC_2\, {}^{n-2}C_{k-2} - ..... + (-1)^k\, {}^nC_k\, {}^{n-k}C_0 = {}^nC_k$$

LHS of above equation can be written as

$$\sum_{r=0}^{k}(-1)^r 2^{k-r}\,{}^nC_r\,{}^{n-r}C_{k-r}$$

$$=\sum_{r=0}^{k}(-1)^r 2^{k-r}\frac{n!}{r!(n-r)!}\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$=\sum_{r=0}^{k}(-1)^r 2^{k-r}\frac{n!k!}{r!k!(n-k)!(k-r)!}$$

$$=\sum_{r=0}^{k}(-1)^r\frac{2^k}{2^r}.\frac{n!}{k!(n-k)!}\frac{k!}{r!(k-r)!}$$

$$=2^k\,{}^nC_k\sum_{r=0}^{k}(-1/2)^r\frac{k!}{r!(k-r)!}$$

$$=2^k\,{}^nC_k\sum_{r=0}^{k}{}^kC_r(-1/2)^r=2^k\,{}^nC_k(1-1/2)^k$$

$$=2^k\,{}^nC_k\frac{1}{2^k}={}^nC_k=\text{R.H.S.}$$ **Hence Proved**

29. We have $\alpha+\beta=1-p$ and $\alpha\beta=-p(1-p)$

For $n=1, p_n=p_1=1$

Also, $A\alpha^n+B\beta^n=A\alpha+B\beta=\frac{(p^2+\beta-1)\alpha}{\alpha\beta-\alpha^2}$

$$+\frac{(p^2+\alpha-1)\beta}{\alpha\beta-\beta^2}=\frac{p^2+\beta-1}{\beta-\alpha}+\frac{p^2+\alpha-1}{\alpha-\beta}$$

$$=\frac{p^2+\beta-1-p^2-\alpha+1}{\beta-\alpha}=\frac{\beta-\alpha}{\beta-\alpha}=1$$

For n = 2, $p_2=1-p^2$

Also, $A\alpha^n+B\beta^n=A\alpha^2+B\beta^2$

$$=\frac{(p^2+\beta-1)\alpha^2}{\alpha\beta-\alpha^2}+\frac{(p^2+\alpha-1)\beta^2}{\alpha\beta-\alpha^2}$$

which is true for $n=2$

Now let result is true for $k<n$ where $n\geq 3$.

$P_n=(1-p)P_{n-1}+p(1-p)P_{n-2}$

$=(1-p)(A\alpha^{n-1}+B\beta^{n-1})+p(1-p)(A\alpha^{n-2}+B\beta^{n-2})$

$=A\alpha^{n-2}\{(1-p)\alpha+p(1-p)\}+B\beta^{n-2}\{(1-p)\beta-p(1-p)$

$=A\alpha^{n-2}\{(\alpha+\beta)\alpha-\alpha\beta\}$

$+B\beta^{n-2}\{(\alpha+\beta)\beta-\alpha\beta\}$ by (1)

$=A\alpha^{n-2}\{\alpha^2+\beta\alpha-\alpha\beta\}+B\beta^{n-2}\{\alpha\beta+\beta^2-\alpha\beta\}$

$=A\alpha^{n-2}(\alpha^2)+B\beta^{n-2}(\beta^2)=A\alpha^n+B\beta^n$

This is true for n. Hence by principle of mathematical induction, the result holds good for all $n\in N$.

I. Integer Value Correct Type

1. **(6)** Let the coefficients of three consecutive terms of $(1+x)^{n+5}$ be ${}^{n+5}C_{r-1}$, ${}^{n+5}C_r$, ${}^{n+5}C_{r+1}$, then we have ${}^{n+5}C_{r-1}:{}^{n+5}C_r:{}^{n+5}C_{r+1}=5:10:14$

$$\frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r}=\frac{5}{10}\Rightarrow\frac{r}{n+6-r}=\frac{1}{2}$$

or $n-3r+6=0$ ...(1)

Also $\frac{{}^{n+1}C_r}{{}^{n+5}C_{r+1}}=\frac{10}{14}\Rightarrow\frac{r+1}{n-r+5}=\frac{5}{7}$

or $5n-12r+18=0$ (2)

Solving (1) and (2) we get $n=6$.

2. **(5)** $(1+x)^2+(1+x)^3+....+(1+x)^{49}+(1+mx)^{50}$

$$=(1+x)^2\left[\frac{(1+x)^{48}-1}{(1+x)-1}\right]+(1+mx)^{50}$$

$$=\frac{1}{x}\left[(1+x)^{50}-(1+x)^2\right]+(1+mx)^{50}$$

Coeff. of x^2 in the above expansion

= Coeff. of x^3 in $(1+x)^{50}$ + Coeff. of x^2 in $(1+mx)^{50}$

$\Rightarrow {}^{50}C_3+{}^{50}C_2\,m^2$

$\therefore (3n+1)\,{}^{51}C_3={}^{50}C_3+{}^{50}C_2\,m^2$

$$\Rightarrow(3n+1)=\frac{{}^{50}C_3}{{}^{51}C_3}+\frac{{}^{50}C_2}{{}^{51}C_3}m^2$$

$$\Rightarrow 3n+1=\frac{16}{17}+\frac{1}{17}m^2\Rightarrow n=\frac{m^2-1}{51}$$

Least positive integer m for which n is an integer is m = 16 and then n = 5

Section-B JEE Main/ AIEEE

1. (a) We have $t_{p+1} = {}^{p+q}C_p x^p$ and $t_{q+1} = {}^{p+q}C_q x^q$
${}^{p+q}C_p = {}^{p+q}C_q$.[Remember ${}^nC_r = {}^nC_{n-r}$]

2. (c) We have $2^n = 4096 = 2^{12} \Rightarrow n = 12$; the greatest coeff = coeff of middle term. So middle term

$= t_7$.; $t_7 = t_{6+1} \Rightarrow$ coeff of $t_7 = {}^{12}C_6 = \frac{12!}{6!6!} = 924$.

3. (d) $(1+0.0001)^{10000} = \left(1+\frac{1}{n}\right)^n, n = 10000$

$$= 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + ...$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right) + \left(1-\frac{2}{n}\right) + ...$$

$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{(9999)!}$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + ...\infty = e < 3$$

4. (c) $t_{r+2} = {}^{2n}C_{r+1}\ x^{r+1}; t_{3r} = {}^{2n}C_{3r-1}\ x^{3r-1}$
Given ${}^{2n}C_{r+1} = {}^{2n}C_{3r-1}$;
$\Rightarrow {}^{2n}C_{2n-(r+1)} = {}^{2n}C_{3r-1}$
$\Rightarrow 2n - r - 1 = 3r - 1 \Rightarrow 2n = 4r \Rightarrow n = 2r$

5. (b) $a_1 = \sqrt{7} < 7$. Let $a_m < 7$

Then $a_{m+1} = \sqrt{7 + a_m} \Rightarrow a^2_{m+1} = 7 + a_m < 7 + 7 < 14$.

$\Rightarrow a_{m+1} < \sqrt{14} < 7$; So by the principle of mathematical induction $a_n < 7\ \forall\ n$.

6. (d) $T_{r+1} = \frac{n(n-1)(n-2).........(n-r+1)}{r!}(x)^r$

For first negative term, $n - r + 1 < 0 \Rightarrow r > n + 1$

$\Rightarrow r > \frac{32}{5}\ \therefore r = 7.\ \left(\because n = \frac{27}{5}\right)$

Therefore, first negative term is T_8.

7. (c) $T_{r+1} = {}^{256}C_r(\sqrt{3})^{256-r}(\sqrt[8]{5})^r = {}^{256}C_r(3)^{\frac{256-r}{2}}(5)^{r/8}$

Terms will be integral if $\frac{256-r}{2}$ & $\frac{r}{8}$ both are +ve integer, which is so if r is an integral multiple of 8. As $0 \le r \le 256$

$\therefore$ r = 0,8,16,24,........256 , total 33 values.

8. (b) $S(k) = 1+3+5+...+(2k-1) = 3 + k^2$

$S(1) : 1 = 3 + 1$, which is not true

$\because S(1)$ is not true.

$\therefore$ P.M.I cannot be applied

Let $S(k)$ is true, i.e.

$$1 + 3 + 5.... + (2k-1) = 3 + k^2$$

$$\Rightarrow 1 + 3 + 5.... + (2k-1) + 2k + 1$$

$$= 3 + k^2 + 2k + 1 = 3 + (k+1)^2$$

$\therefore S(k) \Rightarrow S(k+1)$

9. (c) The middle term in the expansion of

$(1+\alpha x)^4 = T_3 = {}^4C_2(\alpha x)^2 = 6\alpha^2 x^2$

The middle term in the expansion of

$(1-\alpha x)^6 = T_4 = {}^6C_3(-\alpha x)^3 = -20\alpha^3 x^3$

According to the question

$$6\alpha^2 = -20\alpha^3 \Rightarrow \alpha = -\frac{3}{10}$$

10. (b) Coeff of x^n in $(1+x)(1-x)^n$

$= $ Coeff of x^n in $(1-x)^n$ + Coeff of x^{n-1} in $(1-x)^n$

$= (-1)^n\ {}^nC_n + (-1)^{n-1}\ {}^nC_{n-1} = (-1)^n 1 + (-1)^{n-1} n$

$= (-1)^n [1-n]$

11. (d) ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$$\Rightarrow {}^{50}C_4 + \left[\begin{array}{r} {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 \\ + {}^{51}C_3 + {}^{50}C_3 \end{array}\right]$$

We know $\left[{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\right]$

$$\Rightarrow ({}^{50}C_4 + {}^{50}C_3)$$

$$+ {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$$

$$\Rightarrow ({}^{51}C_4 + {}^{51}C_3) + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$$

Proceeding in the same way, we get

$\Rightarrow {}^{55}C_4 + {}^{55}C_3 = {}^{56}C_4$.

12. (a) We observe that

$A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and we can prove by induction that $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

Now $nA-(n-1)I=\begin{bmatrix} n & 0 \\ n & n \end{bmatrix}-\begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$=\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}=A^n$

$\therefore nA-(n-1)I=A^n$

13. (d) T_{r+1} in the expansion

$\left[ax^2+\frac{1}{bx}\right]^{11}={}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$

$={}^{11}C_r(a)^{11-r}(b)^{-r}(x)^{22-2r-r}$

For the Coefficient of x^7, we have

$\Rightarrow 22-3r=7 \Rightarrow r=5$

$\therefore$ Coefficient of $x^7={}^{11}C_5(a)^6\ (b)^{-5}$ ...(1)

Again T_{r+1} in the expansion

$\left[ax-\frac{1}{bx^2}\right]^{11}={}^{11}C_r(ax^2)^{11-r}\left(-\frac{1}{bx^2}\right)^r$

$={}^{11}C_r(a)^{11-r}(-1)^r\times(b)^{-r}(x)^{-2r}(x)^{11-r}$

For the Coefficient of x^{-7}, we have

Now $11-3r=-7 \Rightarrow 3r=18 \Rightarrow r=6$

$\therefore$ Coefficient of $x^{-7}={}^{11}C_6a^5\times1\times(b)^{-6}$

$\therefore$ Coefficient of x^7 = Coefficient of x^{-7}

$\Rightarrow {}^{11}C_5(a)^6(b)^{-5}={}^{11}C_6a^5\times(b)^{-6} \Rightarrow ab=1.$

14. (c) $\because$ x^3 and higher powers of x may be neglected

$\therefore \dfrac{(1+x)^{\frac{3}{2}}-\left(1+\frac{x}{2}\right)^3}{\left(1-x\right)^{\frac{1}{2}}}$

$=(1-x)^{\frac{-1}{2}}\left[\left(1+\frac{3}{2}x+\frac{\frac{3}{2}.\frac{1}{2}}{2!}x^2\right)-\left(1+\frac{3x}{2}+\frac{3.2}{2!}\frac{x^2}{4}\right)\right]$

$=\left[1+\frac{x}{2}+\frac{\frac{1}{2}.\frac{3}{2}}{2!}x^2\right]\left[\frac{-3}{8}x^2\right]=\frac{-3}{8}x^2$

(as x^3 and higher powers of x can be neglected)

15. (d) $(1-ax)^{-1}(1-bx)^{-1}$

$=(1+ax+a^2x^2+...)(1+bx+b^2x^2+...)$

$\therefore$ Coefficient of x^n

$x^n=b^n+ab^{n-1}+a^2b^{n-2}+.......+a^{n-1}b+a^n$

{which is a G.P. with $r=\frac{a}{b}$

$\therefore$ Its sum is $=\dfrac{b^n\left[1-\left(\frac{a}{b}\right)^{n+1}\right]}{1-\frac{a}{b}}\}=\dfrac{b^{n+1}-a^{n+1}}{b-a}$

$\therefore a_n=\dfrac{b^{n+1}-a^{n+1}}{b-a}$

16. (d) $(1-y)^m(1+y)^n$

$=[1-{}^mC_1y+{}^mC_2y^2-......]$

$[1+{}^nC_1y+{}^nC_2y^2+.....]$

$=1+(n-m)+\left\{\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-mn\right\}y^2+.....$

$\therefore a_1=n-m=10$

and $a_2=\dfrac{m^2+n^2-m-n-2mn}{2}=10$

So, $n-m=10$ and $(m-n)^2-(m+n)=20$

$\Rightarrow m+n=80$

$\therefore m=35, n=45$

17. (b) $T_{r+1}=(-1)^r.{}^nC_r(a)^{n-r}.(b)^r$ is an expansion of $(a-b)^n$

$\therefore$ 5th term $=t_5=t_{4+1}$
$=(-1)^4.{}^nC_4(a)^{n-4}.(b)^4={}^nC_4.a^{n-4}.b^4$

6th term $=t_6=t_{5+1}=(-1)^5\,{}^nC_5(a)^{n-5}(b)^5$

Given $t_5+t_6=0$

$\therefore {}^nC_4.a^{n-4}.b^4+(-{}^nC_5.a^{n-5}.b^5)=0$

$\Rightarrow \dfrac{n!}{4!(n-4)!}.\dfrac{a^n}{a^4}.b^4-\dfrac{n!}{5!(n-5)!}.\dfrac{a^nb^5}{a^5}=0$

$\Rightarrow \dfrac{n!.a^nb^4}{4!(n-5)!.a^4}\left[\frac{1}{(n-4)}-\frac{b}{5.a}\right]=0$

or, $\dfrac{1}{n-4}-\dfrac{b}{5a}=0 \Rightarrow \dfrac{a}{b}=\dfrac{n-4}{5}$

18. (d) We know that, $(1+x)^{20}={}^{20}C_0+{}^{20}C_1x+{}^{20}C_2x^2+......{}^{20}C_{10}x^{10}+.....{}^{20}C_{20}x^{20}$

Put $x=-1$, $(0)={}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+......+{}^{20}C_{10}-{}^{20}C_{11}....+{}^{20}C_{20}$

$\Rightarrow 0=2[{}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+.....-{}^{20}C_9]+{}^{20}C_{10}$

$\Rightarrow {}^{20}C_{10}=2[{}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+......-{}^{20}C_9+{}^{20}C_{10}]$

$\Rightarrow {}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+....+{}^{20}C_{10}=\frac{1}{2}\,{}^{20}C_{10}$

19. (b) We have

$\sum_{r=0}^{n}(r+1)\,{}^nC_rx^r=\sum_{r=0}^{n}r.{}^nC_rx^r+\sum_{r=0}^{n}{}^nC_rx^r$

$$=\sum_{r=1}^{n} r\cdot\frac{n}{r}\,{}^{n-1}C_{r-1}x^r+(1+x)^n$$

$$=nx\sum_{r=1}^{n}{}^{n-1}C_{r-1}x^{r-1}+(1+x)^n$$

$= nx\,(1+x)^{n-1}+(1+x)^n=$ RHS

$\therefore$ Statement 2 is correct.

Putting $x=1$, we get

$$\sum_{r=0}^{n}(r+1)^n C_r=n\cdot 2^{n-1}+2^n=(n+2)\cdot 2^{n-1}.$$

$\therefore$ Statement 1 is also true and statement 2 is a correct explanation for statement 1.

20. (a) $(8)^{2n}-(62)^{2n+1}$

$=(64)^n-(62)^{2n+1}=(63+1)^n-(63-1)^{2n+1}$

$$=\left[{}^nC_0(63)^n+{}^nC_1(63)^{n-1}+{}^nC_2(63)^{n-2}+\ldots\ldots+{}^nC_{n-1}(63)+{}^nC_n\right]$$

$$=\left[{}^{2n+1}C_0(63)^{2n+1}-{}^{2n+1}C_1(63)^{2n}+{}^{2n+1}C_2(63)^{2n-1}-\ldots\ldots+(-1)^{2n+1}\,{}^{2n+1}C_{2+1}\right]$$

$$=63\times\left[{}^nC_0(63)^{n-1}+{}^nC_1(63)^{n-2}+{}^nC_2(63)^{n-3}+\ldots\ldots\right]+1$$

$$-63\times\left[{}^{2n+1}C_0(63)^{2n}-{}^{2n+1}C_1(63)^{2n-1}+\ldots\ldots\right]+1$$

$\Rightarrow$ 63 × some integral value + 2

$\Rightarrow$ $8^{2n}-(62)^{2n+1}$ when divided by 9 leaves 2 as the remainder.

21. (b) $S_2=\sum_{j=1}^{10}j\,{}^{10}C_j=\sum_{j=1}^{10}10\,{}^9C_{j-1}$

$$=10\left[{}^9C_0+{}^9C_1+{}^9C_2+\ldots.+{}^9C_9\right]=10\,.\,2^9$$

22. (b) $(1-x-x^2+x^3)^6=[(1-x)-x^2(1-x)]^6$

$=(1-x)^6(1-x^2)^6$

$=(1-6x+15x^2-20x^3+15x^4-6x^5+x^6)$

$\times(1-6x^2+15x^4-20x^6+15x^8-6x^{10}+x^{12})$

Coefficient of $x^7=(-6)(-20)+(-20)(15)+(-6)(-6)$

$=-144$

23. (a) $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$

$$=\left[\left(\sqrt{3}+1\right)^2\right]^n-\left[\left(\sqrt{3}-1\right)^2\right]^n$$

$=\left(4+2\sqrt{3}\right)^n-\left(4-2\sqrt{3}\right)^n$

$=2^n\left[\left(2+\sqrt{3}\right)^n-\left(2-\sqrt{3}\right)^n\right]$

$=2^n\times 2\left[{}^nC_1\,2^{n-1}\sqrt{3}+{}^nC_3\cdot 2^{n-3}3\sqrt{3}+\ldots.\right]$

$=2^{n+1}\sqrt{3}\left[{}^nC_1\cdot 2^{n-1}+{}^nC_3 2^{n-3}\cdot 3+\ldots..\right]$

$=\sqrt{3}\times$ Some integer $\therefore$ irrational number

24. (c) Given expression can be written as

$$\left((x^{1/3}+1)-\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}=\left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}$$

$=(x^{1/3}-x^{-1/2})^{10}$

General term $=T_{r+1}={}^{10}C_r(x^{1/3})^{10-r}(-x^{-1/2})^r$

$$={}^{10}C_r\,x^{\frac{10-r}{3}}\cdot(-1)^r\cdot x^{-\frac{r}{2}}={}^{10}C_r(-1)^r\cdot x^{\frac{10-r}{3}-\frac{r}{2}}$$

Term will be independent of x when $\frac{10-r}{3}-\frac{r}{2}=0$

$\Rightarrow r=4$

So, required term $=T_5={}^{10}C_4=210$

25. (b) Consider $(1+ax+bx^2)(1-2x)^{18}$

$=(1+ax+bx^2)[{}^{18}C_0-{}^{18}C_1(2x)+{}^{18}C_2(2x)^2-{}^{18}C_3(2x)^3+{}^{18}C_4(2x)^4-\ldots\ldots]$

Coeff of $x^3={}^{18}C_3(-2)^3+a.(-2)^2.\,{}^{18}C_2+b(-2).\,{}^{18}C_1=0$

Coeff. of $x^3=-{}^{18}C_3.8+a\times 4.\,{}^{18}C_2-2b\times 18=0$

$$=-\frac{18\times 17\times 16}{6}.8+\frac{4a+18\times 17}{2}-36b=0$$

$=-51\times 16\times 8+a\times 36\times 17-36b=0$

$=-34\times 16+51a-3b=0$

$=51a-3b=34\times 16=544$

$=51a-3b=544$ ....(i)

Only option number (b) satisfies the equation number (i)

26. (c) $(1-2\sqrt{x})^{50}={}^{50}C_0-{}^{50}C_1\,2\sqrt{x}+{}^{50}C_2\left(2\sqrt{x}\right)^2$ ...(1)

$(1+2\sqrt{x})^{50}={}^{50}C_0+{}^{50}C_1\,2\sqrt{x}-{}^{50}C_2(2\sqrt{x})^2+\ldots+{}^{50}C_3(2\sqrt{x})^3-{}^{50}C_4(2\sqrt{x})^4$ ...(2)

Adding equation (1) and (2)

$\left(1-2\sqrt{x}\right)^{50}+\left(1+2\sqrt{x}\right)^{50}$

$=2\left[{}^{50}C_0+{}^{50}C_2 2^2x+{}^{50}C_4 2^3x^2+\ldots\right]$

Putting x = 1, we get above as $\frac{3^{50}+1}{2}$

27. (b) Total number of terms $={}^{n+2}C_2=28$

$(n+2)(n+1)=56$

$x=6$

Sum of coefficients $=(1-2+4)^n=3^6=729$