# Chapter 3

# Stability Analysis

**LEARNING OBJECTIVES**

*After reading this chapter, you will be able to understand:*

- Stability
- Routh–Hurwitz criterion
- Auxiliary equation
- Root locus
- Angle and magnitude conditions
- Break away point
- Frequency response analysis
- Frequency-Domine specifications
- Resonant peak
- Polar plot

## Introduction to Stability

### Stability

A linear time-invariant system is stable if the output of the system is bounded for a bounded input and the output of the system tends towards zero in the absence of the input.

Stability is classified as follows:

1. Absolute stability
2. Conditional stability
3. Marginal stability
4. Unstable

#### *Absolute Stability*

A system is absolutely stable with respect to a parameter, if the system is stable for all values of that parameter.

#### *Conditional Stability*

A system in conditionally stable with respect to a parameter, if the system is stable for only certain bounded ranges of values of this parameter.

#### *Marginal Stability*

A system is marginally stable if the natural response of the system neither decays nor grows but remains constant or oscillates as time approaches infinity.

**Unstable:** A system is unstable if its response is unbounded with a bounded input applied.

### Stability and Poles

The system poles that are in the left half plane yield either pure exponential decay or damped sinusoidal natural response, which is the necessary condition for a system to be stable.

**Note 1:** Stable system have closed loop transfer function with poles only in the left half plane.

**Note 2:** Unstable systems have loop transfer function with at least one pole in the right half plane or poles of multiplicity greater than 1 on the imaginary axis.

**Note 3:** Marginally stable system have closed loop transfer function with only imaginary poles of multiplicity 1 and poles in the left half plane.

### Necessary Conditions for Stability

1. Positiveness of the coefficients of characteristic equation is necessary as well as sufficient condition for stability of first and second-order system.
2. Positiveness and existence of the all coefficients of the characteristic equation is necessary condition for stability of the system.

**Note:** Roots with negative real part indicates all positive coefficients in characteristic equation but all positive coefficients does not indicate proofs with negative real part in the characteristic equation.

## Routh–Hurwitz Criterion

Routh–Hurwitz Criterion gives the necessary and sufficient condition for all roots of polynomial to lie in the left half of the *S*-plane, without actually solving for the roots of the equation.

The characteristic equation of the *n*th-order system is

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

### Routh Array

$$
\begin{array}{l|llllllll}
S^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot & \cdot \\
S^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot & \cdot \\
S^{n-2} & b_1 & b_2 & b_3 & b_4 \\
S^{n-3} & c_1 & c_2 & c_3 \\
S^{n-4} & d_1 & d_2 \\
. \\
. \\
S^2 & e_1 & a_n \\
S^1 & f_1 \\
S^0 & a_n
\end{array}
$$

The coefficients $b_1$, $b_2$ … are evaluated as follows

$$b_1 = (a_1 a_2 - a_0 a_3)/a_1;$$

$$b_2 = (a_1 a_4 - a_0 a_5)/a_1;$$

$$b_3 = (a_1 a_6 - a_0 a_7)/a_1 \ldots$$

This process will continue till we get a zero as the least coefficient in the third row. Similarly the coefficients of the other rows are also evaluated.

The roots of the characteristic equation are all in the left half of *S*-plane if all the Coefficients of the first column of the Routh's tabulation are of the same sign.

The number of changes of signs in the elements of the first column equals the number of roots with positive real parts or in the right half of *S*-plane.

**Special case 1:** When the first term in any row of the Routh array is zero while rest of the row has at least one non-zero term.

In this case, if zero appears as the first element of a row, the elements in the next row will all becomes infinite, to overcome this problem we replace the zero element by an arbitrary small positive number '$\varepsilon$' and then proceed with Routh's tabulation.

Finally substitute the value of $\varepsilon = 0$ and find the values of the elements of the array which are functions of '$\varepsilon$'. The resultant Routh's array is analysed as usual.

**Note:** If there is a single element zero in *s*' row, it is considered as row of all zeros.

**Special case 2:** When all the elements in one row of Routh's tabulation are zeros before the tabulation is properly terminated, it indicates the following:

(i) There are symmetrically located roots in *S*-plane
(ii) Pair of real roots with opposite signs and/or pair of conjugate roots on the imaginary axis and/or complex conjugate roots forming quadrates in the *S*-plane.

### Auxiliary Equation

The polynomial formed by the coefficients of the row just above the row of zeros in the Routh array is called auxiliary equation $[A(s) = 0]$.

**Note 1:** The order of the auxiliary equation is always even
**Note 2:** The roots of the auxiliary equation also satisfy the original characteristic equation.
**Note 3:** Break down in the Routh table due to zero row is overcome by replacing the row of zeros with first derivative of auxiliary equation $\left(\frac{dA(s)}{ds}\right)$ with respect to '$s$'.

**Solved Examples**

**Example 1:** If a system transfer function has some poles lying on the imaginary axis, it is
(A) Unconditionally stable
(B) Conditionally stable
(C) Unstable
(D) Marginally stable

**Solution:** (D)
When the poles are on imaginary axis, system is marginally stable.

**Example 2:** System has some roots with real parts equal to zero, but none with positive real part is
(A) Absolutely unstable (B) Absolutely stable
(C) Relatively stable (D) Marginally stable

**Solution:** (D)
Marginally stable

**Example 3:** Closed loop stability implies that $1 + G(s)H(s)$ has only _________ in the left half of the *S*-plane
(A) Poles (B) Zeros
(C) Poles and zeros (D) Poles or zeros

**Solution:** (B)
Zeros of characteristic equation are poles of the transfer function

**Example 4:** None of the poles of a linear control system lie in the right half of *S*-plane. For a bounded input the output of this system
(A) Could be bounded (B) Always tends to zero
(C) Is always bounded (D) None of the above

**Solution:** (D)
Poles are not on the right half indicates they can be on imaginary axis, so stability cannot be justified

**Example 5:** For the equation $s^3 - 4s^2 + s + 5 = 0$, the number of roots in the left half of $S$-plane will be
(A) Zero (B) One
(C) Two (D) Three

**Solution:** (B)
Routh array for $s^3 - 4s^2 + s + 5 = 0$

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & -4 & 5 \\ s^1 & 2.25 & \\ s^\circ & 5 & \end{array}$$

Sign changes in first column of Routh array are 2($1 \rightarrow -4 \rightarrow 1$) poles on left half = 3 – 2 = 1

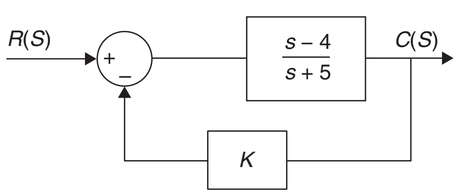
**Example 6:** The number of roots of the equation $2s^4 + s^3 + 5s + 6 = 0$ that lie in the right half of $S$-plane is
(A) Zero (B) One
(C) Two (D) Four

**Solution:** (C)
Routh array for $2s^4 + s^3 + 3s^2 + 5s + 6 = 0$

$$\begin{array}{c|ccc} s^4 & 2 & 3 & 6 \\ s^3 & 1 & 5 & \\ s^2 & -7 & 6 & \\ s^1 & \frac{41}{7} & & \\ s^\circ & 6 & & \end{array}$$

No. of sign changes in Routh array are 2 ($1 \rightarrow -7 \rightarrow \frac{41}{7}$)
No. of poles on the right half = 2

**Example 7:** For what range of $K$ is the following system is asymptotically stable; assume $K \geq 0$




(A) $0 \leq K < \frac{5}{4}$ (B) $0 \leq K < \frac{4}{5}$
(C) $K > -\frac{4}{5}$ (D) $K > -\frac{5}{4}$

**Solution:** (A)
Given system transfer function

$$= \frac{K(s-4)}{(1+K)s + (5-4K)}$$

Characteristic equation of the system is

$$(1 + K)\, s + (5 - 4K) = 0$$

For the system to be stable, all the coefficients of '$s$' in the characteristic equation must be positive

$$1 + K > 0 \qquad 5 - 4K > 0$$
$$K > -1 \qquad -4K > -5$$
$$K < \frac{5}{4}$$

Actual ranges of '$K$' is $-1 < K < \frac{5}{4}$

Given $K \geq 0;\ 0 \leq K < \frac{5}{4}$.

**Example 8:** The open loop transfer function of a unity feedback system is given below

$$G(s) = \frac{K(s+4)}{(s+1)(s+2)}$$

The range of positive values of '$K$' for which the closed loop system will remain stable is
(A) $2 < K < 3$ (B) $\frac{2}{4} < K < 3$
(C) $0 < K < \infty$ (D) $\frac{2}{4} < K < \infty$

**Solution:** (C)
Closed loop transfer function

$$= \frac{G(s)}{1+G(s)} = \frac{K(s+4)}{s^2 + (3+K)s + (2+4K)}$$

Characteristic equation of the system $s^2 + (3 + K)s + (2 + 4K) = 0$

Condition for stability is that all coefficients of '$s$' must be greater than zero in characteristic equation

$$3 + K > 0 \qquad 2 + 4K > 0$$
$$K > -3 \qquad 4K > -2$$
$$K > -\frac{2}{4}$$

∴ System is stable for all value of $K > -\frac{2}{4}$

∴ Range of positive values of '$k$' for stability is $0 < k < \infty$

**Example 9:** A certain closed loop system with unity feedback has the following transfer function given by $G(s) = \frac{k}{s(s+2)(s+4)}$ with the gain set at the ultimate value, the system will oscillate at an angular frequency of
(A) 2 rad/sec (B) 4 rad/sec
(C) 8 rad/sec (D) $2\sqrt{2}$ rad/sec

**Solution:** (D)
Characteristic equation of the system is

$$s^3 + 6s^2 + 8s + K = 0$$

System will oscillate when it is marginally stable/ from Routh array

| | | |
|---|---|---|
| $s^3$ | 1 | 8 |
| $s^2$ | 6 | $K$ |
| $s^1$ | $\frac{48-K}{6}$ | |
| $s^0$ | $K$ | |

System is marginally stable if $48 - K = 0 \Rightarrow K = 48$

Then Auxiliary equation is $6s^2 + 48 = 0$

$$s^2 = -8$$

$$\Rightarrow \quad s = j\,2\sqrt{2}$$

Oscillation frequency = $2\sqrt{2}$ rad/sec

## Root Locus

The root locus is basically the technique of finding the locus of roots as a single gain is changed, by solving for the roots of the characteristic equation, at each gain.

The gain that is to be varied will be open loop gain. Note this does not mean the gain of the open loop system that is typically fixed: this refers to cascading a controller in the forward path. Using the root locus method the control system engineer can predict the effect of varying gain on the open loop poles or what effect will be caused by adding open loop poles or open loop zeros.

### Angle and Magnitude Conditions

Consider the following general system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic equation of the system is obtained by setting the denominator of the closed loop system to zero as follows.

$$1 + G(s)\,H(s) = 0$$

$$\therefore \quad G(s)\,H(s) = -1$$

Since complex variable has both an angle and a magnitude, we can split the above equation into two separate equations as follows.

$$\angle G(s)\,H(s) = \pm 180^\circ\,(2K+1) \quad (K = 0, 1, 2, \ldots)$$

Angle condition

$$|G(s)H(s)| = 1 \qquad \text{Magnitude condition}$$

The values of '$s$' that satisfy the angle and magnitude conditions are the roots of the characteristic equation (The closed loop poles). ONLY these values will be the roots. As we vary the gain, these values of '$s$' that satisfy both conditions will change. The resulting collection of point in $S$-plane are called root locus.

**Note:** Open loop gain '$K$' corresponding to any point on root locus can be calculated using the equation.

$$K = \frac{\text{Product of lengh of vectors from open loop poles to the point}}{\text{Product of lenght of vectors form open loop zeros to the point}}$$

### Rules for Construction of Root Locus

1. The root locus is symmetric about origin
2. Number of branches in a root locus is equal to either the number of poles ($n$) or the number of zeros ($m$) whichever is greater. Each branch of root locus starts form open poles (Assuming number of poles is greater than zero) corresponding to $K = 0$ and terminates at either a finite open loop zero or infinity corresponding to $K = \infty$. '$n$' number of branches will terminate to finite open loop zeros and remaining branches of root locus $(n - m)$ will terminate to infinity.
   $B = P$ if $P > Z \Rightarrow P - Z$ branches will terminate at $\infty$
   $B = Z$ if $Z > P \Rightarrow Z - P$ branches will terminate at $\infty$
   $P$ = Number of poles, $Z$ = number of zeros
   $B$ = Number of branches of root locus
3. A section of real axis lies on root locus if the total number of open loops poles plus zeros to the right of that section is odd.
4. The angle of asymptotes and centroid:
   If $P > Z$, $P - Z$ number of branches will terminate at $\infty$ along straight line (asymptotes) making angle with real axis given by
   $$\phi_A = \frac{180(2q+1)}{P-Z}; \ (q = 0, 1, 2, 3, \ldots (P - Z - 1))$$
   If $Z > P \Rightarrow \phi_A = \frac{180(2q+1)}{Z-P}$; $(q = 0, 1, 2, 3, \ldots (Z - P - 1))$
   The point of intersection of the asymptotes with the real axis is called centroid denoted by '$\sigma$'
   $$\text{Centroid } (\sigma) = \frac{\text{Sum of real part of pole} - \text{Sum of real part of zeros}}{P-Z}$$
5. Breakaway/in point:
   A point on root locus where multiple poles/zeros exist is known as breakaway/ in point.
   The breakaway or breakin point is given by the roots of the equation $\frac{dK}{ds} = 0$, where '$K$' is obtained form $1 + KG(S)\,H(S) = 0$

   **Note 1:** Breakaway point exists if there is a root locus on real axis between two adjacent poles.

   **Note 2:** Breakin point exists if there is a root locus on real axis between two adjacent zeros.

   **Note 3:** Breakin point exists if there is a zero on real axis and left to that there is no root loci or poles or zeros.

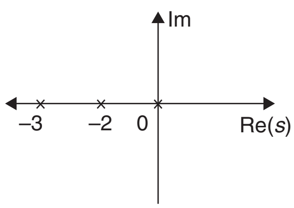
6. The angle of departure/arrival:
   The angle of departure or angle of arrival is given by
   Angle of departure = 180 – $\phi$
   Angle of arrival = 180 + $\phi$
   where $\phi$ =

$$\left[\begin{array}{l}\text{Sum of angles of vectors}\\ \text{to the complex pole/zero from}\\ \text{other poles}\end{array} - \begin{array}{l}\text{Sum of angles of vectors}\\ \text{to the complex pole/zero from}\\ \text{other zeros.}\end{array}\right]$$

7. The intersection of root locus with the imaginary axis can be determined by the use of Routh criterion [finding poles on imaginary axis]

**Example 10:** Plot the root locus for a transfer function $G(s) = \dfrac{K}{s(s+2)(s+3)}$

**Solution:** The number of poles = 3
The poles are at $s = 0$, $s = -2$ and $s = -3$




## Break-away Point

$$\frac{d}{ds}(s^3 + 5s^2 + 6s) = 0$$

$$3s^2 + 10s + 6 = 0$$

$$s = -0.784 \text{ and } s = -2.549$$

$s = -2.549$ does not lie on the root locus

## Asymptotes

$$\theta_1 = \pm\frac{180}{3} = \pm 60°$$

$$\theta_3 = \pm\frac{3\times 180}{3} = \pm 180°$$

## Centroid

$$s = \frac{(0-2-3)-(0)}{3-0} = -1.667$$

## Imaginary Axis Cross-over

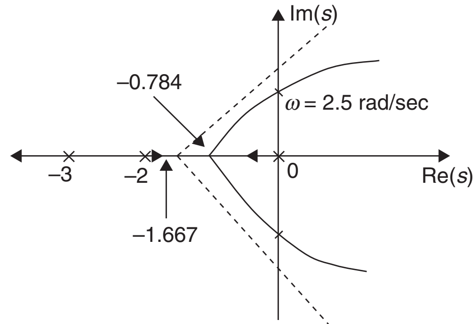
$$G(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+3)}$$

$$G(j\omega) = K\left[\frac{-5\omega^2}{25\omega^4+(6\omega-\omega^3)^2} - j\,\frac{(6\omega-\omega^3)}{25\omega^4+(6\omega-\omega^3)^2}\right]$$

Equating the imaginary part to zero

$$\omega = \pm\, 2.5 \text{ rad/sec.}$$

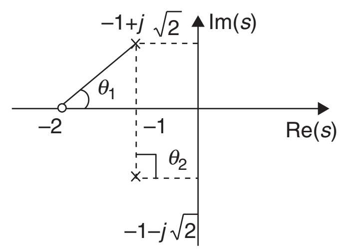
The root locus is drawn as shown in the figure




**Example 11:** $G(s) = \dfrac{K(s+2)}{s^2+2s+3}$, $H(s) = 1$

Sketch the root locus

**Solution:** No of branches of root locus = 2
The poles are at $s = -1 \pm j\sqrt{2}$
The zero is at $s = -2$




The root locus starts from the conjugate poles and break in on the real axis between –2 and –∞. One root locus ends in $s = -2$, the other ends at $s = -\infty$.

## Asymptote

$$\theta_1 = \pm\frac{180}{2-1} = \pm 180$$

## Angle of Departure

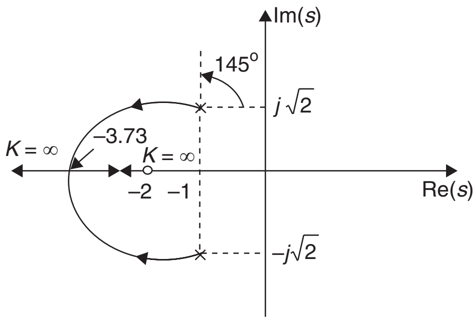
$$= 180 - \left(90 - \tan\frac{\sqrt{2}}{1}\right) = 145°$$

## Break-away Point

$$\frac{dG(s)}{ds} = 0$$

$$s = -3.73$$

The root locus is drawn in the following figure.




1. $N - P = Z$
   Here $P = 1$
   If $K > 1$, $N = 1$,
   $Z = 0$, then the closed loop system is stable
2. If $K < 1$, $N = 0$
   $Z \neq 0$, then the closed loop system is unstable.

**Example 13:** Given $G(s)\,H(s) = \dfrac{K}{s(s+2)(s+5)}$, the point of intersection of the asymptotes of the root locus with the real axis is

(A) 0 (B) – 2 (C) – 2.3 (D) – 3.5

**Solution:** (C)
No. of poles $(P) = 3\ (0, -2, -5)$
No. of zeros $(Z) = 0$
No. of asymptotes = 3
Centroid (Intersection of the asymptotes)

$$= \frac{\Sigma \text{ Real part of all poles } - \Sigma \text{ Real part of all zero}}{P - Z}$$

$$= \frac{0-2-5-0}{3} = \frac{-7}{3} = -2.33.$$

**Example 14:** The open loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K(s+2)}{s(s^2+2s+1)}$$

The centroid and angles of root locus are, respectively,

(A) $-\dfrac{2}{3}$ and +60°, –60° (B) –2 and +90°, –90°
(C) Zero and +90°, –90° (D) – 2 and +60°, –60°

**Solution:** (C)
No. of poles = 3 (0, – 1, – 1)
No. of zeros = 1 (– 2)
No. of asymptotes = 2
Angle of asymptotes = +90° and –90°

$$\text{Centroid} = \frac{\Sigma \text{Real part of all poles} - \Sigma \text{Real part of all zero}}{2}$$

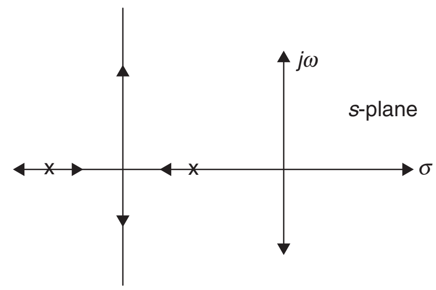
$$= \frac{(0-1-1)-(-2)}{2} = \frac{-2+2}{2} = 0.$$

**Example 15:** Figure shown below gives root locus of the open loop transfer function $G(s)\,H(s)$ of a system.

Consider the following inference drawn from the figure.

(1) It has no zero.
(2) It is a stable system.
(3) It is a second-order system.

Which of these inferences are correct?




(A) 1, 2, and 3 (B) 1 and 2
(C) 2 and 3 (D) 1 and 3

Two poles are terminated to infinity indicates that there are no zeros.

Two poles indicates the order of the system as '2'.

**Example 16:** The characteristic equation of a unity-feedback control system is given by $S^3 + AS^2 + S + B = 0$.

Consider the following statements in this regard.

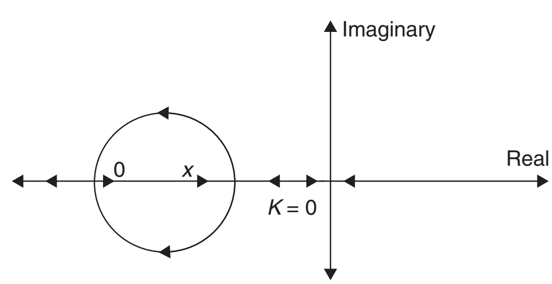
1. For a given value of B, all the root-locus branches will terminate at infinity for the variable 'A' in the positive direction.
2. For a given value of B, only one root locus branch will terminate at infinity for the variable $K$, in the positive direction.
3. For a given value, of A, all the root locus branches will terminate at infinity for the variable 'B' in the positive direction.

Of these statement

(A) 1 and 3 are correct.
(B) 2 and 3 are correct.
(C) Only 2 is correct.
(D) Only 1 is correct.

**Solution:** (B).

**Example 17:** The root locus of a unity feedback system is shown in the following figure. The open loop transfer function is given by



(A) $\dfrac{K}{s(s+2)(s+4)}$  (B) $\dfrac{K(s+4)}{s(s+3)(s+5)}$

(C) $\dfrac{K(s+4)}{s(s+3)}$  (D) $\dfrac{Ks}{(s+3)(s+5)}$

**Solution:** (C)

The given root locus indicates that the open loop transfer function has 2 poles and one zero.

One pole is at origin and another pole location is on the right side to the zero.

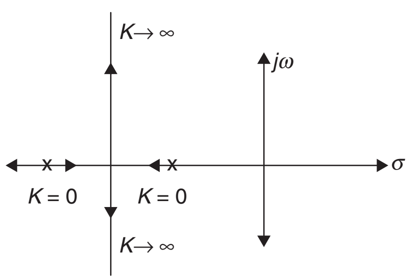
So option 'C' is correct.

**Example 18:** The closed loop transfer function of a feedback system is given by

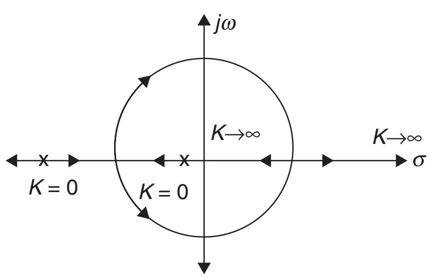
$$\frac{C(S)}{R(S)} = \frac{K}{s^2 + (4-K)s + 3}$$

Which of the following diagrams represents a root locus of the system for $K > 0$?

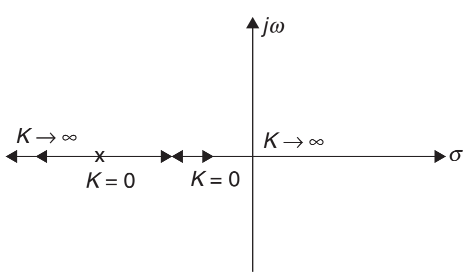
(A)



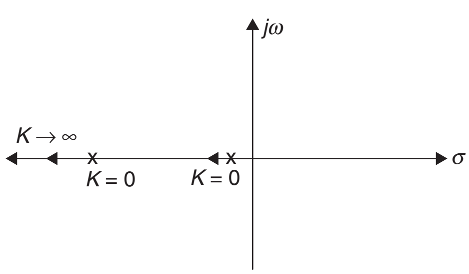

(B)




(C)




(D)




**Solution:** (B)

When the value of '$K$' is increasing the location of poles will tend to right-hand side of the $S$-plane when $K > 4$

→ When $K = 4$ characteristic equation $s^2 + 3 = 0$

$s = \pm j\sqrt{3} =$

(Poles are on the imaginary axis)

→ When $K = 10$ characteristic equation $s^2 - 6s + 3 = 0$

$(s - 3)^2 = 0$

$s = +3, +3$

(Poles are on RHS and equal)

→ When $K > 10$ poles are on RHS but not equal

∴ Option 'B' is the suitable root locus.

**Example 19:** A control system has

$$G(s)\,H(s) = \frac{K(s+5)}{(s+2)(s+3)}$$

The break away and break in points are located, respectively, at

(A) –2 and –1  (B) –1.589 and –7.5

(C) –2.55 and –7.5  (D) –1.5 and –6.89

**Solution:** (C)

Characteristic equation

$$1 + G(s)\,H(s) = 0$$

$$\Rightarrow \quad K = \frac{-(s+2)(s+3)}{(s+5)} = \frac{s^2+5s+6}{s+5}$$

Breakaway or breakin points are roots of $\dfrac{dK}{ds} = 0$

$$\frac{dK}{ds} = \frac{(2s+5)(s+5)-(s^2+5s+6)}{(s+5)^2} = 0$$

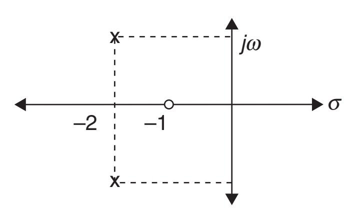
$$2s^2 + 15s + 25 - s^2 - 5s - 6 = 0$$

$$s^2 + 10s - 19 = 0$$

$$s = -2.55, -7.449$$

Breakaway point is –2.55, breakin point –7.449.

**Example 20:** A transfer function $G(s)$ has type pole zero plot as shown in figure. Given that the steady-state gain is 3, the transfer function $G(s)$ will be




(A) $\dfrac{2(s+1)}{s^2+4s+5}$  (B) $\dfrac{5(s+1)}{s^2+4s+5}$

(C) $\dfrac{15(s+1)}{s^2+4s+5}$  (D) $\dfrac{15(s+1)}{(s+2)^2}$

**Solution:** (C)

Form the given pole zero plot transfer function has

1. Zero is at (–1, 0)
2. Complex poles

Only for option 'C' steady-state gain is '3'.

$$\underset{s\to 0}{\text{Lt}}\, G(s) = \underset{s\to 0}{\text{Lt}} \frac{15(s+1)}{s^2+4s+5} = \frac{15}{5} = 3.$$

# Frequency Response Analysis

The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed as frequency response. In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal

when the input of a linear time-invariant system is sinusoidal with amplitude $A$ and frequency $\omega_0$.

$$r(t) = A\sin\omega_0 t.$$

The steady-state output of a system $y(t)$ will be a sinusoidal with the same frequency $\omega_0$ but possibly with different amplitude and phase

$$y(t) = B\sin(\omega_0 t + \phi)$$

$$\frac{B}{A} = |\text{Transfer function}| = \frac{|G(j\omega)|}{1+G(j\omega)\,H(j\omega)} = |M(j\omega)|$$

$\phi = \angle$Transfer function $= \angle G(j\omega) - \angle[1 + G(j\omega)\,H(j\omega)] = \angle M(j\omega)$

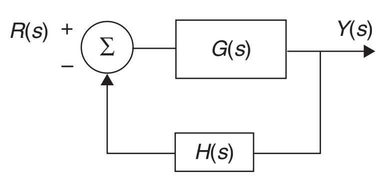
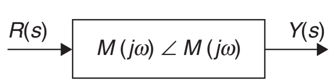



**Figure 1** Closed loop control system




The ease and accuracy of measurements are some of the advantages of the frequency response method. Extraction of transfer function is easy from frequency response test than step response test (time response). The design and parameter adjustment of the open-loop transfer function of a system for specific. Closed loop performance is carried out more easily in frequency domain than in time domain. The effect of noise disturbance and parameter variation are relatively easy to visualize and access through frequency response. Nyquist criterion is a powerful frequency domain method of extracting. The information regarding stability as well as relative stability of a system without the needs to evaluate roots of the characteristic equation.

## Frequency-domain Specifications

### Resonant Peak ($M_r$)

The resonant peak $M_r$ is the maximum value of $|M(j\omega)|$. The magnitude $M_r$ gives indication on the relative stability of a stable closed loop system.

For second-order system,

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \text{ for } \xi \le \frac{1}{\sqrt{2}}$$

$$M_r = 1 \text{ for } \xi > \frac{1}{\sqrt{2}}$$

**Note:** A large $M_r$ corresponds to a large maximum over short of the step response.

### Resonant Frequency ($\omega_r$)

The resonant frequency $\omega_r$ is the frequency at which the peak resonance $M_r$ occurs.

For second-order system, $\omega_r = \omega_n \sqrt{1-2\xi^2}$ for $\xi \le \frac{1}{\sqrt{2}}$

$$\omega_r = 0 \quad \text{for} \quad \xi > \frac{1}{\sqrt{2}}$$

### Bandwidth (BW)

The bandwidth ($BW$) is the frequency at which $|M(j\omega)|$ drops to 70.7% of or 3 dB down from its zero frequency value.

For second-order system

$$\text{BW} = \omega_n\left[\left(1-2\xi^2\right)+\sqrt{4\xi^4-4\xi^2+2}\right]^{1/2}$$

Bandwidth gives an indication of the transient response of a control system, noise filtering characteristics and robustness of the system.

### Gain Margin (GM)

Gain margin is the amount of gain in decibel (dB) that can be added to the open loop before the closed loop system becomes unstable.

$$\text{Gain margin} = \text{GM} = 20\log_{10}\frac{1}{\left[M(j\omega_{pc})\right]}$$

$$= -20\log\left|M(j\omega_{pc})\right| \text{dB}$$

The phase crossover frequency ($\omega_{pc}$) is the frequency at which phase angle becomes –180°.

### Phase Margin

Phase margin (PM) is defined as the angle m degrees through which the $M(j\omega)$ plot must be rotated about the origin so that the gain cross over passes through the (–1. $j$0) point.

$$\text{Phase margin} = \text{PM} = \angle M(j\omega_{gc}) - 180°$$

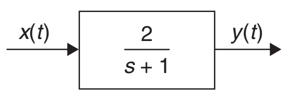
Gain crossover frequency ($\omega$gc) is the frequency at which $M(j\omega)$ becomes 1 or decibel magnitude of

$M(j\omega)$ becomes zero.

## Relation Between Time Domain and Frequency Domain Characteristics

1. The resonant peak $M_r$ of the closed loop frequency response depends on $\xi$ only. when $\xi = 0$, $M_r = \infty$ when $\xi$ is negative, the system is unstable, and the value of $M_r$ ceases to have any meaning. As $\xi$ increases, $M_r$ decreases.
   In comparison to time response, maximum peak overshoot also depends on '$\xi$' only. The peak overshoot is zero if $\xi \geq 1$.
2. Bandwidth is directly proportional to $\omega_n$.
   $\rightarrow B\omega$ increases linearly with $\omega_n$.
   $\rightarrow B\omega$ decreases with increase in $\xi$ for a fixed $\omega_n$.
   For time response, rise time increases as $\omega_n$ decreases.
   $\therefore$ Bandwidth $\alpha \dfrac{1}{\text{Rise time}}$.
3. Band width (BW) and $M_r$ are proportional to each other for $0 \leq \xi \leq \dfrac{1}{\sqrt{2}}$.

**Example 21:** For the system shown in figure, the input $x(t) = \sin t$.
In the steady-state, the response $y(t)$ will be




(A) $\dfrac{1}{\sqrt{2}}\sin(t - 45°)$ (B) $\sqrt{2}\sin(t - 45°)$

(C) $\dfrac{1}{\sqrt{2}}\sin(t - 45°)$ (D) $\sqrt{2}\sin(t - 45°)$

**Solution:** (D)
Transfer function $(T) = \dfrac{2}{s+1} = \dfrac{2}{j\omega+1}$
Input $= \sin t$ [$\therefore$ $\omega = 1$]

$$|T|\angle\theta = \frac{2}{\sqrt{1+1}}\angle -\tan^{-1}\left(\frac{1}{1}\right)$$

$$= \sqrt{2}\ \angle -45°$$

$$y(t) = 1 \times \sqrt{2}\ \sin(t - 45°)$$

$$y(t) = \sqrt{2}\ \sin(t - 45°)$$

**Example 22:** A system with zero initial condition has the closed loop transfer function

$$T(s) = \frac{s^2+16}{(s+2)(s+3)}$$

The system output is zero at the frequency.

(A) 1 rad/s (B) 2 rad/s
(C) 3 rad/s (D) 4 rad/s

**Solution:** (D)
Magnitude of transfer function

$$= \frac{|-\omega^2+16|}{|(j\omega+2)(j\omega+3)|}$$

Magnitude of transfer function will affect the magnitude of the system output. output becomes zero when transfer function magnitude is zero

$$\frac{|-\omega^2+16|}{|(j\omega+2)(j\omega+3)|} = 0$$

$$-\omega^2 + 16 = 0$$

$$\omega = 4 \text{ rad/sec}$$

**Example 23:** The gain margin of a unity feedback control system with the open loop transfer function $G(s) = \dfrac{s+4}{s^2}$ is

(A) 0 (B) $\dfrac{1}{\sqrt{4}}$

(C) $\sqrt{4}$ (D) $\infty$

**Solution:** (A)
Phase crossover frequency $\angle G(s) = -180°$

$$\tan^{-1}\left(\frac{\omega_{pc}}{4}\right) - 180° = -180°$$

$$\tan^{-1}\left(\frac{\omega_{pc}}{4}\right) = 0$$

$$\omega_{pc} = 0$$

Magnitude of transfer function at phase crossover frequency

$$\left[\frac{j\omega_{pc}+4}{(j\omega_{pc})^2}\right]\omega_{pc=0} = \infty$$

$$\text{Gain margin} = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{\infty} = 0$$

**Example 24:** The open loop transfer function of a unit feedback control system B given as $G(s) = \dfrac{sx+1}{s^2}$. The value of '$x$' to give a phase margin of $\dfrac{\pi}{4}$ is equal to

(A) 0.441 (B) 0.141
(C) 1.141 (D) 0.841

**Solution:** (D)
Phase margin = 180° $\angle G(j\omega) = 45°$

$$180° + \tan^{-1}\left(\frac{x\omega_{gc}}{1}\right) - 180° = 45°$$

$$x\omega_{gc} = 1$$

Magnitude phase margin $\left|G\left(j\omega_{pc}\right)\right| = 1$

$$\left|\frac{j\omega_{gc}+1}{\left(j\omega_{gc}\right)^2}\right| = 1$$

$$_{gc} = 12^{1/4}$$

$$x\omega_{gc} = 1$$

$$x = \frac{1}{2^{1/4}} = 0.841$$

**Example 25:** In the $G\,H(s)$ plane, the Nyquist plot of the loop transfer function $G(s)\,H(s) = \dfrac{\pi.e^{-s}}{s}$ passes through the negative real axis at the point

(A) $(-1, j0)$ (B) $(-0.5, j0)$
(C) $(-3, j0)$ (D) $(-2, j0)$

**Solution:** (D)

At the point of intersection with negative real axis,

$$\angle G(s)\,H(s) = -\pi$$

$$-\left(\omega_{pc} + \frac{\pi}{2}\right) = -\pi$$

$$\omega_{pc} = \frac{\pi}{2}$$

Magnitude of the $G(s)\,H(s)$ at $\pi = \omega_{pc}$ in the intersection point with negative real axis

$$\left|G(s)\;H(s)\right| w = w_{pc} = \left|\frac{\pi e^{-s}}{s}\right| = \left|\frac{\pi e^{-j\omega_{pc}}}{j\omega_{pc}}\right|$$

$$= \frac{\pi}{\omega_{pc}} = \frac{\pi}{\frac{\pi}{2}} = 2$$

∴ Nyquist plot passes through (–2, 0).

## Bode Plot

Bode plot is a graph of the transfer function of a linear, time-invariant system frequency plotted with a log-frequency axis, to show the system's frequency response. It is usually a combination of a Bode magnitude plot, expressing the magnitude of the frequency response gain, and a Bode phase plot expressing the frequency response phase shift.

The standard logarithmic magnitude of open loop transfer function of $G(j\omega)$ is $20\log_{10}\left|G\left(j\omega\right)\right|$. The units used in this representation of the magnitude are the decibel, usually denoted as dB.

Generally, a transfer function can be expressed in terms of factors of its poles and zeros. The advantage of the logarithmic plot is the conversion of these multiplicative factors to additive terms.

Consider the general open loop transfer function.

$$G(s) = \frac{K\left(1+sT_{z_1}\right)\left(1+sT_{z_2}\right)\ldots\left(1+sT_{zm}\right)}{s^P\left(1+sT_{P_1}\right)\left(1+sT_{P_2}\right)\ldots\left(1+sT_{pn}\right)}$$

In this example, the transfer function includes '$m$' number of zeros, '$p$' number of poles at origin and in the mentioned part '$n$' number of poles. Let $m = 1$, $n = 2$, $p = 1$.

$$\Rightarrow \qquad G(s) = \frac{K\left(1+sT_{z1}\right)}{s\left(1+sT_{p1}\right)\left(1+sT_{p2}\right)}$$

$$G(j\omega) = \frac{K\left(1+j\omega T_{z1}\right)}{j\omega\left(1+j\omega T_{p1}\right)\left(1+j\omega T_{p2}\right)}$$

$$\text{Magnitude of } G(j\omega) = \frac{K\sqrt{1+\omega^2 T_{z1}^{\ 2}}}{\omega\sqrt{1+\omega^2\,T_{P_1}^{\ 2}}\sqrt{1+\omega^2\;T_{P_2}^{\ 2}}}$$

Magnitude of $G(j\omega)$ is decibels is

$$\left|G\left(j\omega\right)\right| \text{ in dB} = 20\log\left|G(j\omega)\right|$$

$$= 20\log K + 20\log\sqrt{1+\omega^2\,T_{p1}^2} - 20\log\omega$$

$$-20\log\sqrt{1+\omega^2\;T_{P_1}^{\ 2}} - 20\log\sqrt{1+\omega^2\,T_{P_2}^{\ 2}}$$

The phase angle of $G(j\omega) = \angle G(j\omega) = \tan^{-1}\omega T_{z1} - 90° - \tan^{-1}\omega T_{P1} - \tan^{-1}\omega T_{P2}$

**Note:** From the above analysis, it is clear that, when the magnitude is expressed in dB, the multiplication is converted to addition.

Therefore to sketch the magnitude plot, knowledge of the magnitude variation of individual factors of the open loop transfer function is essential. The various factors of open loop transfer function are

1. Constant gain, $K$
2. Poles (or zeros) at origin, $\dfrac{1}{\left(j\omega\right)^n}$ or $\left(j\omega\right)^m$
3. First-order factor, $\dfrac{1}{1+j\omega T_p}$ or $1+j\omega T_z$
4. Quadratic factor, $\dfrac{1}{1+2\xi\left(j\omega/\omega_n\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2}$

   or $1+2\xi\left(\dfrac{j\omega}{\omega_n}\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2$

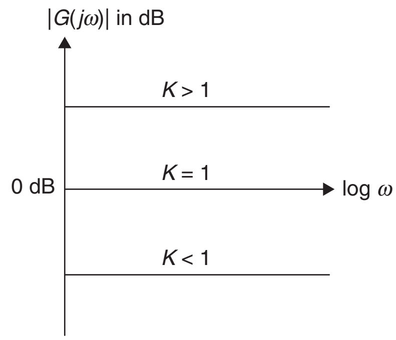
### Constant Gain: *K*

Let $G(s) = K$

$$\therefore \qquad G(j\omega) = K\angle 0°$$

$$\left|G\left(j\omega\right)\right| \text{ in dB} = 20\log K$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{0}{k}\right) = 0°$$




**Note 1:** The magnitude plot and phase plot of a constant '$K$' is independent of frequency and straight line.

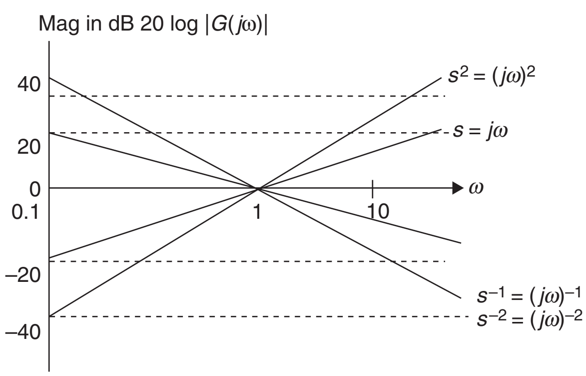
**Note 2:** A constant ($K$) greater than unity has a positive value in decibels, while a number smaller than unity has a negative value in decibels.

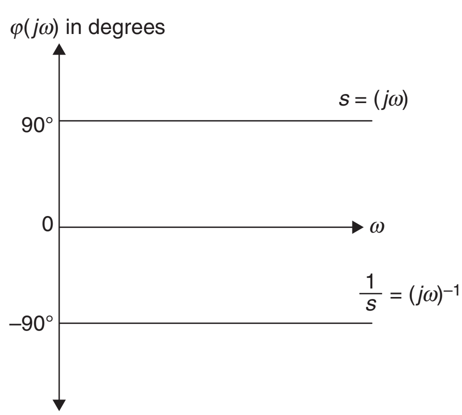
**Note 3:** The change in the value of gain ($K$) of transfer function is increase or decrease in the magnitude plot.

## Poles (zeros) at Origin $(j\omega)^{\pm n}$

Pole: Open loop transfer function $[G(s)] = \frac{1}{s^n}$

$$\text{Phase angle } \phi = -n.\tan^{-1}\left(\frac{\omega}{0}\right) = -n \times 90°$$




Zeros: Open loop transfer function $[G(S)] = s^n$

Log magnitude = $20 \log|(j\omega)^n) = 20n \log\omega$

$$\text{Phase angle } (\phi) = n \times \tan^{-1}\left(\frac{\omega}{0}\right) = n \times 90°$$

**Note 1:** Magnitude plot of $S^{\pm n}$ is a straight line with slop of $\pm 20 \times n$ dB/decade that passes through the point [0 dB, 1 rad/s]

**Note 2:** Phase angle plot of $S^{\pm n}$ is independent of frequency and it is constant angle of value $\pm 90n$ degrees.

## First-order Factor $(1+ j\omega T)^{\pm 1}$

Pole: open loop transfer function $G(s) = \frac{1}{1+sTp}$

$$\text{Log magnitude} = 20\log\left|\frac{1}{1+j\omega T_p}\right| = -20\ \log\sqrt{1+\omega^2 T_p^{\,2}}$$

For $\omega << \frac{1}{T_p}$; the asymptote is 20 log1 = 0 dB

For $\omega >> \frac{1}{T_p}$; the asymptote is $-20 \log \omega T_p$: It is a straight line with slope of –20 dB/decade. This asymptote intersect 0 dB at the break frequency $\omega_c = 1/T_p$, which is known as corner frequency.

$$\text{Phase angle } \phi = -\tan^{-1}\left(\frac{\omega T_p}{1}\right) = -\tan^{-1}\left(\omega T_p\right)$$

At corner frequency $\phi = -\tan^{-1}(\omega T_p)$

$$= -\tan^{-1}1 = 45°$$

The phase angle of the factor $(1 + sT_p)^{-1}$ varies from 0 to –90° as '$\omega$' is varied from 0 to infinity. The phase angle plot crosses –45° at $\omega = \omega_c = \frac{1}{T_p}$

Zero: open loop transfer function

$$G(s) = (1 + sT_z)$$

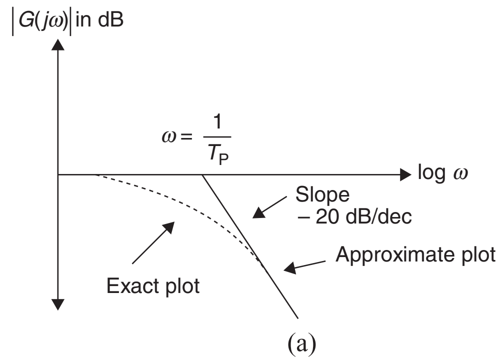
$$\text{Log magnitude} = 20\log\sqrt{1+\omega^2\,T_z^{\,2}}$$

For $\omega << \frac{1}{T_z}$; the asymptote is 20 log 1 = 0 dB

For $\omega >> \frac{1}{T_z}$; the asymptote is $20 \log\omega T_z$: it is a straight line with slope of +20 dB/decade. This asymptote intersect 0 dB at the break frequency $\omega_c = \frac{1}{T_z}$, which is known as corner frequency.

$$\text{Phase angle } \phi = \tan^{-1}\omega T_z$$

The phase angle of the factor $(1 + sT_z)$ varies from 0 to 90° as '$\omega$' is varied from zero to infinity. The phase angle plot crosses 45° at $\omega = \omega_c = \frac{1}{T_z}$.



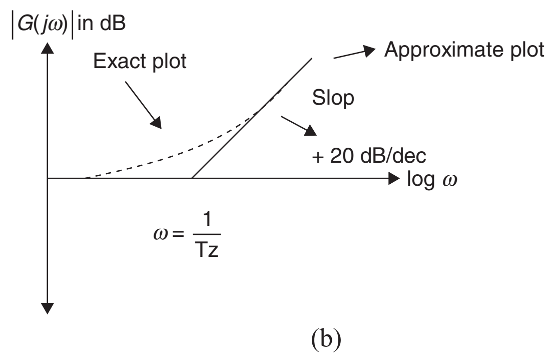




**Figure 2** Magnitude plot for first-order pole (a) and first-order zero (b)

## Quadratic Factor

$$\left[1+2\xi\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]^{\pm 1} = \text{Open loop transfer function} =$$

$$\left\{1+2\xi\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right\}^{\pm 1}$$

The magnitude in decibels is

$$=\pm 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}$$

For $\omega << \omega_n$, the log magnitude is asymptotic to a straight line of constant gain 0 dB and phase angle approaches to 0 deg.

For $\omega >> \omega_n$, the log magnitude approaches $\pm 40\log\left(\frac{\omega}{\omega_n}\right)$; a straight line with slop of ± 40 dB/dec.

Asymptote intersect 0 dB at corner frequency $\omega = \omega_n$.

**Note:** The resonant frequency is given by

$$\omega_r = \omega_n\sqrt{1-2\xi^2} \quad \text{for } \xi < \frac{1}{\sqrt{2}}$$

The maximum magnitude is

$$\text{Mp} = |G(j\omega_r)| = \frac{1}{2\xi\sqrt{1-\xi^2}} \text{ for } \xi < \frac{1}{\sqrt{2}}$$

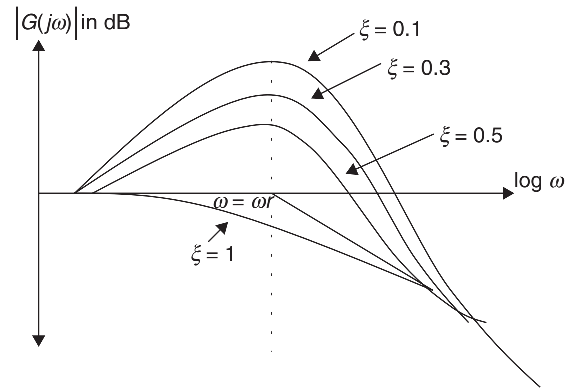



**Figure 3** Bode plot for quadratic factor in denominator

**Example 26:** Draw the Bode plot for a system having

$$G(s)H(s) = \frac{100}{s(s+1)(s+2)}$$

Find

(A) Gain margin

(B) Phase margin

(C) Gain crossover frequency

(D) Phase crossover frequency

**Solution:** $G(j\omega)H(j\omega) = \dfrac{50}{j\omega(1+j\omega)(1+0.5j\omega)}$

The corner frequencies are

$$\omega = 1 \text{ rad/s and } \omega = 2 \text{ rad/s}$$

For $\omega \le 1$ rad/s

$$G(j\omega)H(j\omega) = \frac{50}{j\omega}$$

Slope = –20 dB/decade

$$|G(j\omega)H(j\omega)|_{dB} = 20\log 50 - 20\log\omega$$

at $\omega = 0.1$

$$|G(j\omega)H(j\omega)|_{dB} = 20\log 50 - 20\log(0.1) = 53.98 \text{ dB}$$

At $\omega = 1$

$$|G(j\omega)H(j\omega)| = 20\log 50 = 33.98 \text{ dB}$$

For $1 < \omega \le 2$

$$G(j\omega)H(j\omega) = \frac{50}{j\omega(1+j\omega)}$$

Slope = –20 – 20

= –40 dB/decade

As $\omega$ increases from 1 to 2, the reduction in gain

$$= 40\log\left(\frac{2}{1}\right) = 12.04 \text{ dB}$$

At $\omega = 2$

$$|G(j\omega)H(j\omega)|_{dB} = 21.94 \text{ dB}$$

For $\omega > 2$

$$G(j\omega)H(j\omega) = \frac{50}{j\omega(1 + j\omega)(1 + 0.5j\omega)}$$

Slope = –40 – 20 = –60 dB/decade

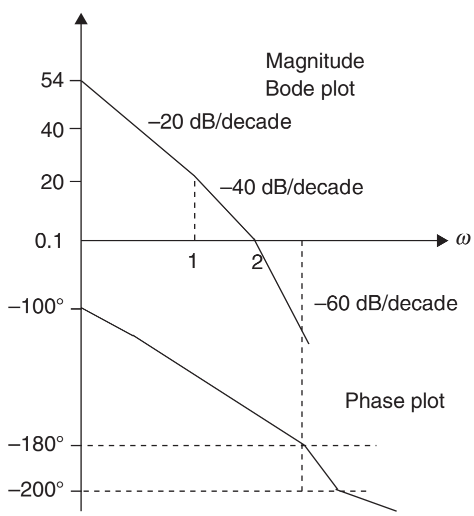
As $\omega$ increases from 2 to 10, the reduction in gain = $60 \log\left(\frac{10}{2}\right) = 41.94$ dB

At $\omega = 10$,

$$|G(j\omega)H(j\omega)|_{\text{dB}} = -19.99 \text{ dB}$$

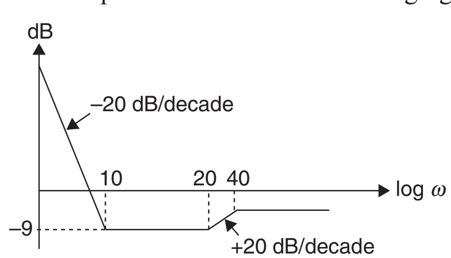
$$\angle G(j\omega)H(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}(0.5\omega)$$

| $\omega$ | $\angle G(j\omega)H(j\omega)$ |
|---|---|
| 0 | –90 |
| 0.1 | –98.6 |
| 0.2 | –107 |
| 0.5 | –130.6 |
| 1 | –161.6 |
| 1.3 | –175.5 |
| 1.4 | –179.5 |
| 1.5 | –183.2 |
| 2 | –198.4 |




Gain crossover frequency = 4.45 rad/s
Phase crossover frequency = 1.40 rad/s
Gain margin = 27 dB
Phase margin = 53°

**Example 27:** Find the transfer function of the system whose asymptotic Bode plot is shown in the following figure.




**Solution:** The line with a slope of –20 dB/decade does not pass through $\omega = 1$ rad/s, i.e. there is a term $\frac{K}{s}$

$$20 \log K = -9$$

$$K = 0.35$$

At $\omega = 1$ rad/sec, slope changes to 0 dB/dec indicating a zero at $\omega = 1$ rad/sec. The term is $(1 + s)$

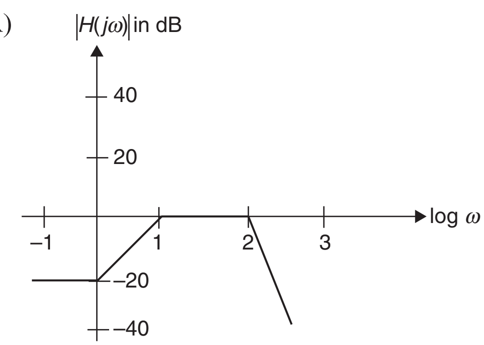
At $\omega = 20$ rad/sec, the slope changes to +20 dB/decade, indicating a term $\left(1 + \frac{s}{20}\right)$ or $(1 + 0.05\,s)$

At $\omega = 40$ rad/sec, the slope changes to 0 dB/dec indicating a term $\left(1 + \frac{s}{40}\right)$ in the denominator.

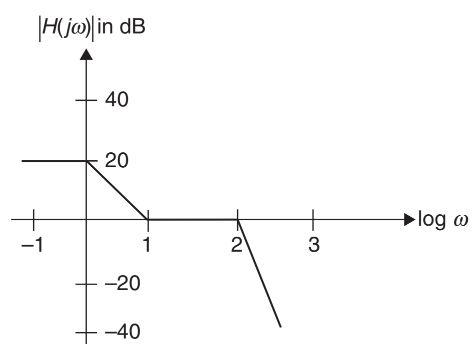
i.e., $$G(s) = \frac{0.35\,(1 + s)(1 + 0.05s)}{s(1 + 0.025\,s)}$$

**Example 28:** The Bode magnitude plot of $H(s) = \frac{10^4(1+s)}{(10+s)(100+s)^2}$

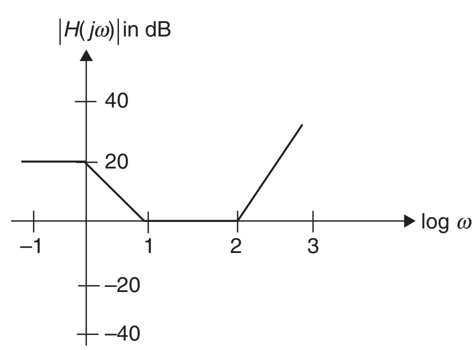
(A)



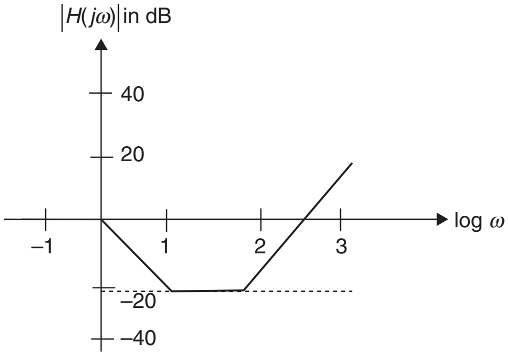

(B)




(C)



(D)




**Solution:** (A)

Given function $H(s) = \dfrac{10^4(1+s)}{(10+s)(10+s)^2}$

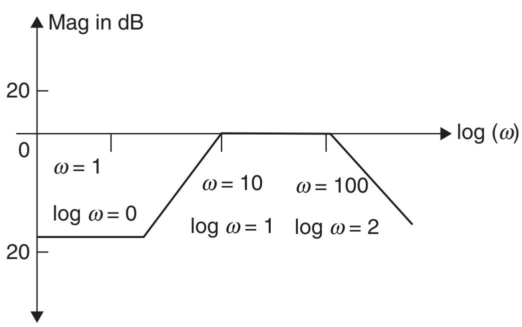
$$H(s) = \frac{10^4(1+s)}{(1+0.1s)(1+0.01s)^2 \times 10 \times 100^2}$$

$$H(s) = \frac{0.1(1+s)}{(1+0.1s)(1+0.01s)^2}$$

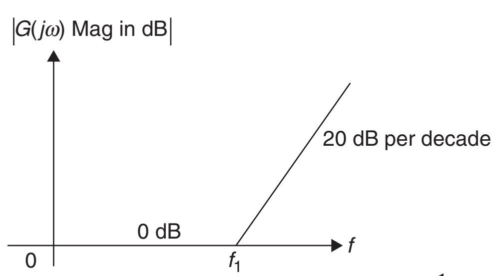
Corner frequencies are 1, 10 and 100

Initial magnitude = 20 log 0.1 = –20 dB

Magnitude starts increasing with slop of +20 dB/dec at $\omega_c$ = 1 rad, constant at $\omega$ = 10 rad and decays with a slope of 20 dB/dec at $\omega$ = 100 rad.




**Example 29:** The function corresponding to the Bode plot of figure is




(A) $G = jf/f_1$

(B) $G = \dfrac{1}{\left(1 + j\dfrac{f}{f_1}\right)}$

(C) $G = \dfrac{1}{\left(1 - j\dfrac{f}{f_1}\right)}$

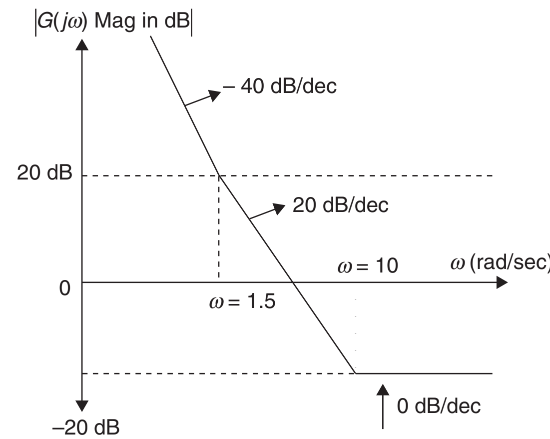
(D) $G = 1 + jf/f_1$

**Solution:** Magnitude plot slop change at frequency '$f_1$' and its increasing. This indicates there is a zero at $f = f_1$.

$$G = (1 + sT_1) = (1 + j\omega T_1)$$

$$= \left(1 + j\frac{2\pi f}{2\pi f_1}\right)\begin{bmatrix}\omega = 2\pi f \\ T_1 = \dfrac{1}{\omega_c} = \dfrac{1}{2\pi f_1}\end{bmatrix} \Rightarrow G = 1 + j\frac{f}{f_1}$$

**Example 30:** The asymptotic Bode magnitude plot of a minimum phase transfer function is shown in figure.




This transfer function has

(A) Two poles and one zero
(B) Two poles and two zeros
(C) One pole and two zeros
(D) Three poles and one zero

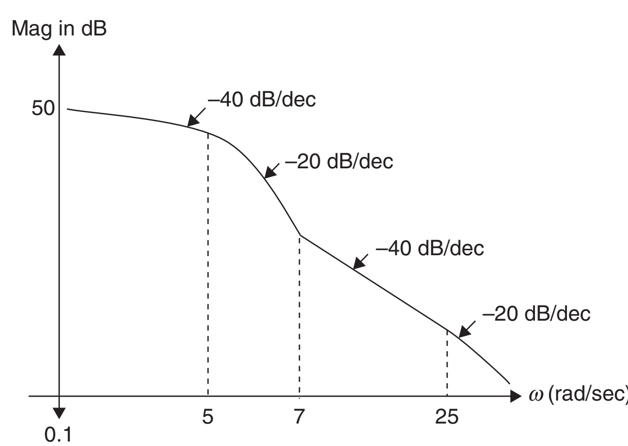
**Solution:** Initial slop of the magnitude plot is –40 dB/dec, indicates 2 poles of the system are at origin

Reduction is slop by 20 dB/dec at $\omega$ = 1.5 indicates a zero.

Reduction in slop by 20 dB/dec at $\omega$ =10 indicates another zero.

∴ Total 2 poles and 2 zeros.

**Example 31:** The asymptotic approximation of the log-magnitude versus frequency plot of a minimum phase system with real poles and one zero is shown in figure. Its transfer function is



(A) $\dfrac{10(s+5)(s+25)}{s^2(s+7)}$ (B) $\dfrac{4.4(s+7)(s+5)}{s^2(s+25)}$

(C) $\dfrac{10(s+7)}{s^2(s+5)(s+25)}$ (D) $\dfrac{4.4(s+5)(s+25)}{s^2(s+7)}$

**Solution:** Transfer function has corner frequencies 5, 7 and 25

Zeros are at $\omega = 5$ and $\omega = 25$

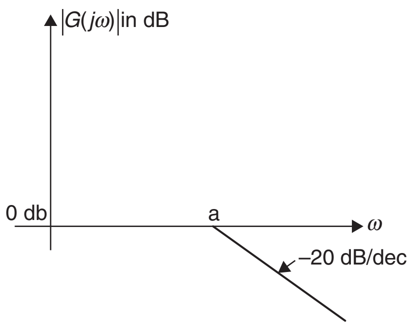
Poles are at $\omega = 7$

$$\text{T. F} = \frac{K\left(1+\frac{s}{5}\right)\left(1+\frac{s}{25}\right)}{s^2(1+s/7)} = \frac{7\times 5}{25}\,\frac{K(s+5)(s+25)}{s^2(s+7)}$$

Initial magnitude = 20 log $K$ = 10 $\Rightarrow K = 3.16$

$$\text{Transfer function} = \frac{4.4(s+5)(s+25)}{s^2(s+7)}$$

**Example 32:** The asymptotic Bode magnitude plot of a transfer function $\dfrac{1}{1+s/a}$ is show in figure. The error in dB gain at a frequency of $\omega = 0.5a$ is




(A) 3 dB
(B) 9 dB
(C) 0.97 dB
(D) 5.4 dB

**Solution:** Actual magnitude of given transfer function = $-20\log\left(1+\dfrac{j\omega}{a}\right)$

At $\omega = 0.5a$

$$\Rightarrow \quad |G(j\omega)| = -20\ \log\left(1+j\frac{0.5a}{a}\right)$$

$$= -20\log\sqrt{1+(0.5)^2}$$

$$|G(j\omega)| = -0.969 = -0.97\text{ dB}$$

Approximated magnitude in given plot at $\omega = 0.5a$ is 0 dB

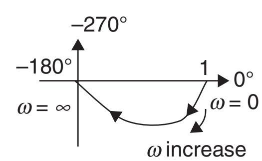
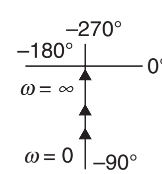
Error = 0 dB – (0.97 dB) = 0.97 dB.

## Polar Plot

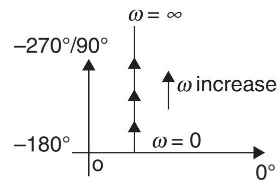
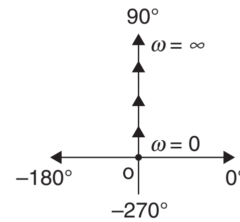
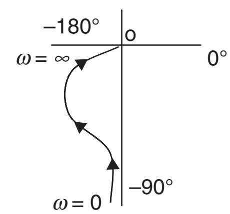
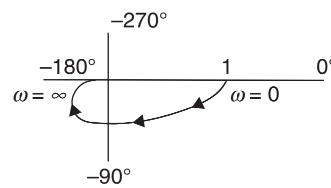
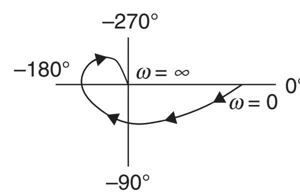
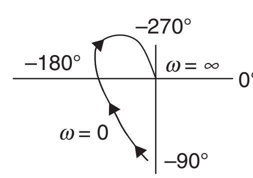
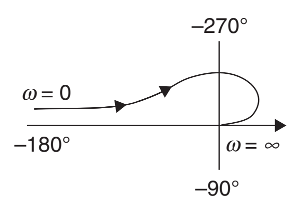
The transfer function $G(s)$ is a complex function and it is given by

$$G(s) = G(j\omega) = |G(j\omega)|\ \angle G(j\omega) = M\angle\phi$$

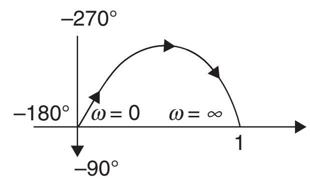
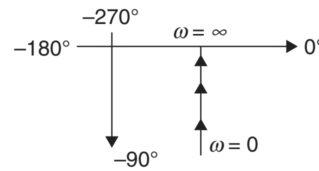
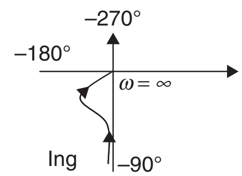
As the input frequency is varied from 0 to ∞, the magnitude $M$ and phase angle '$\phi$' change, the locus traced by the tip of the phasor $G(j\omega)$ is known as polar plot.

| **G(s)** | $\lvert G(j\omega)\rvert$ | $\angle G(j\omega)$ | $\lvert G(j\omega)\rvert\angle G(j\omega)$ value $\omega\to 0$ | $\omega\to\infty$ | **Polar plot** |
|---|---|---|---|---|---|
| $\dfrac{1}{1+sT}$ | $\dfrac{1}{\sqrt{1+\omega^2T^2}}$ | $-\tan^{-1}\omega T$ | $1\angle 0$ | $0\angle -90°$ |   |
| $\dfrac{1}{s}$ | $\dfrac{1}{\omega}$ | –90° | $\infty\angle -90°$ | $0\angle -90°$ |   |

(*Continued*)

| $G(s)$ | $\lvert G(j\omega)\rvert$ | $\angle G(j\omega)$ | $\lvert G(j\omega)\rvert \angle G(j\omega)$ value $\omega \to 0$ | $\omega \to \infty$ | Polar plot |
|---|---|---|---|---|---|
| $1 + sT$ | $1 + \omega T$ | 90° | $(1+0)\angle 90°$ | $(1+\infty)\angle 90°$ |  |
| $s$ | $\omega$ | 90° | $0 \angle 90°$ | $\infty \angle 90°$ |  |
| $\dfrac{1}{s(1+sT)}$ | $\dfrac{1}{\omega\sqrt{1+\omega^2T^2}}$ | $-90° - \tan^{-1}\omega T$ | $\infty \angle -90°$ | $0 \angle -180°$ |  |
| $\dfrac{1}{(1+sT_1)(1+sT_2)}$ | $\dfrac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$ | $-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$ | $1 \angle 0°$ | $0 \angle -180°$ |  |
| $\dfrac{1}{s(1+sT_1)(1+sT_2)}$ | $\dfrac{1}{\omega\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$ | $-90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$ | $\infty\angle -90°$ | $0\angle -270°$ |  |
| $\dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$ | $\dfrac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$ | $-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-}\omega T_3$ | $1\angle 0°$ | $0\angle -270°$ |  |
| $\dfrac{1}{s^2(1+sT_1)(1+sT_2)}$ | $\dfrac{1}{\omega^2\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$ | $-180° - \tan^{1}\omega T_1 - \tan^{-1}\omega T_2$ | $\infty\angle -180°$ | $0\angle -360°$ |  |

(*Continued*)

| $G(s)$ | $\lvert G(j\omega)\rvert$ | $\angle G(j\omega)$ | $\lvert G(j\omega)\rvert\angle G(j\omega)$ value $\omega \to 0$ | $\omega \to \infty$ | Polar plot |
|---|---|---|---|---|---|
| $\frac{s}{1+sT}$ | $\frac{\omega}{\sqrt{1+\omega^2T^2}}$ | $90° - \tan^{-1}\omega T$ | $0\angle 90°$ | $1\angle 0°$ |   |
| $\frac{1+sT}{s}$ | $\frac{\sqrt{1+\omega^2T^2}}{\omega}$ | $-90° + \tan^{-1}\omega T$ | $\infty\angle -90°$ | $1\angle 0°$ |   |
| $\frac{1+sT_1}{s(1+sT_2)(1+sT_3)}$ | $\frac{\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T^2}\sqrt{1+\omega^2T_3^2}}$ | $\tan^{-1}\omega T_1 - 90° - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$ | $\infty\angle -90°$ | $0\angle -180°$ |   |

**Note 1:** Addition of a non-zero pole to a transfer function results in further rotation of the polar plot through an angle of –90° as $\omega \to \infty$ (head of the polar plot shifts).

**Note 2:** Addition of a pole at origin to a transfer function results in rotation of the polar plot at zero and infinite frequency (head and tail of polar plot) by further angle of –90°.

**Note 3:** The effect of addition of a zero to a transfer function is to rotate the high frequency portion of the polar plot by 90° in anti-clockwise direction.

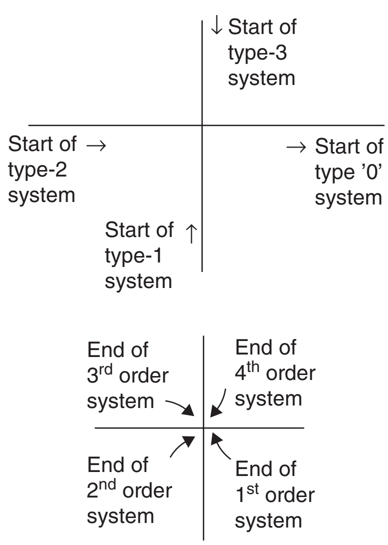



**Figure 4** Start point and end point of polar plot for different system types and orders

## NYQUIST CRITERION

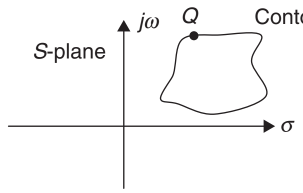
The Nyquist criterion relates the stability of a closed-loop-system to the open-loop frequency response and open-loop pole location. This criterion can tell us how many closed-loop poles are in the right half of $S$-plane.

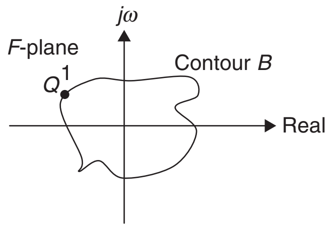
The Nyquist criterion used the following concepts for the establishment of criterion.

1. The poles of $1 + G(s)H(s)$ and the poles of $G(s)H(s)$ are same.
2. The zero of the $1 + G(s)H(s)$ are the poles of the closed loop transfer function $T(s)$ of the system.
3. Mapping: Consider a complex number on the $S$-plane and substitute it into a function $F(s)$, another complex number results. This process is called 'mapping'.
   **Example:** Substituting $s = 4 + j3$ into function $F(s) = s^2 + 2s + 1$, results in $16 + j30$. We say that $4 + j3$ maps into $16 + j30$ through the function $(s^2 + 2s + 1)$.
5. Mapping contours:
   Consider the collection of points, called a contour, shown in figure as contour A. Also assume that

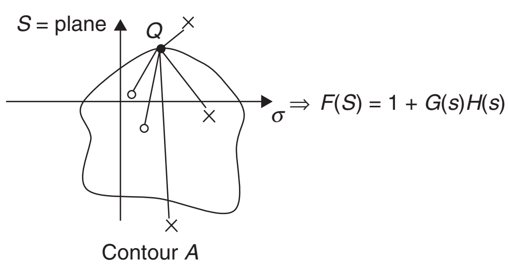
$$F(s) = \frac{(s-Z_1)(s-Z_2)(s-Z_3)\ldots}{(s-P_1)(s-P_2)(s-P_3)\ldots}$$

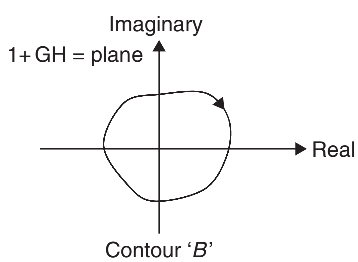
Contour 'A' can be mapped through $F(s)$ into contour 'B' by substituting each point of contour A into the function $F(s)$ and plotting the resulting complex numbers. For example, point 'Q' in $S$-plane maps into point 'Q' through the function $F(s)$.




Let us first assume that $F(s) = 1 + G(s)H(s)$, with the picture of the poles and zeros of $1 + G(s)H(s)$ as shown in figure below. As each point Q of the contour 'A' is substituted into $1 + G(s)H(s)$, a mapped point results on a contour 'B'. As we move round contour 'A' in a clockwise direction, each vector of $F(s)$ that lies inside contour A will appear to undergo a complete rotation or a change in angle of 360°. On the other hand, each vector drawn from the poles and zeros of $1 + G(s)H(s)$ that exist outside contour A will appear to oscillate and return to its previous position, undergoing a net angular change of 0°.




Number of anti-clockwise rotations of contour 'B' about origin $(N) = P - Z$

where, $P$ = Number of poles of $1 + G(s)H(s)$ inside contour A.
$Z$ = Number of zeros of $1 + G(s)H(s)$ inside contour A.

**Note:** Since the poles of $1 + G(s)H(s)$ are the poles of $G(s)H(s)$ and zeros of $1 + G(s)H(s)$ are poles of closed loop system,

$P$ = Number of open poles enclosed
$Z$ = Number of closed loop poles enclosed
$N = Z - P$ = Number of closed loop poles inside the contour.

If we extend the contour to include the entire right half of $S$-plane, we can count the number of right-half-plane closed loop poles inside contour 'A' and determine a system's stability.

**Note:** When we map the entire right half of $S$-plane through $G(s)H(s)$ instead of $1 + G(s)H(s)$, the resulting contour is same as mapping through $1 + G(s)H(s)$, except that it is translated one unit to the left. So we count rotations about $-1 + j0$ instead of rotations about the origin.

## Statement of the Nyquist Stability Criterion

If a contour 'A' that encircles the entire right half-plane is mapped through $G(s)H(s)$, then the number of the closed loop poles ($Z$) in the right half-plane equals the number of open loop poles ($P$) that are in the right half-plane minus the number of contour clockwise revolutions ($N$) around '–1' of the mapping (i.e., $Z = P - N$). The mapping is called the Nyquist diagram/Nyquist plot of $G(s)H(s)$.

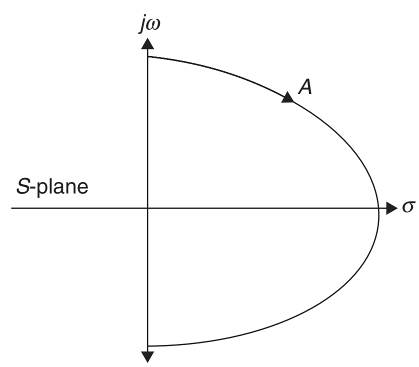



**Figure 5** Contour enclosing right half of $S$-plane of determine stability

**Note 1:** If the contour 'A' of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the $S$-plane encircles the point $(-1 + j0)$ in the anticlockwise direction as many times as the number of right half $S$-plane poles $G(s)H(s)$, the closed loop system is stable.

**Note 2:** No encirclement of $-1 + j0$ implies that the system is stable if there are no poles of $G(s)H(s)$ in the right half of '$s$' plane; otherwise the system is unstable.

**Note 3:** Clockwise encirclements in the Nyquist plane indicate that the system is unstable.

If $G(s)H(s)$ has any poles on $j\omega$ axis, the Nyquist contour defined earlier cannot be used as such. Also, the $S$-plane contour should not pass through a singularity of $1 + G(s)H(s)$. The stability in such cases is studied with modified Nyquist contour which bypasses any $j\omega$-axis poles. This is accomplished by indenting the Nyquist contour around the '$j\omega$' poles along a semicircle of radius '$\varepsilon$', where $\varepsilon \to 0$.

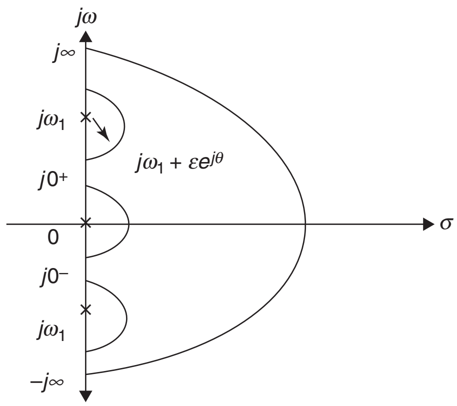



**Figure 6** Indented Nyquist Contour for $j\omega$-axis open loop poles

**Example 33:** Consider a system with an open loop transfer function

$$G(s)H(s)=\frac{(4s+1)}{s^2(s+1)(2s+1)}$$

Find the stability of the system using Nyquist plot.

**Solution:** The given open loop transfer function has a double pole at origin. The Nyquist contour is intended to bypass the origin. The mapping of Nyquist contour is obtained as follows.

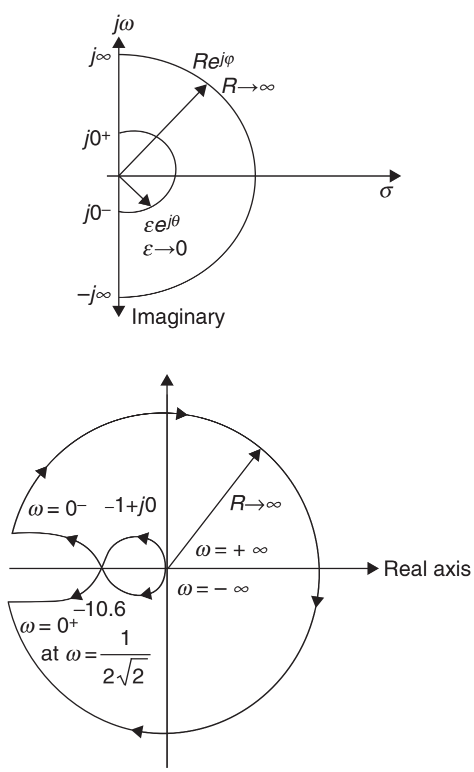



**Figure 7** Nyquist contour and corresponding Nyquist plot

1. Semicircular indent represented by $s=\lim_{\varepsilon\to 0}\varepsilon e^{j\theta}$ (where '$\theta$' varied from –90° through 0° to 90°) is mapped into

$$\lim_{\varepsilon\to 0}\left[\frac{4\varepsilon e^{j\theta}}{\varepsilon^2 e^{j2\theta}(\varepsilon e^{j\theta}+1)(2\varepsilon e^{j\theta}+1)}\right]=\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2 e^{j2\theta}}\right)=\infty e^{-j2\theta}$$

$$= \infty\,(\angle 180^\circ \to 0^\circ \to \angle -180^\circ).$$

This part of map is an infinite circle.

2. Mapping of positive imaginary axis

$$G(j\omega)H(j\omega)=\frac{(1+j4\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

For various values of '$\omega$', $G(j\omega)H(j\omega)$ is calculated and plotted using polar plots.

The $G(j\omega)H(j\omega)$ – locus intersects the real axis at a point where

$\angle G(j\omega)\,H(j\omega) = -180^\circ$

$-180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega + \tan^{-1}4\omega = -180^\circ$

$$\therefore\ \omega=\frac{1}{2\sqrt{2}}=0.354\text{ rad/sec}$$

$$\therefore \left|G(j\omega)H(j\omega)\right|_{\omega}=\frac{1}{2\sqrt{2}}=10.6$$

Further as $\omega \to +j\infty$

$$\Rightarrow \left|G(j\omega)H(j\omega)\right|\angle G(j\omega)H(j\omega)\Rightarrow 0\angle -270^\circ$$

as $\omega \to 0^+ \Rightarrow |G(j\omega)\,H(j\omega)| \Rightarrow \infty\angle -180^\circ$

3. The infinite semicircle of the Nyquist contour represented by $s=\lim_{R\to\infty} \mathrm{Re}^{j\phi}$ ($\phi$ varies from + 90° through 0° to + 90°) is mapped to

$$\lim_{R\to\infty}\frac{(1+4\,\mathrm{Re}^{j\phi})}{R^2 e^{j2\phi}(1+\mathrm{Re}^{j\phi})(1+2\,\mathrm{Re}^{j\phi})}=0e^{-j3\phi}$$

$= 0(\angle -270^\circ \to \angle 0^\circ \to \angle +270^\circ)$

Number of counter clockwise encirclements to origin are '–2'.

Number of right half poles of open loop is zero.

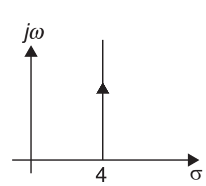
$$Z = P - N = 0 - (-2) = +2$$

∴ Number of poles on right half plane for closed loop transfer function is '2'.

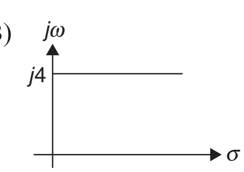
∴ System is unstable.

**Example 34:** Nyquist plot for the transfer function $G(s) = (4 + s)$ for positive frequencies has the form

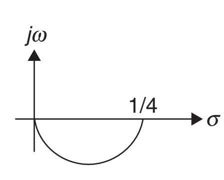
(A)



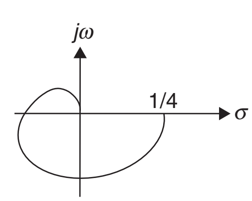

(B)




(C)




(D)



**Solution:** (A)
Given transfer function $G(s) = 4 + s = 4 + j\omega$

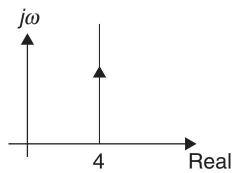
At $\quad \omega = 0 \Rightarrow 4 + j.0$

$$\omega = 10 \Rightarrow 4 + j10$$

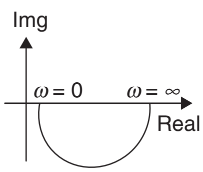
$$\omega = 100 \Rightarrow 4 + j100$$

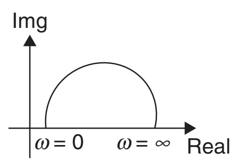
$$\omega = \infty \Rightarrow 4 + j\infty$$

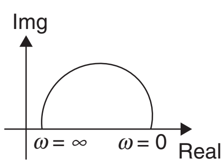
$\therefore$ Nyquist plot is parallel to imaginary axis

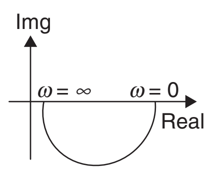



**Example 35:** Which one of the following polar diagram corresponds to a lag network?

(A)



(B)



(C)



(D)



**Solution:** (D)
Lag network offers only negative phase angles and. Let us consider a lag network example.

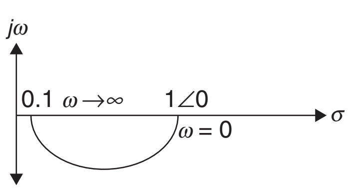
$$G(j\omega) = \frac{s+1}{10s+2}$$

$$|G(j\omega)| \sqrt{\frac{\omega^2+1}{100\omega^2+1}}; \angle G(j\omega) = \tan^{-1}\frac{\omega}{1}\tan^{-1} 10\,\omega.$$

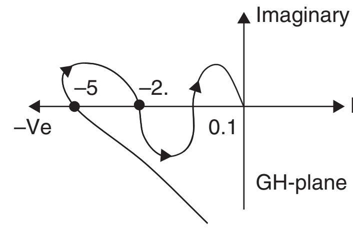
At $\quad \omega = 0 \Rightarrow 1\angle 0$

At $\quad \omega = 5 \Rightarrow 0.103\angle{-10.16}$

At $\quad \omega = \infty \Rightarrow 0.1\ \angle 0^{\circ}$




**Example 36:** The polar plot of a conditionally stable system for open loop gain $K = 1$ is shown in the figure. The open loop transfer function of the system is known to be stable. The closed loop system is stable for




(A) $0.5 < K < 10$ (B) $K < \dfrac{1}{5}$

(C) Both A and B (D) $0.5 > K > 10$

**Solution:** (C)
System gain '$K$' should be adjusted such that the point $(-1 + j0)$ lies in the 0.1 to 0.2 region, because no. of encirclements in this case is zero which results in stable operation of the system.

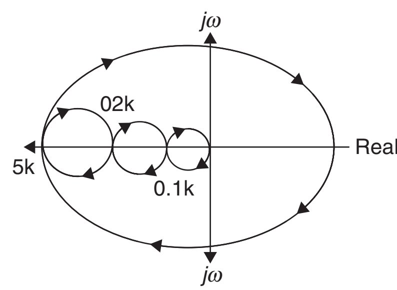
$$\therefore \quad 0.1k < 1 \Rightarrow K < 10$$

$$2K > 1 \Rightarrow K > 0.5$$

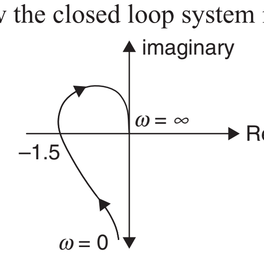
Range of $K$ is $0.5 < K < 10$

System is also stable if $5K < 1$ [no. of encirclements will be zero in this case]

$$K < \frac{1}{5}$$




**Example 37:** The polar plot of an open loop stable system as shown below the closed loop system is




(A) Marginally stable
(B) Always stable
(C) Unstable with one pole on the RH $S$-plane
(D) Unstable with two poles on the RH $S$-plane

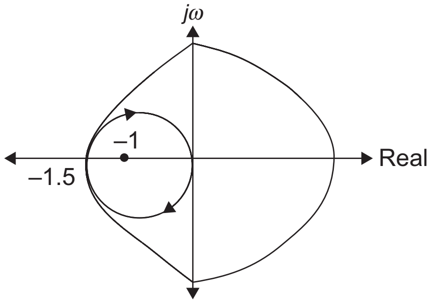
**Solution:** (D)
Complete polar plot of the given system is given in the figure No. of encirclements of $(-1 + j.0)$ are '–2'

$\therefore$ No. of open loop poles the right hand side = 0

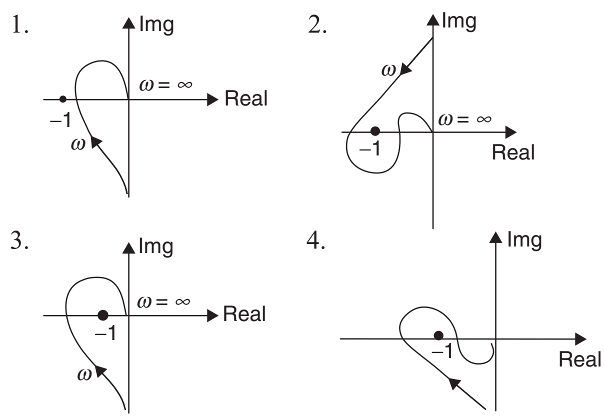
$$N = P - Z \Rightarrow Z = P - N$$

$$Z = 0 - (-2) = 2$$

∴ RHS-plane poles of closed loop system are '2' ⇒ Unstable

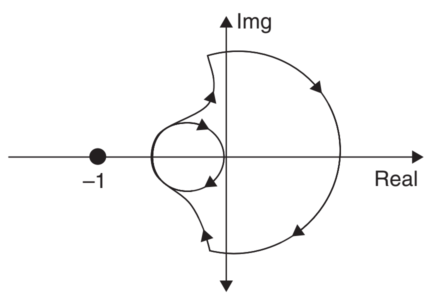



**Example 38:** Consider the following Nyquist plots of loop transfer function over positive frequencies. Which of the plots represents an unstable system?




(A) (1), (2) and (3)
(B) (2), (3) and (4)
(C) (1), (2) and (4)
(D) (3) and (4) only

**Solution:** Plot (1) Nyquist plot is

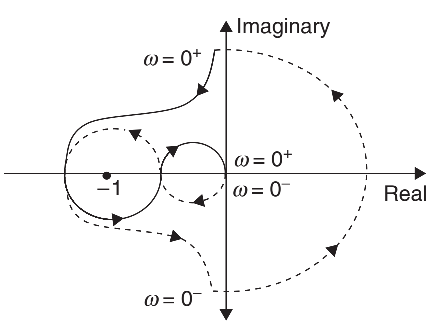



No. of encirclements ($N$) = 0

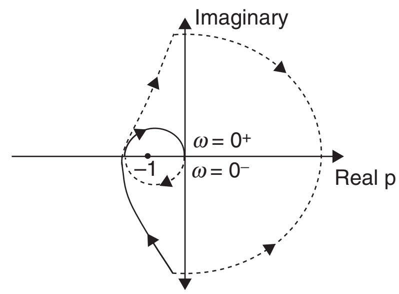
If the open loop poles on RHS = 0

∴ System is stable

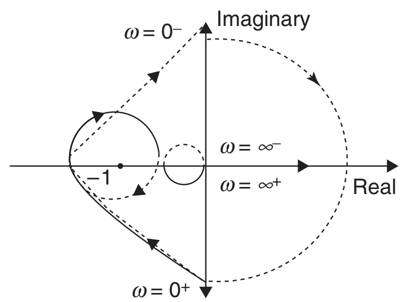
Plot (2) complete Nyquist plot is




No. of anti-clockwise encirclements ($N$) = 2.
No. of anti-clockwise encirclements ($N$) = $P - Z = 2$
No. of open loop poles on RHS side ($p$) = 0
No. of poles of closed loop system = 2
∴ System is unstable.
Plot (3) complete Nyquist plot is




No. of counter clockwise encirclements ($N$) = –2
No. of poles of closed loop system on RHS = 2
∴ Unstable system
Option (4) complete Nyquist plot is




**Solution:** (B)
No. of counter clockwise encirclements ($N$) = – 2

No. of RHS poles of closed loop control system ($Z$) = $P - N = 2$

∴ System is unstable.

**Example 39:** A unity feedback system has the open loop transfer function

$$G(s) = \frac{1}{(s-1)(s+2)(s+3)}$$

The Nyquist plot of 'G' encircles the origin
(A) Once
(B) Twice
(C) Thrice
(D) Never

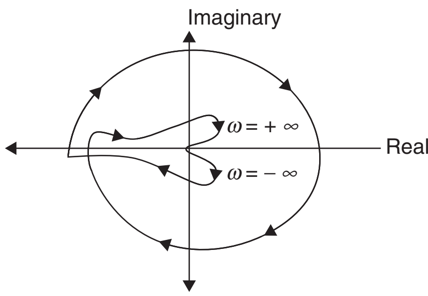
**Solution:** (A)
No. of encirclements equals the difference between no. of right hand side poles of $G(s)$ and zeros.

$$N = P_{\text{OLTF}} - Z_{\text{OLTF}}$$

$$P_{\text{OLTF}} = 1 \text{ and } Z_{\text{OLTF}} = 0$$

$$N = 1.$$

**Example 40:** Which of the following is the transfer function of a system having the Nyquist plot shown in the following figure?




(A) $\frac{K}{s^2(s+4)(s+8)}$ (B) $\frac{K}{s^2(s+4)^2(s+5)}$

(C) $\frac{K(s+2)}{s^2(s+4)(s+8)}$ (D) $\frac{K(s+2)(s+2)}{s^2(s+4)(s+5)}$

**Solution: (A)**

Nyquist plot started at –180° angle. It indicates that the open loop system has two poles at origin.

Magnitude and phase angle at $\omega \to 0$

$$\Rightarrow \infty\angle -180^\circ$$

Magnitude and phase angle at $\omega \to \infty$

$$\Rightarrow 0\angle -360^\circ$$

Angle at the termination of NP is – 360°

Angle of termination –360° indicated system order is '4'.

∴ System is type 2 and order '4' system with no zeros.

**Example 41:** In the GH-plane, the Nyquist plot of the loop transfer function $G(s)H(s) = \frac{2\pi e^{-5s}}{s}$ pass through the negative real axis at the, point

(A) (–5, $j$0) (B) (–2, $j$0)
(C) (–10, $j$0) (D) (–20, $j$0)

**Solution:** (D)

At the point of intersection of Nyquist plot with real axis phase angle $\angle G(s)\,H(s) = -180^\circ = -\pi$

$$\angle\frac{2\pi e^{-5j\omega}}{j\omega} = -\pi$$

$$-5\omega - \tan^{-1}\frac{\omega}{0} = -\pi$$

$$-5\omega - \frac{\pi}{2} = -\pi$$

$$5\omega = \frac{\pi}{2} \Rightarrow \omega = \frac{\pi}{10}$$

$$\left|G(s)H(s)\right|_{\omega=\frac{x}{10}} = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{10}} = 20$$

∴ Nyquist plot passes through (–20, $j$0)

## Exercises

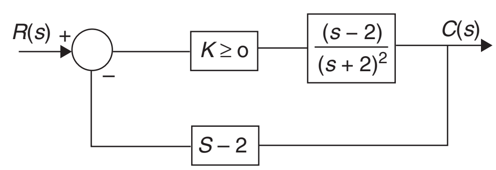
### Practice Problems 1

***Directions for questions 1 to 22***: Select the correct alternative from the given choices.

1. The characteristic equation of a system is given by $s^6 + 3s^5 + 8s^4 + 18s^3 + 37s^2 + 75s + 50 = 0$; the system is
   (A) Stable. (B) Unstable.
   (C) Marginally stable. (D) Conditionally stable.
2. How many roots of the characteristic equation

   $$s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$$

   lie in the left half of $S$-plane?
   (A) 4 (B) 5 (C) 1 (D) 6
3. A system described by the transfer function $H(s) = \frac{1}{s^3 + \alpha s^2 + ks + 2}$ is stable. The constraints on $\alpha$ and $k$ are
   (A) $\alpha > 0$, $\alpha k > 2$ (B) $\alpha > 0$, $\alpha k < 2$
   (C) $\alpha > 0$, $\alpha k > 0$ (D) $\alpha < 0$, $\alpha k < 0$
4. The characteristic equation of a system is given by $s(s^2 + 2s + 2) + K(s + 3) = 0$. The range of $k$ for which the system is stable is
   (A) $0 < k < 30$. (B) $K > 3$.
   (C) $0 < k < 4$. (D) $3 < K < 29$.
5. The feedback control system is fig is stable




   (A) for all $K \geq 0$ (B) only if $K \geq 1$
   (C) only if $0 \leq k < 1$ (D) only if $0 \leq k \leq 1$
6. Consider the points $S_1 = -3 + j4$ and $S_2 = -3-j2$ in the $S$–plane. Then for a system with the open-loop transfer function $G(s)\,H(s) = \frac{k}{(s+1)^4}$ is
   (A) $S_1$ is on the root locus, but not $S_2$.
   (B) Both $S_1$ and $S_2$ are on the root locus.
   (C) $S_2$ is on the root locus, but not $S_1$.
   (D) Neither $S_1$ nor $S_2$ on the root locus.
7. The gain margin (in dB) of a system having the open loop transfer function.

   $$G(s)\,H(s) = \frac{\sqrt{2}}{s(s+1)} \text{ is}$$

   (A) 0 (B) 3.01 (C) –3.01 (D) ∞

**8.** The characteristic equation of a feedback control system is given by $s^3 + 5s^2 + (K + 6)s + K = 0$ In the root loci diagram the asymptotes of the root loci for large '$K$' meet at a point in the $s$-plane whose coordinates are

(A) (2, 0) (B) (–1, 0)
(C) (–2, 0) (D) (–3, 0)

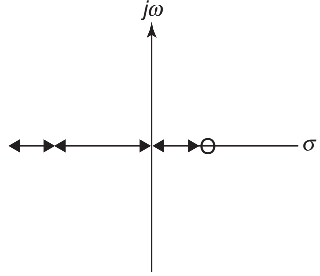
**9.** The open-loop transfer function of a system is given by $G(s) = \dfrac{k}{s(s+1)(s+2)}$ the value of $k$ which will cause sustained oscillations in the closed-loop unity feed book system is

(A) 4 (B) 6 (C) 5 (D) 3

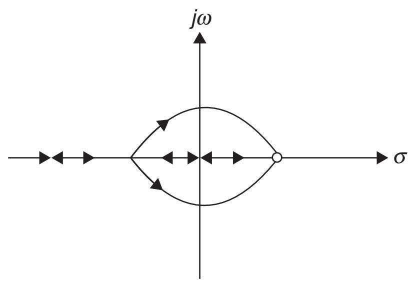
**10.** A unity feedback system is given as

$G(s) = \dfrac{k(1-s)}{s(s+3)}$ Indicate the correct root Locus diagram.

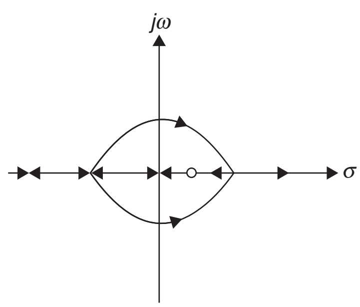
(A)



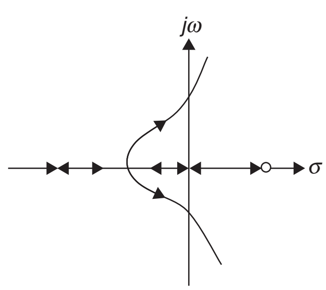

(B)




(C)

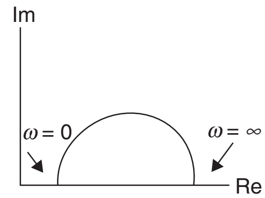



(D)

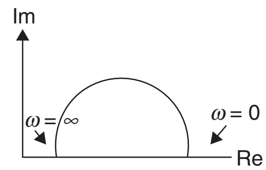



**11.** Which one of the following polar diagram corresponds to a lag network?

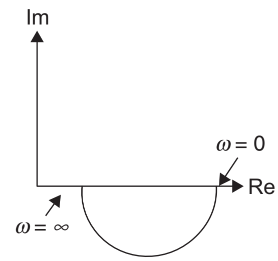
(A)



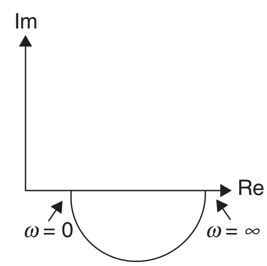

(B)




(C)




(D)




**Common Data for Questions 12 and 13:**

The open-loop transfer function of a unity feedback system is given by $G(S) = \dfrac{3e^{-2s}}{s(s+2)}$

**12.** The gain and phase cross – over frequencies in rad/s, respectively,

(A) 0.485 and 0.632. (B) 1.26 and 0.632.
(C) 0.632 and 1.26. (D) 0.632 and 0.485.

**13.** Based on the above results, the gain and phase margins of the system will be

(A) –7.09 dB and 87.5°.
(B) 7.09 dB and 87.5°.
(C) 7.09 dB and –87.5°.
(D) –7.09 dB and –87.5°.

**14.** The loop transfer function of a closed-loop control system is given as

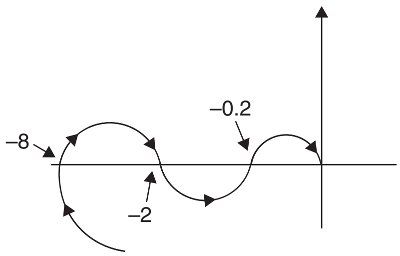
$G(S)H(S) = \dfrac{k(s+1)}{s(s+2)(s+3)}$. The centroid of the asymptotes is

(A) (–4, 0) (B) (–1, 0)
(C) (–2, 0) (D) (–3, 0)

**15.** A system has 10 poles and 2 zeroes. The slope of its highest frequency asymptote in its magnitude plot is

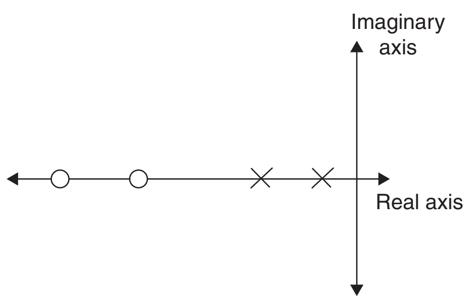
(A) –100 dB/dec (B) –120 dB/dec
(C) –160 dB/dec (D) –240 dB/dec

**16.** The polar diagram of a conditionally stable system for open loop gain $K=1$ is shown in figure. The open loop transfer function of the system is known to be stable. The closed loop system is stable for



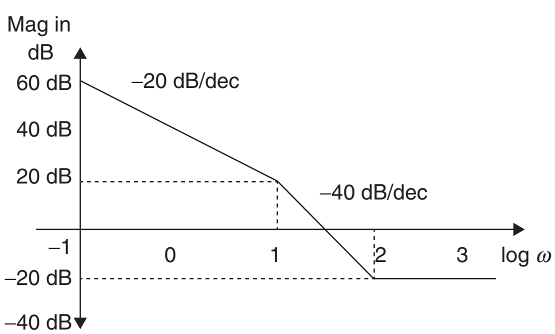

(A) $K<5$ and $\frac{1}{2}<K<\frac{1}{8}$ (B) $K<\frac{1}{8}$ and $\frac{1}{2}<K<5$
(C) $K<\frac{1}{8}$ and $5<K$ (D) $K>\frac{1}{8}$ and $K<5$

**17.** Pole zero plot of a loop transfer function is shown in figure below, the breakaway/ breakin points in the root locus diagram is




(A) 1 (B) 2 (C) 0 (D) 3

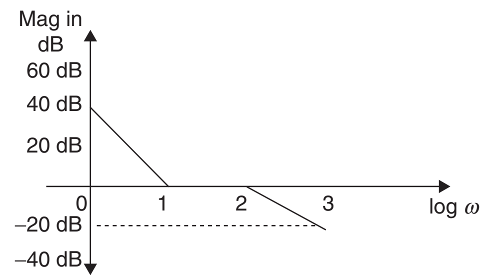
**18.** Loop transfer function $G(s)H(s)$ of the magnitude plot shown in the figure




(A) $\frac{100(S+2)^2}{S(S+1)}$ (B) $\frac{10(.01s+1)}{s(.1s+1)}$

(C) $\frac{100(.01S+1)^2}{S(.1S+1)}$ (D) $\frac{(.01S+1)^2}{S(.1S+1)}$

**19.** Loop transfer function $G(s)H(s)$ of the magnitude plot shown in the figure



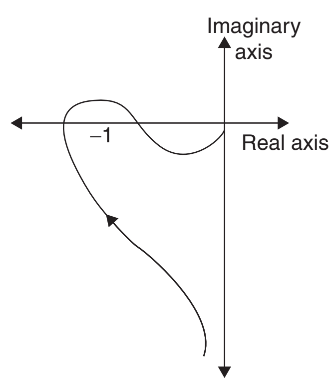

(A) $\frac{100(0.1s+1)^2}{s^2(0.01s+1)}$ (B) $\frac{100(s+10)^2}{(s+100)(s+1)^2}$

(C) Both A and B (D) None of the above

**20.** closed loop control system with transfer function $\frac{G(s)}{1+G(s)H(s)}$ is stable when

(A) Poles of the transfer function are on the left hand side of the $S$-plane.
(B) Zeros of the characteristic equation are on the left half of the $S$-plane.
(C) Poles of the characteristic equation are on the left half of the $S$-plane.
(D) Both A and B.

**21.** Polar plot of an open-loop stable system is shown in the figure. The system is




(A) System is stable.
(B) System is unstable with one pole on the right-hand side of $S$-plane.
(C) System is unstable with two poles on the right-hand side of $S$-plane.
(D) System is marginally stable.

**22.** Which of the flowing are effects of PD controller on system?

1. Reduces peak overshoot.
2. Reduces raise time.
3. Improves damping.
4. Reduces steady-state error.

(A) 1, 2, 3 (B) 2, 3, 4
(C) 2, 3, 4 (D) 1, 3, 4

## Practice Problems 2

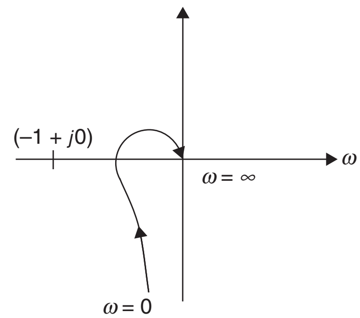
***Directions for questions 1 to 15:*** Select the correct alternative from the given choices.

**1.** Which of the following statements are 'true'?
   (i) Root Locus is a frequency response plot.
   (ii) The roots of characteristic equation are not a function of open Loop gain *K*.
   (iii) Root Locus technique is a tool for adjusting the location of closed loop poles to achieve the desired system performance.
   (iv) The exact root– locus is sketched by trial-and-error procedure.
   (A) i and ii
   (B) ii and iii
   (C) iii and iv
   (D) ii, iii, and iv

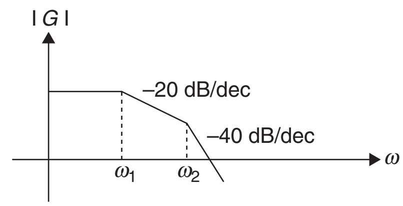
**2.** The following statements refer to the equation $P(s) + KQ(s) = 0$, where $P(s)$ and $Q(s)$ are polynomials of $s$ with constant coefficients. Identify the statements which are 'true'.
   (i) The intersection of the asymptotes must always be on the real axis.
   (ii) The breakaway points of the root loci must always be on the real axis.
   (iii) Given the equation $1 + KG_1(s)H_1(s) = 0$, where $G_1(s)H_1(s)$ is a rational function of $s$ and does not contain $K$, the roots of $\frac{dG_1(s)H_1(s)}{ds}$ are all break away points on the root loci $(-\infty < K < \infty)$
   (iv) At the break away points on the root loci the root sensitivity is infinite.
   (A) i and iv
   (B) i, ii, and iv
   (C) ii and iii
   (D) ii, iii and iv

**3.** Which of the following statements are true?
   (i) Adding a zero to the function $G(s)H(s)$ tends to push the root loci to the left.
   (ii) Adding a zero to the forward-path transfer function will generally improve the system damping, and thus always reduce the maximum over shoot of the system.
   (iii) Adding a pole to $G(s)H(s)$ has the effect of pushing the root loci to the right.
   (iv) Complementary root locus (CRL) refers to root loci with positive '*k*'.
   (A) i, ii, and iii
   (B) i, ii and iv
   (C) ii, iii and iv
   (D) ii and iv

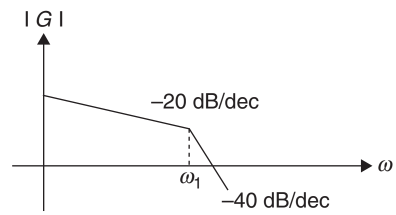
**4.** The Nyquist plot for a control system is shown in figure. The bode plot for the same system will be



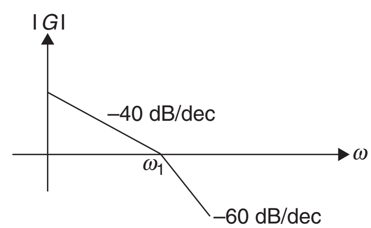

(A)



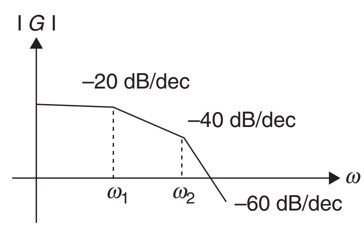

(B)



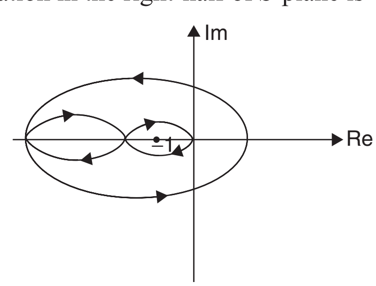

(C)




(D)




**5.** The Nyquist plot for the open- loop transfer function $G(s)$ of a unity negative feedback system is shown in the figure, if $G(s)$ has no pole in the right half of $S$-plane, the number of roots of the system characteristic equation in the right-half of $S$-plane is



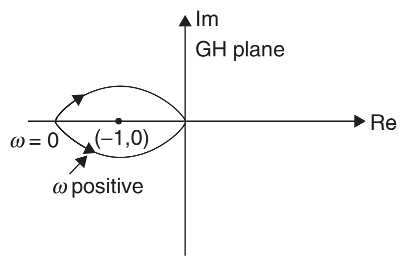

(A) 0 (B) 1 (C) 2 (D) 3

**6.** Which of the following points is NOT on the root locus of a system with the open-loop transfer function?

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)}$$

(A) $s = -j\sqrt{3}$ (B) $s = -1.5$
(C) $s = -3$ (D) $s = -\infty$

**7.** The figure shows the Nyquist plot of the open-loop transfer function $G(s)H(s)$ of a system; If $G(s)H(s)$ has one right-hand pole, the closed loop system is

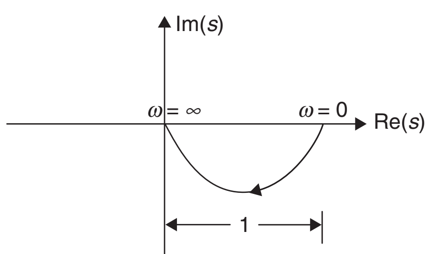



(A) Always stable
(B) Unstable with one closed loop right-hand pole
(C) Unstable with two closed loop right-hand poles
(D) Unstable with three closed-loop right-hand poles

**8.** Given $G(s)H(s) = \frac{K}{s(s+1)(s+3)}$, the point of intersection of the asymptotes of the root loci with the real axis is

(A) –4 (B) 1.33 (C) –1.33 (D) 4

**9.** The polar plot shown in the figure represents the transfer function:




(A) $G(s) = \frac{1}{s}$

(B) $G(s) = \frac{1}{s(1+sT)}$

(C) $G(s) = \frac{1}{1+sT}$

(D) $G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$

**10.** The open loop transfer function of a unity gain feedback control system is given by

$$G(s) = \frac{K}{(s+1)(s+3)}$$

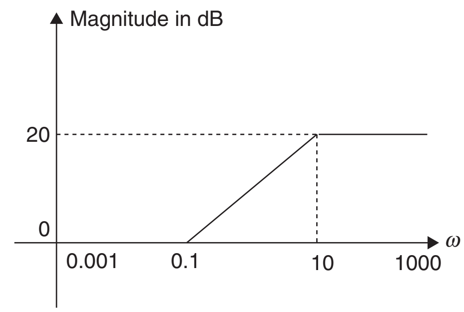
The gain margin of the system is dB is given by

(A) ∞ (B) 1 (C) 20 (D) 0

**11.** If the closed loop transfer function of a control system is given by $T(s) = \frac{s-5}{(s+2)(s+3)}$, then it is

(A) An unstable system
(B) An uncontrollable system
(C) A minimum-phase system
(D) A non-minimum phase system

**12.** For the asymptotic Bode magnitude plot shown in the following figure, the system transfer function can be



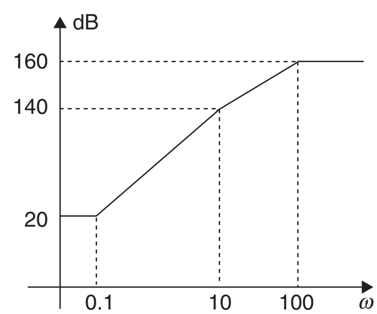

(A) $\frac{10s+1}{0.1s+1}$ (B) $\frac{100s+1}{0.1s+1}$

(C) $\frac{100s}{10s+1}$ (D) $\frac{0.1s+1}{10s+1}$

**13.** The root locus of the system $G(s)H(s) = \frac{K}{s(s+2)(s+3)}$ has the break-away point located at

(A) (–0.5, 0) (B) (–2.548, 0)
(C) (–4, 0) (D) (–0.784, 0)

**14.**




The approximate Bode-Magnitude plot of a minimum-phase system is shown in the figure. The transfer function of the system is

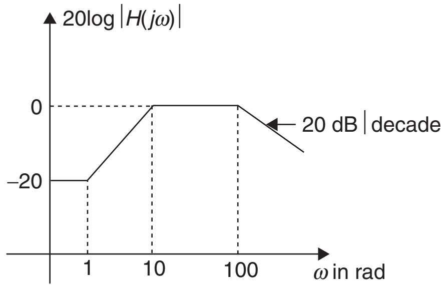
(A) $10^8 \frac{(s+0.1)^3}{(s+10)^2(s+100)}$

(B) $10^7 \frac{(s+0.1)^3}{(s+10)(s+100)}$

(C) $10^8 \frac{(s+0.1)^2}{(s+10)^2(s+100)}$

(D) $10^9 \frac{(s+0.1)^3}{(s+10)(s+100)^2}$

**15.** Consider the Bode magnitude plot shown in the figure. The transfer function $H(s)$ is




(A) $\dfrac{(s+10)}{(s+1)(s+100)}$

(B) $\dfrac{10(s+1)}{(s+10)(s+100)}$

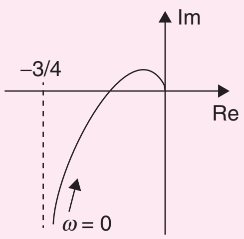
(C) $\dfrac{10^2(s+1)}{(s+10)(s+100)}$

(D) $\dfrac{10^3(s+100)}{(s+1)(s+10)}$

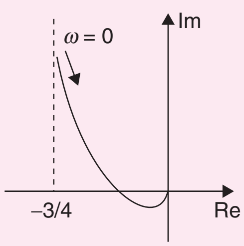
## Previous Years' Questions

**1.** The frequency response of $G(s) = 1/[s\,(s + 1)\,(s + 2)]$ plotted in the complex $G(j\omega)$ plane (for $0 < \omega < \infty$) is **[2010]**

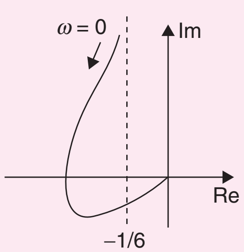
(A)



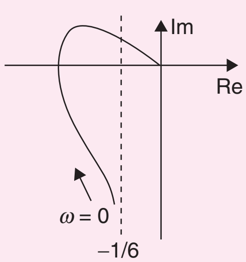

(B)




(C)




(D)




**2.** The frequency response of a linear system $G(j\omega)$ is provided in the tabular form below

| $\lvert G(j\omega)\rvert$ | 1.3 | 1.2 | 1.0 | 0.8 | 0.5 | 0.3 |
|---|---|---|---|---|---|---|
| $\angle G(j\omega)$ | –130° | –140° | –150° | –160° | –180° | –200° |

The gain margin and phase margin of the system are **[2011]**

(A) 6 dB and 30° (B) 6 dB and –30°
(C) –6 dB and 30° (D) –6 dB and –30°

**3.** The open loop transfer function $G(s)$ of a unity feedback control system is given as, $G(s) = \dfrac{k\left(s+\frac{2}{3}\right)}{s^2(s+2)}$

From the root locus, it can be inferred that when $k$ tends to positive infinity. **[2011]**

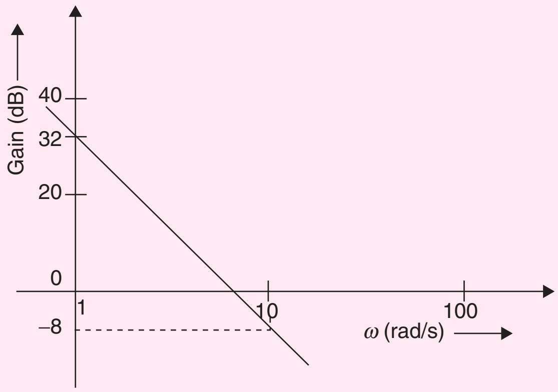
(A) Three roots with nearly equal real parts exist on the left half of the $S$-plane.

(B) One real root is found on the right half of the $S$-plane.

(C) The root loci cross the $j\omega$ axis for a finite value of $k$; $k \neq 0$.

(D) Three real roots are found on the right half of the $S$-plane.

**4.** The Bode plot of a transfer function $G(s)$ is shown in the below figure. The gain (20 log|$G(s)$|) is 32 dB and –8 dB at 1 rad/s and 10 rad/s, respectively. The phase is negative for all $\omega$. The $G(s)$ is **[2013]**

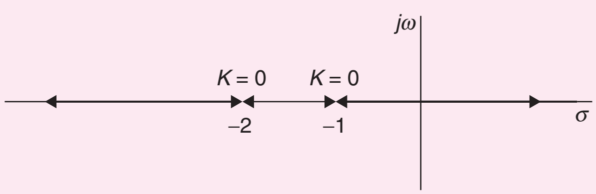



(A) $\frac{39.8}{s}$ (B) $\frac{39.8}{s^2}$ (C) $\frac{32}{s}$ (D) $\frac{32}{s^2}$

**5.** In the formation of Routh–Hurwitz array for a polynomial all the elements of a row have zero values. This premature termination of the array indicates the presence of **[2014]**

(A) Only one root at the origin
(B) Imaginary roots
(C) Only positive real roots
(D) Only negative real roots

**6.** The root locus of a unity feedback system is shown in the figure **[2014]**




The closed loop transfer function of the system is

(A) $\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+2)}$

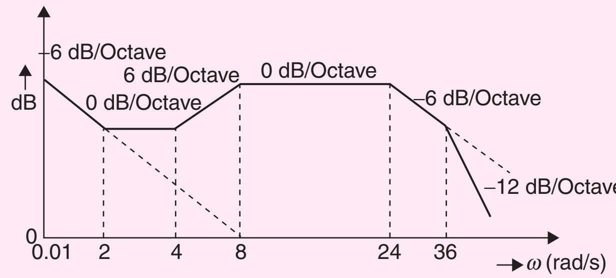
(B) $\frac{C(s)}{R(s)} = \frac{-K}{(s+1)(s+2)+K}$

(C) $\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+2)-K}$

(D) $\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+2)+K}$

**7.** For the given system, it is desired that the system be stable. The minimum value of $\alpha$ for this condition is ________. **[2014]**

**8.** The Bode magnitude plot of the transfer function $G(s) = \frac{K(1+0.5s)(1+as)}{s\left(1+\frac{s}{8}\right)(1+bs)\left(1+\frac{s}{36}\right)}$ is shown below: Note that-6 dB/octave = –20 dB/decade. The value of $\frac{a}{bK}$ is ________. **[2014]**




**9.** A system with the open loop transfer function $G(s) = \frac{K}{s(s+2)(s^2+2s+2)}$ is connected in a negative feedback configuration with a feedback again of unity. For the closed loop system to be marginally stable, the value of $K$ is _____ **[2014]**

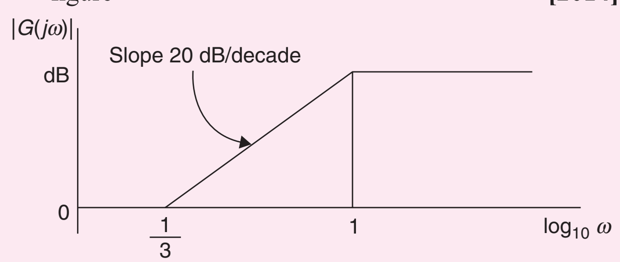
**10.** For the transfer function

$$G(s) = \frac{5(s+4)}{s(s+0.25)(s^2+4s+25)}$$

The values of the constant gain term and the highest corner frequency of the Bode plot, respectively, are **[2014]**

(A) 3.2, 5.0 (B) 16.0, 4.0
(C) 3.2, 4.0 (D) 16.0, 5.0

**11.** The magnitude Bode plot of a network is shown in the figure **[2014]**

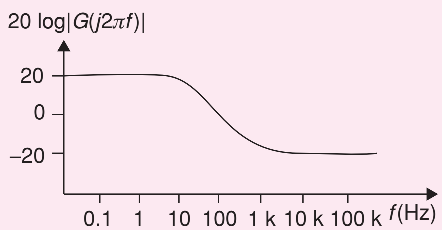



The maximum phase angle $\Phi_m$ and the corresponding gain $G_m$, respectively, are

(A) −30° and 1.73 dB
(B) −30° and 4.77 dB

(C) +30° and 4.77 dB
(D) +30° and 1.73 dB

**12.** A Bode magnitude plot for the transfer function $G(s)$ of a plant is shown in the figure. Which one of the following transfer functions best describes the plant? **[2015]**




(A) $\dfrac{1000(s+10)}{s+1000}$

(B) $\dfrac{10(s+10)}{s(s+1000)}$

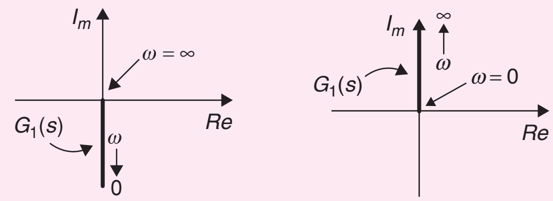
(C) $\dfrac{s+1000}{10s(s+10)}$

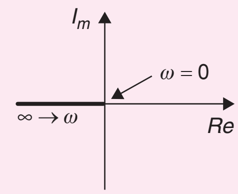
(D) $\dfrac{s+1000}{10(s+10)}$

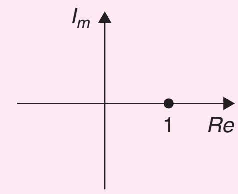
**13.** The transfer function of a second order real system with a perfectly flat magnitude response of unity has a pole at $(2-j3)$. List all the poles and zeroes. **[2015]**
(A) Poles at $(2 \pm j3)$, no zeroes.
(B) Poles at $(\pm 2 - j3)$, one zero at origin.
(C) Poles at $(2-j3)$, $(-2+j3)$, zeroes at $(-2-j3)$, $(2+j3)$
(D) Poles at $(2 \pm j3)$, zeroes at $(-2 \pm j3)$

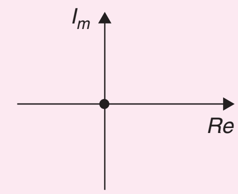
**14.** The open loop poles of a third order unity feedback system are at 0, –1, –2. Let the frequency corresponding to the point where the root locus of the system transits to unstable region be K. Now suppose we introduce a zero in the open loop transfer function at –3, while keeping all the earlier open loop poles intact. Which one of the following is TRUE about the point where the root locus of the modified system transits to unstable region? **[2015]**
(A) It corresponds to a frequency greater than $K$
(B) It corresponds to a frequency less than $K$
(C) It corresponds to a frequency $K$
(D) Root locus of modified system never transits to unstable region.

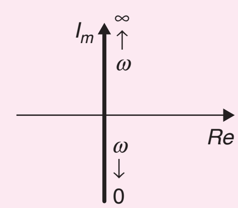
**15.** Nyquist plots of two functions $G_1(s)$ and $G_2(s)$ are shown in figure. **[2015]**




(a)



(b)



(c)



(d)



**16.** An open loop transfer function $G(s)$ of a system is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

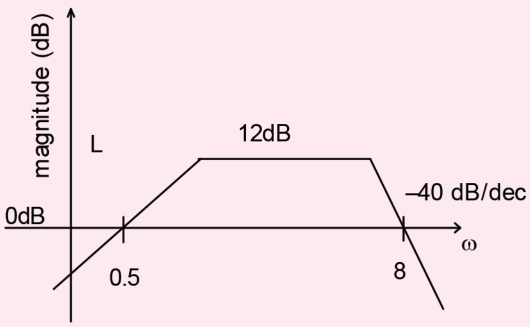
For a unity feedback system, the breakaway point of the root loci on the real axis occurs at, **[2015]**
(A) –0.42
(B) –1.58
(C) –0.42 and –1.58
(D) none of the above

**17.** The transfer function of a system is $\dfrac{Y(s)}{R(s)} = \dfrac{s}{S+2}$ the steady state output $y(t)$ is $A\cos(2t+\phi)$ for the input $\cos(2t)$the values of A and $\phi$ respectively are **[2016]**

(A) $\frac{1}{\sqrt{2}}$ ,–45°
(B) $\frac{1}{\sqrt{2}}$ ,+ 45°
(C) $\sqrt{2}$ ,–45°
(D) $\sqrt{2}$ ,+45°

**18.** The phase cross-over frequency of the transfer function $G(S) = \frac{100}{(S+1)^3}$ in rad/s is **[2016]**

**19.** Consider the following asymptotic Bode magnitude plot (ω is in rad/s).




Which one of the following transfer functions is best represented by the above Bode magnitude plot? **[2016]**

(A) $\frac{2S}{(1+0.5S)(1+0.25S)^2}$

(B) $\frac{4(1+0.5S)}{S(1+0.25S)}$

(C) $\frac{2S}{(1+2S)(1+4S)}$

(D) $\frac{4S}{(1+2S)(1+4S)^2}$

**20.** Loop transfer function of a feedback system is $G(s) H(s) = \frac{s+3}{s^3(s-3)}$ Take the Nyquist contour in the clockwise direction. Then, the Nyquist plot of $G(s) H(s)$ encircles $-1 + j\,0$. **[2016]**

(A) once in clockwise direction
(B) twice in clockwise direction
(C) once in anticlockwise direction
(D) twice in anticlockwise direction

**21.** Given the following polynomial equation $S^3 + 5.5S^2 + 8.5^S + 3 = 0$, the number of roots of the polynomial, which have real parts strictly less than –1 is ______. **[2016]**

**22.** Consider a linear time - invariant system with transfer function

$$H(S) = \frac{1}{(S+1)}$$

If the input is Cos(t) and the steady state output is $A\text{Cos}(t + \alpha)$, then the value of A is _________. **[2016]**

**23.** The open loop transfer function of a unity feedback control system is given by

$$G(S) = \frac{K(S+1)}{S(1+Ts)(1+2S)}, K > 0, T > 0$$

The closed loop system will be stable if, **[2016]**

(A) $0 < T < \frac{4(K+1)}{K-1}$

(B) $0 < K < \frac{4(T+2)}{T-2}$

(C) $0 < K < \frac{T+2}{T-2}$

(D) $0 < T < \frac{8(K+1)}{K-1}$

**24.** The gain at the breakaway point of the root locus of a unity feedback system with open loop transfer function $G(S) = \frac{KS}{(S-1)(S-4)}$ is **[2016]**

(A) 1
(B) 2
(C) 5
(D) 9

## Answer Keys

# Exercises

## Practice Problems 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** B | **2.** B | **3.** A | **4.** C | **5.** C | **6.** C | **7.** D | **8.** C | **9.** B | **10.** C |
| **11.** D | **12.** B | **13.** D | **14.** C | **15.** C | **16.** B | **17.** B | **18.** C | **19.** A | **20.** C |
| **21.** A | | | | | | | | | |

## Practice Problems 2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** C | **2.** A | **3.** A | **4.** D | **5.** A | **6.** B | **7.** A | **8.** C | **9.** C | **10.** A |
| **11.** D | **12.** A | **13.** D | **14.** A | **15.** C | | | | | |

## Previous Years' Questions

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** A | **2.** A | **3.** A | **4.** B | **5.** B | **6.** C | **7.** 0.618 | **8.** 0.75 | **9.** 5 | **10.** A |
| **11.** C | **12.** D | **13.** D | **14.** D | **15.** B | **16.** A | **17.** B | **18.** A | **19.** A | **20.** A |
| **21.** 2 | **22.** 0.707 | **23.** C | **24.** A | | | | | | |