UNIT - 1 : PROGRESSION AND SERIES [JEE – MAIN CRASH COURSE]

Arithmetic Progression (AP)

If a is the first term and d is the common difference, then AP can be written as

$$a, a + d, a + 2d, \ldots, a + (n - 1) d, \ldots$$

nth term: $T_n = a + (n - 1) d = l$ (last term), where $d = T_n - T_{n-1}$.

nth term from last: $T_n' = l - (n - 1)d$.

Sum of *n* terms of an AP

The sum S_n of n terms of an AP with first term "a" and common difference "d" is

$$S_n = \frac{n}{2}[2a + (n - 1)d] \text{ or } S_n = \frac{n}{2}[a + l]$$

where l = last term = $a + (n - 1)d$.

Thus, in general, sum of n terms of AP is $S_n = An^2 + Bn$.

Some important facts about AP

1. If a fixed number is added or subtracted to each term of a given AP, then the resulting sequence is also an AP, and its common difference remains same.
2. If each term of an AP is multiplied by a fixed constant or divided by a fixed non-zero constant, then the resulting sequence is also an AP.
3. If $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$, are two AP's then $x_1 \pm y_1, x_2 \pm y_2, x_3 \pm y_3 + \ldots$ are also AP's
4. Three terms in AP should preferably be taken as $a - d, a, a + d$ and four terms as $a - 3d, a - d, a + d, a + 3d$.
5. In AP, $a_n = (1/2)(a_{n-k} + a_{n+k})$, for $k \le n$.
6. In AP, $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots$

Insertion of Arithmetic Means

If between two given numbers a and b we have to insert n numbers $A_1, A_2, \ldots, A_n$ such that $a, A_1, A_2, \ldots, A_n, b$ form an AP, then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

Geometric Progression (GP)

If a is the first term and r is the common ratio, then GP can be written as $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}$.

nth term: $T_n = ar^{n-1} = l$ (last term),

where $r = \frac{T_n}{T_{n-1}}$, $n \ge 2$.

nth term from last: $T_n' = \frac{l}{r^{n-1}}$.

Sum of *n* terms of a GP

The sum of n terms of a GP with first term "a" and common ratio "r" is given by

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ or } S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \ne 1$$

Some important facts about GP

1. If each term of a GP is multiplied or divided by some fixed non-zero number, then the resulting sequence is also a GP.
2. If $x_1, x_2, x_3, \ldots$, and $y_1, y_2, y_3, \ldots$, are two GP's then $x_1y_1, x_2y_2, x_3y_3, \ldots$, and $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \ldots$, are also GP's.

3. If $x_1, x_2, x_3, \ldots$ is a GP of positive terms then $\log x_1, \log x_2, \log x_3, \ldots$ is an AP and vice versa.
4. Three numbers in GP can be taken as a/r, a, ar and four terms in GP as a/r^3 a/r, ar, ar^3. This presentation is useful if product of terms is involved in the problem, otherwise terms should be taken as $a, ar, ar^2 \ldots$.
5. In GP, $a_1, a_2, a_3, \ldots, a_{n-1}, a_n : a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = \cdots$ or product of equidistant terms from start and end is same.

Insertion of Geometric Means

Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a GP. Then the numbers $G_1, G_2, \ldots, G_n$ are known as n geometric means (GM's) between a and b.

Harmonic Progression (HP)

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of non-zero numbers is called a harmonic progression or a harmonic sequence, if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}, \ldots$ is an arithmetic progression.

nth term of an HP The nth term of an HP is the reciprocal of the nth term of the corresponding AP. Thus, if $a_1, a_2, a_3, \ldots, a_n$, is an HP and the common difference of the corresponding AP is d, i.e., $d = \frac{1}{a_{n+1}} - \frac{1}{a_n}$, then the nth term of the HP is given by $a_n = \dfrac{1}{\frac{1}{a_1} + (n-1)d}$

In other words, nth term of an HP is the reciprocal ot the nth term of the corresponding AP.

Insertion of Harmonic Means

Let a, b be two given non-zero numbers. If n numbers $H_1, H_2, \ldots, H_n$ are inserted between a and b such that the sequence $a, H_1, H_2, H_3, \ldots, H_n, b$ is an HP, then $H_1, H_2, \ldots, H_n$ are called n harmonic means between a and b.

Now, $a, H_1, H_2, \ldots, H_n, b$ are in HP.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_n}, \frac{1}{b}$ are in AP

Harmonic means of two given numbers a and b is $H = \dfrac{2ab}{a+b}$.

AM, GM, and HM of two positive real numbers

Let A, G, and H be arithmetic, geometric, and harmonic means of two positive numbers a and b, Then,

$$A = \frac{a+b}{2},\ G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

These three means possess the following properties:

1. $A \geq G \geq H$.
2. A, G, H form a GP, i.e., $G^2 = AH$.
3. If A and G be the AM and GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.
4. The equation having a and b as its roots in $x^2 - 2Ax + G^2 = 0$.
5. If A, G, H are arithmetic, geometric, and harmonic means of three given numbers a, b, and c, then the equation having a, b, c as its roots is $x^3 - 3Ax^2 + \frac{3G^3}{H} x - G^3 = 0$.

Sigma (Σ) Operator

Properties of sigma notations

1. $\sum_{r=1}^{n} (T_r \pm T_r') = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} T_r'$ (sigma operator is distributive over addition and subtraction)
2. $\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \cdots n \text{ times} = n$
3. $\sum_{r=1}^{n} aT_r = a\sum_{r=1}^{n} T_r$ (where a is constant)