Continuity and Differentiability Exercise 1 :
Single Option Correct Type Questions

1. If $f(x)=\begin{cases}\sin\dfrac{\pi x}{2}, & x<1\\ [x], & x\geq 1\end{cases}$, where $[x]$ denotes the greatest integer function, then

(a) $f(x)$ is continuous at $x=1$
(b) $f(x)$ is discontinuous at $x=1$
(c) $f(1^{+})=0$
(d) $f(1^{-})=-1$

2. Consider $f(x)=\begin{cases}\dfrac{8^x-4^x-2^x+1}{x^2}, & x>0\\ e^x\sin x+\pi x+k\log 4, & x<0\end{cases}$

Then, $f(0)$ so that $f(x)$ is continuous at $x=0$, is

(a) $\log 4$ (b) $\log 2$
(c) $(\log 4)(\log 2)$ (d) None of these

3. Let $f(x)=\begin{cases}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2}, & x<0\\ 3, & x=0\\ \left[1+\left(\dfrac{cx+dx^3}{x^2}\right)\right]^{1/x}, & x>0\end{cases}$

If f is continuous at $x=0$, then $(a+b+c+d)$ is

(a) 5 (b) -5 (c) $\log_e 3-5$ (d) $5-\log_e 3$

4. If $f(x)=\begin{cases}\cos^{-1}\{\cot x\}, & x<\pi/2\\ \pi[x]-1, & x\geq \pi/2\end{cases}$, then the jump of discontinuity, (where $[\cdot]$ denotes greatest integer and $\{\}$ denotes fractional part function) is

(a) 1 (b) $\pi/2$ (c) $\dfrac{\pi}{2}-1$ (d) 2

5. Let $f:[0,1]\xrightarrow{\text{onto}}[0,1]$ be a continuous function, then $f(x)=x$ holds for

(a) at least one $x\in[0,1]$ (b) at least one $x\in[1,2]$
(c) at least one $x\in[-1,0]$ (d) can't be discussed

6. If $f(x)=\dfrac{x+1}{x-1}$ and $g(x)=\dfrac{1}{x-2}$, then $(fog)(x)$ is discontinuous at

(a) $x=3$ only (b) $x=2$ only
(c) $x=2$ and 3 (d) $x=1$ only

7. Let $y_n(x)=x^2+\dfrac{x^2}{(1+x^2)}+\dfrac{x^2}{(1+x^2)^2}+\ldots+\dfrac{x^2}{(1+x^2)^{n-1}}$ and $y(x)=\lim\limits_{n\to\infty}y_n(x)$, then $y_n(x)$, $n=1,2,3,\ldots,n$ and $y(x)$ is

(a) continuous for $x\in R$ (b) continuous for $x\in R-\{0\}$
(c) continuous for $x\in R-\{1\}$ (d) data unsufficient

8. If $g(x)=\begin{cases}\dfrac{1+a^x\cdot x\cdot\log a-a^x}{a^x\cdot x^2}, & \text{for } x\leq 0\\ \dfrac{2^x\cdot a^x-x\log 2-x\log a-1}{x^2}, & \text{for } x>0\end{cases}$

where $a>0$, then a for which $g(x)$ is continuous, is

(a) $-\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) 2 (d) -2

9. Let $f(x)=\begin{cases}\dfrac{\left(\dfrac{\pi}{2}-\sin^{-1}(1-\{x\}^2)\right)\cdot\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)}, & x\neq 0,\\ \dfrac{\pi}{2}, & x=0\end{cases}$ where $\{\}$ is fractional part of x, then

(a) $f(0^{+})=-\dfrac{\pi}{2}$
(b) $f(0^{-})=\dfrac{\pi}{4\sqrt{2}}$
(c) $f(x)$ is continuous at $x=0$
(d) None of the above

10. Let $f(x)=\begin{cases}\text{sgn}(x)+x, & -\infty<x<0\\ -1+\sin x, & 0\leq x\leq \pi/2\\ \cos x, & \pi/2\leq x<\infty\end{cases}$, then the number of points, where $f(x)$ is not differentiable, is/are

(a) 0 (b) 1
(c) 2 (d) 3

11. Let $f(x)=\begin{cases}\dfrac{1}{|x|}, & |x|\geq 1\\ ax^2+b, & |x|<1\end{cases}$ be continuous and differentiable for all x. Then, a and b are

(a) $-\dfrac{1}{2},\dfrac{3}{2}$ (b) $\dfrac{1}{2},-\dfrac{3}{2}$
(c) $\dfrac{1}{2},\dfrac{3}{2}$ (d) None of these

12. If $f(x)=\begin{cases}A+Bx^2, & x<1\\ 3Ax-B+2, & x\geq 1\end{cases}$, then A and B, so that $f(x)$ is differentiable at $x=1$, are

(a) $-2, 3$ (b) $2, -3$
(c) $2, 3$ (d) $-2, -3$

13. If $f(x)=\begin{cases}|x-1|([x]-x), & x\neq 1\\ 0, & x=1\end{cases}$, then

(a) $f'(1^{+})=0$ (b) $f'(1^{-})=0$
(c) $f'(1^{-})=-1$ (d) $f(x)$ is differentiable at $x=1$

14. If $f(x)=\begin{cases}[\cos\pi x], & x\le 1\\ 2\{x\}-1, & x>1\end{cases}$, where $[\cdot]$ and $\{\cdot\}$ denote greatest integer and fractional part of x, then

(a) $f'(1^-)=2$ (b) $f'(1^+)=2$

(c) $f'(1^-)=-2$ (d) $f'(1^+)=0$

15. If $f(x)=\begin{cases}x-3, & x<0\\ x^2-3x+2, & x\ge 0\end{cases}$, then $g(x)=f(|x|)$ is

(a) $g'(0^+)=-3$

(b) $g'(0^-)=-3$

(c) $g'(0^+)=g'(0^-)$

(d) $g(x)$ is not continuous at $x=0$

16. If $f(x)=\begin{cases}\left\{x+\dfrac{1}{3}\right\}[\sin\pi x], & 0\le x<1\\ [2x]\operatorname{sgn}\left(x-\dfrac{4}{3}\right), & 1\le x\le 2\end{cases}$, where $[\cdot]$ and $\{\cdot\}$ denote greatest integer and fractional part of x respectively, then the number of points of non-differentiability, is

(a) 3 (b) 4 (c) 5 (d) 6

17. Let f be differentiable function satisfying $f\left(\dfrac{x}{y}\right)=f(x)-f(y)$ for all $x,y>0$. If $f'(1)=1$, then $f(x)$ is

(a) $2\log_e x$ (b) $3\log_e x$ (c) $\log_e x$ (d) $\dfrac{1}{2}\log_e x$

18. Let $f(x+y)=f(x)+f(y)-2xy-1$ for all x and y. If $f'(0)$ exists and $f'(0)=-\sin\alpha$, then $f\{f'(0)\}$ is

(a) -1 (b) 0

(c) 1 (d) 2

19. A derivable function $f:R^+\to R$ satisfies the condition

$$f(x)-f(y)\ge\log\left(\frac{x}{y}\right)+x-y,\ \forall\ x,y\in R^+.$$

If g denotes the derivative of f, then the value of the sum $\sum_{n=1}^{100} g\left(\dfrac{1}{n}\right)$ is

(a) 5050 (b) 5510 (c) 5150 (d) 1550

20. If $\dfrac{d(f(x))}{dx}=e^{-x}f(x)+e^{x}f(-x)$, then $f(x)$ is, (given $f(0)=0$)

(a) an even function

(b) an odd function

(c) neither even nor odd function

(d) can't say

21. Let $f:(0,\infty)\to R$ be a continuous function such that

$$f(x)=\int_0^x t\ f(t)\,dt.$$

If $f(x^2)=x^4+x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to

(a) 216 (b) 219

(c) 222 (d) 225

22. For $x>0$, let $h(x)=\begin{cases}\dfrac{1}{q}, & \text{if } x=\dfrac{p}{q}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$, where p and $q>0$ are relatively prime integers, then which one of the following does not hold good?

(a) $h(x)$ is discontinuous for all x in $(0,\infty)$

(b) $h(x)$ is continuous for each irrational in $(0,\infty)$

(c) $h(x)$ is discontinuous for each rational in $(0,\infty)$

(d) $h(x)$ is not derivable for all x in $(0,\infty)$

23. Let $f(x)=\dfrac{g(x)}{h(x)}$, where g and h are continuous functions on the open interval (a,b). Which of the following statements is true for $a<x<b$?

(a) f is continuous at all x for which $x\ne 0$

(b) f is continuous at all x for which $g(x)=0$

(c) f is continuous at all x for which $g(x)\ne 0$

(d) f is continuous at all x for which $h(x)\ne 0$

24. If $f(x)=\dfrac{2\cos x-\sin 2x}{(\pi-2x)^2}$, $g(x)=\dfrac{e^{-\cos x}-1}{8x-4\pi}$ and

$$h(x)=\begin{cases}f(x), & \text{for } x<\pi/2\\ g(x), & \text{for } x>\pi/2\end{cases},$$

then which of the following holds?

(a) h is continuous at $x=\pi/2$

(b) h has an irremovable discontinuity at $x=\pi/2$

(c) h has a removable discontinuity at $x=\pi/2$

(d) $f\left(\dfrac{\pi^+}{2}\right)=g\left(\dfrac{\pi^-}{2}\right)$

25. If $f(x)=\dfrac{x-e^x+\cos 2x}{x^2}$, $x\ne 0$ is continuous at $x=0$, then

(a) $f(0)=\dfrac{5}{2}$ (b) $[f(0)]=-2$

(c) $\{f(0)\}=-0.5$ (d) $[f(0)]\cdot\{f(0)\}=-1.5$

where, $[x]$ and $\{x\}$ denote the greatest integer and fractional part function.

26. Consider the function $f(x)=\begin{cases}x\{x\}+1, & \text{if } 0\le x<1\\ 2-\{x\}, & \text{if } 1\le x\le 2\end{cases}$, where $\{x\}$ denotes the fractional part function. Which one of the following statements is not correct?

(a) $\lim_{x\to 1} f(x)$ exists

(b) $f(0)\ne f(2)$

(c) $f(x)$ is continuous in $[0,2]$

(d) Rolle's theorem is not applicable to $f(x)$ in $[0,2]$

27. Let $f(x)=\begin{cases}\dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}, & \text{if } x>2\\ \dfrac{x^2-4}{x-\sqrt{3x-2}}, & \text{if } x<2\end{cases}$, then

(a) $f(2)=8 \Rightarrow f$ is continuous at $x=2$
(b) $f(2)=16 \Rightarrow f$ is continuous at $x=2$
(c) $f(2^-)\neq f(2^+) \Rightarrow f$ is discontinuous
(d) f has a removable discontinuity at $x=2$

28. Let $[x]$ denotes the integral part of $g(x)=x-[x], x\in R$. Let $f(x)$ be any continuous function with $f(0)=f(1)$, then the function $h(x)=f(g(x))$

(a) has finitely many discontinuities
(b) is discontinuous at some $x=c$
(c) is continuous on R
(d) is a constant function

29. Let f be a differentiable function on the open interval (a, b).

I. f is continuous on the closed interval $[a, b]$.
II. f is bounded on the open interval (a, b).
III. If $a<a_1<b_1<b$ and $f(a_1)<0<f(b_1)$, then there exists a number c such that $a_1<c<b_1$ and $f(c)=0$.

Which of the above statements must be true?

(a) I and II
(b) I and III
(c) II and III
(d) Only III

30. Number of points, where the function $f(x)=(x^2-1)|x^2-x-2|+\sin(|x|)$ is not differentiable, is

(a) 0 (b) 1
(c) 2 (d) 3

31. Consider function $f: R-\{-1, 1\}\to R$. $f(x)=\dfrac{x}{1-|x|}$.

Then, which of the statements is incorrect?

(a) It is continuous at the origin
(b) It is not derivable at the origin
(c) The range of the function is R
(d) f is continuous and derivable in its domain

32. If the functions $f: R\to R$ and $g: R\to R$ are such that $f(x)$ is continuous at $x=\alpha$ and $f(\alpha)=a$ and $g(x)$ is discontinuous at $x=a$ but $g(f(x))$ is continuous at $x=\alpha$, where $f(x)$ and $g(x)$ are non-constant functions.

(a) $x=\alpha$ is a extremum of $f(x)$ and $x=a$ is an extremum of $g(x)$
(b) $x=\alpha$ may not be an extremum of $f(x)$ and $x=a$ is an extremum of $g(x)$
(c) $x=\alpha$ is an extremum of $f(x)$ and $x=a$ may not be an extremum of $g(x)$
(d) None of the above

33. The total number of points of non-differentiability of $f(x)=\min\left[|\sin x|, |\cos x|, \dfrac{1}{4}\right]$ in $(0, 2\pi)$ is

(a) 8 (b) 9 (c) 10 (d) 11

34. The function $f(x)=[x]^2-[x^2]$, where $[y]$ is the greatest integer less than or equal to y, is discontinuous at

(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

35. The function $f(x)=(x^2-1)|x^2-6x+5|+\cos|x|$ is not differentiable at

(a) -1 (b) 0 (c) 1 (d) 5

36. Let $f(x)=\begin{cases}e^{\frac{-1}{x^2}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

Then, $f'(0)$ is equal to

(a) 0 (b) 1
(c) -1 (d) doesn't exist

37. Given $f(x)=\dfrac{e^x-\cos 2x-x}{x^2}$, for $x\in R-\{0\}$

$g(x)=\begin{cases} f(\{x\}), & \text{for } n<x<n+\frac{1}{2}\\ f(1-\{x\}), & \text{for } n+\frac{1}{2}\le x<n+1, n\in I\\ \frac{5}{2}, & \text{otherwise}\end{cases}$

$\left\{\begin{array}{l}\text{where } \{x\} \text{ denotes}\\ \text{fractional part function,}\end{array}\right.$ then $g(x)$ is

(a) discontinuous at all integral values of x only
(b) continuous everywhere except for $x=0$
(c) discontinuous at $x=n+\frac{1}{2}, n\in I$ and at some $x\in I$
(d) continuous everywhere

38. The function $g(x)=\begin{cases}x+b, & x<0\\ \cos x, & x\ge 0\end{cases}$ cannot be made differentiable at $x=0$,

(a) if b is equal to zero (b) if b is not equal to zero
(c) if b takes any real value (d) for no value of b

39. The graph of function f contains the point $P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is $y=\left(\dfrac{s^2+2s-3}{s-1}\right)x-1-s$. The value of $f'(1)$ is

(a) 2 (b) 3
(c) 4 (d) non-existent

40. Consider

$$f(x)=\left[\frac{2(\sin x-\sin^3 x)+|\sin x-\sin^3 x|}{2(\sin x-\sin^3 x)-|\sin x-\sin^3 x|}\right], x\neq\frac{\pi}{2} \text{ for}$$

$x\in(0,\pi)$, $f(\pi/2)=3$, where [ ] denotes the greatest integer function, then

(a) f is continuous and differentiable at $x=\pi/2$
(b) f is continuous but not differentiable at $x=\pi/2$
(c) f is neither continuous nor differentiable at $x=\pi/2$
(d) None of the above

41. If $f(x+y)=f(x)+f(y)+|x|y+xy^2, \forall\ x,y\in R$ and $f'(0)=0$, then

(a) f need not be differentiable at every non-zero x
(b) f is differentiable for all $x\in R$
(c) f is twice differentiable at $x=0$
(d) None of the above

42. Let $f(x)=\max\{|x^2-2|x||,|x|\}$ and $g(x)=\min\{|x^2-2|x||,|x|\}$, then

(a) both $f(x)$ and $g(x)$ are non-differentiable at 5 points
(b) $f(x)$ is not differentiable at 5 points whether $g(x)$ is non-differentiable at 7 points
(c) number of points of non-differentiability for $f(x)$ and $g(x)$ are 7 and 5 points, respectively
(d) both $f(x)$ and $g(x)$ are non-differentiable at 3 and 5 points, respectively

43. Let $g(x)=\begin{cases}3x^2-4\sqrt{x}+1, & \text{for } x<1\\ ax+b, & \text{for } x\geq 1\end{cases}$. If $g(x)$ is the continuous and differentiable for all numbers in its domain, then

(a) $a=b=4$
(b) $a=b=-4$
(c) $a=4$ and $b=-4$
(d) $a=-4$ and $b=4$

44. Let $f(x)$ be continuous and differentiable function for all reals and $f(x+y)=f(x)-3xy+f(y)$.
If $\lim_{h\to 0}\frac{f(h)}{h}=7$, then the value of $f'(x)$ is

(a) $-3x$
(b) 7
(c) $-3x+7$
(d) $2f(x)+7$

45. Let $[x]$ be the greatest integer function and

$$f(x)=\frac{\sin\frac{1}{4}\pi[x]}{[x]}$$

. Then, which one of the following does not hold good?

(a) Not continuous at any point
(b) Continuous at 3/2
(c) Discontinuous at 2
(d) Differentiable at 4/3

46. If $f(x)=\begin{cases}b([x]^2+[x])+1, & \text{for } x\geq -1\\ \sin(\pi(x+a)), & \text{for } x<-1\end{cases}$, where $[x]$ denotes the integral part of x, then for what values of a, b the function is continuous at $x=-1$?

(a) $a=2n+(3/2);\ b\in R;\ n\in I$
(b) $a=4n+2;\ b\in R;\ n\in I$
(c) $a=4n+(3/2);\ b\in R^+;\ n\in I$
(d) $a=4n+1;\ b\in R^+;\ n\in I$

47. If both $f(x)$ and $g(x)$ are differentiable functions at $x=x_0$, then the function defined as, $h(x)=\max\{f(x),g(x)\}$

(a) is always differentiable at $x=x_0$
(b) is never differentiable at $x=x_0$
(c) is differentiable at $x=x_0$ when $f(x_0)\neq g(x_0)$
(d) cannot be differentiable at $x=x_0$, if $f(x_0)=g(x_0)$

48. Number of points of non-differentiability of the function

$$g(x)=[x^2]\{\cos^2 4x\}+\{x^2\}[\cos^2 4x]+x^2\sin^2 4x +[x^2][\cos^2 4x]+\{x^2\}\{\cos^2 4x\} \text{ in } (-50,50),$$

where $[\cdot]$ and $\{\cdot\}$ are greatest integer function and fractional part of x, is equal to

(a) 98
(b) 99
(c) 100
(d) 0

49. If $f(x)=\frac{\{x\}g(x)}{\{x\}g(x)}$ is a periodic function with period $\frac{1}{4}$, where $g(x)$ is a differentiable function, then (where $\{\cdot\}$ denotes fractional part of x).

(a) $g'(x)$ has exactly three roots in $\left(\frac{1}{4},\frac{5}{4}\right)$
(b) $g(x)=0$ at $x=\frac{k}{4}$, where $k\in I$
(c) $g(x)$ must be non-zero function
(d) $g(x)$ must be periodic function

50. If $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ for all $x,y\in R$, $y\neq 0$ and $f'(x)$ exists for all x, $f(2)=4$. Then, $f(5)$ is

(a) 3
(b) 5
(c) 25
(d) None of these

Continuity and Differentiability Exercise 2 :
More than One Option Correct Type Questions

51. Indicate the correct alternative, if $f(x)=\frac{x}{2}-1$, then on the interval $[0,\pi]$,

(a) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both continuous

(b) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both discontinuous

(c) $\tan(f(x))$ and $f^{-1}(x)$ are both continuous

(d) $\tan(f(x))$ is continuous but $\frac{1}{f(x)}$ is not continuous

52. On the interval $I=[-2,2]$, if the function

$$f(x)=\begin{cases}(x+1)\,e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0\\ 0, & x=0\end{cases}$$ then which of the following hold good?

(a) $f(x)$ is continuous for all values of $x\in I$

(b) $f(x)$ is continuous for $x\in I-\{0\}$

(c) $f(x)$ assumes all intermediate values from $f(-2)$ to $f(2)$

(d) $f(x)$ has a maximum value equal to $3/e$

53. If $f(x)=\begin{cases}3-\left[\cot^{-1}\left(\dfrac{2x^3-3}{x^2}\right)\right], & \text{for } x>0\\ \{x^2\}\cos(e^{1/x}), & \text{for } x<0\end{cases}$, where $\{x\}$ and $[x]$ denote fractional part and the greatest integer function respectively, then which of the following statements does not hold good?

(a) $f(0^-)=0$

(b) $f(0^+)=0$

(c) $f(0)=0 \Rightarrow$ Continuous at $x=0$

(d) Irremovable discontinuity at $x=0$

54. If $f(x)=\begin{cases}b([x]^2+[x])+1, & \text{for } x>-1\\ \sin(\pi(x+a)), & \text{for } x<-1\end{cases}$, where $[x]$ denotes the integral part of x, then for what values of a and b, the function is continuous at $x=-1$?

(a) $a=2n+\frac{3}{2}$; $b\in R$, $n\in I$

(b) $a=4n+2$; $b\in R$, $n\in I$

(c) $a=4n+\frac{3}{2}$; $b\in R^+$, $n\in I$

(d) $a=4n+1$; $b\in R^+$, $n\in I$

55. Let $[x]$ be the greatest integer function, then

$$f(x)=\frac{\sin\frac{1}{4}\pi[x]}{[x]}$$ is

(a) not continuous at any point (b) continuous at $x=\frac{3}{2}$

(c) discontinuous at $x=2$ (d) differentiable at $x=\frac{4}{3}$

56. If $f(x)=\begin{cases}(\sin^{-1}x)^2\cos\left(\dfrac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$, then $f(x)$ is

(a) continuous nowhere in $-1\le x\le 1$

(b) continuous everywhere in $-1\le x\le 1$

(c) differentiable nowhere in $-1\le x\le 1$

(d) differentiable everywhere in $-1\le x\le 1$

57. Let $f(x)=\cos x$ and

$$H(x)=\begin{cases}\min(f(t):0\le t<x), & \text{for } 0\le x\le \pi/2\\ \dfrac{\pi}{2}-x, & \text{for } \dfrac{\pi}{2}<x\le 3\end{cases},$$ then

(a) $H(x)$ is continuous and derivable in $[0,3]$

(b) $H(x)$ is continuous but not derivable at $x=\frac{\pi}{2}$

(c) $H(x)$ is neither continuous nor derivable at $x=\frac{\pi}{2}$

(d) maximum value of $H(x)$ in $[0,3]$ is 1

58. If $f(x)=3(2x+3)^{2/3}+2x+3$, then

(a) $f(x)$ is continuous but not differentiable at $x=-\frac{3}{2}$

(b) $f(x)$ is differentiable at $x=0$

(c) $f(x)$ is continuous at $x=0$

(d) $f(x)$ is differentiable but not continuous at $x=-\frac{3}{2}$

59. If $f(x)=\begin{cases}\dfrac{x\cdot\ln(\cos x)}{\ln(1+x^2)}, & x\neq 0\\ 0, & x=0\end{cases}$, then

(a) f is continuous at $x=0$

(b) f is continuous at $x=0$ but not differentiable at $x=0$

(c) f is differentiable at $x=0$

(d) f is not continuous at $x=0$

60. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x)=[x \sin \pi x]$, then $f(x)$ is
(a) continuous at $x=0$
(b) continuous in $(-1, 0)$
(c) differentiable at $x=1$
(d) differentiable in $(-1, 1)$

61. The function $f(x)=[|x|]-|[x]|$, where $[x]$ denotes greatest integer function
(a) is continuous for all positive integers
(b) is discontinuous for all non-positive integers
(c) has finite number of elements in its range
(d) is such that its graph does not lie above the X-axis

62. The function $f(x)=\sqrt{1-\sqrt{1-x^2}}$
(a) has its domain $-1 \le x \le 1$
(b) has finite one sided derivates at the point $x=0$
(c) is continuous and differentiable at $x=0$
(d) is continuous but not differentiable at $x=0$

63. Consider the function $f(x)=|x^3+1|$. Then,
(a) domain of $f(x) \in R$
(b) range of f is R^+
(c) f has no inverse
(d) f is continuous and differentiable for every $x \in R$

64. f is a continuous function in $[a, b]$, g is a continuous function in $[b, c]$. A function $h(x)$ is defined as
$$h(x)=\begin{cases} f(x) \text{ for } x \in [a, b) \\ g(x) \text{ for } x \in (b, c] \end{cases}.$$ If $f(b)=g(b)$, then
(a) $h(x)$ has a removable discontinuity at $x=b$
(b) $h(x)$ may or may not be continuous in $[a, c]$
(c) $h(b^-)=g(b^+)$ and $h(b^+)=f(b^-)$
(d) $h(b^+)=g(b^-)$ and $h(b^-)=f(b^+)$

65. Which of the following function(s) has/have the same range?
(a) $f(x)=\dfrac{1}{1+x}$
(b) $f(x)=\dfrac{1}{1+x^2}$
(c) $f(x)=\dfrac{1}{1+\sqrt{x}}$
(d) $f(x)=\dfrac{1}{\sqrt{3-x}}$

66. If $f(x)=\sec 2x+\operatorname{cosec} 2x$, then $f(x)$ is discontinuous at all points in
(a) $\{n\pi, n \in N\}$
(b) $\left\{(2n \pm 1)\dfrac{\pi}{4}, n \in I\right\}$
(c) $\left\{\dfrac{n\pi}{4}, n \in I\right\}$
(d) $\left\{(2n \pm 1)\dfrac{\pi}{8}, n \in I\right\}$

67. Let $f(x)=\begin{cases} x^n \sin\left(\dfrac{1}{x^2}\right), & x \ne 0 \\ 0, & x=0 \end{cases}$, $(n \in I)$, then
(a) $\lim\limits_{x \to 0} f(x)$ exists for every $n>1$
(b) f is continuous at $x=0$ for $n>1$
(c) f is differentiable at $x=0$ for every $n>1$
(d) None of the above

68. A function is defined as $f(x)=\begin{cases} e^x, & x \le 0 \\ |x-1|, & x>0 \end{cases}$, then $f(x)$ is
(a) continuous at $x=0$
(b) continuous at $x=1$
(c) differentiable at $x=0$
(d) differentiable at $x=1$

69. Let $f(x)=\int_{-2}^{x} |t+1|\, dt$, then
(a) $f(x)$ is continuous in $[-1, 1]$
(b) $f(x)$ is differentiable in $[-1, 1]$
(c) $f'(x)$ is continuous in $[-1, 1]$
(d) $f'(x)$ is differentiable in $[-1, 1]$

70. A function $f(x)$ satisfies the relation $f(x+y)=f(x)+f(y)+xy(x+y), \forall\, x, y \in R$. If $f'(0)=-1$, then
(a) $f(x)$ is a polynomial function
(b) $f(x)$ is an exponential function
(c) $f(x)$ is twice differentiable for all $x \in R$
(d) $f'(3)=8$

71. If $f(x)=\begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2<x \le 3 \end{cases}$, then
(a) $f(x)$ is increasing on $[-1, 2]$
(b) $f(x)$ is continuous on $[-1, 3]$
(c) $f'(2)$ doesn't exist
(d) $f(x)$ has the maximum value at $x=2$

72. If $f(x)=0$ for $x<0$ and $f(x)$ is differentiable at $x=0$, then for $x>0$, $f(x)$ may be
(a) x^2
(b) x
(c) $-x$
(d) $-x^{3/2}$

Continuity and Differentiability Exercise 3 :
Statements I and II Type Questions

- **Directions** (Q. Nos. 73 to 82) *For the following questions, choose the correct answer from the codes* (a), (b), (c) *and* (d) *defined as follows :*
  (a) Statement I is correct, Statement II is also correct; Statement II is the correct explanation of Statement I
  (b) Statement I is correct, Statement II is also correct; Statement II is not the correct explanation of Statement I
  (c) Statement I is correct, Statement II is incorrect
  (d) Statement I is incorrect, Statement II is correct

73. **Statement I** $f(x) = \sin x + [x]$ is discontinuous at $x = 0$.

Statement II If $g(x)$ is continuous and $f(x)$ is discontinuous, then $g(x) + f(x)$ will necessarily be discontinuous at $x = a$.

74. Consider $f(x) = \begin{cases} 2\sin(a\cos^{-1} x), & \text{if } x \in (0, 1) \\ \sqrt{3}, & \text{if } x = 0 \\ ax + b, & \text{if } x < 0 \end{cases}$

Statement I If $b = \sqrt{3}$ and $a = \frac{2}{3}$, then $f(x)$ is continuous in $(-\infty, 1)$.

Statement II If a function is defined on an interval *I* and limit exists at every point of interval *I*, then function is continuous in *I*.

75. Let $f(x) = \begin{cases} \frac{\cos x - e^{x^2/2}}{x^3}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then

Statement I $f(x)$ is continuous at $x = 0$.

Statement II $\lim_{x \to 0} \frac{\cos x - e^{-x^2/2}}{x^3} = -\frac{1}{12}$

76. **Statement I** The equation $\frac{x^3}{4} - \sin \pi x + \frac{2}{3} = 0$ has atleast one solution in $[-2, 2]$.

Statement II Let $f : [a, b] \to R$ be a function and c be a number such that $f(a) < c < f(b)$, then there is at least one number $n \in (a, b)$ such that $f(n) = c$.

77. **Statement I** Range of $f(x) = x\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$ is not *R*.

Statement II Range of a continuous even function cannot be *R*.

78. Let $h(x) = f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x)$, where $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ are real valued functions of x.

Statement I $f(x) = |\cos|x|| + \cos^{-1}(\text{sgn } x) + |\ln x|$ is not differentiable at 3 points in $(0, 2\pi)$.

Statement II Exactly one function , is $f_i(x), i = 1, 2, \ldots, n$ is not differentiable and the rest of the function is differentiable at $x = a$ makes $h(x)$ not differentiable at $x = a$.

79. **Statement I** $f(x) = |x| \sin x$ is differentiable at $x = 0$.

Statement II If $g(x)$ is not differentiable at $x = a$ and $h(x)$ is differentiable at $x = a$, then $g(x) \cdot h(x)$ cannot be differentiable at $x = a$.

80. **Statement I** $f(x) = |\cos x|$ is not derivable at $x = \frac{\pi}{2}$.

Statement II If $g(x)$ is differentiable at $x = a$ and $g(a) = 0$, then $|g(x)|$ is non-derivable at $x = a$.

81. Let $f(x) = x - x^2$ and $g(x) = \{x\}, \forall x \in R$, where { } denotes fractional part function.

Statement I $f(g(x))$ will be continuous, $\forall x \in R$.

Statement II $f(0) = f(1)$ and $g(x)$ is periodic with period 1.

82. Let $f(x) = \begin{cases} -ax^2 - b|x| - c, & -\alpha \le x < 0 \\ ax^2 + b|x| + c, & 0 \le x \le \alpha \end{cases}$, where *a*, *b*, *c* are positive and $\alpha > 0$, then

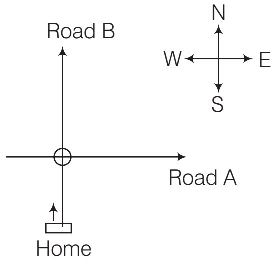
Statement I The equation $f(x) = 0$ has at least one real root for $x \in [-\alpha, \alpha]$.

Statement II Values of $f(-\alpha)$ and $f(\alpha)$ are opposite in sign.

Continuity and Differentiability Exercise 4 :
Passage Based Questions

Passage I (Q. Nos. 83 to 85)

A man leaves his home early in the morning to have a walk. He arrives at a junction of roads A and B as shown in figure. He takes the following steps in later journies :



(i) *1 km in North direction.*

(ii) *Changes direction and moves in North-East direction for $2\sqrt{2}$ km.*

(iii) *Changes direction and moves Southwards for distance of 2 km.*

(iv) *Finally, he changes the direction and moves in South-East direction to reach road A again.*

Visible/invisible path *The path traced by the man in the direction parallel to road A and road B is called invisible path, the remaining path is called visible.*

Visible points *The point about which the man changes direction are called visible points, except the point from where he changes direction last time.*

Now, if roads A and B are taken as X-axis and Y-axis, then visible point representing the graph of $y = f(x)$.

83 The value of x at which the function is discontinuous, is

(a) 2 (b) 0 (c) 1 (d) 3

84. The value of x for which *fof* is discontinuous, is

(a) 0 (b) 1 (c) 2 (d) 3

85. If $f(x)$ is periodic with period 3, then $f(19)$ is

(a) 2 (b) 3
(c) 19 (d) None of these

Passage II (Q. Nos. 86 to 89)

Let f be a function that is differentiable everywhere and that has the following properties :

(i) $f(x) > 0$ (ii) $f'(0) = -1$

(iii) $f(-x) = \dfrac{1}{f(x)}$ and $f(x+h) = f(x) \cdot f(h)$

A standard result is $\dfrac{f'(x)}{f(x)} dx = \log|f(x)| + C$.

86. Range of $f(x)$ is

(a) R (b) $R - \{0\}$
(c) R^+ (d) $(0, e)$

87. The range of the function $\Delta = f(|x|)$ is

(a) $[0, 1]$ (b) $[0, 1)$
(c) $(0, 1]$ (d) None of these

88. The function $y = f(x)$ is

(a) odd (b) even
(c) increasing (d) decreasing

89. If $h(x) = f'(x)$, then $h(x)$ is given by

(a) $-f(x)$ (b) $\dfrac{1}{f(x)}$ (c) $f(x)$ (d) $e^{f(x)}$

Passage III (Q. Nos. 90 to 92)

Let $y = f(x)$ be defined in $[a, b]$, then

(i) *Test of continuity at $x = c, a < c < b$*

(ii) *Test of continuity at $x = a$*

(iii) *Test of continuity at $x = b$*

***Case* I** *Test of continuity at $x = c, a < c < b$*

If $y = f(x)$ be defined at $x = c$ and its value $f(c)$ be equal to limit of $f(x)$ as $x \to c$, i.e. $f(c) = \lim_{x \to c} f(x)$

or $$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

or $$\text{LHL} = f(c) = \text{RHL}$$

Then, $y = f(x)$ is continuous at $x = c$.

***Case* II** *Test of continuity at $x = a$*

If $RHL = f(a)$

Then, $f(x)$ is said to be continuous at the end point $x = a$.

***Case* III** *Test of continuity at $x = b$, if $LHL = f(b)$*

Then, $f(x)$ is continuous at right end $x = b$.

90. If $f(x) = \begin{cases} \sin x, & x \le 0 \\ \tan x, & 0 < x < 2\pi \\ \cos x, & 2\pi \le x < 3\pi \\ 3\pi, & x = 3\pi \end{cases}$, then $f(x)$ is discontinuous at

(a) $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$ (b) $0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 3\pi$

(c) $\dfrac{\pi}{2}, 2\pi$ (d) None of these

91. Number of points of discontinuity of $[2x^3 - 5]$ in $[1, 2)$ is (where $[\cdot]$ denotes the greatest integral function)

(a) 14 (b) 13
(c) 10 (d) None of these

92. Max $([x], |x|)$ is discontinuous at

(a) $x = 0$
(b) ϕ
(c) $x = n, n \in I$
(d) None of the above

Passage IV (Q. Nos. 93 to 95)

$f(x) = \cos x$ *and* $H_1(x) = \min\{f(t), 0 \le t < x\};$

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

$f(x) = \cos x$ *and* $H_2(x) = \max\{f(t), 0 \le t \le x\};$

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

$g(x) = \sin x$ *and* $H_3(x) = \min\{g(t), 0 \le t \le x\};$

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

$g(x) = \sin x$ *and* $H_4(x) = \max\{g(t), 0 \le t \le x\};$

$0 \le x \le \frac{\pi}{2} = \frac{\pi}{2} - x; \frac{\pi}{2} < x \le \pi$

93. Which of the following is true for $H_2(x)$?

(a) Continuous and derivable in $[0, \pi]$

(b) Continuous but not derivable at $x = \frac{\pi}{2}$

(c) Neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

94. Which of the following is true for $H_3(x)$?

(a) Continuous and derivable in $[0, \pi]$

(b) Continuous but not derivable at $x = \frac{\pi}{2}$

(c) Neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

95. Which of the following is true for $H_4(x)$?

(a) Continuous and derivable in $[0, \pi]$

(b) Continuous but not derivable at $x = \frac{\pi}{2}$

(c) Neither continuous nor derivable at $x = \frac{\pi}{2}$

(d) None of the above

Passage V (Q. Nos. 96 to 99)

Let $f(x)$ be a real valued function not identically zero, which satisfied the following conditions

I. $f(x + y^{2n+1}) = f(x) + (f(y))^{2n+1}, n \in N, x, y$ *are any real numbers.*

II. $f'(0) \ge 0$

96. The value of $f(1)$ is

(a) 0 (b) 1
(c) 2 (d) Not defined

97. The value of $f(x)$ is

(a) $2x$ (b) $x^2 + x + 1$
(c) x (d) None of these

98. The value of $f'(10)$ is

(a) 10 (b) 0 (c) $2n + 1$ (d) 1

99. The function $f(x)$ is

(a) odd
(b) even
(c) neither even nor odd
(d) both even as well as odd

Passage VI (Q. Nos. 100 to 101)

If $f : R \to (0, \infty)$ is a differentiable function $f(x)$ satisfying $f(x + y) - f(x - y) = f(x) \cdot \{f(y) - f(-y)\}, \forall\ x, y \in R,$ *$(f(y) \ne f(-y)$ for all $y \in R)$ and $f'(0) = 2010$. Now, answer the following questions*

100. Which of the following is true for $f(x)$?

(a) $f(x)$ is one-one and into
(b) $\{f(x)\}$ is non-periodic, where $\{\cdot\}$ denotes fractional part of x
(c) $f(x) = 4$ has only two solutions
(d) $f(x) = f'(x)$ has only one solution

101. The value of $\frac{f'(x)}{f(x)}$ is

(a) 2016 (b) 2014 (c) 2012 (d) 2010

Continuity and Differentiability Exercise 5 :
Matching Type Questions

102. Match the column.

	Column I	Column II	
(A)	If $f(x) = \begin{cases} \sin\{x\}; & x < 1 \\ \cos x + a; & x \geq 1 \end{cases}$, where $\{x\}$ denotes the fractional part function, such that $f(x)$ is continuous at $x = 1$. If $\lvert K \rvert = \dfrac{a}{\sqrt{2}\sin\dfrac{(4-\pi)}{4}}$, then K is	(p)	1
(B)	If the function $f(x) = \dfrac{1 - \cos(\sin x)}{x^2}$ is continuous at $x = 0$, then $f(0)$ is	(q)	0
(C)	If $f(x) = \begin{cases} x, & x \in \theta \\ 1 - x, & x \notin \theta \end{cases}$, then the values of x at which $f(x)$ is continuous, is	(r)	-1
(D)	If $f(x) = x + \{-x\} + [x]$, where $[x]$ and $\{x\}$ represent greatest integer and fractional part of x, then the values of x at which $f(x)$ is discontinuous, is	(s)	$\dfrac{1}{2}$

103. Match the column.

	Column I	Column II	
(A)	Number of points where the function $f(x) = \begin{cases} 1 + \left[\cos\dfrac{\pi x}{2}\right], & 1 < x \leq 2 \\ 1 - \{x\}, & 0 \leq x < 1 \text{ and } f(1) = 0 \\ \lvert\sin \pi x\rvert, & -1 \leq x < 0 \end{cases}$ is continuous but non-differentiable, where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ denotes fractional part of x, is	(p)	0
(B)	If $f(x) = \begin{cases} x^2 e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(0^-)$ is	(q)	1
(C)	The number of points at which $f(x) = \dfrac{1}{1 + 2/f(x)}$ is not differentiable, where $f(x) = \dfrac{1}{1 + 1/x}$, is	(r)	2
(D)	Number of points where tangent does not exist for the curve $y = \operatorname{sgn}(x^2 - 1)$, is	(s)	3

104. Match the column.

	Column I	Column II	
(A)	The number of values of x in $(0, 2\pi)$, where the function $f(x) = \dfrac{\tan x + \cot x}{2} - \left\lvert\dfrac{\tan x - \cot x}{2}\right\rvert$ is continuous but not differentiable, is	(p)	2
(B)	The number of points where the function $f(x) = \min\{1, 1 + x^3, x^2 - 3x + 3\}$ is non-derivable, is	(q)	0
(C)	The number of points where $f(x) = (x + 4)^{1/3}$ is non-differentiable, is	(r)	4
(D)	Consider $f(x) = \begin{cases} \dfrac{-\pi}{2}\log\left(\dfrac{2x}{\pi}\right) + \dfrac{\pi}{2}, & 0 < x \leq \dfrac{\pi}{2} \\ \sin^{-1}(\sin x), & \dfrac{\pi}{2} < x \leq \dfrac{3\pi}{2} \end{cases}$ Number of points in $\left(0, \dfrac{3\pi}{2}\right)$, where $f(x)$ is non-differentiable, is	(s)	1

105. Match the entries of the following two columns.

	Column I	Column II	
(A)	$f(x) = \begin{cases} x + 1, & \text{if } x < 0 \\ \cos x, & \text{if } x \geq 0 \end{cases}$ at $x = 0$ is	(p)	continuous
(B)	For every $x \in R$, the function $g(x) = \dfrac{\sin(\pi[x - \pi])}{1 + [x]^2}$, where $[x]$ denotes the greatest integer function, is	(q)	differentiability
(C)	$h(x) = \sqrt{\{x\}^2}$, where $\{x\}$ denotes fractional part function for all $x \in I$, is	(r)	discontinuous
(D)	$k(x) = \begin{cases} x^{\frac{1}{\ln x}}, & \text{if } x \neq 1 \\ e, & \text{if } x = 1 \end{cases}$ at $x = 1$ is	(s)	non-derivable

106. Match the entries of the following two columns.

	Column I	Column II	
(A)	$\lim\limits_{x \to \infty}\left(e^{\sqrt{x^4 + 1}} - e^{(x^2 + 1)}\right)$ is	(p)	e
(B)	For $a > 0$, let $f(x) = \begin{cases} \dfrac{a^x + a^{-x} - 2}{x^2}, & \text{if } x > 0 \\ 3\ln(a - x) - 2, & \text{if } x \leq 0 \end{cases}$. If f is continuous at $x = 0$, then a equals	(q)	e^2
(C)	Let $L = \lim\limits_{x \to a}\dfrac{x^x - a^a}{x - a}$ and $M = \lim\limits_{x \to a}\dfrac{x^x - a^x}{x - a}$ $(a > 0)$. If $L = 2M$, then the value of a is equal to	(r)	non-existent

Continuity and Differentiability Exercise 6 :
Single Integer Answer Type Questions

107. Number of points of discontinuity of $f(x) = \tan^2 x - \sec^2 x$ in $(0, 2\pi)$ is ....

108. Number of point(s) of discontinuity of the function $f(x) = [x^{1/x}]$, $x > 0$, (where [ ] denotes the greatest integral function) is ....

109. Let $f(x) = x + \cos x + 2$ and $g(x)$ be the inverse function of $f(x)$, then $g'(3)$ equals to ....

110. Let $f(x) = x \tan^{-1}(x^2) + x^4$. Let $f^k(x)$ denotes kth derivative of $f(x)$ w. r. t. x, $k \in N$.

If $f^{2m}(0) \neq 0$, $m \in N$, then m equals to ....

111. Let $f_1(x)$ and $f_2(x)$ be twice differentiable functions, where $F(x) = f_1(x) + f_2(x)$ and $G(x) = f_1(x) - f_2(x), \forall x \in R$, $f_1(0) = 2$ and $f_2(0) = 1$. If $f_1'(x) = f_2(x)$ and $f_2'(x) = f_1(x), \forall x \in R$, then the number of solutions of the equation $(F(x))^2 = \dfrac{9x^4}{G(x)}$ is ....

112. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x) - 10x$ at $x = 1$ is equal to ....

113. In a ΔABC, angles A, B, C are in AP.

If $f(x) = \lim\limits_{A \to C} \dfrac{\sqrt{3 - 4 \sin A \sin C}}{|A - C|}$, then $f'(x)$ is equal to ....

114. Let $f(x) = \begin{cases} x \dfrac{\left(\frac{3}{4}\right)^{1/x} - \left(\frac{3}{4}\right)^{-1/x}}{\left(\frac{3}{4}\right)^{1/x} + \left(\frac{3}{4}\right)^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

If $P = f'(0^-) - f'(0^+)$, then

$$4\left(\lim_{x \to P-} \frac{(\exp((x+2)\log 4))^{\frac{[x+1]}{4}} - 16}{4^x - 16}\right) \text{ is ....}$$

(where $[x]$ denotes greatest integer function.)

115. Let $f(x) = -x^3 + x^2 - x + 1$ and

$g(x) = \begin{cases} \min(f(t)), 0 \leq t \leq x \text{ and } 0 \leq x \leq 1 \\ x - 1, \quad 1 < x \leq 2 \end{cases}$

Then, the value of $\lim\limits_{x \to 1} g(g(x))$ is...... .

116. The number of points at which the function $f(x) = (x - |x|)^2 (1 - x + |x|)^2$ is not differentiable in the interval $(-3, 4)$ is ....

117. If $f(x) = \begin{cases} \dfrac{\frac{\pi}{2} - \sin^{-1}(1 - \{x\}^2) \sin^{-1}(1 - \{x\})}{\sqrt{2}(\{x\} - \{x\}^3)}, & x > 0 \\ k, & x = 0 \\ \dfrac{A \sin^{-1}(1 - \{x\}) \cos^{-1}(1 - \{x\})}{\sqrt{2\{x\}}(1 - \{x\})}, & x < 0 \end{cases}$

is continuous at $x = 0$, then the value of $\sin^2 k + \cos^2\left(\dfrac{A\pi}{\sqrt{2}}\right)$, is...... (where $\{\cdot\}$ denotes fractional part of x).

Continuity and Differentiability Exercise 7 :
Subjective Type Questions

118. Discuss the continuity of the function $f(x) = [[x]] - [x - 1]$, where [ ] denotes the greatest integral function.

119. Examine the continuity or discontinuity of the following :

(i) $f(x) = [x] + [-x]$ (ii) $g(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$

120. Examine the continuity and differentiability at points $x = 1$ and $x = 2$.

The function f defined by

$f(x) = \begin{cases} x[x], & 0 \leq x < 2 \\ (x - 1)[x], & 2 \leq x \leq 3 \end{cases}$ (where $[\cdot]$ denotes the greatest integral function less than or equal to x).

121. Let f be twice differentiable function, such that
$f''(x) = -f(x)$ and $f'(x) = g(x)$,
$h(x) = [f(x)]^2 + [g(x)]^2$. Find $h(10)$, if $h(5) = 11$.

122. A function $f : R \to R$ satisfies the equation $f(x+y) = f(x) \cdot f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R. Hence, determine $f(x)$.

123. A function $f : R \to R$, where R is a set of real numbers, satisfying the equation $f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y) + f(0)}{3}$ for all x, y in R. If the function is differentiable at $x = 0$, then show that it is differentiable for all x in R.

124. Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all real x, y and $f(x)$ be differentiable functions. If $f'(0) = \cos \alpha$, then prove that $f(x) > 0, \forall x \in R$.

Continuity and Differentiability Exercise 8 :
Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

125. For every pair of continuous function $f, g : [0,1] \to R$ such that $\max\{f(x) : x \in [0,1]\} = \max\{g(x) : x \in [0,1]\}$. The correct statement(s) is/are

[More than One Correct Option, 2014]

(a) $[f(c)]^2 + 3f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0,1]$

(b) $[f(c)]^2 + f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0,1]$

(c) $[f(c)]^2 + 3f(c) = [g(c)]^2 + g(c)$ for some $c \in [0,1]$

(d) $[f(c)]^2 = [g(c)]^2$ for some $c \in [0,1]$

126. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \leq 0 \\ \min\{f(x), g(x)\}, & \text{if } x > 0 \end{cases}.$$

The number of points at which $h(x)$ is not differentiable is

[Integer Answer Type, 2014]

127. Let $f(x) = \begin{cases} x^2 \left|\cos \frac{\pi}{x}\right|, & x \neq 0\ x \in R, \\ 0, & x = 0, \end{cases}$ then f is

[One Correct Option, 2012]

(a) differentiable both at $x = 0$ and at $x = 2$
(b) differentiable at $x = 0$ but not differentiable at $x = 2$
(c) not differentiable at $x = 0$ but differentiable at $x = 2$
(d) differentiable neither at $x = 0$ nor at $x = 2$

128. For every integer n, let a_n and b_n be real numbers. Let function $f : R \to R$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases},$$

for all integers n.

If f is continuous, then which of the following hold(s) for all n?

[More than One Correct Option, 2012]

(a) $a_{n-1} - b_{n-1} = 0$ (b) $a_n - b_n = 1$
(c) $a_n - b_{n+1} = 1$ (d) $a_{n-1} - b_n = -1$

129. Let $f : R \to R$ be a function such that $f(x+y) = f(x) + f(y), \forall x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then

[More than One Correct Option, 2011]

(a) $f(x)$ is differentiable only in a finite interval containing zero
(b) $f(x)$ is continuous for all $x \in R$
(c) $f'(x)$ is constant for all $x \in R$
(d) $f(x)$ is differentiable except at finitely many points

130. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \leq 0 \\ x - 1, & 0 < x \leq 1 \\ \ln x, & x > 1 \end{cases}$, then

[More than One Correct Option, 2011]

(a) $f(x)$ is continuous at $x = -\frac{\pi}{2}$
(b) $f(x)$ is not differentiable at $x = 0$
(c) $f(x)$ is differentiable at $x = 1$
(d) $f(x)$ is differentiable at $x = -\frac{3}{2}$

131. For the function $f(x) = x \cos \frac{1}{x}, \ x \geq 1$,

[More than One Correct Option, 2009]

(a) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$
(b) $\lim_{x \to \infty} f'(x) = 1$
(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$
(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

132. Let $g(x) = \frac{(x-1)^n}{\log \cos^m (x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then

[One Correct Option, 2008]

(a) $n = 1, m = 1$ (b) $n = 1, m = -1$

(c) $n = 2, m = 2$ (d) $n > 2, m = n$

133. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x) \sin x$.

Statement I $\lim_{x \to 0} [g(x) \cos x - g(0)] \operatorname{cosec} x = f''(0)$.

Statement II $f'(0) = g(0)$.

For the above question, choose the correct answer from the codes (a), (b), (c) and (d) defined as follows. **[Assertion and Reason, 2008]**

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I

(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

134. In the following, $[x]$ denotes the greatest integer less than or equal to x. **[Matching type Question, 2007]**

	Column I		Column II
A.	$x\|x\|$	p.	continuous in $(-1, 1)$
B.	$\sqrt{\|x\|}$	q.	differentiable in $(-1, 1)$
C.	$x + [x]$	r.	strictly increasing in $(-1, 1)$
D.	$\|x-1\|+\|x+1\|$	s.	not differentiable atleast at one point in $(-1, 1)$

135. If $f(x) = \min\{1, x^2, x^3\}$, then **[More than One Correct, 2006]**

(a) $f(x)$ is continuous everywhere

(b) $f(x)$ is continuous and differentiable everywhere

(c) $f(x)$ is not differentiable at two points

(d) $f(x)$ is not differentiable at one point

136. Let $f(x) = ||x| - 1|$, then points where, $f(x)$ is not differentiable is/are **[One Correct Option, 2005]**

(a) $0, \pm 1$ (b) ± 1

(c) 0 (d) 1

137. If f is a differentiable function satisfying $f\left(\frac{1}{n}\right) = 0, \forall$ $n \geq 1, n \in I$, then **[One Correct Option, 2005]**

(a) $f(x) = 0, x \in (0, 1]$

(b) $f'(0) = 0 = f(0)$

(c) $f(0) = 0$ but $f'(0)$ not necessarily zero

(d) $|f(x)| \leq 1, x \in (0, 1]$

138. The domain of the derivative of the functions

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1 \\ \frac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases} \text{ is}$$

[One Correct Option, 2002]

(a) $R - \{0\}$ (b) $R - \{1\}$

(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

139. The left hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k is an integer, is **[One Correct Option, 2001]**

(a) $(-1)^k (k-1)\pi$

(b) $(-1)^{k-1}(k-1)\pi$

(c) $(-1)^k k\pi$

(d) $(-1)^{k-1} k\pi$

140. Which of the following functions is differentiable at $x = 0$? **[One Correct Option, 2001]**

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

(ii) JEE Main & AIEEE

141. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

(a) g is not differentiable at $x = 0$ **[2016, JEE Main]**

(b) $g'(0) = \cos(\log 2)$

(c) $g'(0) = -\cos(\log 2)$

(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

142. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is **[2015, JEE Main]**

(a) 2 (b) $\frac{16}{5}$

(c) $\frac{10}{3}$ (d) 4

143. If f and g are differentiable functions in $(0, 1)$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$ **[2014, JEE Main]**

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$

(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$

144. If $f : R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is **[2012 AIEEE]**

(a) continuous for every real x

(b) discontinuous only at $x = 0$

(c) discontinuous only at non-zero integral values of x

(d) continuous only at $x = 0$

Answers

Chapter Exercises

1. (a) **2.** (c) **3.** (c) **4.** (c) **5.** (a) **6.** (c)
7. (b) **8.** (b) **9.** (b) **10.** (b) **11.** (a) **12.** (c)

13. (a) **14.** (b) **15.** (a) **16.** (c) **17.** (c) **18.** (c)
19. (c) **20.** (b) **21.** (b) **22.** (b) **23.** (d) **24.** (b)
25. (d) **26.** (c) **27.** (c) **28.** (c) **29.** (d) **30.** (c)
31. (b) **32.** (c) **33.** (d) **34.** (d) **35.** (d) **36.** (a)
37. (d) **38.** (d) **39.** (c) **40.** (a) **41.** (b) **42.** (b)
43. (c) **44.** (c) **45.** (c) **46.** (a) **47.** (c) **48.** (d)
49. (b) **50.** (c) **51.** (c,d) **52.** (b,c,d) **53.** (a,b,c) **54.** (a,c)

55. (b,c,d) **56.** (b,d) **57.** (a,d) **58.** (a,b,c) **59.** (a,c) **60.** (a, b, c, d)
61. (a, b, c, d) **62.** (a, b, d) **63.** (a, c) **64.** (a,c)
65. (b, c) **66.** (a, b, c) **67.** (a, b, c)
68. (a, b) **69.** (a, b, c, d) **70.** (a, c, d)
71. (a, b, c, d) **72.** (a,d) **73.** (a) **74.** (c)
75. (a) **76.** (c) **77.** (a) **78.** (a) **79.** (c) **80.** (c)
81. (a) **82.** (d) **83.** (a) **84.** (b,c) **85.** (a) **86.** (c)
87. (a) **88.** (d) **89.** (a) **90.** (a) **91.** (b) **92.** (b)
93. (c) **94.** (b) **95.** (c) **96.** (b) **97.** (c) **98.** (d)
99. (a) **100.** (b) **101.** (d)
102. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (s); (C) $\rightarrow$ (s); (D) $\rightarrow$ (p, q, r)
103. (A) $\rightarrow$ (q); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (p)
104. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (q)
105. (A) $\rightarrow$ (p, s); (B) $\rightarrow$ (p, q); (C) $\rightarrow$ (r, s); (D) $\rightarrow$ (p, q)
106. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p, q); (C) $\rightarrow$ (p)

107. (2) **108.** (1) **109.** (1) **110.** (2) **111.** (2)
112. (9) **113.** (0) **114.** (2) **115.** (1) **116.** (0) **117.** (2)
125. (a,d) **126.** (3) **127.** (b) **128.** (b, d) **129.** (b,c) **130.** (a, b, c, d)
131. (b, c, d) **132.** (c) **133.** (b)
134. (A) $\rightarrow$ (p, q, r); (B) $\rightarrow$ (p, s); (C) $\rightarrow$ (r, s); (D) $\rightarrow$ (p, q)
135. (a,d) **136.** (a) **137.** (b) **138.** (d) **139.** (a) **140.** (d)
141. (b) **142.** (a) **143.** (d) **144.** (a)

Solutions

1. $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x<1 \\ [x], & x\geq 1 \end{cases}$

$f(1)=1,\ f(1^+)=\lim\limits_{h\to 0}[1+h]=1$

$f(1^-)=\lim\limits_{h\to 0}\sin\dfrac{\pi}{2}(1-h)=\lim\limits_{h\to 0}\sin\left(\dfrac{\pi}{2}-\dfrac{\pi h}{2}\right)=\lim\limits_{h\to 0}\cos\dfrac{\pi h}{2}=1$

$\therefore$ $f(x)$ is continuous at $x=1$.

2. $f(0^+)=\lim\limits_{h\to 0}\dfrac{8^h-4^h-2^h+1}{h^2}=\lim\limits_{h\to 0}\dfrac{(4^h-1)}{h}\dfrac{(2^h-1)}{h}$

$= (\log 4)(\log 2)$

$f(0^-)=\lim\limits_{h\to 0}(e^{-h}\sin(-h)+\pi(-h)+k\log 4)=k\log 4$

Since, $f(x)$ is continuous at $x=0$

$\Rightarrow$ $k\log 4=(\log 4)(\log 2)=f(0)$.

$\therefore$ $f(0)=(\log 4)(\log 2)$, when $k=\log 2$

3. $f(0^-)=\lim\limits_{x\to 0^-}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2}=f(0)$

$=\lim\limits_{x\to 0^-}\dfrac{(a+b+5)+\left(-a-\dfrac{b}{2}\right)x^2+\ldots}{x^2}=3$

$\Rightarrow$ $a+b+5=0$ and $-a-\dfrac{b}{2}=3$

$\Rightarrow$ $a=-1,\ b=-4$

$\Rightarrow$ $f(0^+)=\lim\limits_{x\to 0^+}\left[1+\left(\dfrac{cx+dx^3}{x^2}\right)\right]^{1/x}$ exists.

$\Rightarrow$ $\lim\limits_{x\to 0^+}\dfrac{cx+dx^3}{x^2}=0 \Rightarrow c=0$

Now, $\lim\limits_{x\to 0^+}(1+dx)^{1/x}=\lim\limits_{x\to 0^+}((1+dx)^{1/dx})^d=e^d$

So, $e^d=3 \Rightarrow d=\log_e 3$

$\therefore$ $a+b+c+d=\log_e 3-5$

4. $f\left(\dfrac{\pi^+}{2}\right)=\lim\limits_{h\to 0}\pi\left[\dfrac{\pi}{2}+h\right]-1=(\pi-1)$

$f\left(\dfrac{\pi^-}{2}\right)=\lim\limits_{h\to 0}\cos^{-1}\left\{\cot\left(\dfrac{\pi}{2}-h\right)\right\}=\lim\limits_{h\to 0}\cos^{-1}\{\tan h\}=\dfrac{\pi}{2}$

$\therefore$ Jump of discontinuity $=(\pi-1)-\dfrac{\pi}{2}=\dfrac{\pi}{2}-1$

5. Consider $g(x)=f(x)-x$

$g(0)=f(0)-0=f(0)\geq 0$ $\quad[\because 0\leq f(x)\leq 1]$

$g(1)=f(1)-1\leq 0$

$\Rightarrow$ $g(0)\cdot g(1)\leq 0$

$\Rightarrow$ $g(x)=0$ has at least one root in [0, 1].

$\Rightarrow$ $f(x)=x$ for at least one root in [0, 1].

6. Here, $f(x)=\dfrac{x+1}{x-1}$, discontinuous at $x=1$

$g(x)=\dfrac{1}{x-2}$, discontinuous at $x=2$

$f(g(x))=\dfrac{g(x)+1}{g(x)-1}$, discontinuous at $g(x)=1$

$\Rightarrow$ $\dfrac{1}{x-2}=1$

$\Rightarrow$ $x=3$

$\therefore$ $(fog)(x)$ is discontinuous at $x=2$ and 3.

7. Here, $y_n(x)=\begin{cases} \dfrac{x^2\left[\dfrac{1}{(1+x^2)^n}-1\right]}{\dfrac{1}{1+x^2}-1}=(1+x^2)\cdot\left[1-\dfrac{1}{(1+x^2)^n}\right], & \text{when } x\neq 0,\ n\in N \\ 0, & \text{when } x=0, n\in N \end{cases}$

$\therefore$ $y(x)=\lim\limits_{n\to\infty}y_n(x)=\begin{cases}(1+x^2), & x\neq 0\\ 0, & x=0\end{cases}$

$\therefore$ $y(x)$ is discontinuous at $x=0$.

8. $g(0^-)=\lim\limits_{h\to 0}\dfrac{a^h-1-h\log a}{h^2}$

$=\lim\limits_{h\to 0}\dfrac{\left[1+h\log a+\dfrac{h^2}{2!}(\log a)^2+\ldots\right]-1-h\log a}{h^2}$

$=\lim\limits_{h\to 0}\left[\dfrac{(\log a)^2}{2!}\cdot 1+\dfrac{(\log a)^3}{3!}\cdot h\ldots\right]=\dfrac{(\log a)^2}{2}$

$g(0^+)=\lim\limits_{h\to 0}\dfrac{2^h a^h-h\log 2-h\log a-1}{h^2}$

$=\lim\limits_{h\to 0}\dfrac{\left[1+h\log(2a)+\dfrac{h^2}{2!}(\log(2a))^2\ldots\right]-h\log 2-h\log a-1}{h^2}$

$=\dfrac{(\log(2a))^2}{2}$

Since, $g(x)$ is continuous.

$\Rightarrow$ $\dfrac{(\log a)^2}{2}=\dfrac{(\log(2a))^2}{2} \Rightarrow (\log a)^2=(\log 2+\log a)^2$

$\Rightarrow$ $\log a=-\dfrac{1}{2}\log 2=\log 2^{-1/2}$

$\Rightarrow$ $a=\dfrac{1}{\sqrt{2}}$

9. $f(0^+)=\lim\limits_{h\to 0}\dfrac{\left(\dfrac{\pi}{2}-\sin^{-1}(1-h^2)\right)\cdot\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$

$=\lim\limits_{h\to 0}\dfrac{(\cos^{-1}(1-h^2))\cdot\sin^{-1}(1-h)}{\sqrt{2}(1-h^2)\cdot h}=\dfrac{\pi}{2}$

$$f(0^-) = \lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-(1-h^2))\right)\cdot\sin^{-1}(1-(1-h))}{\sqrt{2}((1-h)-(1-h)^3)}$$

$$= \lim_{h\to 0}\frac{\frac{\pi}{2}\sin^{-1}h}{\sqrt{2}(1-h)(2-h)h} = \frac{\pi}{4\sqrt{2}}$$

10. $f(x) = \begin{cases} -1+x, & -\infty<x<0 \\ -1+\sin x, & 0\le x<\pi/2 \\ \cos x, & \pi/2\le x<\infty \end{cases}$

$$f'(x) = \begin{cases} 1, & -\infty<x<0 \\ \cos x, & 0<x<\pi/2 \\ -\sin x, & \pi/2<x<\infty \end{cases}$$

$f'(0^-) = 1, f'(0^+) = 1$

$\Rightarrow f(x)$ is differentiable at $x=0$

$$f'\left(\frac{\pi^-}{2}\right) = 0, \quad f'(\pi/2^+) = -1$$

$\Rightarrow f(x)$ is not differentiable at $x=\frac{\pi}{2}$.

$\therefore$ Number of points of non-differentiability is 1.

11. $f(x) = \begin{cases} -\frac{1}{x}, & x\le -1 \\ ax^2+b, & -1<x<1 \\ \frac{1}{x}, & x\ge 1 \end{cases}$

Since, function is continuous everywhere.

$\therefore$ LHL = RHL at $x=-1$

$\Rightarrow f(-1^-) = 1, f(-1^+) = a+b$

$\Rightarrow a+b=1$ ...(i)

$f(x)$ is differentiable at $x=-1$

$$f'(x) = \begin{cases} \frac{1}{x^2}, & x<-1 \\ 2ax, & -1<x<1 \\ -\frac{1}{x^2}, & x>1 \end{cases}$$

$f'(-1^-) = 1, f'(-1^+) = -2a \Rightarrow -2a=1, a=-1/2$ ...(ii)

From Eqs. (i) and (ii), we get

$$a = -\frac{1}{2}, b = \frac{3}{2}$$

12. $f(1^-) = A+B, f(1^+) = 3A-B+2$

Since, continuous $A+B=3A-B+2$

$\Rightarrow 2A-2B=-2 \Rightarrow A-B=-1$ ...(i)

$$f'(x) = \begin{cases} 2Bx, & x<1 \\ 3A, & x>1 \end{cases}$$

$$f'(1^+) = 3A, f'(1^-) = 2B$$

$\Rightarrow 3A=2B$ ...(ii)

On solving Eqs. (i) and (ii), we get

$A=2, B=3$

13. $f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$$=\lim_{h\to 0}\frac{|1+h-1|([1+h]-(1+h))}{h} = 0 = \lim_{h\to 0}\frac{h(1-1-h)}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{|-h|([1-h]-(1-h))}{-h}$$

$$= \lim_{h\to 0}\frac{h(0-1+h)}{-h} = 1$$

$\Rightarrow f'(1^+)=0, f'(1^-)=1$

$\therefore f(x)$ is not differentiable at $x=1$.

14. $f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[\cos\pi(1-h)]+1}{-h}$

$$= \lim_{h\to 0}\frac{-1+1}{-h} = 0$$

$$f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{2\{1+h\}-1+1}{h}$$

$$= \lim_{h\to 0}\frac{2h}{h} = 2$$

$\therefore f'(1^-) = 0, f'(1^+) = 2$

15. $f(x) = \begin{cases} x-3, & x<0 \\ x^2-3x+2, & x\ge 0 \end{cases}$

$$f(|x|) = \begin{cases} |x|-3, & |x|<0 \text{ not possible} \\ |x|^2-3|x|+2, & |x|\ge 0 \end{cases}$$

$\Rightarrow g(x) = \begin{cases} x^2+3x+2, & x\le 0 \\ x^2-3x+2, & x\ge 0 \end{cases}$

$\Rightarrow g(x)$ is continuous at $x=0$.

$$g'(x) = \begin{cases} 2x+3, & x\le 0 \\ 2x-3, & x\ge 0 \end{cases}$$

$$g'(0^+) = -3, g'(0^-) = 3$$

16. $f(x) = \begin{cases} 0, & 0\le x<1/2 \\ 5/6, & x=1/2 \\ 0, & 1/2<x<1 \\ -2, & 1\le x<4/3 \\ 0, & x=4/3 \\ 2, & 4/3<x<3/2 \\ 3, & 3/2\le x<2 \\ 4, & x=2 \end{cases}$

Hence, $f(x)$ is neither continuous nor differentiable at $x=\frac{1}{2}, 1, \frac{4}{3}, \frac{3}{2}, 2$.

$\therefore$ Number of points is 5.

17. Put $x=y=1$ in given rule

$\Rightarrow f(1) = f(1) - f(1) = 0$

$$\therefore f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \left(\frac{f\left(1 + \frac{h}{x}\right) - f(1)}{\frac{h}{x}} \right) \frac{1}{x} = \frac{f'(1)}{x} = \frac{1}{x}$$

[given, $f'(1) = 1$]

On integrating both sides, $f(x) = \log x + C$

On putting $x = 1$, we get $C = 0 \Rightarrow f(x) = \log_e x$

18. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) - 2xh - 1 - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \left(-2x + \frac{f(h) - 1}{h} \right) = -2x + f'(0)$$

On integrating both sides, we get

$\Rightarrow \quad f(x) = -x^2 + f'(0)x + c$

$\Rightarrow \quad f(x) = -x^2 - (\sin\alpha)x + c \quad$ [given $f'(0) = -\sin\alpha$]

$\Rightarrow \quad f(0) = c \Rightarrow c = 1$

So, $f\{f'(0)\} = f\{-\sin\alpha\} = -\sin^2\alpha + \sin^2\alpha + 1$

$= 1$

19. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq \lim_{h \to 0} \frac{\log\left(\frac{x+h}{x}\right) + x + h - x}{h}$

$$= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \frac{1}{x} + 1 \geq \frac{1}{x} + 1 \quad \text{...(i)}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \leq \lim_{h \to 0} \frac{\log\left(\frac{x-h}{x}\right) + x - h - x}{-h}$$

$$= \lim_{h \to 0} \frac{\log\left(1 - \frac{h}{x}\right)}{-h} + 1 \leq \frac{1}{x} + 1 \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii), we get $f'(x) = \frac{1}{x} + 1 = g(x)$

$$\Rightarrow \sum_{n=1}^{100} g\left(\frac{1}{n}\right) = g\left(\frac{1}{1}\right) + g\left(\frac{1}{2}\right) + \dots + g\left(\frac{1}{100}\right)$$

$= (1 + 1) + (2 + 1) + (3 + 1) + \dots + (100 + 1)$

$= (1 + 2 + \dots + 100) + 100$

$= 5050 + 100 = 5150$

20. Here, $\frac{d(f(x))}{dx} = e^{-x} f(x) + e^x f(-x)$

Let $\frac{d(f(x))}{dx} = g(x)$, where $g(x) = e^{-x} f(x) + e^x f(-x)$

$\therefore \quad g(-x) = e^x f(-x) + e^{-x} f(x) = g(x)$

$\Rightarrow$ $g(x)$ is even function.

Hence, $f(x)$ should be an odd function as $f(0) = 0$.

21. We have, $f(x^2) = \int_0^{x^2} t\, f(t)\, dt = x^4 + x^5 \quad$ ...(i)

On differentiating both the sides w.r.t. x, we get

$$x^2 f(x^2) \cdot 2x = 4x^3 + 5x^4$$

$$\Rightarrow \quad f(x^2) = 2 + \frac{5}{2}x \quad \text{...(ii)}$$

$$\therefore \sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12} \left(2 + \frac{5}{2}r\right) = 24 + \frac{5}{2} \cdot \frac{(12)(13)}{2} = 24 + 195 = 219$$

Hence, $\sum_{r=1}^{12} f(r^2) = 219$

22. Let $x = \frac{2}{3}$, which is rational.

$$\Rightarrow \quad h\left(\frac{2}{3}\right) = \frac{1}{3}$$

$\lim_{t \to 0} h\left(\frac{2}{3} + t\right) = 0 \Rightarrow$ Discontinuous at $x \in Q$

Let $\quad x = \sqrt{2} \notin Q$

$h(\sqrt{2}) = 0$, consider $\sqrt{2} = 1.4142135624$

$$h(\sqrt{2}) = h\left(\frac{14142135624}{10^{10}}\right) = \frac{1}{10^{10}} \to 0$$

Hence, h is continuous for all irrational.

23. By theorem, if g and h are continuous functions on the open interval (a, b), then g/h is also continuous at all x in the open interval (a, b), where $h(x) \neq 0$.

24. $h(x) = \begin{cases} \frac{2\cos x - \sin 2x}{(\pi - 2x)^2}, & x < \frac{\pi}{2} \\ \frac{e^{-\cos x} - 1}{8x - 4\pi}, & x > \frac{\pi}{2} \end{cases}$

LHL at $x = \pi/2$

$$\text{LHL} = \lim_{h \to 0} \frac{2\sin h - \sin 2h}{4h^2} = \lim_{h \to 0} \frac{2\sin h\,(1 - \cos h)}{4h^2} = 0$$

$$\text{RHL} = \lim_{h \to 0} \frac{e^{\sin h} - 1}{8((\pi/2) + h) - 4\pi}$$

$$= \lim_{h \to 0} \frac{e^{\sin h} - 1}{8h} \cdot \frac{\sin h}{\sin h} = \frac{1}{8}$$

$\Rightarrow h(x)$ is discontinuous at $x = \pi/2$.

Irremovable discontinuity at $x = \pi/2$.

$$f\left(\frac{\pi^+}{2}\right) = 0 \text{ and } g\left(\frac{\pi^-}{2}\right) = \frac{1}{8}$$

$$\Rightarrow \quad f\left(\frac{\pi^+}{2}\right) \neq g\left(\frac{\pi^-}{2}\right)$$

25. $\lim_{x \to 0} \frac{x - e^x + 1 - (1 - \cos 2x)}{x^2} = -\frac{1}{2} - 2 = -\frac{5}{2}$

Hence, for continuity, $f(0) = -\frac{5}{2}$

$\therefore [f(0)] = -3$; $\{f(0)\} = \left\{-\frac{5}{2}\right\} = \frac{1}{2}$

Hence, $\quad [f(0)] \cdot \{f(0)\} = -\frac{3}{2} = -1.5$

26. $f(1^+) = f(1^-) = f(1) = 2$

$f(0) = 1, \qquad f(2) = 2$

$f(2^-) = 1,$

$\Rightarrow$ f is not continuous at $x = 2$.

27. $f(2^+) = 8, f(2^-) = 16$

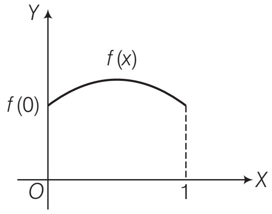
28. $g(x) = x - [x] = \{x\}$

f is continuous with $f(0) = f(1)$

$h(x) = f(g(x)) = f(\{x\})$

Let the graph of f is as shown in the figure satisfying

$$f(0) = f(1)$$



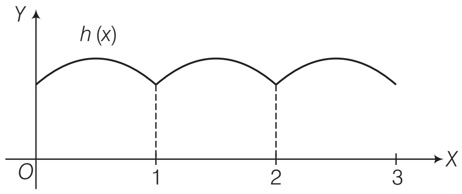
Now,

$h(0) = f(\{0\}) = f(0) = f(1)$

$h(0.2) = f(\{0.2\}) = f(0.2)$

$h(1.5) = f(\{1.5\}) = f(0.5)$ etc.

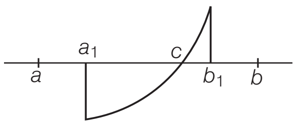
Hence, the graph of $h(x)$ will be a periodic graph as shown



$\Rightarrow h$ is continuous in R.

29. Statements I and II are false. The function $f(x) = 1/x$, $0 < x < 1$ is a counter example.

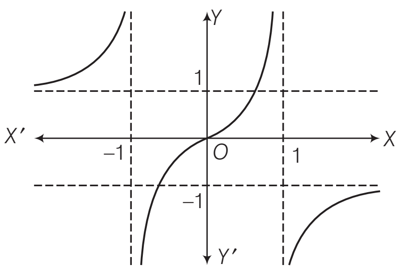
Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$.



30. Since, the given function is not differentiable, at $x = 0$ and 2.

Hence, the number of points is 2.

31. $$f(x) = \begin{cases} \dfrac{x}{1-x}, & \text{if } x \geq 0, x \neq 1 \\ \dfrac{x}{1+x}, & \text{if } x < 0, x \neq -1 \end{cases}$$



and $$f'(x) = \begin{cases} \dfrac{1}{(1-x)^2}, & \text{if } x > 0, x \neq 1 \\ \dfrac{1}{(1+x)^2}, & \text{if } x < 0, x \neq -1 \end{cases}$$

32. $g[f(x)]$ is continuous at $x = \alpha$

$\Rightarrow \quad g[f(\alpha)] = g(a)$

Also, $\quad \lim_{x \to \alpha} g(f(x)) = g(a)$

$\Rightarrow \quad g[f(\alpha^-)] = g[f(\alpha^+)] = g(a)$

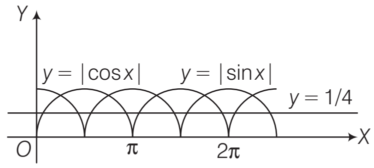
$\Rightarrow g(x)$ takes same limiting values at

$f(\alpha^-)$, $f(\alpha^+)$ and $f(\alpha)$.

$\Rightarrow f(\alpha^-) = f(\alpha^+) \Rightarrow x = \alpha$ is an extremum of $f(x)$

and $x = a$ may not be an extremum of $g(x)$.

33. From the graph, number of points of non-differentiability = 11



34. Let n be any integer other than 1.

$$\lim_{x \to n^-} f(x) = \lim_{h \to 0} ([n-h]^2 - [(n-h)^2])$$

$$= (n-1)^2 - (n^2 - 1) = 2 - 2n$$

$$\lim_{x \to n^+} f(x) = \lim_{h \to 0} [n+h]^2 - [(n+h)^2] = n^2 - n^2 = 0$$

$\therefore$ LHL $\neq$ RHL unless $n = 1$.

Hence, $f(x)$ is discontinuous at all integral values except 1.

35. At $x = 5$, $$f'(x) = \lim_{x \to 5} \frac{\{(x-1)^2 (x+1)|x-5| + \cos|x|\} - \cos 5}{x-5}$$

$$= \lim_{x \to 5} \frac{96|x-5|}{x-5} = +96, \text{ if } x > 5 \text{ and } -96, \text{ if } x < 5$$

Hence, $f'(5)$ doesn't exist.

This ambiguity doesn't occur at other points.

$\therefore f(x)$ is not differentiable at $x = 5$.

36. $$f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{e^{\frac{-1}{x^2}} - 0}{x} = \lim_{x \to 0^+} \frac{1/x}{e^{1/x^2}}$$

$$= \lim_{x \to 0^+} \frac{-1/x^2}{e^{1/x^2} \cdot \left(-\dfrac{2}{x^3}\right)} = \lim_{x \to 0^+} \frac{x}{2e^{1/x^2}} = 0$$

As, f is even, so $f'(0^-) = f'(0^+) = 0$. Thus, $f'(0) = 0$

37. $$\lim_{h \to 0} g(n+h) = \lim_{h \to 0} \frac{e^h - \cos 2h - h}{h^2}$$

$$= \lim_{h \to 0} \frac{e^h - h - 1}{h^2} + \lim_{h \to 0} \frac{(1 - \cos 2h)}{4h^2} \cdot 4 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$\lim_{h \to 0} g(n-h) = \frac{e^{1-\{n-h\}} - \cos 2(1-\{n-h\}) - (1-\{n-h\})}{(1-\{n-h\})^2}$$

$$= \lim_{h \to 0} \frac{e^h - \cos 2h - h}{h^2} \quad [\{n-h\} = \{-h\} = 1-h] = \frac{5}{2}$$

$g(n) = \dfrac{5}{2}$. Hence, $g(x)$ is continuous, $\forall \ x \in I$.

Hence, $g(x)$ is continuous, $\forall \ x \in R$.

38. $$g'(0^+) = \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

$$g'(0^-) = \lim_{h \to 0} \frac{-h + b - 1}{-h}$$ for existence of limit $b = 1$

Thus, $g'(0^-) = 1$

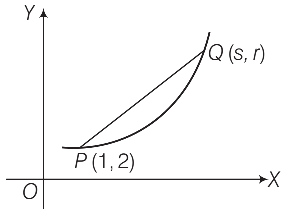
Hence, g cannot be made differentiable for no value of b.

39. By definition $f'(1)$ is the limit of the slope of the secant line when $s \to 1$.

Thus, $f'(1) = \lim_{s \to 1} \dfrac{s^2 + 2s - 3}{s - 1}$

$= \lim_{s \to 1} \dfrac{(s-1)(s+3)}{s-1}$

$= \lim_{s \to 1} (s + 3) = 4$



Aliter

By substituting $x = s$ into the equation of the secant line and cancelling by $s - 1$. Again, we get $y = s^2 + 2s - 1$.

This is $f(s)$ and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.

40. In the immediate neighbourhood of $x = \pi/2, \sin x > \sin^3 x \Rightarrow |\sin x - \sin^3 x| = \sin x - \sin^3 x$

Hence, for $x \neq \pi/2$,

$$f(x) = \left[\frac{2(\sin x - \sin^3 x) + \sin x - \sin^3 x}{2(\sin x - \sin^3 x) - \sin x + \sin^3 x}\right]$$

$$= \frac{3\sin x - 3\sin^3 x}{\sin x - \sin^3 x} = 3$$

Hence, f is continuous and differentiable at $x = \pi/2$.

41. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{f(h) + |x|h + xh^2}{h}$

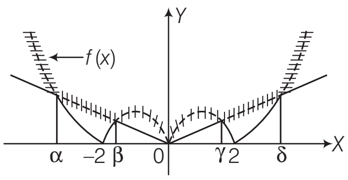
$\because$ $f(0) = 0$

Hence, $f'(x) = \lim_{h \to 0} \left[\dfrac{f(h) - f(0)}{h} + |x| + xh\right]$

$\Rightarrow$ $f'(x) = f'(0) + |x| = |x|$

42. $f(x)$ is non-differentiable at $x = \alpha, \beta, 0, \gamma, \delta$

and $g(x)$ is non-differentiable at $x = \alpha, -2, \beta, 0, \gamma, 2, \delta$.



43. We have, $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1, & \text{for } x < 1 \\ ax + b, & \text{for } x \geq 1 \end{cases}$

For differentiability at $x = 1$, $g'(1^+) = g'(1^-)$

$$a = 6 - \frac{4}{2\sqrt{1}}$$

$\Rightarrow$ $a = 6 - 2 = 4$

For continuity at $x = 1$, $g(1^+) = g(1^-)$

$$a + b = 3 - 4 + 1 \Rightarrow a + b = 0$$

$\Rightarrow$ $b = -4$

Hence, $a = 4$ and $b = -4$

44. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{-3xh + f(h)}{h}$

$= \lim_{h \to 0} \left\{-3x + \dfrac{f(h)}{h}\right\} = -3x + \lim_{h \to 0} \dfrac{f(h)}{h}$

$= -3x + 7$

45. $f(x) = \dfrac{\sin \dfrac{\pi[x]}{4}}{[x]}$

Obviously, continuity at $x = 3/2$

$$f(2^-) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

At $x = 2$,

$$f(2) = \frac{\sin\dfrac{\pi}{2}}{2} = \frac{1}{2}$$

Hence, $f(x)$ is discontinuous at $x = 2$.

46. $f(-1) = b(1-1) + 1 = 1$

and $\lim_{h \to 0} f(-1+h) = 1$

$\lim_{h \to 0} f(-1-h) = \lim_{h \to 0} \sin((-1-h+a)\pi) = -\sin \pi a$

For continuity, $\sin \pi a = -1 = \sin\left(2n\pi + \dfrac{3\pi}{2}\right)$

$\Rightarrow$ $\pi a = 2n\pi + \dfrac{3\pi}{2}$

$\Rightarrow$ $a = 2n + \dfrac{3}{2}$

Hence, $a = 2n + \dfrac{3}{2}$, $n \in I$ and $b \in R$

47. Consider the graph of $h(x) = \max(x, x^2)$ at $x = 0$ and $x = 1$

For (d) : $h(x) = \max(x^2, -x^2)$

48. Here, $g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2 \sin^2 4x$

$+ [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$

$= [x^2](\{\cos^2 4x\} + [\cos^2 4x]) + \{x^2\}([\cos^2 4x]$

$+ \{\cos^2 4x\}) + x^2 \sin^2 4x$

$= ([x^2] + \{x^2\})(\{\cos^2 4x\} + [\cos^2 4x]) + x^2 \sin^2 4x$

$= x^2 \cos^2 4x + x^2 \sin^2 4x$

$= x^2$

Clearly, $g(x)$ is always differentiable.

$\therefore$ Number of points of non-differentiability is 0.

49. Here, $f(x) = \dfrac{\{x\}g(x)}{\{x\}g(x)} = 1$, when $\{x\}g(x) \neq 0$

If $f(x)$ is periodic with period $\dfrac{1}{4}$, then $\{x\}g(x) \neq 0$ with period $\dfrac{1}{4}$.

$\Rightarrow$ $g(x) = 0$ at $x = \dfrac{k}{4}$, where $k \in I$.

50. Here, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f\left(x\left(1 + \dfrac{h}{x}\right)\right) - f\left(\dfrac{1}{1/x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(\dfrac{1 + h/x}{1/x}\right) - f\left(\dfrac{1}{1/x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{f(1+h/x)}{f(1/x)} - \dfrac{f(1)}{f(1/x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h/x)-f(1)}{h/x} \cdot \frac{1}{xf\left(\dfrac{1}{x}\right)} = f'(1)\cdot\frac{1}{x}\cdot\frac{f(x)}{f(1)}$$

$\therefore \quad \dfrac{dy}{dx} = k\cdot\dfrac{y}{x}$, where $\dfrac{f'(1)}{f(1)} = k$, $f(x) = y$.

$\Rightarrow \quad \displaystyle\int \frac{dy}{y} = \int k\frac{dx}{x}$

$\Rightarrow \quad \log y = k\log x + \log C$

$\Rightarrow \quad \log y = \log(x^k\cdot C)$

$\Rightarrow \quad y = C\cdot x^k$

$\therefore \quad f(x) = C\cdot x^k$

Put $x = 2$, $y = 1$ in $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$

$\Rightarrow \quad f(2) = \dfrac{f(2)}{f(1)} \Rightarrow f(1) = 1$

Also, $f(x) = C\cdot x^k$, put $x = 1$

$\Rightarrow \quad f(1) = C \Rightarrow C = 1$

$\therefore \quad f(x) = x^k$, now $f(2) = 4$

$\Rightarrow \quad f(2) = 2^k$

$4 = 2^k \Rightarrow k = 2$

$\therefore \quad f(x) = x^2 \Rightarrow f(5) = 25$

51. $\tan(f(x)) = \tan\left(\dfrac{x}{2} - 1\right), \quad x \in [0, \pi]$

$$0 \le x \le \pi \Rightarrow -1 \le \frac{x}{2} - 1 \le \frac{\pi}{2} - 1$$

$\therefore \tan(f(x))$ is continuous in $[0, \pi]$.

$\dfrac{1}{f(x)} = \dfrac{2}{x-2}$ is not defined at $x = 2 \in [0, \pi]$.

$y = \dfrac{x-2}{2} \Rightarrow f^{-1}(x) = 2x + 2$ is continuous in R.

52. $\displaystyle\lim_{x\to 0^+} (x+1)\, e^{-\frac{2}{x}} = \lim_{x\to 0^+} \frac{(x+1)}{e^{2/x}} = 0$

$\displaystyle\lim_{x\to 0^-} (x+1)\, e^0 = 1$

Hence, continuous for $x \in I - \{0\}$, assumes all intermediate values from $f(-2)$ to $f(2)$ and maximum value $\dfrac{3}{e}$ at $x = 2$.

53. RHL $= \displaystyle\lim_{x\to 0^+} \left(3 - \left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right]\right)$

$= 3 - [\cot^{-1}(-\infty)] = 3 - 3 = 0$

LHL $= \displaystyle\lim_{x\to 0^-} \{x^2\} \cos\left(e^{-\frac{1}{x}}\right)$

$= \displaystyle\lim_{h\to 0} h^2 \cos\left(e^{\frac{1}{h}}\right) = 0$

and $f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

54. $\displaystyle\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (b([x]^2 + [x]) + 1)$

$= \displaystyle\lim_{h\to 0} (b([-1+h]^2 + [-1+h]) + 1)$

$= \displaystyle\lim_{h\to 0} (b((-1)^2 - 1) + 1) = 1$

$\Rightarrow \quad b \in R$

$\displaystyle\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \sin(\pi(x+a))$

$= \displaystyle\lim_{h\to 0} \sin(\pi((-1-h)+a))$

$\therefore \quad \sin\pi a = -1$

$\Rightarrow \quad \pi a = 2n\pi + \dfrac{3\pi}{2} \Rightarrow a = 2n + \dfrac{3}{2}$

55. $f(x) = \begin{cases} \dfrac{\sin\frac{\pi}{4}}{1} = \dfrac{1}{\sqrt{2}}, & 1 \le x < 2 \\ \dfrac{\sin\frac{\pi}{2}}{1} = \dfrac{1}{2}, & 2 \le x < 3 \end{cases}$

Hence, $f(x)$ is continuous at $\dfrac{3}{2}$, differentiable at $\dfrac{4}{3}$ and discontinuous at 2.

56. Since, $\sin^{-1} x$ and $\cos(1/x)$ are continuous and differentiable in $x \in [-1, 1] - \{0\}$.

Now, at $x = 0$

$$f'(0^-) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h} = 0$$

$$= \lim_{h\to 0} \frac{(\sin^{-1}(-h))^2 \cos\left(-\frac{1}{h}\right) - 0}{-h} = 0$$

$$f'(0^+) = \lim_{h\to 0} \frac{(\sin^{-1} h)^2 \cos\left(\frac{1}{h}\right) - 0}{h} = 0$$

Hence, LHD = RHD. So, $f(x)$ is continuous and differentiable everywhere in $-1 \le x \le 1$.

57. $H(x) = \begin{cases} \cos x, & 0 \le x < \dfrac{\pi}{2} \\ \dfrac{\pi}{2} - x, & \dfrac{\pi}{2} < x \le 3 \end{cases}$

$$H'\left(\frac{\pi}{2}^-\right) = -\sin x = -1$$

$$H'\left(\frac{\pi}{2}^+\right) = -1$$

Hence, $H(x)$ is continuous and derivable in $[0, 3]$ and has maximum value 1 in $[0, 3]$.

58. Here, $f(x) = 3(2x+3)^{2/3} + 2x + 3$

$$\Rightarrow \quad f'(x) = \frac{4}{(2x+3)^{1/3}} + 2$$

Now, $\quad 2x + 3 \neq 0 \Rightarrow x \neq -\frac{3}{2}$

Hence, $f(x)$ is continuous but not differential at $x = -3/2$.

Also, $f(x)$ is differentiable and continuous at $x = 0$.

59. $$f'(0^+) = \lim_{h \to 0} \frac{h \ln (\cos h)}{h \ln (1 + h^2)} = \lim_{h \to 0} \frac{\ln (\cos h)^{1/h^2}}{\frac{\ln (1+h^2)}{h^2}}$$

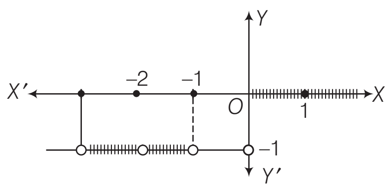
$$= \lim_{h \to 0} \frac{1}{h^2} (\cos h - 1) = -\frac{1}{2}; \text{ similarly } f'(0^-) = -\frac{1}{2}$$

Hence, f is continuous and derivable at $x = 0$.

60. $$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 0, & x = 0 \text{ or } 1 \text{ or } -1 \\ 0, & -1 < x < 0 \end{cases} \Rightarrow f(x) = 0 \text{ for all in } [-1, 1]$$

61. $$[|x|] - |[x]| = \begin{cases} 0, & x = -1 \\ -1, & -1 < x < 0 \\ 0, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$$

$\Rightarrow$ Range is $\{0, -1\}$. The graph is



62. $f'(0^+) = \frac{1}{\sqrt{2}}, f'(0^-) = -\frac{1}{\sqrt{2}}$;

$$f(x) = \frac{\sqrt{x^2}}{\sqrt{1 + \sqrt{1 - x^2}}} = \frac{|x|}{\sqrt{1 + \sqrt{1 - x^2}}}$$

63. Range is $R^+ \cup \{0\} \Rightarrow$ Option (b) is not correct.

f is not differentiable at $x = -1$

As, $$f(x) = \begin{cases} x^3 + 1, & \text{if } x \ge -1 \\ -(x^3 + 1), & \text{if } x < -1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} 3x^2, & \text{if } x > -1 \\ -3x^2, & \text{if } x < -1 \end{cases}$$

$f'(-1^+) = 3, f'(-1^-) = -3 \Rightarrow f$ is not differentiable at $x = -1$.

Also, f is not bijective, hence it has no inverse.

64. Given, f is continuous in $[a, b]$ ...(i)

$\Rightarrow$ g is continuous in $[b, c]$ ...(ii)

$\Rightarrow$ $f(b) = g(b)$ ...(iii)

$\Rightarrow$ $\left.\begin{aligned} h(x) &= f(x) \text{ for } x \in [a, b) \\ &= g(x) \text{ for } x \in (b, c] \end{aligned}\right\}$ ...(iv)

$\Rightarrow h(x)$ is continuous in $[a, b) \cup (b, c]$ [using Eqs. (i) and (ii)]

Also, $\quad f(b^-) = f(b), g(b^+) = g(b)$

[using Eqs. (i) and (ii)] ...(iv)

$\therefore \quad h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$

[using Eqs. (iv) and (v)]

Now, verify each alternative. Of course! $g(b^-)$ and $f(b^+)$ are undefined.

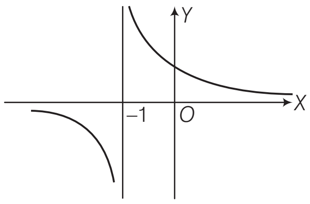
$$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$$

and $\quad h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$

Hence, $\quad h(b^-) = h(b^+) = f(b) = g(b)$

and $h(b)$ is not defined.

65. (a) Domain is $R - \{-1\}$; Range $= R - \{0\}$



(b) Domain is $x \in R$; Range $= (0, 1]$

(c) Domain is $[0, \infty)$; Range $= (0, 1]$

(d) Domain is $(-\infty, 3)$; Range $= (0, \infty)$

66. $$f(x) = \sec 2x + \operatorname{cosec} 2x = \frac{2(\sin 2x + \cos 2x)}{2 \cos 2x \sin 2x}$$

$$= \frac{2(\sin 2x + \cos 2x)}{\sin 4x}$$

is discontinuous, where $4x = n\pi, n \in I$ or $x = \frac{n\pi}{4}$.

Options (a) and (b) also satisfy the condition, since they are subsets of option (c).

67. $$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^n \cdot \sin\left(\frac{1}{x^2}\right) = 0, \text{ if } n > 0$$

and hence true for $n > 1$.

Since, $f(0) = 0$, $f(x)$ is continuous at $x = 0$.

Now, $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^n \sin \frac{1}{x^2}}{x}$$

$$= \lim_{x \to 0} x^{n-1} \sin\left(\frac{1}{x^2}\right) = 0, \text{ if } n > 1.$$

Hence, $f(x)$ is differentiable at $x = 0$, if $n > 1$.

68. $$f(x) = \begin{cases} e^x, & x \le 0 \\ 1 - x, & 0 < x < 1 \\ x - 1, & x \ge 1 \end{cases}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = 1, \lim_{x \to 0^+} f(x) = 1$$

and $$\lim_{x \to 1^-} f(x) = 0, \lim_{x \to 1^+} f(x) = 0$$

Hence, $f(x)$ is continuous at $x = 0$ and 1.

$$f'(0^-) = 1 \text{ and } f'(0^+) = -1.$$

Hence, $f(x)$ is not differentiable at $x = 1$.

69. $f(x) = \int_{-2}^{x} |t+1|\, dt = -\int_{-2}^{-1}(t+1)\, dt + \int_{-1}^{x}(t+1)\, dt$

$$= \frac{1}{2} + \left(\frac{t^2}{2} + t\right)_{-1}^{x} = \frac{x^2}{2} + x + 1, \text{ for } x \geq -1$$

$f(x)$ is a quadratic polynomial.

$\therefore$ $f(x)$ is continuous as well as differentiable in $[-1, 1]$.

Also, $f'(x)$ is continuous as well as differentiable in $[-1, 1]$.

70. We have, $f(x+y) = f(x) + f(y) + xy\,(x+y)$

$f(0) = 0$

$\therefore \quad \lim_{h \to 0} \frac{f(h)}{h} = -1$

$\therefore \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + xh\,(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} x\,(x+h) = -1 + x^2$$

$\therefore \quad f'(x) = -1 + x^2$

$\therefore \quad f(x) = \frac{x^3}{3} - x + c$

$\therefore$ $f(x)$ is a polynomial function,

$f(x)$ is twice differentiable for all $x \in R$ and

$f'(3) = 3^2 - 1 = 8$.

71. We have, $f'(x) = 6x + 12$

For $f(x)$ is increasing, $f'(x) \geq 0 \Rightarrow x \geq -2$

Hence, $f(x)$ is increasing in $[-1, 2]$

$\lim_{x \to 2^+} f(x) = 35$, $\lim_{x \to 2^-} f(x) = 35$ and $f(2) = 35$

Hence, $f(x)$ is continuous on $[-1, 3]$, $f'(2^-) = 24$ and $f'(2^+) = -1$.

Hence, $f'(2)$ doesn't exist for maximum, $f(2) = 35$

$f(-1) = -10, \qquad f(3) = 34$

Hence, $f(x)$ has maximum value at $x = 2$.

72. Since, $f(x) = 0$, $x < 0$ and differentiable at $x = 0$, LHD $= 0$ (function is on X-axis for $x < 0$). If $f(x)$,

(a) x^2, $x > 0$

RHD at $x = 0$,

$f'(0) = 2 \times 0 = 0$ (possible)

(d) $-x^{3/2}$, $x > 0$

RHD at $x = 0$,

$f'(0) = -3/2\, x^{1/2} = -3/2 \times 0 = 0$ (possible)

73. We know that, $\sin x$ is periodic function in $[0, 2\pi]$.

$\therefore$ $\sin x$ is continuous at $x = 0$

Now, $\lim_{x \to 0} [x]$

RHL $= \lim_{x \to 0^+} [x] = \lim_{h \to 0} [x+h] = x = 0 \qquad [\because [x+h] = x]$

LHL $= \lim_{x \to 0^-} [x] = \lim_{h \to 0} [x-h] = (x-1) \qquad [\because [x-h] = (x-1)]$

$= 0 - 1 = -1$

$\therefore$ RHL $\neq$ LHL

So, $[x]$ is discontinuous at $x = 0$.

We know that, sum of continuous and discontinuous functions is discontinuous.

74. We have,

$$f(x) = \begin{cases} 2\sin(a\cos^{-1} x), & \text{if } x \in (0, 1) \\ \sqrt{3}, & \text{if } x = 0 \\ ax + b, & \text{if } x < 0 \end{cases}$$

Continuity at $x = 0$

(LHL at $x = 0$) $= \lim_{x \to 0^-} ax + b$

$= \lim_{h \to 0} a(0-h) + b = \lim_{h \to 0} -ah + b$

$= b$

(RHL at $x = 0$) $= \lim_{x \to 0^+} 2\sin(a\cos^{-1} x)$

$= \lim_{h \to 0} 2\sin(a\cos^{-1}(0+h))$

$= \lim_{h \to 0} 2\sin a\cos^{-1} h$

$= 2\sin a\cos^{-1} h = 2\sin a\cos^{-1} 0$

$= 2\sin\frac{a\pi}{2}$

$f(0) = \sqrt{3}$

For $f(x)$ to be continuous at $x = 0$,

LHL $=$ RHL $= f(0)$

$\therefore \quad b = 2\sin\frac{a\pi}{a} = \sqrt{3}$

$\therefore \quad b = \sqrt{3}$ and $\sin\frac{a\pi}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad b = \sqrt{3}$ and $\frac{a\pi}{2} = \frac{\pi}{3}$

$\Rightarrow \quad b = \sqrt{3}$ and $a = \frac{2}{3}$

So, Statement I is correct.

Since, for $x < 0$, $f(x) = ax + b$

which is a polynomial function and will be continuous for $(-\infty, 0)$.

Again, for $x \in (0, 1)$,

$f(x) = 2\sin(a\cos^{-1} x)$, which is trigonometric function will be continuous for $x \in (0, 1)$.

$\because$ $f(x)$ is continuous in $(-\infty, 1)$.

$\therefore$ Statement II is also correct.

75. We have, $f(x) = \begin{cases} \dfrac{\cos x - e^{-\frac{x^2}{2}}}{x^3}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Clearly, $f(0) = 0$

Now consider, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\cos x - e^{-\frac{x^2}{2}}}{x^3} \qquad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{-\sin x + e^{-\frac{x^2}{2}} \cdot x}{3x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-\cos x + e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \cdot x^2}{6x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\sin x - e^{-\frac{x^2}{2}} \cdot x - 2xe^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}}{6}$$

$$= 0$$

Thus, $\lim_{x \to 0} f(x) = f(0)$

$\Rightarrow f$ is continuous at $x = 0$

Hence, option (c) is correct.

76. Let $f(x) = \frac{x^3}{4} - \sin \pi x + \frac{2}{3}$. Then, $f(x)$ will be continuous function. ($\because$ Sum and difference of two continuous function is continuous)

Here, $f(2) = \frac{8}{3}$ and $f(-2) = -\frac{4}{3}$ $\qquad [\because \sin n\pi = 0, \forall\ n \in Z]$

Now, as $f(-2) < 0 < f(2)$, therefore by intermediate value theorem we can say that there exists atleast one point $n \in [-2, 2]$. Such that $f(n) = 0$

Hence, $f(x) = 0$, i.e. $\frac{x^3}{4} - \sin \pi x + \frac{2}{3} = 0$ has atleast one solution in $[-2, 2]$.

Clearly, Statement II is wrong. Because for this to be true, $f(x)$ should be a continuous function (by intermediate value theorem).

Hence, option (c) is correct.

77. We have,

$$f(x) = x\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$$

$$\therefore f(-x) = (-x)\left(\frac{e^{2(-x)} - e^{-2(-x)}}{e^{2(-x)} + e^{-2(-x)}}\right) + (-x)^2 + (-x)^4$$

$$= -x\left(\frac{e^{-2x} - e^{+2x}}{e^{-2x} + e^{+2x}}\right) + x^2 + x^4$$

$$= -x\left[-\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right)\right] + x^2 + x^4$$

$$= x\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) + x^2 + x^4$$

$$= f(x)$$

$\therefore f(x)$ is even function and even function can't has range equal to R.

78. $y = |\ln x|$ not differentiable at $x = 1$.

$y = |\cos|x||$ is not differentiable at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

$y = \cos^{-1}(\text{sgn } x) = \cos^{-1}(1) = 0$ differentiable, $\forall\ x \in (0, 2\pi)$.

79. $f'(0^+) = \lim_{h \to 0} \frac{h \sin h - 0}{h} = 0$

$f'(0^-) = \lim_{h \to 0} \frac{h \sin(-h) - 0}{-h} = 0$

$f(x)$ is differentiable at $x = 0$.

e.g. $x|x|$ is derivable at $x = 0$.

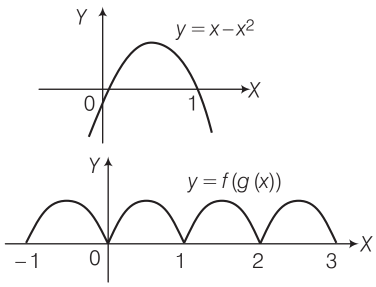
80. Consider $g(x) = x^3$ at $x = 0$, $g(0) = 0$

$|g(x)|$ is derivable at $x = 0$.

Actually nothing definite can be said. Also, for $g(x) = x - 1$ with $g(1) = 0$.

Then, $|g(x)|$ is not derivable at $x = 1$.

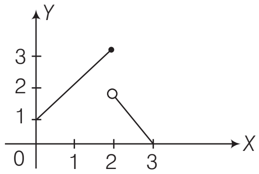
81.



82. $f(x)$ is discontinuous at $x = 0$ and $f(x) < 0, \forall\ x \in [-\alpha, 0)$ and $f(x) > 0, \forall\ x \in [0, \alpha]$.

Sol. (Q. Nos. 83 to 85)

$$f(x) = \begin{cases} x + 1, & 0 \le x \le 2 \\ -x + 3, & 2 < x < 3 \end{cases}$$



$\therefore f(x)$ is discontinuous at $x = 2$.

$$(fof)(x) = \begin{cases} x + 2, & 0 \le x \le 1 \\ -x + 2, & 1 < x \le 2 \\ -x + 4, & 2 < x < 3 \end{cases}$$

$\Rightarrow (fof)(x)$ is discontinuous at $x = 1, 2$

and $\qquad f(19) = f(3 \times 6 + 1) = f(1) = 2$

83. (a) **84.** (b, c) **85.** (a)

Sol. (Q. Nos. 86 to 89)

Since, $f(-x) = \frac{1}{f(x)}$

$\therefore$ At $x = 0$

$\Rightarrow f(0) = \frac{1}{f(0)} \Rightarrow f^2(0) = 1 \Rightarrow f(0) = +1$, as $f(x) > 0$.

$$\Rightarrow \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h}$$

$$\therefore \qquad f'(x) = f(x) \cdot \left(\lim_{h \to 0} \frac{f(h) - 1}{h}\right) = f(x) \cdot f'(0)$$

$$\Rightarrow \qquad \int \frac{f'(x)}{f(x)} = \int -1\, dx \Rightarrow \log f(x) = -x + c$$

$$f(x) = e^{-x} \cdot \lambda \text{ at } x = 0,\ \lambda = 1 \Rightarrow f(x) = e^{-x}$$

$\Rightarrow$ Range of $f(x) \in R^+$. $\Rightarrow$ Range of $f(|x|)$ is $[0, 1]$.

$\Rightarrow f(x)$ is decreasing function

and $\qquad f'(x) = -e^{-x} = -f(x)$.

86. (c) **87.** (a) **88.** (d) **89.** (a)

90. $f(x)=\begin{cases}\sin x, & x\le 0\\ \tan x, & 0<x<2\pi\\ \cos x, & 2\pi\le x<3\pi\\ 3\pi, & x=3\pi\end{cases}$

$f(x)$ is discontinuous at $\frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$.

91. $f(x)=[2x^3-5]$

Since $1\le x<2 \Rightarrow 1\le x^3<8$

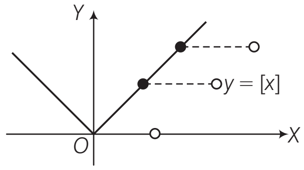
$-3\le 2x^3-5<11$

Now, $2x^3-5$ is discontinuous at integer points.

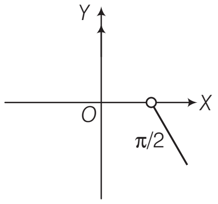
$\therefore\ -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$

Hence, number of points of discontinuity = 13

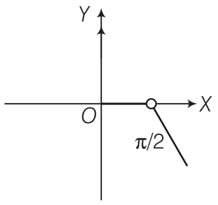
92. Max $([x], |x|)$, hence discontinuity at $x=\phi$.



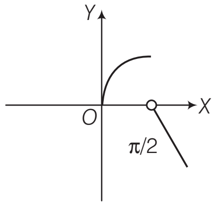
93. From figure, option (c) is correct.



94. From figure, option (b) is correct.



95. From figure, option (c) is correct.



Sol. (Q. Nos. 96 to 99)

Here, $f(x+y^{2n+1})=f(x)+(f(y))^{2n+1}$

On differentiating, we get

$$f'(x+y^{2n+1})=f'(x)$$

$$\left[\because\ x \text{ and } y \text{ are independent, so } \frac{dy}{dx}=0\right]$$

$\Rightarrow\ f'(x)$ is constant, say $f'(x)=k$.

On integrating, we get $f(x)=kx+c$

Now, $f(0)=0 \Rightarrow c=0$ and $f(1)=1 \Rightarrow k=1$

$\therefore\ f(x)=x \Rightarrow f'(x)=1$, for all $x\in R$

$\therefore\ f(x)=x$ is an odd function.

96. (b) **97.** (c) **98.** (d) **99.** (a)

We know, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

and $f'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$

On adding, we get

$$2f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \quad \text{...(i)}$$

$$\Rightarrow\ 2f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h}$$

$$\Rightarrow\ 2f'(x)=\lim_{h\to 0} f(x)\cdot\frac{\{f(h)-f(-h)\}}{h}$$

[using $f(x+y)-f(x-y)=f(x)\{f(y)-f(-y)\}$]...(ii)

Also, $2f'(0)=\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}+\frac{f(-h)-f(0)}{-h}\right)$

[using Eq. (i)]

$$=\lim_{h\to 0}\frac{f(h)-f(-h)}{h} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$2f'(x)=f(x)\cdot 2f'(0) \Rightarrow \frac{f'(x)}{f(x)}=f'(0)$$

$$\therefore\ \frac{f'(x)}{f(x)}=2010 \quad \text{...(iv)}$$

On integrating both sides, we get

$\log(f(x))=2010x+c$, as $f(0)=1$

$\therefore\ c=0 \Rightarrow f(x)=e^{2010x}$

Thus, $\{f(x)\}$ is non-periodic. ...(v)

100. (b) **101.** (d)

102. (A) $\lim_{h\to 0}\sin\{1-h\}=\cos 1+a$

$\Rightarrow\ \lim_{h\to 0}\sin(1-h)-\cos 1=a$

$\Rightarrow\ a=\sin 1-\cos 1$

Now, $|K|=\dfrac{\sin 1-\cos 1}{\sqrt{2}\left(\sin 1\cdot\frac{1}{\sqrt{2}}-\cos 1\cdot\frac{1}{\sqrt{2}}\right)}=1 \Rightarrow K=\pm 1$

(B) $f(0)=\lim_{x\to 0}\dfrac{2\sin^2\left(\frac{\sin x}{2}\right)}{x^2\cdot\left(\frac{\sin x}{2}\right)^2}\cdot\left(\frac{\sin x}{2}\right)^2=\frac{1}{2}$

(C) Function should have same rule for θ and $\theta' \Rightarrow x=1-x$

$\Rightarrow\ x=\frac{1}{2}$

(D) $f(x)=x+\{-x\}+[x]$

x is continuous at $x\in R$.

Check at $x=I$ (where I is integer)

$f(I^+)=2I+1$ or $f(I^-)=2I-1$

So, $f(x)$ is discontinuous at every integer, i.e. $\{1, 0, -1\}$.

103. (A) $f(x)\begin{cases} 1-1, & 1<x\le 2 \\ 0, & x=1 \\ 1-x, & 0\le x<1 \\ -\sin\pi x, & -1\le x<0 \end{cases}$

At $x=0$, $f(x)$ is not continuous and not differentiable. At $x=1$, $f(x)$ is continuous and non-differentiable. At $x=2$ and -1, $f(x)$ is continuous and differentiable.

(B) $f(0^-)=\lim\limits_{h\to 0} h^2 e^{-\frac{1}{h}}=\lim\limits_{h\to 0}\dfrac{h^2}{e^{1/h}}=0$

(C) $f(x)=\dfrac{x}{x+1}$, not defined at $x=-1$.

$$g(x)=\frac{f(x)}{f(x)+2}$$

$g(x)$ is not defined at $f(x)=-2$

$\Rightarrow \quad \dfrac{x}{x+1}=-2 \Rightarrow x=-\dfrac{2}{3}$

Also, $x=0$ is not in domain.

$\therefore$ $f(x)$ is not differentiable at 3 points.

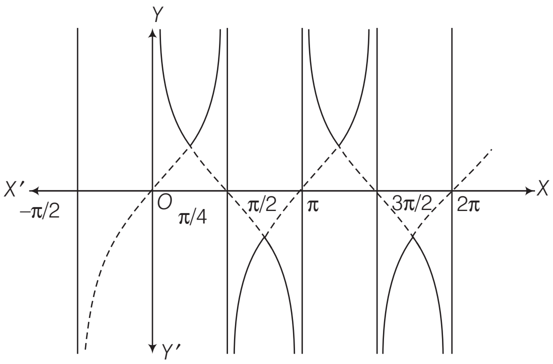
(D) $y=\operatorname{sgn}(x^2-1)=\begin{cases} 1, & x^2-1>0 \\ 0, & x^2-1=0 \\ -1, & x^2-1<0 \end{cases}=\begin{cases} 1, & |x|>1 \\ 0, & |x|=1 \\ -1, & |x|<1 \end{cases}$

$\therefore$ Tangent exists for all x.

$\therefore$ Number of points where tangent does not exists is 0.

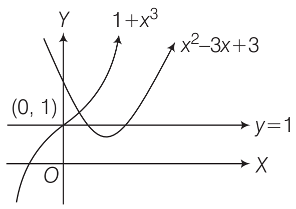
104. (A) $f(x)=\dfrac{\tan x+\cot x}{2}-\left|\dfrac{\tan x-\cot x}{2}\right|$

$$=\begin{cases} \cot x, & \tan x\ge \cot x \\ \tan x, & \cot x>\tan x \end{cases}$$



There are 4 points where the function is continuous but not differentiable in $(0, 2\pi)$.

(B) $f(x)=\min\{1, 1+x^3, x^2-3x+3\}$ can be shown as



$$\therefore \qquad f(x)=\begin{cases} 1+x^3, & x\le 0 \\ 1, & 0\le x\le 1 \text{ or } x\ge 2 \\ x^2-3x+3, & 1\le x\le 2 \end{cases}$$

Clearly, $f(x)$ is not differentiable at 2 points.

(C) $f(x)=(x+4)^{\frac{1}{3}}$

$\Rightarrow f'(x)=\dfrac{1}{3(x+4)^{\frac{2}{3}}}$

$\therefore$ Functions non-derivable at $x=-4$, i.e., at one point.

(D) $f(x)=\begin{cases} -\dfrac{\pi}{2}\cdot\log\left(\dfrac{2x}{\pi}\right)+\dfrac{\pi}{2}, & 0<x\le\dfrac{\pi}{2} \\ \pi-x, & \dfrac{\pi}{2}<x<\dfrac{3\pi}{2} \end{cases}$

$$f'(x)=\begin{cases} -\dfrac{\pi}{2x}, & 0<x<\dfrac{\pi}{2} \\ -1, & \dfrac{\pi}{2}<x<\dfrac{3\pi}{2} \end{cases}$$

$$f'\left(\frac{\pi^-}{2}\right)=f'\left(\frac{\pi^+}{2}\right)=-1$$

$\therefore$ $f(x)$ is differentiable for all $x\in\left(0, \dfrac{3\pi}{2}\right)$.

$\therefore$ Number of points of non-differentiable is 0.

105. (A) $Rf'(0)=\lim\limits_{h\to 0}\dfrac{\cos h-1}{h}=0$ and $Lf'(0)=\lim\limits_{h\to 0}\dfrac{-h+1-1}{-h}=-1$

Obviously,

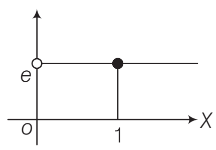
$f(0)=f(0^-)=f(0^+)=1$

Hence, continuous and not derivable.

(B) $g(x)=0$ for all x, hence continuous and derivable.

(C) As $0\le\{x\}<1$, hence $h(x)=\sqrt{\{x\}^2}=\{x\}$ which is discontinuous, hence non-derivable all $x\in I$.

(D) $\lim\limits_{x\to 1} x^{\frac{1}{\ln x}}=\lim\limits_{x\to 1} x^{\log_x e}=e=f(1)$



Hence, $k(x)$ is constant for all $x>0$, hence continuous and differentiable at $x=1$.

106. (A) $l=\lim\limits_{x\to\infty} e^{x^2+1}\left[e^{\sqrt{x^4+1}-(x^2+1)}-1\right]$

Consider, $\lim\limits_{x\to\infty}[\sqrt{x^4+1}-(x^2+1)]$

$$=\lim_{x\to\infty}\frac{x^4+1-(x^4+1+2x^2)}{\sqrt{x^4+1}+(x^2+1)}$$

$$=\lim_{x\to\infty}\frac{-2x^2}{x^2(\sqrt{1+(1/x^4)}+1+1/x^2)}=-1$$

Now, as $x \to \infty$, $\sqrt{x^4 + 1} - (x^2 + 1) \to -1$

$\infty \times \left(\frac{1}{e} - 1\right) \to -\infty$ and hence limit doesn't exist.

(B) $f(0^+) = \lim\limits_{h \to 0} \frac{a^{2h} - 2a^h + 1}{h^2} = \lim\limits_{h \to 0} \left(\frac{a^h - 1}{h}\right)^2 = \ln^2 a$

$f(0^-) = \lim\limits_{h \to 0} 3 \ln(a + h) - 2 = 3 \ln a - 2 = f(0)$

For continuous

$\ln^2 a = 3 \ln a - 2 \qquad \ln^2 a - 3 \ln a + 2 = 0$

$\Rightarrow \quad (\ln a - 2)(\ln a - 1) = 0;\; a = e^2$ or $a = e$

(C) $L = a^a \ln ae$

$\Rightarrow \quad M = a^a$

$\therefore \quad a^a \ln ae = 2a^a$

$\therefore \; \ln ae = 2 \Rightarrow ae = e^2 \Rightarrow a = e$

107. $\tan^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sec^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow$ Number of discontinuities = 2

108. Let $g(x) = x^{1/x}, g'(x) = x^{1/x} \frac{1 - \ln x}{x^2}$

+ | −
e

$\Rightarrow \quad g_{\max} = e^{1/e} \in (1, 2)$

$\Rightarrow \quad \lim\limits_{x \to 0} x^{1/x} = 0, \; \lim\limits_{x \to \infty} x^{1/x} = 1$

So, $\quad f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 \le x < \infty \end{cases}$

109. $f(x) = x + \cos x + 2, f(0) = 3 \Rightarrow g(3) = 0$

$g(f(x)) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1$, putting $x = 0$

$g'(3) \cdot f'(0) = 1$

Now, $\quad f'(x) = 1 - \sin x \Rightarrow f'(0) = 1 \Rightarrow \quad g'(3) = 1$

110. Let $g(x) = x \tan^{-1}(x^2)$. It is an odd function.

So, $\quad g^{2m}(0) = 0.$

Let $\quad h(x) = x^4$

So, $\quad f(x) = g(x) + h(x)$

$\Rightarrow \quad f^{2m}(0) = g^{2m}(0) + h^{2m}(0)$

$= h^{2m}(0) \ne 0$

It happens when $\; 2m = 4 \Rightarrow m = 2$

111. $F(x) = 3e^x$ and $G(x) = e^{-x}$

The equation $9x^4 = (F(x))^2 G(x)$ becomes $x^4 = e^x$

Hence, number of solutions = 2

112. $y' = f'(x) - 2f'(2x)$

$y'(1) = f'(1) - 2f'(2) = 5 \quad$ ...(i)

and $\quad y'(2) = f'(2) - 2f'(4) = 7 \quad$ ...(ii)

Now, let $\quad y = f(x) - f(4x) - 10x$

$y' = f'(x) - 4f'(4x) - 10$

$y'(1) = f'(1) - 4f'(4) - 10 \quad$ ...(iii)

On substituting the value of $f'(2) = 7 + 2f'(4)$ in Eq. (i), we get

$f'(1) - 2[7 + 2f'(4)] = 5$

$f'(1) - 4f'(4) = 19$

$\Rightarrow \quad f'(1) - 4f'(4) - 10 = 9$

113. A, B, C are in AP.

$\therefore \quad 2B = A + C$ and $A + B + C = 180°$

$\therefore \quad B = 60°$

$\therefore \quad \cos B = \frac{1}{2} = \frac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 + c^2 = b^2 + ac$

$\Rightarrow \quad (a - c)^2 = b^2 - ac$

or $\quad |\sin A - \sin C| = \sqrt{\sin^2 B - \sin A \sin C}$

$\Rightarrow \; 2\cos\left(\frac{A + C}{2}\right)\left|\sin\left(\frac{A - C}{2}\right)\right| = \sqrt{\frac{3}{4} - \sin A \sin C}$

$\Rightarrow \quad 2\left|\sin\left(\frac{A - C}{2}\right)\right| = \sqrt{3 - 4\sin A \sin C}$

$\therefore \quad \lim\limits_{A \to C} \frac{\sqrt{3 - 4\sin A \sin C}}{|A - C|} = \lim\limits_{A \to C} \frac{2\left|\sin\left(\frac{A - C}{2}\right)\right|}{|A - C|}$

$= \lim\limits_{A \to C} \left|\frac{\sin\left(\frac{A - C}{2}\right)}{\left(\frac{A - C}{2}\right)}\right|$

$= |1| = 1 \Rightarrow f(x) = 1$

$\therefore \quad f'(x) = 0$

114. $\text{RHD} = \lim\limits_{h \to 0} h \frac{\left(\left(\frac{3}{4}\right)^{1/h} - \left(\frac{3}{4}\right)^{-1/h}\right) - 0}{\left(\left(\frac{3}{4}\right)^{1/h} + \left(\frac{3}{4}\right)^{-1/h}\right) \cdot h} = -1$

$\text{LHD} = \lim\limits_{h \to 0} (-h) \frac{\left(\left(\frac{3}{4}\right)^{-1/h} - \left(\frac{3}{4}\right)^{1/h}\right) - 0}{\left(\left(\frac{3}{4}\right)^{-1/h} + \left(\frac{3}{4}\right)^{1/h}\right) \cdot (-h)} = 1$

$\therefore \quad P = f'(0^-) - f'(0^+) = 1 - (-1) = 2$

Now, $4 \cdot \left(\lim\limits_{x \to 2^-} \frac{(\exp((x + 2)\log 4))^{\frac{[x+1]}{4}} - 16}{4^x - 16}\right)$

$= 4 \cdot \left[\lim\limits_{x \to 2^-} \frac{(4^{x+2})^{\frac{[x+1]}{4}} - 16}{4^x - 16}\right] = 4 \times \frac{1}{2} = 2$

115. Here, $f(x) = -x^3 + x^2 - x + 1 \Rightarrow f'(x) = -3x^2 + 2x - 1$

$\Rightarrow \quad D = 4 - 12 = -8 < 0$

$\therefore \quad f(x)$ is decreasing.

Thus, min $f(t) = f(x)$, as $f(x)$ is decreasing and $0 \le t \le x$.

$$\therefore \quad g(x)=\begin{cases}-x^3+x^2-x+1, & 0<x\le 1\\ x-1, & 1<x\le 2\end{cases}$$

$$\Rightarrow \quad \lim_{x\to 1^+} g(g(x)) = \lim_{x\to 1^+} g(0^+) = 1$$

$$\Rightarrow \quad \lim_{x\to 1^-} g(g(x)) = \lim_{x\to 1^-} g(0^+) = 1$$

$$\therefore \quad \lim_{x\to 1} g(g(x)) = 1$$

116. We have, $f(x)=(x-|x|)^2(1-x+|x|)^2$

$$\Rightarrow \quad f(x)=\begin{cases}(2x)^2\,(1-2x)^2, & x<0\\ 0, & x\ge 0\end{cases}$$

$$=\begin{cases}16x^4-16x^3+4x^2, & x<0\\ 0, & x\ge 0\end{cases}$$

Clearly, $f(x)$ is continuous as well as derivable for all $x\in R$.

$\therefore$ Number of points of non-differentiable = 0.

117. $$\text{RHL} = \lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-h^2)\right)\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$$

$$=\lim_{h\to 0}\frac{\cos^{-1}(1-h^2)\sin^{-1}(1-h)}{\sqrt{2}h(1-h^2)}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}\sqrt{2h^2-h^4}\cdot\sin^{-1}(1-h)}{\sqrt{2}h(1+h)(1-h)}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}(h\sqrt{2-h^2})\sin^{-1}(1-h)}{\sqrt{2}h(1+h)(1-h)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{2}}\cdot\frac{\sin^{-1}(h\sqrt{2-h^2})}{h\cdot\sqrt{2-h^2}}\cdot\frac{\sqrt{2-h^2}}{1+h}\cdot\frac{\sin^{-1}(1-h)}{(1-h)}$$

$$=\frac{1}{\sqrt{2}}\cdot\sqrt{2}\cdot\frac{\sin^{-1}(1)}{1}=\frac{\pi}{2} \Rightarrow k=\frac{\pi}{2} \quad \text{...(i)}$$

$$\text{LHL}=\lim_{h\to 0}\frac{A\sin^{-1}(1-(1-h))\cdot\cos^{-1}(1-(1-h))}{\sqrt{2(1-h)}\cdot(1-(1-h))}$$

$$=\lim_{h\to 0}\frac{A\cdot\sin^{-1}(h)\cdot\cos^{-1}(h)}{\sqrt{2(1-h)}\cdot h}=\frac{A\cdot\pi/2}{\sqrt{2}}=\frac{A\pi}{2\sqrt{2}}$$

$$\Rightarrow \quad \frac{A\pi}{2\sqrt{2}}=\frac{\pi}{2} \Rightarrow A=\sqrt{2}$$

Hence, $\sin^2 k+\cos^2\left(\frac{A\pi}{\sqrt{2}}\right)=\sin^2\left(\frac{\pi}{2}\right)+\cos^2(\pi)=1+1=2.$

118. $\quad f(x)=[[x]]-[x-1]$

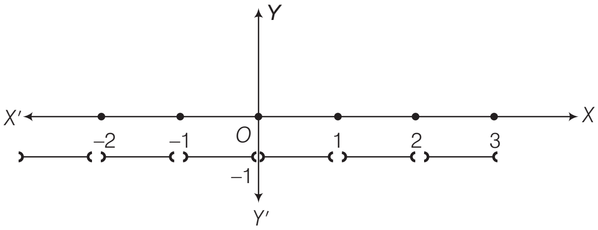
or $\quad f(x)=[x]-[x]+1$

or $f(x)=1$, which is constant function and which is continuous for all real numbers.

119. (i) $f(x)=[x]+[-x]=\begin{cases}x-x, & x\in \text{integers}\\ [x]-1-[x], & x\notin \text{integers}\end{cases}$

$$\therefore\ f(x)=\begin{cases}0, & x\in\text{integers}\\ -1, & x\notin\text{integers}\end{cases}$$

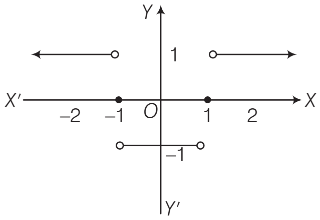
which shows the graph of $f(x)$ as shown in figure.



Thus, $f(x)$ is discontinuous at $x\in$ integers.

(ii) $g(x)=\lim_{n\to\infty}\dfrac{x^{2n}-1}{x^{2n}+1}=\begin{cases}-1, & |x|<1\\ 0, & |x|=1\\ 1, & |x|>1\end{cases}$

which can be shown as in the figure.



Thus, $g(x)$ is discontinuous at $x=\pm 1$.

120. $f(x)=\begin{cases}x\,[x], & 0\le x<2\\ (x-1)\,[x], & 2\le x\le 3\end{cases}$

To check continuity at $x = 1$

RHL (at $x=1$) $=\lim_{h\to 0}(1+h)\,[1+h]=1$

LHL (at $x=1$) $=\lim_{h\to 0}(1-h)\,[1-h]=0$

Hence, the function is discontinuous at $x=1$.

To check continuity at $x = 2$

RHL (at $x=2$) $=\lim_{h\to 0}(2+h-1)\,[2+h]=2$

LHL (at $x=2$) $=\lim_{h\to 0}(2-h)\,[2-h]=2 \Rightarrow f(2)=(2-1)\,[2]=2$

Hence, the function is continuous at $x=2$.

To check differentiability at $x = 2$

$$\text{RHD (at } x=2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h-1)\,[2+h]-2}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)\,2-2}{h}=2$$

$$\text{LHD (at } x=2) = \lim_{h\to 0}\frac{(2-h)\,[2-h]-2}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{2-h-2}{-h}=1$$

which shows $f(x)$ is not differentiable at $x=2$.

Also, $f(x)$ is not differentiable at $x=1$, as $f(x)$ at $x=1$ is not continuous.

121. Differentiating both the sides, we get

$$h'(x) = 2\, f(x) \cdot f'(x) + 2g(x) \cdot g'(x)$$

$$h'(x) = 2\, f(x) \cdot g(x) + 2g(x) \cdot f''(x)$$

[as $g(x) = f'(x)$, $g'(x) = f''(x)$]

$$h'(x) = 2\, f(x) \cdot g(x) - 2g(x) \cdot f(x) \quad [\because f''(x) = -f(x)]$$

$$h'(x) = 0$$

$\therefore$ $h(x)$ must be constant function.

Given, $h(5) = 11$

Hence, $h(10) = 11$

122. Given that,

$$f(x + y) = f(x) \cdot f(y) \text{ for all } x \in R \quad \text{...(i)}$$

On putting $x = y = 0$ in Eq. (i), we get

$$f(0)\{f(0) - 1\} = 0$$

$\Rightarrow$ $f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$, then

$$f(x) = f(x + 0) = f(x) \cdot f(0)$$

$\Rightarrow$ $f(x) = 0$ for all $x \in R$

which is not true, so $f(0) = 1$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h} \quad \text{[from Eq. (i)]}$$

$$= \lim_{h \to 0} f(x) \left\{ \frac{f(h) - 1}{h} \right\}$$

$$= f(x) \cdot f'(0)$$

$$= 2\, f(x) \quad [\text{given } f'(0) = 2]$$

or $\dfrac{f'(x)}{f(x)} = 2$

On integrating both the sides, we get $\log_e f(x) = 2x + C$

At $x = 0$, $f(0) = 1$

Hence, $\log f(1) = 2(0) + C$

$\Rightarrow$ $\log 1 = 0 + C$

$\Rightarrow$ $C = 0$

$\Rightarrow$ $\log_e f(x) = 2x + 0$

$\therefore$ $f(x) = e^{2x}$

123. Since, $f(x)$ is differentiable at $x = 0$.

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = p \text{ (say)} \quad \text{...(i)}$$

Then, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

or $f'(x) = \lim_{h \to 0} \dfrac{f\left\{\dfrac{3x + 3h}{3}\right\} - f\left(\dfrac{3x + 3 \cdot 0}{3}\right)}{h}$

or $f'(x) = \lim_{h \to 0} \dfrac{f(3x) + f(3h) + f(0) - f(3x) - f(0) - f(0)}{3h}$

or $f'(x) = \lim_{h \to 0} \dfrac{f(3h) - f(0)}{3h}$

$$\left[\text{using } f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y) + f(0)}{3}\right]$$

or $f'(x) = f'(0)$ or $f'(x) = p$ (let) [from Eq. (i)]

$\therefore$ $f(x) = px + q$

which shows $f(x)$ is differentiable for all x in R.

124. We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + 2hx - 1 - f(x)}{h}$$

[using given definition]

$$= \lim_{h \to 0} \left\{ 2x + \frac{f(h) - 1}{h} \right\}$$

Now, substituting $x = y = 0$ in the given functional relation, we get

$$f(0) = f(0) + f(0) + 0 - 1$$

$\Rightarrow$ $f(0) = 1$

$$\therefore \quad f'(x) = 2x + \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= 2x + f'(0)$$

$\Rightarrow$ $f'(x) = 2x + \cos\alpha$

On integrating, $f(x) = x^2 + x \cos \alpha + C$

Here, at $x = 0$, $f(0) = 1$

$\therefore$ $1 = C \Rightarrow f(x) = x^2 + x \cos \alpha + 1$

It is a quadratic in x with discriminant,

$$D = \cos^2 \alpha - 4 < 0$$

and coefficient of $x^2 = 1 > 0$

$\therefore$ $f(x) > 0, \ \forall\, x \in R$

125. **Plan** If a continuous function has values of opposite sign inside an interval, then it has a root in that interval.

$$f, g : [0, 1] \to R$$

We take two cases.

***Case* I** Let f and g attain their common maximum value at p.

$\Rightarrow$ $f(p) = g(p)$,

where $p \in [0,1]$

***Case* II** Let f and g attain their common maximum value at different points.

$\Rightarrow$ $f(a) = M$ and $g(b) = M$

$\Rightarrow$ $f(a) - g(a) > 0$ and $f(b) - g(b) < 0$

$\Rightarrow$ $f(c) - g(c) = 0$ for some $c \in [0,1]$ as f and g are continuous functions.

$\Rightarrow$ $f(c) - g(c) = 0$ for some $c \in [0,1]$ for all cases. ...(i)

Option (a) $\Rightarrow f^2(c) - g^2(c) + 3[f(c) - g(c)] = 0$

which is true from Eq. (i).

Option (d) $\Rightarrow f^2(c) - g^2(c) = 0$, which is true from Eq. (i).

Now, if we take $f(x) = 1$ and $g(x) = 1$, $\forall x \in [0,1]$

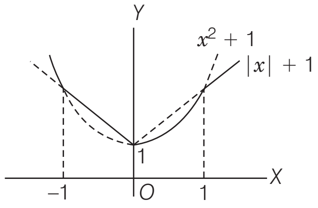
Options (b) and (c) does not hold.

Hence, options (a) and (d) are correct.

126. **Plan**

(i) In these type of questions, we draw the graph of the function.

(ii) The points at which the curve taken a sharp turn, are the points of non-differentiability. Curve of $f(x)$ and $g(x)$ are



$h(x)$ is not differentiable at $x = \pm 1$ and 0.

As, $h(x)$ take sharp turns at $x = \pm 1$ and 0.

Hence, number of points of non-differentiability of $h(x)$ is 3.

127. **Plan** To check differentiability at a point we use RHD and LHD at a point and if RHD = LHD, then $f(x)$ is differentiable at the point.

Description of Situation

As, $R\{f'(x)\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

and $L\{f'(x)\} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$

To check differentiable at $x = 0$,

$$R\{f'(0)\} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \left|\cos \frac{\pi}{h}\right| - 0}{h} = \lim_{h \to 0} h \cdot \left|\cos \frac{\pi}{h}\right| = 0$$

$$L\{f'(0)\} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{h^2 \left|\cos\left(-\frac{\pi}{h}\right)\right| - 0}{-h} = 0$$

So, $f(x)$ is differentiable at $x = 0$.

To check differentiability at $x = 2$,

$$R\{f'(2)\} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 \left|\cos\left(\frac{\pi}{2+h}\right)\right| - 0}{h} = \lim_{h \to 0} \frac{(2+h)^2 \cdot \cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 \cdot \sin\left(\frac{\pi h}{2(2+h)}\right)}{h \cdot \frac{\pi}{2(2+h)}} \cdot \frac{\pi}{2(2+h)} = \pi$$

and $L\{f'(2)\} = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$

$$= \lim_{h \to 0} \frac{(2-h)^2 \cdot \left|\cos \frac{\pi}{2-h}\right| - 2^2 \cdot \left|\cos \frac{\pi}{2}\right|}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)^2 \left(-\cos \frac{\pi}{2-h}\right) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-(2-h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2-h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(2-h)^2 \cdot \sin\left(-\frac{\pi h}{2(2-h)}\right)}{h \times \frac{-\pi}{2(2-h)}} \times \frac{-\pi}{2(2-h)} = -\pi$$

Thus, $f(x)$ is differentiable at $x = 0$ but not differentiable at $x = 2$.

128. $f(2n) = a_n, \quad f(2n^+) = a_n$

$f(2n^-) = b_n + 1 \Rightarrow a_n - b_n = 1$

$f(2n + 1) = a_n \quad \Rightarrow \quad f\{(2n + 1)^-\} = a_n$

$f\{(2n + 1)^+\} = b_{n+1} - 1$

$\Rightarrow \quad a_n = b_{n+1} - 1$ or $a_n - b_{n+1} = -1$

or $\quad a_{n-1} - b_n = -1$

129. $f(x + y) = f(x) + f(y)$, as $f(x)$ is differentiable at $x = 0$.

$\Rightarrow \quad f'(0) = k$ ...(i)

Now, $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) - f(x)}{h} = \lim_{h \to 0} \frac{f(h)}{h} \quad \left[\frac{0}{0} \text{form}\right]$$

$\left[\begin{array}{l}\text{given, } f(x + y) = f(x) + f(y), \forall\ x, y \\ \therefore \quad f(0) = f(0) + f(0), \\ \text{when} \quad x = y = 0 \Rightarrow f(0) = 0\end{array}\right]$

Using L'Hospital's rule,

$$\lim_{h \to 0} \frac{f'(h)}{1} = f'(0) = k \quad \text{...(ii)}$$

$\Rightarrow \quad f'(x) = k$, integrating both sides, we get

$f(x) = kx + C$, as $f(0) = 0 \Rightarrow C = 0$

$\therefore \quad f(x) = kx$

$\therefore f(x)$ is continuous for all $x \in R$ and $f'(x) = k$, i.e. constant for all $x \in R$. Hence, (b) and (c) are correct.

130. We have, $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \le -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \log x, & x > 1 \end{cases}$

Continuity at $x = -\frac{\pi}{2}$, $f\left(-\frac{\pi}{2}\right) = -\left(-\frac{\pi}{2}\right) - \frac{\pi}{2} = 0$

$$\text{RHL} = \lim_{h \to 0} -\cos\left(-\frac{\pi}{2} + h\right) = 0$$

$\therefore$ Continuous at $x = \left(-\frac{\pi}{2}\right)$.

Continuity at $x = 0 \Rightarrow f(0) = -1$

$$\text{RHL} = \lim_{h \to 0} (0 + h) - 1 = -1$$

$\therefore$ Continuous at $x = 0$. Continuity at $x = 1$, $f(1) = 0$

$$\text{RHL} = \lim_{h \to 0} \log(1 + h) = 0$$

$\therefore$ Continuous at $x = 1$

$$f'(x) = \begin{cases} -1, & x \le -\frac{\pi}{2} \\ \sin x, & -\frac{\pi}{2} < x \le 0 \\ 1, & 0 < x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

Differentiable at $x = 0$, LHD = 0, RHD = 1
$\therefore$ Not differentiable at $x = 0$
Differentiable at $x = 1$, LHD = 1, RHD = 1
$\therefore$ Differentiable at $x = 1$.
Also, for $x = -\frac{3}{2} \Rightarrow f(x) = -\cos x$
$\therefore$ Differentiable at $x = -\frac{3}{2}$

131. Given, $f(x) = x\cos\frac{1}{x},\ x \geq 1$

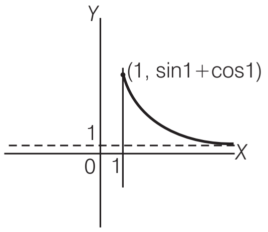
$$\Rightarrow \quad f'(x) = \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x}$$

$$\Rightarrow \quad f''(x) = -\frac{1}{x^3}\cos\left(\frac{1}{x}\right)$$

Now $\lim_{x\to\infty} f'(x) = 0 + 1 = 1 \Rightarrow$ Option (b) is correct.

Now, $x \in [1, \infty) \Rightarrow \frac{1}{x} \in (0, 1] \Rightarrow f''(x) < 0$

Option (d) is correct.
As, $f'(1) = \sin 1 + \cos 1 > 1$
$f'(x)$ is strictly decreasing and $\lim_{x\to\infty} f'(x) = 1$
So, graph of $f'(x)$ is shown as below



Now, in $[x, x+2]$, $x \in [1, \infty)$, $f(x)$ is continuous and differentiable so by LMVT, $f'(x) = \frac{f(x+2) - f(x)}{2}$

As, $\quad f'(x) > 1$
For all $x \in [1, \infty)$

$$\Rightarrow \quad \frac{f(x+2) - f(x)}{2} > 1 \quad \Rightarrow \quad f(x+2) - f(x) > 2$$

For all $x \in [1, \infty)$.

132. Given, $g(x) = \dfrac{(x-1)^n}{\log \cos^m(x-1)}$; $0 < x < 2$, $m \neq 0$, n are integers

and $|x - 1| = \begin{cases} x - 1, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$

The left hand derivative of $|x - 1|$ at $x = 1$ is $p = -1$.

Also, $\lim_{x\to 1^+} g(x) = p = -1$

$$\Rightarrow \quad \lim_{h\to 0} \frac{(1 + h - 1)^n}{\log \cos^m (1 + h - 1)} = -1$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{h^n}{\log \cos^m h} = -1 \Rightarrow \lim_{h\to 0} \frac{h^n}{m \log \cos h} = -1$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{n \cdot h^{n-1}}{m \dfrac{1}{\cos h}(-\sin h)} = -1 \quad \text{[using L'Hospital's rule]}$$

$$\Rightarrow \quad \lim_{h\to 0} \left(-\frac{n}{m}\right) \cdot \frac{h^{n-2}}{\left(\dfrac{\tan h}{h}\right)} = -1 \Rightarrow \left(\frac{n}{m}\right) \lim_{h\to 0} \frac{h^{n-2}}{\left(\dfrac{\tan h}{h}\right)} = 1$$

$$\Rightarrow \quad n = 2 \text{ and } \frac{n}{m} = 1$$

$$\therefore \quad m = n = 2$$

133. We have, $\lim_{x\to 0} \dfrac{g(x)\cos x - g(0)}{\sin x}$ $\quad \left[\dfrac{0}{0} \text{ form}\right]$

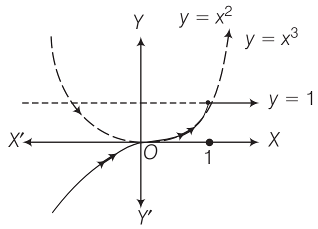
$$= \lim_{x\to 0} \frac{g'(x)\cos x - g(x)\sin x}{\cos x} = 0$$

Statement I
Since, $\quad f(x) = g(x)\sin x$
$\quad f'(x) = g'(x)\sin x + g(x)\cos x$
and $\quad f''(x) = g''(x)\sin x + 2g'(x)\cos x - g(x)\sin x$
$\Rightarrow \quad f''(0) = 0$
Thus, $\lim_{x\to 0}[g(x)\cos x - g(0)] \operatorname{cosec} x = 0 = f''(0)$
$\Rightarrow$ Statement I is true.
Statement II $f'(x) = g'(x)\sin x + g(x)\cos x \Rightarrow f'(0) = g(0)$
Statement II is not a correct explanation of Statement I.

134. Match the Columns

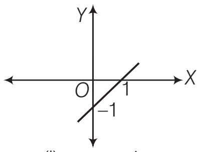
A. $x|x|$ is continuous, differentiable and strictly increasing in $(-1, 1)$.
B. $\sqrt{|x|}$ is continuous in $(-1, 1)$ and not differentiable at $x = 0$.
C. $x + [x]$ is strictly increasing in $(-1, 1)$ and discontinuous at $x = 0$
$\Rightarrow$ not differentiable at $x = 0$.
D. $|x - 1| + |x + 1| = 2$ in $(-1, 1)$
$\Rightarrow$ The function is continuous and differentiable in $(-1, 1)$.

135. Here, $f(x) = \min\{1, x^2, x^3\}$ which could be graphically shown as

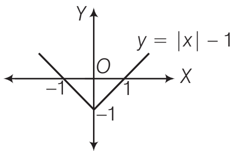


$\Rightarrow f(x)$ is continuous for $x \in R$ and not differentiable at $x = 1$ due to sharp edge. Hence, (a) and (d) are correct answers.

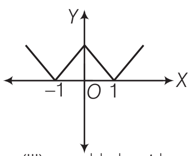
136. Using graphical transformation. As, we know that, the function is not differentiable at sharp edges.



(i) $y = x - 1$



(ii) $y = |x| - 1$



(iii) $y = ||x| - 1|$

In function, $y = ||x| - 1|$, we have 3 sharp edges at $x = -1, 0, 1$.
Hence, $f(x)$ is not differentiable at $\{0, \pm 1\}$.

137. Given, $f(1) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = \ldots = \lim_{n \to \infty} f\left(\frac{1}{n}\right) = 0$

as $f\left(\frac{1}{n}\right) = 0$; $n \in$ integers and $n \ge 1$.

$\Rightarrow \quad \lim_{n \to \infty} f\left(\frac{1}{n}\right) = 0$

$\Rightarrow \quad f(0) = 0$

Since, there are infinitely many points in neighbourhood of $x = 0$.

$\therefore \quad f(x) = 0$

$\Rightarrow \quad f'(x) = 0 \Rightarrow f'(0) = 0$

Hence, $\quad f(0) = f'(0) = 0$

138. Given, $f(x) = \begin{cases} \frac{1}{2}(-x-1), & \text{if } x < -1 \\ \tan^{-1} x, & \text{if } -1 \le x \le 1 \\ \frac{1}{2}(x-1), & \text{if } x > 1 \end{cases}$

$f(x)$ is discontinuous at $x = -1$ and $x = 1$.

$\therefore$ Domain of $f'(x) \in R - \{-1, 1\}$

139. Given, $f(x) = [x] \sin \pi x$

If x is just less than k, $[x] = k - 1$

$\Rightarrow \quad f(x) = (k-1) \sin \pi x.$

LHD of $f(x) = \lim_{x \to k} \frac{(k-1)\sin \pi x - k \sin \pi k}{x - k}$

$= \lim_{x \to k} \frac{(k-1)\sin \pi x}{x - k},$

$= \lim_{h \to 0} \frac{(k-1)\sin \pi (k-h)}{-h}$ [where $x = k - h$]

$= \lim_{h \to 0} \frac{(k-1)(-1)^{k-1} \cdot \sin h\pi}{-h} = (-1)^k (k-1)\pi$

140. RHD of $\sin(|x|) - |x| = \lim_{h \to 0} \frac{\sin h - h}{h} = 1 - 1 = 0$ [$\because f(0) = 0$]

LHD of $\sin(|x|) - |x|$

$= \lim_{h \to 0} \frac{\sin|-h| - |-h|}{-h}$

$= \lim_{h \to 0} \frac{\sin h - h}{-h} = 0$

Therefore, (d) is the answer.

141. We have, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x)),\ x \in R$

Note that, for $x \to 0$, $\log 2 > \sin x$

$\therefore \quad f(x) = \log 2 - \sin x$

$\Rightarrow \quad g(x) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is diferentiable.

Now, $\quad g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$

$= \cos x \cdot \cos(\log 2 - \sin x)$

$\Rightarrow \quad g'(0) = 1 \cdot \cos(\log 2)$

142. Since, $g(x)$ is differentiable $\Rightarrow g(x)$ must be continuous.

$\therefore \ g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$

At $x = 3$, $\quad$ RHL $= 3m + 2$

and at $x = 3$, $\quad$ LHL $= 2k$

$\therefore \quad 2k = 3m + 2$ ...(i)

Also, $\quad g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}}, & 0 \le x < 3 \\ m, & 3 < x \le 5 \end{cases}$

$\therefore \quad L\{g'(3)\} = \frac{k}{4}$ and $R\{g'(3)\} = m$

$\Rightarrow \quad \frac{k}{4} = m$, i.e. $k = 4m$ ...(ii)

On solving Eqs. (i) and (ii), we get

$k = \frac{8}{5},\ m = \frac{2}{5}$

$\Rightarrow \quad k + m = 2$

143. Given, $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$

f and g are differentiable in (0, 1).

Let $\quad h(x) = f(x) - 2g(x)$ ...(i)

$h(0) = f(0) - 2g(0) = 2 - 0 = 2$

and $\quad h(1) = f(1) - 2g(1) = 6 - 2(2) = 2$

$\Rightarrow \quad h(0) = h(1) = 2$

Hence, using Rolle's theorem,

$h'(c) = 0$, such that $c \in (0, 1)$

Differentiating Eq. (i) at c, we get

$\Rightarrow \quad f'(c) - 2g'(c) = 0$

$\Rightarrow \quad f'(c) = 2g'(c)$

144. **Given** A function $f : R \to R$ defined by

$f(x) = [x] \cos \pi\left(x - \frac{1}{2}\right)$, where [] denotes the greatest integer function.

To discuss The continuity of function f.

Now, $\cos x$ is continuous, $\forall x \in R$.

$\Rightarrow \cos \pi\left(x - \frac{1}{2}\right)$ is also continuous, $\forall x \in R$.

Hence, the continuity of f depends upon the continuity of $[x]$.

Since, $[x]$ is discontinuous, $\forall x \in I$.

So, we should check the conitnuity of f at $x = n, \forall n \in I$.

LHL at $x = n$ is given by

$f(n^-) = \lim_{x \to n^-} f(x)$

$= \lim_{x \to n^-} [x] \cos \pi\left(x - \frac{1}{2}\right) = (n-1)\cos\frac{(2n-1)\pi}{2} = 0$

RHL at $x = n$ is given by

$f(n^+) = \lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] \cos \pi\left(x - \frac{1}{2}\right)$

$= (n)\cos\frac{(2n-1)\pi}{2} = 0$

Also, value of the function at $x = n$ is

$f(n) = [n]\cos \pi\left(n - \frac{1}{2}\right) = (n)\cos\frac{(2n-1)\pi}{2} = 0$

$\therefore \quad f(n^+) = f(n^-) = f(n)$

Hence, f is continuous at $x = n, \forall n \in I$.