CHAPTER 5

# Continuity and Differentiability

Syllabus

Continuity and differentiability, derivative of composite functions, chain rule, derivative of inverse trigonometric functions, derivative of implicit functions. Concept of exponential and logarithmic functions. Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.

## CASE STUDY / PASSAGE BASED QUESTIONS

### 1

Let $f(x)$ be a real valued function, then its

- Left Hand Derivative (L.H.D.) : $Lf'(a) = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$
- Right Hand Derivative (R.H.D.) : $Rf'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and are equal.

For the function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$, answer the following questions.

(i) R.H.D. of $f(x)$ at $x = 1$ is

(a) 1 (b) –1 (c) 0 (d) 2

(ii) L.H.D. of $f(x)$ at $x = 1$ is

(a) 1 (b) –1 (c) 0 (d) 2

(iii) $f(x)$ is non-differentiable at

(a) $x = 1$ (b) $x = 2$ (c) $x = 3$ (d) $x = 4$

(iv) Find the value of $f'(2)$.

(a) 1 (b) 2 (c) 3 (d) –1

(v) The value of $f'(-1)$ is

(a) 2 (b) 1 (c) –2 (d) –1

### 2

Let $x = f(t)$ and $y = g(t)$ be parametric forms with $t$ as a parameter, then

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$, where $f'(t) \neq 0$.

On the basis of above information, answer the following questions.

(i) The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x=\frac{\pi}{4}$, where $f'(1)=2$ and $g'(\sqrt{2})=4$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) 1 (d) 0

(ii) The derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is

(a) –1 (b) 1 (c) 2 (d) 4

(iii) The derivative of $e^{x^3}$ with respect to $\log x$ is

(a) $e^{x^3}$ (b) $3x^2 2e^{x^3}$ (c) $3x^3 e^{x^3}$ (d) $3x^2 e^{x^3}+3x$

(iv) The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1}x$ is

(a) 2 (b) $\frac{-1}{2\sqrt{1-x^2}}$ (c) $\frac{2}{x}$ (d) $1-x^2$

(v) If $y=\frac{1}{4}u^4$ and $u=\frac{2}{3}x^3+5$, then $\frac{dy}{dx}=$

(a) $\frac{2}{27}x^2(2x^3+15)^3$ (b) $\frac{2}{7}x^2(2x^3+15)^3$ (c) $\frac{2}{27}x(2x^3+5)^3$ (d) $\frac{2}{7}(2x^3+15)^3$

## 3

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions defined on non-empty sets $A$, $B$, $C$, then $gof: A \rightarrow C$ be is called the composition of $f$ and $g$ defined as, $gof(x) = g\{f(x)\}\ \forall\ x \in A$.

Consider the functions $f(x)=\begin{cases}\sin x, & x\geq 0\\ 1-\cos x, & x\leq 0\end{cases}$, $g(x)=e^x$ and then answer the following questions.

(i) The function $gof(x)$ is defined as

(a) $gof(x)=\begin{cases}e^x & ,x\geq 0\\ 1-e^{\cos x} & ,x\leq 0\end{cases}$

(b) $gof(x)=\begin{cases}e^{\sin x} & ,x\leq 0\\ e^{1-\cos x} & ,x\geq 0\end{cases}$

(c) $gof(x)=\begin{cases}e^{\sin x} & ,x\leq 0\\ 1-e^{\cos x} & ,x\geq 0\end{cases}$

(d) $gof(x)=\begin{cases}e^{\sin x} & ,x\geq 0\\ e^{1-\cos x} & ,x\leq 0\end{cases}$

(ii) $\frac{d}{dx}\{gof(x)\}=$

(a) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\geq 0\\ e^{1-\cos x}\cdot\sin x & ,x\leq 0\end{cases}$

(b) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\geq 0\\ -\sin x\cdot e^{1-\cos x} & ,x\leq 0\end{cases}$

(c) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\geq 0\\ \sin x\cdot(1-\cos x) & ,x\leq 0\end{cases}$

(d) $[gof(x)]'=\begin{cases}\cos x\cdot e^{\sin x} & ,x\geq 0\\ (1-\sin x)\cdot e^{1-\cos x} & ,x\leq 0\end{cases}$

(iii) R.H.D. of $gof(x)$ at $x=0$ is

(a) 0 (b) 1 (c) –1 (d) 2

(iv) L.H.D. of $gof(x)$ at $x=0$ is

(a) 0 (b) 1 (c) –1 (d) 2

(v) The value of $f'(x)$ at $x=\frac{\pi}{4}$ is

(a) 1/9 (b) $1/\sqrt{2}$ (c) 1/2 (d) not defined

## 4

The function $f(x)$ will be discontinuous at $x = a$ if $f(x)$ has

- Discontinuity of first kind : $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ both exist but are not equal. If is also known as irremovable discontinuity.
- Discontinuity of second kind : If none of the limits $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ exist.
- Removable discontinuity : $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ both exist and equal but not equal to $f(a)$.

Based on the above information, answer the following questions.

(i) If $f(x)=\begin{cases}\frac{x^2-9}{x-3}, & \text{for } x\neq 3\\ 4, & \text{for } x=3\end{cases}$, then at $x=3$

(a) $f$ has removable discontinuity (b) $f$ is continuous
(c) $f$ has irremovable discontinuity (d) none of these

(ii) Let $f(x)=\begin{cases}x+2, & \text{if } x\leq 4\\ x+4, & \text{if } x>4\end{cases}$ then at $x=4$

(a) $f$ is continuous (b) $f$ has removable discontinuity
(c) $f$ has irremovable discontinuity (d) none of these

(iii) Consider the function $f(x)$ defined as $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 5, & \text{for } x=2\end{cases}$, then at $x=2$

(a) $f$ has removable discontinuity (b) $f$ has irremovable discontinuity
(c) $f$ is continuous (d) $f$ is continuous if $f(2) = 3$

(iv) If $f(x)=\begin{cases}\frac{x-|x|}{x}, & x\neq 0\\ 2, & x=0\end{cases}$, then at $x=0$

(a) $f$ is continuous (b) $f$ has removable discontinuity
(c) $f$ has irremovable discontinuity (d) none of these

(v) If $f(x)=\begin{cases}\frac{e^x-1}{\log(1+2x)}, & \text{if } x\neq 0\\ 7, & \text{if } x=0\end{cases}$, then at $x=0$

(a) $f$ is continuous if $f(0) = 2$ (b) $f$ is continuous
(c) $f$ has irremovable discontinuity (d) $f$ has removable discontinuity

## 5

If a real valued function $f(x)$ is finitely derivable at any point of its domain, it is necessarily continuous at that point. But its converse need not be true.
For example, every polynomial, constant function are both continuous as well as differentiable and inverse trigonometric functions are continuous and differentiable in its domains etc.

Based on the above information, answer the following questions.

(i) If $f(x)=\begin{cases} x, \text{ for } x\le 0 \\ 0, \text{ for } x>0 \end{cases}$, then at $x=0$

(a) $f(x)$ is differentiable and continuous
(b) $f(x)$ is neither continuous nor differentiable
(c) $f(x)$ is continuous but not differentiable
(d) none of these

(ii) If $f(x) = |x - 1|, x \in R$, then at $x = 1$

(a) $f(x)$ is not continuous
(b) $f(x)$ is continuous but not differentiable
(c) $f(x)$ is continuous and differentiable
(d) none of these

(iii) $f(x) = x^3$ is

(a) continuous but not differentiable at $x = 3$
(b) continuous and differentiable at $x = 3$
(c) neither continuous nor differentiable at $x = 3$
(d) none of these

(iv) If $f(x) = [\sin x]$, then which of the following is true?

(a) $f(x)$ is continuous and differentiable at $x = 0$.
(b) $f(x)$ is discontinuous at $x = 0$.
(c) $f(x)$ is continuous at $x = 0$ but not differentiable.
(d) $f(x)$ is differentiable but not continuous at $x = \pi/2$.

(v) If $f(x) = \sin^{-1}x, -1 \le x \le 1$, then

(a) $f(x)$ is both continuous and differentiable
(b) $f(x)$ is neither continuous nor differentiable.
(c) $f(x)$ is continuous but not differentiable
(d) None of these

## 6

Derivative of $y = f(x)$ w.r.t. $x$ (if exists) is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is called the first order derivative of $y$.

If we take derivative of $\frac{dy}{dx}$ again, then we get $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$ or $f''(x)$ and is called the second order derivative of $y$. Similarly, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is denoted and defined as $\frac{d^3y}{dx^3}$ or $f'''(x)$ and is known as third order derivative of $y$ and so on.

Based on the above information, answer the following questions.

(i) If $y=\tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then $\frac{d^2y}{dx^2}$ is equal to

(a) 2
(b) 1
(c) 0
(d) –1

(ii) If $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$, then $\frac{d^2u}{ds^2}$ is equal to

(a) 12
(b) 32
(c) 36
(d) 10

(iii) If $f(x) = 2 \log \sin x$, then $f''(x)$ is equal to

(a) $2\text{cosec}^3 x$
(b) $2\cot^2 x - 4x^2 \text{cosec}^2 x^2$
(c) $2x \cot x^2$
(d) $-2\text{cosec}^2 x$

(iv) If $f(x) = e^x \sin x$, then $f'''(x) =$

(a) $2e^x(\sin x + \cos x)$
(b) $2e^x(\cos x - \sin x)$
(c) $2e^x(\sin x - \cos x)$
(d) $2e^x \cos x$

(v) If $y^2 = ax^2 + bx + c$, then $\frac{d}{dx}\left(y^3 y_2\right)=$

(a) 1
(b) –1
(c) $\frac{4ac-b^2}{a^2}$
(d) 0

## 7

- A function $f(x)$ is said to be continuous in an open interval $(a, b)$, if it is continuous at every point in this interval.
- A function $f(x)$ is said to be continuous in the closed interval $[a, b]$, if $f(x)$ is continuous in $(a, b)$ and $\lim_{h\to 0} f(a+h) = f(a)$ and $\lim_{h\to 0} f(b-h) = f(b)$

If function $f(x) = \begin{cases} \dfrac{\sin (a+1)x + \sin x}{x} & , x < 0 \\ c & , x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , x > 0 \end{cases}$ is continuous at $x = 0$, then answer the following questions.

**(i)** The value of $a$ is

(a) $-3/2$ (b) 0 (c) 1/2 (d) $-1/2$

**(ii)** The value of $b$ is

(a) 1 (b) $-1$ (c) 0 (d) any real number

**(iii)** The value of $c$ is

(a) 1 (b) 1/2 (c) $-1$ (d) $-1/2$

**(iv)** The value of $a + c$ is

(a) 1 (b) 0 (c) $-1$ (d) $-2$

**(v)** The value of $c - a$ is

(a) 1 (b) 0 (c) $-1$ (d) 2

## 8

Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$, where both $u(x)$ and $v(x)$ are differentiable functions and $f$ and $u$ need to be positive functions.

Let function $y = f(x) = (u(x))^{v(x)}$, then $y' = y\left[\dfrac{v(x)}{u(x)}u'(x) + v'(x)\cdot\log[u(x)]\right]$

On the basis of above information, answer the following questions.

**(i)** Differentiate $x^x$ w.r.t. $x$

(a) $x^x(1 + \log x)$ (b) $x^x(1 - \log x)$ (c) $-x^x(1 + \log x)$ (d) $x^x \log x$

**(ii)** Differentiate $x^x + a^x + x^a + a^a$ w.r.t. $x$

(a) $(1 + \log x) + (a^x \log a + ax^{a-1})$ (b) $x^x(1 + \log x) + \log a + ax^{a-1}$

(c) $x^x(1 + \log x) + x^a \log x + ax^{a-1}$ (d) $x^x(1 + \log x) + a^x \log a + ax^{a-1}$

**(iii)** If $x = e^{x/y}$, then find $\dfrac{dy}{dx}$.

(a) $-\dfrac{(x+y)}{x\log x}$ (b) $-\dfrac{(x-y)}{x\log x}$ (c) $\dfrac{(x+y)}{x\log x}$ (d) $\dfrac{x-y}{x\log x}$

**(iv)** If $y = (2-x)^3(3+2x)^5$, then find $\dfrac{dy}{dx}$.

(a) $(2-x)^3(3+2x)^5\left[\dfrac{15}{3+2x} - \dfrac{8}{2-x}\right]$ (b) $(2-x)^3(3+2x)^5\left[\dfrac{15}{3+2x} + \dfrac{3}{2-x}\right]$

(c) $(2-x)^3(3+2x)^5\left[\frac{10}{3+2x}-\frac{3}{2-x}\right]$ (d) $(2-x)^3(3+2x)^5\cdot\left[\frac{10}{3+2x}+\frac{3}{2-x}\right]$

**(v)** If $y = x^x \cdot e^{(2x+5)}$, then find $\frac{dy}{dx}$.

(a) $x^x e^{2x+5}$ (b) $x^x e^{2x+5}(3-\log x)$ (c) $x^x e^{2x+5}(1-\log x)$ (d) $x^x e^{2x+5}\cdot(3+\log x)$

## 9

If $y = f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then $y = f(g(x)]$ is a differentiable function of $x$ and $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$. This rule is also known as CHAIN RULE.

Based on the above information, find the derivative of functions w.r.t. $x$ in the following questions.

**(i)** $\cos\sqrt{x}$

(a) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$ (b) $\frac{\sin\sqrt{x}}{2\sqrt{x}}$ (c) $\sin\sqrt{x}$ (d) $-\sin\sqrt{x}$

**(ii)** $7^{x+\frac{1}{x}}$

(a) $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$ (b) $\left(\frac{x^2+1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$ (c) $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x-\frac{1}{x}}\cdot\log 7$ (d) $\left(\frac{x^2+1}{x^2}\right)\cdot 7^{x-\frac{1}{x}}\cdot\log 7$

**(iii)** $\sqrt{\frac{1-\cos x}{1+\cos x}}$

(a) $\frac{1}{2}\sec^2\frac{x}{2}$ (b) $-\frac{1}{2}\sec^2\frac{x}{2}$ (c) $\sec^2\frac{x}{2}$ (d) $-\sec^2\frac{x}{2}$

**(iv)** $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(a) $\frac{-1}{x^2+b^2}+\frac{1}{x^2+a^2}$ (b) $\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$ (c) $\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}$ (d) none of these

**(v)** $\sec^{-1}x+\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

(a) $\frac{2}{\sqrt{x^2-1}}$ (b) $\frac{-2}{\sqrt{x^2-1}}$ (c) $\frac{1}{|x|\sqrt{x^2-1}}$ (d) $\frac{2}{|x|\sqrt{x^2-1}}$

## 10

If a relation between $x$ and $y$ is such that $y$ cannot be expressed in terms of $x$, then $y$ is called an implicit function of $x$. When a given relation expresses $y$ as an implicit function of $x$ and we want to find $\frac{dy}{dx}$, then we differentiate every term of the given relation w.r.t. $x$, remembering that a term in $y$ is first differentiated w.r.t. $y$ and then multiplied by $\frac{dy}{dx}$.

Based on the above information, find the value of $\frac{dy}{dx}$ in each of the following questions.

**(i)** $x^3+x^2y+xy^2+y^3=81$

(a) $\frac{(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$ (b) $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$ (c) $\frac{(3x^2+2xy-y^2)}{x^2-2xy+3y^2}$ (d) $\frac{3x^2+xy+y^2}{x^2+xy+3y^2}$

**(ii)** $x^y = e^{x-y}$

(a) $\frac{x-y}{(1+\log x)}$ (b) $\frac{x+y}{(1+\log x)}$ (c) $\frac{x-y}{x(1+\log x)}$ (d) $\frac{x+y}{x(1+\log x)}$

**(iii)** $e^{\sin y} = xy$

(a) $\frac{-y}{x(y\cos y-1)}$ (b) $\frac{y}{y\cos y-1}$ (c) $\frac{y}{y\cos y+1}$ (d) $\frac{y}{x(y\cos y-1)}$

**(iv)** $\sin^2 x + \cos^2 y = 1$

(a) $\frac{\sin 2y}{\sin 2x}$ (b) $-\frac{\sin 2x}{\sin 2y}$ (c) $-\frac{\sin 2y}{\sin 2x}$ (d) $\frac{\sin 2x}{\sin 2y}$

**(v)** $y = (\sqrt{x})^{\sqrt{x}^{\sqrt{x}\ldots\infty}}$

(a) $\frac{-y^2}{x(2-y\log x)}$ (b) $\frac{y^2}{2+y\log x}$ (c) $\frac{y^2}{x(2+y\log x)}$ (d) $\frac{y^2}{x(2-y\log x)}$

## HINTS & EXPLANATIONS

**1.** We have, $f(x)=\begin{cases} x-3 & ,x\ge 3 \\ 3-x & ,1\le x<3 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & ,x<1 \end{cases}$

**(i)** **(b)**: $Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0}\frac{3-(1+h)-2}{h} = \lim_{h\to 0} -\frac{h}{h} = -1$

**(ii)** **(b)**: $Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0}\frac{-1}{h}\left[\frac{(1-h)^2}{4}-\frac{3(1-h)}{2}+\frac{13}{4}-2\right]$

$= \lim_{h\to 0}\left(\frac{1+h^2-2h-6+6h+13-8}{-4h}\right)$

$= \lim_{h\to 0}\left(\frac{h^2+4h}{-4h}\right) = -1$

**(iii)** **(c)**: Since, R.H.D. at $x = 3$ is 1 and L.H.D. at $x = 3$ is $-1$

$\therefore$ $f(x)$ is non-differentiable at $x = 3$.

**(iv)** **(d)**

**(v)** **(c)**: From above, we have

$f'(x) = \frac{x}{2}-\frac{3}{2}, x<1$

$\therefore f'(-1) = \frac{-1}{2}-\frac{3}{2} = -2$

**2.** **(i)** **(a)**: Now, $\frac{df(\tan x)}{dg(\sec x)} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$

$= \frac{f'(\tan x)\sec x}{g'(\sec x)\tan x}$

$\therefore \left[\frac{df(\tan x)}{dg(\sec x)}\right]_{x=\pi/4} = \frac{f'(1)\sqrt{2}}{g'(\sqrt{2})\cdot 1} = \frac{2\sqrt{2}}{4\cdot 1} = \frac{1}{\sqrt{2}}$

**(ii)** **(b)**

**(iii)** **(c)**: Let $y = e^{x^3}$, $z = \log x$

Differentiating w.r.t. $x$, we get

$\frac{dy}{dx} = e^{x^3}(3x^2) = 3x^2e^{x^3}$ and $\frac{dz}{dx} = \frac{1}{x}$

$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{3x^2e^{x^3}}{\left(\frac{1}{x}\right)} = 3x^3e^{x^3}$

**(iv)** **(a)**: Let $y = \cos^{-1}(2x^2-1) = 2\cos^{-1}x$

Differentiating w.r.t. $\cos^{-1}x$, we get

$\frac{dy}{d(\cos^{-1}x)} = \frac{2d(\cos^{-1}x)}{d(\cos^{-1}x)} = 2$

**(v)** **(a)**: We have, $y = \frac{1}{4}u^4 \Rightarrow \frac{dy}{du} = \frac{1}{4}\cdot 4u^3 = u^3$

and $u = \frac{2}{3}x^3+5 \Rightarrow \frac{du}{dx} = \frac{2}{3}\cdot 3x^2 = 2x^2$

$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = u^3\cdot 2x^2 = \left(\frac{2}{3}x^3+5\right)^3(2x^2)$

$= \frac{2}{27}x^2(2x^3+15)^3$

**3.** **(i)** (d) **(ii)** (a) **(iii)** (b)

**(iv)** (a) **(v)** (b)

**4.** **(i)** (a) : $f(3) = 4$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x\to 3}(x+3) = 6 \because \lim_{x\to 3} f(x) \neq f(3)$$

$\therefore$ $f(x)$ has removable discontinuity at $x = 3$.

**(ii)** (c) : $\lim_{x\to 4^-} f(x) = \lim_{x\to 4}(x+2) = 4+2 = 6$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4}(x+4) = 4+4 = 8$$

$$\therefore \lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$$

$\therefore$ $f(x)$ has an irremovable discontinuity at $x = 4$.

**(iii)** (a) : $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x^2-4)}{(x-2)} = \lim_{x\to 2}(x+2) = 4$

and $f(2) = 5$ (given) $\therefore \lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 2$.

**(iv)** (c) : $f(0) = 2$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{x+x}{x} = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{x-x}{x} = 0$$

$$\because \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

$\therefore$ $f(x)$ has an irremovable discontinuity at $x = 0$.

**(v)** (d) : $f(0) = 7$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x-1}{\log(1+2x)} = \lim_{x\to 0} \frac{\left(\frac{e^x-1}{x}\right)}{\frac{\log(1+2x)}{2x}\cdot 2} = \frac{1}{2}$$

$$\because \lim_{x\to 0} f(x) \neq f(0)$$

$\therefore$ $f(x)$ has removable discontinuity at $x = 0$.

**5.** **(i)** (c) **(ii)** (b) **(iii)** (b)

**(iv)** (b) **(v)** (a)

**6.** **(i)** (c) : Given, $y = \tan^{-1}\left(\frac{\log\left(\frac{e}{x^2}\right)}{\log ex^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

$$= \tan^{-1}\left(\frac{1-\log x^2}{1+\log x^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$

$= \tan^{-1}(1) - \tan^{-1}(2\log x) + \tan^{-1}(3) + \tan^{-1}(2\log x)$

$\Rightarrow$ $y = \tan^{-1}(1) + \tan^{-1}(3)$

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$$

**(ii)** (d) : Given, $x = s + 3t$, $y = 2s - t \Rightarrow \frac{dx}{ds} = 1, \frac{dy}{ds} = 2$

Now, $u = x^2 + y^2 \Rightarrow \frac{du}{ds} = 2x\frac{dx}{ds} + 2y\frac{dy}{ds} = 2x + 4y$

$$\Rightarrow \frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right) + 4\left(\frac{dy}{ds}\right) \Rightarrow \frac{d^2u}{ds^2} = 2(1) + 4(2) = 10$$

**(iii)** (d) : We have, $f(x) = 2\log\sin x$

$$\Rightarrow f'(x) = 2\cdot\frac{1}{\sin x}\cdot\cos x = 2\cot x \Rightarrow f''(x) = -2\,\text{cosec}^2 x$$

**(iv)** (b) : We have, $f(x) = e^x\sin x$

$\Rightarrow$ $f'(x) = e^x\cos x + e^x\sin x = e^x(\cos x + \sin x)$

$\Rightarrow$ $f''(x) = e^x(\cos x - \sin x) + e^x(\cos x + \sin x) = 2e^x\cos x$

$\Rightarrow$ $f'''(x) = 2[e^x\cos x - e^x\sin x] = 2e^x[\cos x - \sin x]$

**(v)** (d) : Given $y^2 = ax^2 + bx + c$

$\Rightarrow$ $2yy_1 = 2ax + b$ ...(i)

$\Rightarrow$ $2yy_2 + y_1(2y_1) = 2a$

$$\Rightarrow yy_2 = a - y_1^2 \Rightarrow yy_2 = a - \left(\frac{2ax+b}{2y}\right)^2 \quad \text{(Using (i))}$$

$$= \frac{4y^2a - (4a^2x^2 + b^2 + 4abx)}{4y^2}$$

$$\Rightarrow y^3y_2 = \frac{4a(ax^2+bx+c) - (4a^2x^2 + b^2 + 4abx)}{4}$$

$$= \frac{4ac - b^2}{4}$$

$$\Rightarrow \frac{d}{dx}(y^3y_2) = 0$$

**7.** L.H.L. (at $x = 0$) $= \lim_{x\to 0} \frac{\sin(a+1)x + \sin x}{x} \left(\frac{0}{0} \text{ form}\right)$

Using L′ Hospital rule, we get

L.H.L. (at $x = 0$)

$= \lim_{x\to 0}(a+1)\cos(a+1)x + \cos x = a + 2$ ...(i)

R.H.L. (at $x = 0$) $= \lim_{x\to 0} \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} = \lim_{x\to 0} \frac{\sqrt{1+bx}-1}{bx}$

$= \lim_{x\to 0} \frac{1}{\sqrt{1+bx}+1} = \frac{1}{2}$ ...(ii)

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ From (i) and (ii), we get

$$a + 2 = c = \frac{1}{2} \Rightarrow a = -\frac{3}{2}, c = \frac{1}{2}$$

Also, value of $b$ does not affect the continuity of $f(x)$, so $b$ can be any real number.

**(i)** (a) **(ii)** (d) **(iii)** (b)

**(iv)** (c) : $a + c = -\frac{3}{2} + \frac{1}{2} = -1$

**(v)** (d) : $c - a = \frac{1}{2} + \frac{3}{2} = 2$

8. (i) (a): Let $y = x^x \Rightarrow \log y = x \log x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x\log x) \Rightarrow \frac{dy}{dx} = x^x[1\times\log x + x\times\frac{1}{x}]$$
$$= x^x\,[1+\log x]$$

(ii) (d)

(iii) (d): Given $x = e^{x/y} \Rightarrow \log x = \frac{x}{y}\log e \Rightarrow y\log x = x$

$$\Rightarrow y\frac{1}{x} + (\log x)\frac{dy}{dx} = 1$$
$$\Rightarrow \frac{dy}{dx} = \left(1-\frac{y}{x}\right)\frac{1}{\log x} \Rightarrow \frac{1}{x\log x}(x-y)$$

(iv) (c): $y = (2-x)^3\,(3+2x)^5$

$$\Rightarrow \log y = \log(2-x)^3 + \log(3+2x)^5$$
$$= 3\log(2-x) + 5\log(3+2x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3\times(-1)}{2-x} + \frac{5}{3+2x}\times(2)$$
$$\Rightarrow \frac{dy}{dx} = (2-x)^3\,(3+2x)^5\left[\frac{10}{3+2x} - \frac{3}{2-x}\right]$$

(v) (d): $y = x^x.e^{(2x+5)}$

$$\Rightarrow \log y = x\log x + (2x+5)$$
$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \left(x\cdot\frac{1}{x} + \log x\right) + 2$$
$$\Rightarrow \frac{dy}{dx} = x^x\cdot e^{2x+5}\cdot(3+\log x)$$

9. (i) (a): Let $y = \cos\sqrt{x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x}) = -\sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$
$$= -\sin\sqrt{x}\times\frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

(ii) (a): Let $y = 7^{x+\frac{1}{x}}$ $\therefore$ $\frac{dy}{dx} = \frac{d}{dx}\left(7^{x+\frac{1}{x}}\right)$

$$= 7^{x+\frac{1}{x}}\cdot\log 7\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right) = 7^{x+\frac{1}{x}}\cdot\log 7\cdot\left(1-\frac{1}{x^2}\right)$$
$$= \left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$$

(iii) (a): Let $y = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1+1}} = \tan\left(\frac{x}{2}\right)$

$$\therefore \frac{dy}{dx} = \sec^2\frac{x}{2}.\frac{1}{2} = \frac{1}{2}\sec^2\frac{x}{2}$$

(iv) (b): Let $y = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{b}\times\frac{1}{1+\frac{x^2}{b^2}}\times\frac{1}{b} + \frac{1}{a}\times\frac{1}{1+\frac{x^2}{a^2}}\times\frac{1}{a}$$
$$= \frac{1}{b^2+x^2} + \frac{1}{a^2+x^2}$$

(v) (d): Let $y = \sec^{-1}x + \operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

Put $x = \sec\theta \Rightarrow \theta = \sec^{-1}x$

$$\therefore y = \sec^{-1}(\sec\theta) + \operatorname{cosec}^{-1}\left(\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}\right)$$
$$= \theta + \sin^{-1}\left[\sqrt{1-\cos^2\theta}\right]$$
$$= \theta + \sin^{-1}(\sin\theta) = \theta + \theta = 2\theta = 2\sec^{-1}x$$
$$\therefore \frac{dy}{dx} = 2\frac{d}{dx}(\sec^{-1}x) = 2\times\frac{1}{|x|\sqrt{x^2-1}} = \frac{2}{|x|\sqrt{x^2-1}}$$

10. (i) (b): $x^3 + x^2y + xy^2 + y^3 = 81$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$$
$$\Rightarrow (x^2+2xy+3y^2)\frac{dy}{dx} = -3x^2 - 2xy - y^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$$

(ii) (c): $x^y = e^{x-y} \Rightarrow y\log x = x - y$

$$\Rightarrow y\times\frac{1}{x} + \log x\cdot\frac{dy}{dx} = 1 - \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx}[\log x + 1] = 1 - \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{x-y}{x[1+\log x]}$$

(iii) (d): $e^{\sin y} = xy \Rightarrow \sin y = \log x + \log y$

$$\Rightarrow \cos y\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[\cos y - \frac{1}{y}\right] = \frac{1}{x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x(y\cos y - 1)}$$

(iv) (d): $\sin^2 x + \cos^2 y = 1$

$$\Rightarrow 2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-\sin 2x}{-\sin 2y} = \frac{\sin 2x}{\sin 2y}$$

(v) (d): $y = \left(\sqrt{x}\right)^{\sqrt{x}^{\sqrt{x}\ldots\infty}} \Rightarrow y = \left(\sqrt{x}\right)^y$

$$\Rightarrow \log y = y\left(\log\sqrt{x}\right) \Rightarrow \log y = \frac{1}{2}(y\log x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[y\times\frac{1}{x} + \log x\left(\frac{dy}{dx}\right)\right]$$
$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y} - \frac{1}{2}\log x\right\} = \frac{1}{2}\frac{y}{x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x}\times\frac{2y}{(2-y\log x)} = \frac{y^2}{x(2-y\log x)}$$