CHAPTER TEN

Differentiability and Differentiation

DIFFERENTIABILITY

Let f be a real-valued function defined on an open interval containing a point x_0.
then f is said to be *differentiable* at x_0 if

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists. If this limit exists we call it the derivative of $f(x)$ at x_0 and denote it by $f'(x_0)$ or by

$$\left.\frac{dy}{dx}\right|_{x = x_0}$$

If this limit does not exist, we say that f is not *differentiable* at $x = x_0$.

Equivalently $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. Graphically the graph of f has a non vertical tangent line at $x = x_0$.

Using one-sided limits, we define the *right-hand derivative*, denoted by $f'(x_0+)$, at $x = x_0$ as

$$\lim_{\Delta x \to 0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and the *left-hand derivative*, denoted by $f'(x_0 -)$, at $x = x_0$ as

$$\lim_{\Delta x \to 0-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Thus, a function $f(x)$ is derivable at $x = x_0$ if $f'(x_0+) = f'(x_0-)$.

Every differentiable function is continuous but the converse may not be true. That is, a continuous function need not be differentiable. e.g.

Illustration 1

$f(x) = |x|$ at $x = 0$,

$$f'(0+) = \lim_{\Delta x\to 0+} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0+} \frac{|\Delta x|}{\Delta x} = 1$$

$$f'(0-) = \lim_{\Delta x\to 0-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x\to 0-} \frac{-\Delta x}{\Delta x}$$

(as $\Delta x \to 0-$, we have $\Delta x < 0$)

$= -1$

Thus f is not differentiable at $x = 0$.

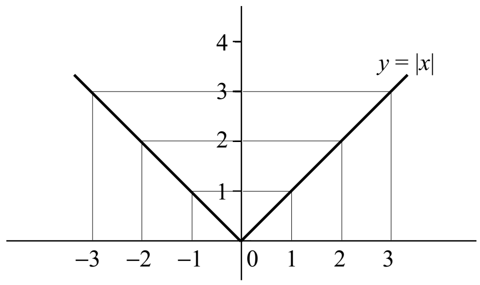


Fig. 10.1

The *absolute value* function is not differentiable at $x = 0$. Clearly f is a continuous function. Similarly if $f(x) = |x - a|$, then f is not differentiable at $x = a$, $f'(a+) = 1$ and $f'(a-) = -1$.

Geometrically, we interpret

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

as the slope of the graph at the point $(x, f(x))$. The line through $(x, f(x))$ which has this slope is called the tangent line at $(x, f(x))$. Thus, if there is no tangent line at a certain point or has a vertical tangent line, the function is not differentiable at that point. In other words, a function is not differentiable at a corner point of a curve, i.e., a point where the curve suddenly changes direction (Fig. 10.2).

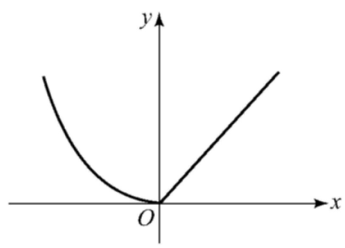


Fig. 10.2

DIFFERENTIABILITY ON AN INTERVAL

A function $f(x)$, defined on $[a, b]$, is said to be differentiable on $[a, b]$ if it is differentiable at every $c \in (a, b)$ and both

$$\lim_{h \to 0+} \frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h \to 0-} \frac{f(b+h)-f(b)}{h}$$

exist.

Continuously Differentiable

A function f is said to be continuously differentiable if the derivative $f'(x)$ exists and is itself a continuous function. e.g.

Illustration 2

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h \sin\frac{1}{h} = 0 \,.$$

For $x \neq 0$ $f'(x) = 2x \sin\dfrac{1}{x} - \cos\dfrac{1}{x}$

Since $\lim\limits_{x\to 0} f'(x)$ does not exist. So f is differentiable but not continuously differentiable. All polynomials are continuously differentiable.

HIGHER ORDER DIFFERENTIATION

Let $y = f(x)$ be a differentiable function such that $z = f'(x)$ is also differentiable. Then the second derivative of $y = f(x)$ is denoted by $y_2(x)$, $f''(x)$ or d^2y/dx^2, and is defined by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \left(= \frac{dz}{dx}\right)$$

Similarly, we can define

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x) \text{ and } f^{(iv)}(x) = \frac{d^4y}{dx^4}$$

or, in general, $\dfrac{d^ny}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$ for any $n \in \mathbf{N}$, $n > 1$.

Many functions, like polynomial, trigonometric, exponential and logarithmic functions, are differentiable at each point of their domain.

SOME FORMULAE OF DIFFERENTIATION

Let $f(x)$ and $g(x)$ be differentiable functions and $\alpha \in \mathbf{R}$.

1. *Sum and Difference Rule*

   $$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

2. *Scalar Multiple Rule*

   $$\frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx} f(x)$$

3. *Product rule.*

   $$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(g(x))$$

4. *Quotient rule*

   $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$

   provided $g(x) \neq 0$

5. *Chain rule* If $y = h(u)$ and $u = f(x)$, then

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

6. *Test of constancy* If at all points of a certain interval, $f'(x) = 0$, then the function $f(x)$ has a constant value within this interval.

DIFFERENTIATION OF SOME ELEMENTARY FUNCTIONS

1. The trigonometric functions have the following derivatives:

   $$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

   $$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

   $$\frac{d}{dx}(\sec x) = \sec x \tan x$$

   $$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

2. If $f(x)$ is a differentiable function

   $\dfrac{d}{dx}(x^n) = nx^{n-1}$ and in general

   $$\frac{d}{dx}(f(x)^n) = n(f(x))^{n-1} f'(x)$$

3. If $f(x)$ is a differentiable function

   $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$ and in general

   $$\frac{d}{dx}(\log f(x))) = \frac{1}{f(x)} f'(x)$$

4. $\frac{d}{dx}(a^x) = a^x \log a$. In particular,

$$\frac{d}{dx}(e^x) = e^x \quad \text{and} \quad \frac{d}{dx}(e^{f(x)}) = e^{f(x)} f'(x)$$

5. $\frac{d}{dx}(\sin^{-1}x) = -\frac{d}{dx}(\cos^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

At the points $x = \pm 1$, $\sin^{-1} x$ and $\cos^{-1} x$ are not differentiable.

6. $\frac{d}{dx}(\tan^{-1} x) = -\frac{d}{dx}(\cot^{-1} x) = \frac{1}{1+x^2}$, for $x \in \mathbf{R}$

7. $\frac{d}{dx}(\sec^{-1}x) = -\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ for $|x|>1$.

DIFFERENTIATION OF IMPLICIT FUNCTIONS

To find dy/dx when a differentiable function y = $y(x)$ satisfies the equation $F(x, y) = 0$, we differentiate F with respect to x, considering y as a function of x, and solve the resulting equation to obtain for dy/dx.

Illustration 3

To find $\frac{dy}{dx}$ if y satisfies $x^3 + y^3 - 3\,axy = 0$.

Differentiating w.r.t. x, we have

$$3x^2 + 3y^2 \frac{dy}{dx} - 3a\left[x\frac{dy}{dx} + y\right] = 0$$

$$\Rightarrow \quad [3y^2 - 3ax]\frac{dy}{dx} = 3ay - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

DIFFERENTIATION OF FUNCTIONS REPRESENTED PARAMETRICALLY

Let there be given two functions $x = \varphi(t)$ and $y = \psi(t)$, where $t \in (\alpha, \beta)$. If the function $x = \varphi(t)$ has an inverse, $t = \varphi^{-1}(x)$ on the interval (α, β), then the new function defined by $y(x) = \psi(\varphi^{-1}(x))$ is said to represent y parametrically. We find its derivative by using the relation

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

Illustration 4

$$x = e^t \sin t,\ y = e^t \cos t$$

To find $\frac{dy}{dx}$, we compute first $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = e^t \cos t + e^t \sin t = e^t(\sin t + \cos t)$$

$$\frac{dy}{dt} = -e^t \sin t + e^t \cos t = e^t(\cos t - \sin t)$$

So, $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\cos t - \sin t}{\cos t + \sin t}.$$

LOGARITHMIC DIFFERENTIATION

The process of taking logarithms of both sides and then differentiating them is called logarithmic differentiation. It is useful in the following cases:

1. If the given function consists of three or more factors which are functions of x.
2. If the given function is of the form $(f(x))^{h(x)}$
3. If the given function is of the form $h_1(x)h_2(x)/h_3(x)h_4(x)$.

If $h(x) = (f_2(x))^{f_1(x)} f_3(x)$, then take logarithum of both the sides. The right hand side will be take the form

$$f_1(x)\log f_2(x) + \log f_3(x)$$

The last expression can be differentiated easily.

Illustration 5

$$y = (\sin x)^{\cos x}$$

Taking logarithum of both the sides, we have

$$\log y = \cos x \log \sin x$$

Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}(\log\sin x) + (\log\sin x)\frac{d}{dx}\cos x$$

$$= \cos x\frac{\cos x}{\sin x} - (\log\sin x)\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - (\log\sin x)\sin x\right]$$

Illustration 6

$$y = \sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}}$$

Taking logarithum, we obtain

$$\log y = \frac{1}{3}\left[\log x + \log(x^2+1) - 2\log(x^2-1)\right]$$

Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{4x}{x^2-1}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}} \times \frac{1}{3}\left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{4x}{x^2-1}\right]$$

LEIBNITZ THEOREM AND *n*th DERIVATIVES

Let $f(x)$ and $g(x)$ be functions both possessing derivatives up to nth order. Then,

$$\frac{d^n}{dx^n}(f(x)\,g(x)) = f^n(x)\,g(x) + {}^nC_1 f^{n-1}(x)\,g^1(x) + {}^nC_2 f^{n-2}(x)\,g^2(x) + \ldots + {}^nC_r f^{n-r}(x)\,g^r(x) + \ldots + {}^nC_n f(x)\,g^n(x).$$

where for any $k > 1$, $f^k(x) = \frac{d^k}{dx^k}(f(x))$ i.e. kth derivative of $f(x)$

$$\frac{d^n}{dx^n}(x^n) = n!;\ \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{(-1)^n\, n!}{x^{n+1}};$$

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(x + n\frac{\pi}{2}\right),$$

$$\frac{d^n}{dx^n}(\cos x) = \cos\left(x + n\frac{\pi}{2}\right);$$

$$\frac{d^n}{dx^n}(e^{mn}) = m^n e^{mx}.$$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: If $x = k(t - \sin t)$, $y = k(1 - \cos t)$ $(k \neq 0)$ then $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$

(a) $-\frac{1}{k}$ (b) $\frac{1}{2k}$

(c) $\frac{1}{k^2}$ (d) $2k$

Ans. (a)

Solution:

$$\frac{dx}{dt} = k(1 - \cos t),\quad \frac{dy}{dt} = k\sin t$$

so $$\frac{dy}{dx} = \frac{k\sin t}{k(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{1}{2}\left(-\operatorname{cosec}^2\left(\frac{t}{2}\right)\right)\frac{1}{k\left(2\sin^2\left(\frac{t}{2}\right)\right)}$$

$$= -\frac{1}{4k}\operatorname{cosec}^4\frac{t}{2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/2} = -\frac{1}{4k}(\sqrt{2})^4 = -\frac{1}{k}$$

Example 2: If $x = 2t + 3t^2$, $y = t^2 + 2t^3$ then $y'^2 + 2y'^3$ $\left(y' = \frac{dy}{dx}\right)$ is equal to

(a) $2y$ (b) y^2

(c) y (d) $3y + 2$

Ans. (c)

Solution: $\frac{dx}{dt} = 2 + 6t$, $\frac{dy}{dt} = 2t + 6t^2$

So that $$y' = \frac{dy}{dx} = \frac{2t+6t^2}{2+6t} = t$$

$$y'^2 + 2y'^3 = t^2 + 2t^3 = y$$

Example 3: If function $y = \frac{x-3}{x+4}$ satisfies the relationship $Ay'^2 = (y-1)y''$ then A is equal to

(a) 1 (b) 3

(c) 4 (d) 2

Ans. (d)

Solution: $y = 1 - \frac{7}{x+4}$

$\Rightarrow$ $$y' = \frac{7}{(x+4)^2} \text{ and } y'' = -\frac{14}{(x+4)^3}$$

Therefore, the L.H.S. $= A\,y'^2 = \frac{49A}{(x+4)^4}$

and R.H.S. $= (y-1)y'' = \left(\frac{x-3-x-4}{x+4}\right)\left(-\frac{14}{(x+4)^3}\right)$

$$= \frac{98}{(x+4)^4}$$

Hence $A = 2$

Example 4: Let $f(x) = \left|x-\left(\frac{\pi}{2}\right)\right|^3 + \sin^2 x$, then $f''\left(\frac{\pi}{2}\right)$

(a) does not exist (b) is equal to 2
(c) is equal to –2 (d) is equal to 0

Ans. (c)

Solution: $f(x) = \begin{cases} \left(x-\frac{\pi}{2}\right)^3 + \sin^2 x & \text{if } x \ge \frac{\pi}{2} \\ -\left(x-\frac{\pi}{2}\right)^3 + \sin^2 x & \text{if } x < \frac{\pi}{2} \end{cases}$

Let $u(x) = \begin{cases} \left(x-\frac{\pi}{2}\right)^3 & x \ge \frac{\pi}{2} \\ -\left(x-\frac{\pi}{2}\right)^3 & x < \frac{\pi}{2} \end{cases}$

$$u'\left(\frac{\pi}{2}+\right) = \lim_{h\to 0+} \frac{\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^3 - 0}{h}$$

$$= \lim_{h\to 0+} \frac{h^3}{h} = \lim_{h\to 0+} h^2 = 0$$

$$u'\left(\frac{\pi}{2}-\right) = \lim_{h\to 0-} \frac{-\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^3}{h} = 0$$

$$u'(x) = \begin{cases} 3\left(x-\frac{\pi}{2}\right)^2, & x \ge \frac{\pi}{2} \\ -3\left(x-\frac{\pi}{2}\right)^2, & x < \frac{\pi}{2} \end{cases}$$

$$u''(x) = \begin{cases} 6\left(x-\frac{\pi}{2}\right), & x > \frac{\pi}{2} \\ -6\left(x-\frac{\pi}{2}\right), & x < \frac{\pi}{2} \end{cases}$$

$$u''\left(\frac{\pi}{2}+\right) = \lim \frac{3\left(h+\frac{\pi}{2}-\frac{\pi}{2}\right)^2 - 0}{h} = 0$$

If $v(x) = \sin^2 x$, then $v'(x) = 2 \sin x \cos x = \sin 2x$

so $v''(x) = 2\cos 2x$ and $v''\left(\frac{\pi}{2}\right) = -2$

Hence $f''\left(\frac{\pi}{2}\right) = 0 - 2 = -2$

Example 5: If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, then the value of $\frac{\left(1+y'^2\right)^{3/2}}{y''}$ is equal to

(a) $\frac{a-b}{a+b}$ (b) $\frac{c+a}{c-a}$
(c) 1 (d) $-c$

Ans. (d)

Solution: $2(x-a) + 2(y-b)\,\frac{dy}{dx} = 0$ (i)

$$\Rightarrow \quad y' = \frac{dy}{dx} = -\frac{x-a}{y-b}$$

Differentiating (i) again, we get

$1 + (y-b)\,y'' + y'^2 = 0$

$$\Rightarrow \quad y'' = -\frac{1+y'^2}{(y-b)}$$

So, $$\frac{(1+y'^2)^{3/2}}{y''} = \frac{\left(1+\left(\frac{x-a}{y-b}\right)^2\right)^{1/2}\left(1+y'^2\right)}{-\frac{\left(1+y'^2\right)}{y-b}}$$

$$= -(y-b)\frac{\left((y-b)^2+(x-a)^2\right)^{1/2}}{(y-b)}$$

$$= -c$$

Example 6: If $y = \sin(n \sin^{-1}x)$ satisfies $(1-x^2)y'' - xy' + Ay = 0$ then A is equal to

(a) 1 (b) 2
(c) n^2 (d) n

Ans. (c)

Solution: $y' = \cos\left(n\sin^{-1}x\right)\frac{n}{\sqrt{1-x^2}}$

Squaring both the sides, we have

$$(1-x^2)\,y'^2 = n^2\cos^2(n\sin^{-1}x)$$
$$= n^2\,[1-\sin^2(n\sin^{-1}x)]$$
$$= n^2\,[1-y^2]$$

Differentiating again

$$-2xy'^2 + (1-x^2)\,2y'y'' = -n^2\,2yy'$$

$$\Rightarrow \quad (1-x^2)\,y'' - xy' = -n^2 y$$

Hence $A = n^2$

Example 7: If $y = (x^2+1)^{\sin x}$, then $y'(0)$ is equal to

(a) 0 (b) 1
(c) 2 (d) –1

Ans. (a)

Solution: Using logarithmic differentiation, we have

$$\log y = \sin x\,(\log(x^2+1))$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x\,(\log(x^2+1)) + \frac{(\sin x)(2x)}{x^2+1}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + \cos x\left(\log(x^2+1)\right)\right]$$

So, $\left.\dfrac{dy}{dx}\right|_{x=0} = 1\,[0 + \text{l log l}]$

$= 0$

Example 8: If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$ then $\dfrac{dy}{dx}$ at $x = 0$ is

(a) 1 (b) -1
(c) 0 (d) 2

Ans. (b)

Solution: Differentiating the given relation implies

$$x\frac{1}{2\sqrt{1+y}}\frac{dy}{dx} + \sqrt{1+y} + \frac{dy}{dx}\sqrt{1+x} + \frac{y}{2\sqrt{1+x}} = 0$$

$$\Rightarrow \left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right)\frac{dy}{dx} = -\left(\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}\right)$$

Since $y(0) = 0$ So putting these values, we have

$$(0+\sqrt{1})\left.\frac{dy}{dx}\right|_{x=0} = -(\sqrt{1}+0)$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = -1$$

Example 9: If $y = \sin\left(e^{x^2+3x-2}\right)$ then $\left.\dfrac{dy}{dx}\right|_{x=1}$ is equal to

(a) $5e^2 \cos e^2$ (b) $3e^2 \cos e^2$
(c) $e^2 \cos e^2$ (d) $\cos e^2$

Ans. (a)

Solution:

$$\frac{dy}{dx} = \cos\left(e^{x^2+3x-2}\right)\frac{d}{dx}\left(e^{x^2+3x-2}\right)$$

$$= \cos\left(e^{x^2+3x-2}\right)e^{x^2+3x-2}\frac{d}{dx}(x^2+3x-2)$$

$$= \cos\left(e^{x^2+3x-2}\right)e^{x^2+3x-2}(2x+3)$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 5e^2 \cos(e^2)$$

Example 10: If $x = e^{\sin^{-1} y}$ then $\left.\dfrac{dy}{dx}\right|_{x=1}$

(a) -1 (b) 2
(c) 1/2 (d) 1

Ans. (d)

Solution: $x = e^{\sin^{-1} y} \Rightarrow \log x = \sin^{-1} y$

$\Rightarrow y = \sin(\log x)$

$$\frac{dy}{dx} = \cos(\log x)\frac{1}{x},$$

so $\left.\dfrac{dy}{dx}\right|_{x=1} = \cos(\log 1) = \cos 0 = 1$

LEVEL 1

Straight Objective Type Questions

Example 11: If $f'(3) = 2$, then $\displaystyle\lim_{h\to 0}\frac{f(3+h^2)-f(3-h^2)}{2h^2}$ is

(a) 1 (b) 2
(c) 3 (d) 1/2

Ans. (b)

Solution: $\displaystyle\lim_{l-0}\frac{f\left(3+h^2\right)-f\left(3-h^2\right)}{2h^2}$

$$= \frac{1}{2}\lim_{h\to 0}\frac{f(3+h^2)-f(3)}{h^2} - \frac{1}{2}\lim_{h\to 0}\frac{f(3-h^2)-f(3)}{h^2}$$

$$= \frac{1}{2}\,[f'(3)+f'(3)] = 2.$$

Example 12: The set of all points where the function $f(x) = \sqrt{1-e^{-x^2}}$ is differentiable is

(a) $(0, \infty)$ (b) $(-\infty, \infty)$
(c) $(-\infty, \infty) \sim \{0\}$ (d) none of these.

Ans. (c)

Solution: For $x \neq 0$, $f'(x) = \dfrac{1}{2}\dfrac{1}{\sqrt{1-e^{-x^2}}}\,[-(-2x)e^{-x^2}]$

$$= \frac{x\,e^{-x^2}}{\sqrt{1-e^{-x^2}}}$$

Also $\displaystyle f'(0+) = \lim_{h\to 0+}\frac{f(h)-f(0)}{h}$

$$= \lim_{h \to 0+} \left(\frac{e^{-h^2} - 1}{-h^2} \right)^{1/2} = 1$$

and $$f'(0-) = -\lim_{h \to 0-} \left(\frac{e^{-h^2} - 1}{-h^2} \right)^{1/2} = -1.$$

Hence the set of all points of differentiability is $(-\infty, \infty) \sim \{0\}$.

Example 13: If $f(x) = |\log_{10} x|$ then at $x = 1$

(a) f is not continuous
(b) f is continuous but not differentiable
(c) f is differentiable
(d) the derivative is 1.

Ans. (b)

Solution: Since $g(x) = |x|$ is continuous but not differentiable at $x = 0$, so f is continuous but not differentiable for $\log_{10} x = 0$ i.e. for $x = 1$.

Example 14: $f(x) = \begin{cases} |x-4| & \text{for } x \ge 1 \\ x^3/2 - x^2 + 3x + 1/2 & \text{for } x < 1, \end{cases}$

then

(a) $f(x)$ is continuous at $x = 1$ and at $x = 4$
(b) $f(x)$ is differentiable at $x = 4$
(c) $f(x)$ is continuous and differentiable at $x = 1$
(d) $f(x)$ is only continuous at $x = 1$.

Ans. (a)

Solution: Since $g(x) = |x|$ is a continuous function and $\lim_{x \to 1+} f(x) = 3 = \lim_{x \to 1-} f(x)$ so f is a continuous function. In particular f is continuous at $x = 1$ and $x = 4$. f is clearly not differentiable at $x = 4$. Since $g(x) = |x|$ is not differentiable at $x = 0$. Now

$$f'(1+) = \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0+} \frac{|-3+h| - 3}{h} = -1,$$

$$f'(1-) = \lim_{h \to 0-} \frac{(1/2)(1+h)^3 - (1+h)^2 + 3(1+h) + 1/2 - 3}{h}$$

$$= \lim_{h \to 0-} \frac{(1/2)\left(h^3 + 3h^2 + 3h\right) - \left(h^2 + 2h\right) + 3h}{h}$$

$$= \frac{5}{2}.$$

Hence f is not differentiable at $x = 1$.

Example 15: If $f(x) = |x - a| + |x + b|$, $x \in \mathbf{R}$, $b > a > 0$. Then

(a) $f'(a+) = 1$ (b) $f'(a+) = 0$
(c) $f'(-b+) = 0$ (d) $f'(-b+) = 1$

Ans. (c)

Solution: $$f'(a+) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0+} \frac{|h| + |a+b+h| - |a+b|}{h}$$

$$= \lim_{h \to 0+} \frac{h + a + b + h - (a+b)}{h} \text{ as } h > 0, a+b > 0$$

$$= 2$$

$$f'(-b+) = \lim_{h \to 0+} \frac{f(-b+h) - f(-b)}{h}$$

$$= \lim_{h \to 0+} \frac{|-b+h-a| + |h| - |-b-a|}{h}$$

$$= \lim_{h \to 0+} \frac{|a+b-h| + |h| - |a+b|}{h}$$

$$= \lim_{h \to 0+} \frac{a+b-h+h-(a+b)}{h} = 0.$$

Example 16: The set of all points where the function $f(x) = 2x|x|$ is differentiable is

(a) $(-\infty, \infty)$ (b) $(-\infty, 0) \cup (0, \infty)$
(c) $(0, \infty)$ (d) $[0, \infty)$

Ans. (a)

Solution: $f(x) = \begin{cases} 2x^2, & x \ge 0 \\ -2x^2, & x < 0 \end{cases}$

Since x^2 and $-x^2$ are differentiable functions, $f(x)$ is differentiable except possible at $x = 0$

Now $$f'(0+) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{f(h)}{h}$$

$$= \lim_{h \to 0+} \frac{2h^2}{h} = 0$$

and $$f'(0-) = \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{f(h)}{h}$$

$$= \lim_{h \to 0-} \frac{-2h^2}{h} = 0.$$

Hence f is differentiable everywhere.

Example 17: The function $f(x) = \sin^{-1}(\tan x)$ is not differentiable at

(a) $x = 0$ (b) $x = -\pi/6$
(c) $x = \pi/6$ (d) $x = \pi/4$

Ans. (d)

Solution: The function $f(x) = \sin^{-1}(\tan x)$ $(-\pi/4 \le x < \pi/4)$ is not differentiable for those x for which $\tan x = 1$ or -1. This happens if $x = \pi/4$ or $-\pi/4$.

Example 18: Let $[\cdot]$ denote the greatest integer function and $f(x) = [\tan^2 x]$. Then

(a) $\lim_{x\to 0} f(x)$ does not exist

(b) $f(x)$ is continuous at $x = 0$

(c) $f(x)$ is not differentiable at $x = 0$

(d) $f'(0) = 1$.

Ans. (b)

Solution: For $0 < x < \pi/4$, $0 < \tan^2 x < 1 \Rightarrow [\tan^2 x] = 0$ for $0 < x < \pi/4$. As $\tan^2 x$ is an even function, so $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} f(x) = 0$. So f is continuous at $x = 0$. Now

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-(0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Hence f is differentiable at $x = 0$ and $f'(0) = 0$.

Example 19: If $f(x)$ differentiable everywhere then

(a) $|f(x)|$ is differentiable everywhere

(b) $|f|^2$ is differentiable everywhere

(c) $f|f|$ is not differentiable at some point

(d) none of these

Ans. (b)

Solution: If f is differentiable then $|f|^2(x) = f(x)^2$ which is differentiable. For $f(x) = x$, (a) and (c) do not hold.

Example 20: Suppose f is differentiable at $x = 1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h) = 5$ then

(a) $f'(1) = 4$ (b) $f'(1) = 3$

(c) $f'(1) = 6$ (d) none of these

Ans. (d)

Solution: Since f is differentiable so it is continuous also. Thus

$$f(1) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} h\frac{f(1+h)}{h} = (0)(5) = 0$$

Hence $$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{f(1+h)}{h} = 5.$$

Example 21: If $f(a) = a^2$, $\phi(a) = b^2$ $(a > 0, b > 0)$ and $f'(a) = 3\phi'(a)$ then $\lim_{x\to a}\frac{\sqrt{f(x)}-a}{\sqrt{\phi(x)}-b}$ is

(a) b^2/a^2 (b) b/a

(c) $2b/a$ (d) none of these

Ans. (d)

Solution: $$\lim_{x\to a}\frac{\sqrt{f(x)}-a}{\sqrt{\phi(x)}-b}$$

$$= \lim_{x\to a}\frac{f(x)-a^2}{\phi(x)-b^2}\times\frac{\sqrt{\phi(x)}+b}{\sqrt{f(x)}+a}$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{\phi(x)-\phi(a)}\times\frac{\sqrt{\phi(x)}+b}{\sqrt{f(x)}+a}$$

$$= \frac{f'(a)}{\phi'(a)}\times\frac{\sqrt{\phi(a)}+b}{\sqrt{f(a)}+a} = \frac{3b}{a}.$$

Example 22: Let f and g be differentiable function satisfying $g'(a) = 2$, $g(a) = b$ and $g = f^{-1}$ then $f'(b)$ is equal to

(a) 1/2 (b) 2

(c) 2/3 (d) none of these

Ans. (a)

Solution: Since $g = f^{-1}$ so $f\,o\,g\,(x) = x$. Therefore, $f'(g(x))\,g'(x) = 1$. Thus, $f'(g(a))\,g'(a) = 1$

$\Rightarrow \quad f'(b)2 = 1 \quad \Rightarrow \quad f'(b) = 1/2.$

Example 23: Let $f(x) = |x - a|\,\varphi(x)$, where φ is a continuous function and $\varphi(a) \neq 0$. Then

(a) $f'(a+) = \varphi'(a)$

(b) f is differentiable at $x = a$

(c) $f'(a+) = \varphi'(a)$

(d) $f'(a-) = -\varphi(a)$

Ans. (d)

Solution: $$f'(a+) = \lim_{h\to 0+}\frac{|a+h-a|\,\varphi(a+h)-0}{h}$$

$$= \lim_{h\to 0+}\frac{|h|\,\varphi(a+h)}{h}$$

$$= \lim_{h\to 0+}\varphi(a+h) = \varphi(a)$$

Similarly $f'(a-) = -\varphi(a)$.

Example 24: If $f'(c)$ exists and non-zero then $\lim_{h\to 0}\frac{f(c+h)+f(c-h)-2f(c)}{h}$ is equal to

(a) $f'(c)$ (b) 0

(c) $2f'(c)$ (d) none of these

Ans. (b)

Solution: $$\lim_{h\to 0}\frac{f(c+h)+f(c-h)-2f(c)}{h}$$

$$= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}+\frac{f(c-h)-f(c)}{h}$$

$$= f'(c) - \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

$$= f'(c) - f'(c) = 0$$

Example 25: Let $f: \mathbf{R} \to \mathbf{R}$ be any function. Define $g: \mathbf{R} \to \mathbf{R}$ by $g(x) = |f(x)|$ for all x. Then g is

(a) g may be bounded even if f is unbounded

(b) one-one if f is one

(c) continuous if f is continuous

(d) differentiable if f is differentiable

Ans. (c)

Solution: Take $f(x) = x$. Since g: $\mathbf{R} \to \mathbf{R}$ given by $g(x) = |x|$ is not one-one so (b) is violated. Also g is not differentiable at $x = 0$. Let $u(x) = |x|$ then u is continuous function and $g(x) = u(f(x)) = u \circ f(x)$. Hence g is continuous if f is continuous. It is easy to see g is bounded if and only if f is bounded.

Example 26: The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos|x|$ is not differentiable at

(a) -1 (b) 0
(c) 1 (d) 2

Ans. (d)

Solution: $f(x) = (x+1)(x-1)|x-1||x-2| + \cos x$, because

$$\cos|x| = \begin{cases} \cos x, & if\ x \ge 0 \\ \cos(-x), & if\ x < 0 \end{cases} = \cos x.$$

Since $h(x) = x|x|$ is differentiable at $x = 0$, so $(x-1)|x-1|$ is differentiable at $x = 1$. Also $f(x)$ is not differentiable at $x = 2$.

Example 27: Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = xf(x)$ then

(a) g is a differentiable function
(b) g is differentiable at 0 if f is continuous at 0
(c) g is one-one if f is one-one
(d) none of these

Ans. (b)

Solution: Take $f(x) = \operatorname{sgn} x$ then $g(x) = |x|$, hence g need not be differentiable. Take $f(x) = x$, then $g(x) = x^2$ which is not one-one.

Let f be continuous at 0, then for $h \ne 0$.

$$\frac{g(0+h) - g(0)}{h} = \frac{hf(h)}{h} = f(h). \text{ Therefore, } g'(0) = f(0).$$

Example 28: If $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2} - 1}{x}\right)$, then $y'(0)$ is

(a) 1/2 (b) 0
(c) 1 (d) doesn't exist.

Ans. (a)

Solution: Putting $x = \tan\theta$, we get

$$\frac{\sqrt{1+x^2} - 1}{x} = \frac{\sec\theta - 1}{\tan\theta} = \frac{1 - \cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right)$$

Hence $y = \dfrac{1}{2}\tan^{-1} x$ so $y'(x) = \dfrac{1}{2(1+x^2)}$ and thus $y'(0) = 1/2$.

Example 29: If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$ then $(f(24))^2 + (g(24))^2$ is

(a) 32 (b) 24
(c) 64 (d) 48

Ans. (a)

Solution: $\dfrac{d}{dx}((f(x))^2 + (g(x))^2) = 2[f(x)f'(x) + g(x)g'(x)] = 2[f(x)g(x) - g(x)f(x)] = 0.$

Hence $(f(x))^2 + (g(x))^2$ is constant. Thus $(f(24))^2 + (g(24))^2 = (f(2))^2 + (g(2))^2 = 16 + 16 = 32.$

Example 30: If $f(x) = \log_{x^2}(\log x)$ then $f'(x)$ at $x = e$ is

(a) 0 (b) 1
(c) e^{-1} (d) $(2e)^{-1}$.

Ans. (d)

Solution: $f(x) = \dfrac{\log(\log x)}{\log x^2} = \dfrac{1}{2}\dfrac{\log(\log x)}{\log x}$

$$\text{Therefore } f'(x) = \frac{1}{2}\,\frac{\log x \cdot \dfrac{1}{x\log x} - \dfrac{1}{x}\log(\log x)}{(\log x)^2}$$

and thus $f'(e) = \dfrac{1}{2}\left[\dfrac{1}{e} - 0\right] = (2e)^{-1}$.

Example 31: If $x = 2\sin t - \sin 2t$, $y = 2\cos t - \cos 2t$, then the value of $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{2}$ is

(a) 2 (b) $-1/2$
(c) $-3/4$ (d) $-3/2$.

Ans. (b)

Solution: $\dfrac{dx}{dt} = 2\cos t - 2\cos 2t = 2\,[2\sin(3t/2)\sin t/2]$

and $\dfrac{dy}{dt} = -2\sin t + 2\sin 2t = 2\left[\sin\dfrac{3t}{2}\cos\dfrac{t}{2}\right]$.

Hence $\dfrac{dy}{dx} = \cot(t/2)$ and $\left(\dfrac{d^2y}{dx^2}\right) = -\dfrac{1}{2}\operatorname{cosec}^2(t/2) \times \dfrac{dt}{dx}$

Therefore, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{2}\operatorname{cosec}^2(t/2)\,x \times \dfrac{1}{4\sin(3t/2)\sin(t/2)}$

and $\left.\dfrac{d^2y}{dx^2}\right|_{t=\pi/2} = \left(-\dfrac{1}{2}\right) \times \left(\sqrt{2}\right)^2 \times \dfrac{1}{4 \times (1/2)} = -\dfrac{1}{2}$.

Example 32: Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log x$. If $F(x) = h(f(g(x)))$, then $F''(x)$ is

(a) $2\operatorname{cosec}^3 x$
(b) $2\cot x^2 - 4x^2\operatorname{cosec}^2 x^2$
(c) $2x\cot x^2$
(d) $-2\operatorname{cosec}^2 x$.

Ans (b)

Solution: $F(x) = h(f(g(x))) = h(f(x^2)) = h(\sin x^2) = \log\sin x^2$.

Hence $F'(x) = 2x\cot x^2$. and $F''(x) = 2\cot x^2 - 4x^2\operatorname{cosec}^2 x^2$.

Example 33: Let $y = e^{2x}$. Then $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{d^2x}{dy^2}\right)$ is

(a) 1 (b) e^{-2x}
(c) $2e^{-2x}$ (d) $-2e^{-2x}$.

Ans. (d)

Solution: $\dfrac{dy}{dx} = 2e^{2x}$ and $\dfrac{d^2y}{dx^2} = 4e^{2x}$. Also $x = \dfrac{1}{2}\log y$,

so $$\frac{dx}{dy} = \frac{1}{2y} \text{ and } \frac{d^2x}{dy^2} = -\frac{1}{2y^2} = -\frac{1}{2}e^{-4x}.$$

Hence $$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = -2e^{-2x}.$$

Example 34: Let $f(x + y) = f(x) f(y)$ for all x and y. If $f(5) = 2$ and $f'(0) = 3$ then $f'(5)$ is equal to

(a) 5 (b) 8
(c) 0 (d) none of these

Ans. (d)

Solution: Putting $x = 0$, $y = 5$ in the given equation, we get

$$f(0+5) = f(0) f(5) \Rightarrow f(5)[f(1) - 0] = 0 \Rightarrow f(0) = 1$$

$$f'(5) = \lim_{h\to 0}\frac{f(5+h) - f(5)}{h} = \lim_{h\to 0}\frac{f(5)(f(h)-1)}{h}$$

$$= f'(5)\lim_{h\to 0}\frac{f(h) - f(0)}{h} = (2)(3) = 6.$$

Example 35: If $f''(x)$ is continuous at $x = 0$ and $f''(0) = 4$, then the value of

$$\lim_{x\to 0}\frac{2f(x) - 3f(2x) + f(4x)}{x^2} \text{ is}$$

(a) 4 (b) 8
(c) 12 (d) none of these

Ans. (c)

Solution: Required limit

$$= \lim_{x\to 0}\frac{2f'(x) - 6f'(2x) + 4f'(4x)}{2x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{2f''(x) - 12f''(2x) + 16f''(4x)}{2}$$

$$= 3f''(0) = 12.$$

Example 36: The solution set of $f'(x) > g'(x)$ where $f(x) = (1/2)\,5^{2x+1}$ and $g(x) = 5^x + 4x\log 5$ is

(a) $(1, \infty)$ (b) $(0, 1)$
(c) $[0, \infty)$ (d) $(0, \infty)$

Ans. (d)

Solution: $f'(x) = (1/2)\,5^{2x+1}(\log 5)(2) = \log 5 \cdot (5^{2x+1})$
also $g'(x) = 5^x\log 5 + 4\log 5$.

So $\{x : f'(x) > g'(x)\} = \{x : \log 5 \cdot 5^{2x+1} > \log 5\, 5^x + 4\log 5\}$
$= \{x : 5^{2x+1} > 5^x + 4\}$
$= \{t = 5^x : 5t^2 - t - 4 > 0\}$
$= \{t = 5^x : (5t + 4)(t - 1) > 0\}$
$= \{t = 5^x : t > 1 \text{ or } t < -4/5\}$
$= \{t = 5^x : t > 1\} = (0, \infty)$.

Example 37: The values of a and b such that the function f defined as

$$f(x) = \begin{cases} ax^2 - b, & |x| < 1 \\ -1/|x|, & |x| \ge 1 \end{cases} \text{ is differentiable are}$$

(a) $a = 1, b = -1$
(b) $a = 1/2, b = 1/2$
(c) $a = 1/2, b = 3/2$
(d) none of these

Ans. (c)

Solution: Since every differentiable function is continuous, so we must have

$$\lim_{x\to 1-} f(x) = f(1) \Rightarrow a - b = -1.$$

For f to be differentiable, $f'(1-) = f'(1+)$

$$\Rightarrow \lim_{h\to 0-}\left[\frac{a(1+h)^2 - b + 1}{h}\right] = \lim_{h\to 0+}\left[\frac{-1/|1+h| + 1}{h}\right]$$

$$\Rightarrow \lim_{h\to 0-}\left[\frac{a(2h+h^2)}{h}\right] = \lim_{h\to 0+}\frac{h}{h(1+h)} \quad (\text{as } a - b = -1)$$

$\Rightarrow$ $2a = 1$. Hence $a = 1/2$ and $b = 3/2$.

Example 38: For a whole number n, if $f(x) = x^{n-1}\sin(1/x)$, $x \ne 0$ and $f(0) = 0$ then in order that f is differentiable at all x, the value of n can be

(a) 1 (b) 2
(c) 3 (d) 0

Ans. (c)

Solution: f is clearly differentiable except possibly $x = 0$.

$$f'(0) = \lim_{h\to 0}\frac{h^{n-1}\sin(1/h)}{h}$$

$$= \lim_{h\to 0} h^{n-2}\sin(1/h)$$

The last limit exists and is equal to 0 only if $n - 2 \ge 1$, i.e. $n \ge 3$. [Using Sandwich Theorem]

Example 39: $\dfrac{d}{dx}(\log(ax)^x)$, where a is a constant is equal to

(a) 1 (b) $\log ax$
(c) $1/a$ (d) $\log(ax) + 1$

Ans. (d)

◎ **Solution:** $\frac{d}{dx}(\log(ax)^x) = \frac{d}{dx}(x\log ax) = \log ax + 1.$

◉ **Example 40:** If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log a$ then $\frac{dy}{dx}$ is equal to

(a) y/x (b) x/y
(c) x^2/y^2 (d) y^2/x^2

Ans. (a)

◎ **Solution:** $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log\ a \Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = \cos$

$\log a = A$ (say)

Putting $u = y/x$ and applying componendo and dividendo, we have

$$(y/x)^2 = u^2 = (1 - A)/(1 + A)$$
$$\Rightarrow y/x = \sqrt{(1 - A)/(1 + A)}$$
$$\Rightarrow \quad x\,dy/dx - y = 0$$
$$\Rightarrow dy/dx = y/x$$

◉ **Example 41:** If $(\sin x)(\cos y) = 1/2$ then d^2y/dx^2 at $(\pi/4, \pi/4)$ is equal to

(a) -4 (b) -2
(c) -6 (d) 0

Ans. (a)

◎ **Solution:** Differentiating the given equation, we have $\cos x \cos y - \sin x \sin y\ dy/dx = 0$. Putting $x = y = \pi/4$, we have $\left.\frac{dy}{dx}\right|_{(\pi/4,\ \pi/4)} = 1$. Differentiating again, we get $-\sin x \cos y - \cos x \sin y\ dy/dx - \cos x \sin y\ dy/dx - \sin x \sin y\ (dy/dx)^2 - \sin x \sin y\ d^2y/dx^2 = 0$. Putting $x = y = \pi/4$, we have

$$\left.\frac{d^2 y}{dx^2}\right|_{(\pi/4,\pi/4)} = -4.$$

◉ **Example 42:** The set onto which the derivative of the function $f(x) = x(\log x - 1)$ maps the ray $[1, \infty)$ is

(a) $[1, \infty)$ (b) $(0, \infty)$
(c) $[0, \infty)$ (d) none of these

Ans. (c)

◎ **Solution:** $f'(x) = \log x - 1 + x\left(\frac{1}{x}\right) = \log x$

Since $\log x$ is an increasing function so f' maps $[1, \infty)$ onto $[0, \infty)$.

◉ **Example 43:** Let $f(x)$ be a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbf{R}$ and $f(x) = 1 + x\ g(x)$ where $\lim_{x\to 0} g(x) = 1$ then $f'(x)$ is equal to

(a) $g'(x)$ (b) $g(x)$
(c) $f(x)$ (d) none of these

Ans. (c)

◎ **Solution:** $f'(x)$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h) - f(x)}{h}$$
$$= f(x)\lim_{h\to 0}\frac{f(h)-1}{h} = f(x)\lim_{h\to 0}\frac{1+hg(h)-1}{h}$$
$$= f(x)\lim_{h\to 0} g(h) = f(x)$$

◉ **Example 44:** Let $g = f^{-1}$ and $f'(x) = \frac{1}{1+x^4}$ then $g'(x)$ is equal to

(a) $(1 + (g(x)^4)^{-1}$ (b) $(1 + (f(x))^4)^{-1}$
(c) $1 + (f(x))^4$ (d) $1 + (g(x))^4$

Ans. (d)

◎ **Solution:** Since $g(f(x)) = x$ so $g'(f(x)) f'(x) = 1$ $\Rightarrow g'(f(x)) = 1 + x^4$. Putting $u = f(x)$ so that $g(u) = x$, we have $g'(u)$

$= 1 + (g(u))^4$.

◉ **Example 45:** Suppose for a differentiable function f, $f(0) = 0, f(1) = 1$ and $f'(0) = 4 = f'(1)$. If $g(x) = f(e^x)\,e^{f(x)}$ then $g'(0)$ is equal to

(a) 4 (b) 8
(c) 2 (d) none of these

Ans. (b)

◎ **Solution:** $g'(x) = f'(e^x)\,e^x\,e^{f(x)} + f(e^x)\,e^{f(x)} f'(x)$,
so $g'(0) = f'(1)\,e^{f(0)} + f(1)\,e^{f(0)} f'(0) = f'(1) + f'(0) = 8$

◉ **Example 46:** If the function f is differentiable and strictly increasing in a neighbourhood of 0, then

$\lim_{x\to 0}\frac{f(x^5) - f(x)}{f(x) - f(0)}$ is equal to

(a) -1 (b) 0
(c) 1 (d) 5/2

Ans. (a)

◎ **Solution:** $\frac{f(x^5) - f(x)}{f(x) - f(0)}$

$$= \frac{f(x^5) - f(0) - (f(x) - f(0))}{f(x) - f(0)}$$
$$= \frac{f(x^5) - f(0)}{f(x) - f(0)} - 1$$
$$= \frac{\frac{f(x^5) - f(0)}{x^5 - 0}\times x^5}{\frac{f(x) - f(0)}{x - 0}\times x} - 1$$

So $$\lim_{x\to 0}\frac{f(x^5) - f(x)}{f(x) - f(0)} = \lim_{x\to 0} x^4 \times \frac{f'(0)}{f'(0)} - 1$$
$$= 0 - 1 = -1\ .\ (f'(0) > 0)$$

Example 47: Let $f(x) = ||x-1|-1|$, then all the points where $f(x)$ is not differentiable is (are)

(a) 1 (b) 1, 2
(c) ± 1 (d) 0, 1, 2

Ans. (d)

Solution: $f(x) = \begin{cases} |x-1|-1 & \text{if } |x-1| \geq 1 \\ 1-|x-1| & \text{if } |x-1| < 1 \end{cases}$

$$= \begin{cases} -x & x \leq 0 \\ x & 0 < x \leq 1 \\ 2-x & 1 < x < 2 \\ x-2 & x \geq 2 \end{cases}$$

f is not differentiable at 0, 1, 2 as these are corner points on the graph.

Example 48: $\dfrac{d^2x}{dy^2}$ equals

(a) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

(c) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (d) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

Ans. (d)

Solution: $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$-\frac{d}{dx}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)^{-2}\left(\frac{dy}{dx}\right)^{-1}$$

$$= -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}$$

Example 49: If $x^2 + y^2 = t + \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$ $(xy > 0)$ then $\dfrac{d^2y}{dx^2}$ is equal

(a) $1/x^3$ (b) $2/x^3$
(c) $2\sqrt{2}/x^3$ (d) $2\sqrt{2}/x^4$

Ans. (b)

Solution: $(x^2 + y^2)^2 = \left(t+\dfrac{1}{t}\right)^2 = t^2 + \dfrac{1}{t^2} + 2 = x^4 + y^4 + 2$

$$\Rightarrow \quad 2x^2y^2 = 2 \quad \Rightarrow \quad x^2y^2 = 1 \quad \Rightarrow \quad xy = 1$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{x^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{x^3}.$$

Example 50: If $2f(\sin x) + f(\cos x) = x$ for all $x \in \mathbf{R}$ then $y = f(x)$ satisfies

(a) $(1 - x^2)\, y'' - xy' = 0$
(b) $x^2 y'' - (1 - x^2)y' = 0$
(c) $(1 + x^2)\, y'' + xy' = 0$
(d) $(1 - x^2)\, y'' + xy' = 0$

Ans. (a)

Solution: Replacing x by $\pi/2 - x$ in the given equation, we get

$$2f(\cos x) + f(\sin x) = \pi/2 - x$$

Solving this equation with the given equation, we obtain

$$f(\cos x) = \pi/3 - x \Rightarrow y = f(x) = \pi/3 - \cos^{-1}x$$

$$\Rightarrow \quad y' = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad y'^2(1 - x^2) = 1$$

Differentiating again we have

$$2\, y'\, y''(1 - x^2) - 2x\, y'^2 = 0$$

$$\Rightarrow \quad (1 - x^2)\, y'' - xy' = 0.$$

Example 51: Let $f(x) = (1 + \sin x)(2 + \sin^2 x) \ldots + (n + \sin^n x)$ then $f'(0)$ is equal to

(a) 1 (b) $n!$
(c) $n!/2$ (d) n

Ans. (b)

Solution: $\log f(x) = \displaystyle\sum_{k=1}^{n} \log\left(k + \sin^k x\right)$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{k \sin^{k-1} x \cos x}{k + \sin^k x}$$

$f'(0) = f(0)\,.\,1$ (If $k = 1$, then at $x = 0$ the corresponding term in summation is 1 otherwise it is 0)

$f'(0) = 1.\,2.\,\ldots n = n!$ as $f(0) = n!$

Example 52: Let f_α be a function defined by $f_\alpha(x) = \begin{cases} x^\alpha \sin 1/x & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

A value of α so that f_α is differentiable on $\mathbf{R}$ and f'_α is differentiable on $\mathbf{R}$ is

(a) 2 (b) 5/2
(c) 7/2 (d) 1

Ans. (c)

Solution: For $x \neq 0$

$$f'_\alpha(x) = x^{\alpha-2} \sin 1/x + \alpha x^{\alpha-1} \sin 1/x$$

$$\frac{f_\alpha(x) - f_\alpha(0)}{x - 0} = x^{\alpha-1} \sin 1/x$$

$$\lim_{x \to 0} \frac{f_\alpha(x) - f_\alpha(0)}{x - 0} \text{ exists and is equal 0 if } \alpha > 1$$

For $x \neq 0$, $f''_\alpha(x) = (\alpha - 2)\, x^{\alpha-3} \sin 1/x - x^{\alpha-4} \cos 1/x + \alpha(\alpha - 1)\, x^{\alpha-2} \sin 1/x - \alpha\, x^{\alpha-3} \cos 1/x$

$$\lim_{x\to 0}\frac{f'_\alpha(x)-f'_\alpha(0)}{x-0} = \lim_{x\to 0}\left(x^{\alpha-3}\sin 1/x+\alpha x^{\alpha-2}\sin 1/x\right)$$

The last limit exists if $\alpha > 3$ So $\alpha = 7/2$.

Example 53: If function $f(x)$ is differentiable at $x = a$ then $\lim_{x\to a}\frac{x^2 f(a)-a^2 f(x)}{x-a}$ is.

(a) $-a^2 f'(a)$

(b) $a f(a) - a^2 f'(a)$

(c) $2a f(a) - a^2 f'(a)$

(d) $2a f(a) + a^2 f'(a)$

Ans. (c)

Solution: For $x \neq a$, $\frac{x^2 f(a)-a^2 f(x)}{x-a}$

$$= \frac{(x^2-a^2) f(a)+a^2 (f(a)-f(x))}{x-a}$$

$$= (x + a) f(a) - a^2\frac{f(x)-f(a)}{x-a}$$

Therefore, $\lim_{x\to a}\frac{x^2 f(a)-a^2 f(x)}{x-a} = 2a f(a) - a^2 f'(a)$.

Example 54: Let f: $(0, 1) \to \mathbf{R}$ be defined by $f(x) = \frac{b-x}{1-bx}$ where b is a constant such that $0 < b < 1$. Then

(a) f is not invertible on (0,1)

(b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(c) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(d) f^{-1} is differentiable on (0, 1).

Ans. (a)

Solution: As $f(0)$ and hence $f'(0)$ are not defined so (b) and (c) are not possible. Note that $0 < bx < 1$, for $0 < x < 1$. Thus $f(x) > 0$, for $0 < x < b$, $f(b) = 0$ and $f(x) < 0$, for $b < x < 1$. For $y_1, y_2 \in (0, 1)$

$$f(y_1) = f(y_2) \Rightarrow \frac{b-y_1}{1-by_1} = \frac{b-y_2}{1-by_2}$$

$\Rightarrow$ $b - y_1 - b^2y_2 + by_1 y_2 = b - b^2y^1 - y_2 + by_1y_2$

$\Rightarrow$ $(y_2 - y_1)(1 - b^2) = 0$

Since $0 < b < 1$ so $y_2 = y_1$

Thus f is one – one, but f is not onto. The pre-image of any $y \in (1, \infty)$ is $\frac{b-y}{1-by}$ but $\frac{b-y}{1-by} \in (0,1)$ if $0 < b < 1/y$.

Example 55: If $y = \text{cosec}(\cot^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) 1

(c) $-\sqrt{2}$ (d) $-\frac{1}{\sqrt{2}}$

Ans. (a)

Solution:

$$\frac{dy}{dx} = -\text{cosec}(\cot^{-1}x)\cot(\cot^{-1}x)\frac{d}{dx}(\cot^{-1}x)$$

$$= \text{cosec}(\cot^{-1}x)\times\frac{1}{1+x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \text{cosec}\frac{\pi}{4}\times\frac{1}{2} = \sqrt{2}\times\frac{1}{2} = \frac{1}{\sqrt{2}}.$$

Example 56: Let $f(x) = \begin{cases}\frac{1}{|x|} & \text{if } |x|>2\\ A+Bx^2 & \text{if } |x|\le 2\end{cases}$

then $f(x)$ is differentiable at $x = -2$ for:

(a) $A = \frac{3}{4},\ B = -\frac{1}{16}$

(b) $A = -\frac{1}{4},\ B = \frac{1}{16}$

(c) $A = \frac{1}{4},\ B = -\frac{1}{16}$

(d) $A = \frac{3}{4},\ B = \frac{1}{16}$

Ans. (a)

Solution: $f(x) = \begin{cases}-\frac{1}{x}, & x<-2\\ A+Bx^2, & -2\le x\le 2\\ \frac{1}{x}, & x>2\end{cases}$

$$f'(-2+) = \lim_{h\to 0+}\frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0+}\frac{A+B(-2+h)^2-(A+4B)}{h}$$

$$= B\lim_{h\to 0+}\frac{(4+h^2-4h)-4}{h}$$

$$= B\lim_{h\to 0+}(h-4) = -4B$$

$$f'(2-) = \lim_{h\to 0-}\frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0-}\frac{-\frac{1}{-2+h}-(A+4B)}{h}$$

$$= -\lim_{h\to 0-} \frac{1+(A+4B)(-2)+(A+4B)h}{(-2+h)h}$$

The last limit exists if $1 + (-2)(A + 4B) = 0$ (i)

In case (i) is satisfied then the last limit is equal to

$$-\lim_{h\to 0-} \frac{A+4B}{(-2+h)} = \frac{1}{2}(A+4B) = \frac{1}{4}$$

Hence $B = -\frac{1}{16}$. Putting this value in (i)

$$1 + (-2)\left(A - \frac{1}{4}\right) = 0 \Rightarrow A = \frac{3}{4}$$

Example 57: If $f''(x) = -f(x)$, $h(x) = f'(x)$ and $F(x) = (f(x/2))^2 + (h(x/2))^2$. Given that $F(2) = 4$, the value $F(1)$ is

(a) 2 (b) 4

(c) 1 (d) 0

Ans. (b)

Solution: $h'(x) = f''(x) = -f(x)$.

$$F'(x) = 2f\left(\frac{x}{2}\right)f'\left(\frac{x}{2}\right)\frac{1}{2} + 2h\left(\frac{x}{2}\right)h'\left(\frac{x}{2}\right)\frac{1}{2}$$

$$= f\left(\frac{x}{2}\right)f'\left(\frac{x}{2}\right) + f'\left(\frac{x}{2}\right)\left(-f\left(\frac{x}{2}\right)\right) = 0$$

Hence F is constant function. Therefore, $F(1) = F(2) = 4$.

Example 58: Let f be a differentiable function such that $8f(x) + 6f(1/x) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = 1$ is

(a) $\frac{15}{14}$ (b) $\frac{17}{14}$

(c) $\frac{19}{14}$ (d) $-\frac{17}{14}$

Ans. (c)

Solution: Differentiating the given expression, we get

$$8f'(x) + 6f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - 1 = 0$$

$\Rightarrow$ $8f'(1) + 6f'(1)(-1) = 1$

$\Rightarrow$ $f'(1) = 1/2$

Also $\frac{dy}{dx} = 2xf(x) + x^2 f'(x)$

so, $\left.\frac{dy}{dx}\right|_{x=1} = 2f(1) + f'(1)$

Putting $x = 1$ in the given equation, we obtain

$8f(1) + 6f(1) - 1 = 5$

$\Rightarrow$ $14f(1) = 6$

$\Rightarrow$ $f(1) = \frac{3}{7}$

Hence $\left.\frac{dy}{dx}\right|_{x=1} = 2, \frac{3}{7} + \frac{1}{2} = \frac{19}{14}$

Example 59: Let $f(x) = \begin{cases} Ax & , x < 1 \\ Ax^2 + Bx + 4 & , x \geq 1 \end{cases}$

If f is differentiable for all x, then

(a) $A = 2, B = -2$ (b) $A = -4, B = 4$

(c) $A = 1, B = -1$ (d) $A = 4, B = -4$

Ans. (d)

Solution:

$$f'(1+) = \lim_{h\to 0+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0+} \frac{A(1+h)^2 + B(1+h) + 4 - (A+B+4)}{h}$$

$$= \lim_{h\to 0+} \frac{A(h^2+2h) + Bh}{h} = 2A + B$$

$$f'(1-) = \lim_{h\to 0-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0-} \frac{A(1+h) - (A+B+4)}{h}$$

$$= \lim_{h\to 0-} \frac{Ah - (B+4)}{h}$$

The last limit exist if $B + 4 = 0 \Rightarrow B = -4$. In this case, the last limit is equal to A.

Hence $A = 2A + B \Rightarrow A = 4$

Example 60: A twice differentiable function $f : \mathbf{R} \to \mathbf{R}$ satisfies $\sin x \cos y\,(f(2x + 2y) - f(2x - 2y)) = \cos x \sin y\,(f(2x + 2y) + f(2x - 2y))$ for all $x, y \in \mathbf{R}$ then

(a) $f''(x) = f(x) = 0$ (b) $4f''(x) + f(x) = 0$

(c) $f''(x) + f(x) = 0$ (d) $4f''(x) - f(x) = 0$

Ans. (b)

Solution:

$f(2x + 2y)(\sin x \cos y - \cos x \sin y)$

$= f(2x - 2y)[\sin x \cos y + \cos x \sin y]$

$\Rightarrow$ $f(2x + 2y)\sin(x - y) = f(2x - 2y)\sin(x + y)$

$$\Rightarrow \quad \frac{f(2(x+y))}{\sin(x+y)} = \frac{f(2x-2y)}{\sin(x-y)}$$

Putting $2(x + y) = u$, $2(x - y) = v$

$$\Rightarrow \quad \frac{f(u)}{\sin\frac{u}{2}} = \frac{f(v)}{\sin\frac{v}{2}} = k \text{ for all } u, v \in \mathbf{R}$$

$$\Rightarrow \quad f(u) = k \sin\frac{u}{2}$$

$$\Rightarrow \quad f'(u) = \frac{k}{2}\cos\frac{u}{2} \text{ and } f''(u) = \frac{-k}{4}\sin\frac{u}{2}$$

so $4f''(u) = -k \sin\frac{u}{2} = -f(u)$

$\Rightarrow$ $4f''(u) + f(u) = 0$

Assertion-Reason Type Questions

Example 61: Let $y = \sqrt{2x - x^2}$, $0 \le x \le 2$

Statement-1: y satisfies $y^3 y'' + 1 = 0$
Statement-1: y is a differentiable function of x on [0, 2]

Ans. (c)

Solution: $y' = \frac{1}{2}\frac{2-2x}{\sqrt{2x-x^2}} = \frac{1-x}{y}$ for $x \ne 0, 2$

$\Rightarrow \quad yy' = 1 - x$. Differentiating again, we have

$$\Rightarrow \quad y'^2 + yy'' = -1 \quad \Rightarrow \quad y^3y'' = -y^2 - (yy')^2$$
$$= -y^2 - (1-x)^2$$
$$= -(2x - x^2) - (1 + x^2 - 2x) = -1$$

y is not differentiable at $x = 0, 2$.

Example 62: Let $x = \log t$, $y = t^2 - 1$

Statement-1: $\frac{d^2y}{dx^2}$ at $x = 0$ is 4

Statement-1: $\frac{d^2y}{dx^2} = 4t^2$

Ans. (a)

Solution: $\frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = 2t$ so $\frac{dy}{dx} = 2t^2$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = 4t \,.\, t = 4t^2$$

If $x = 0$ then $t = 1$ so $\frac{d^2y}{dx^2}$ at $x = 0$ is 4.

Example 63: If $e^y + xy = e$ then

Statement-1: $y''(0) = 1/e^2$
Statement-2: $y'(0) = -1$

Ans. (c)

Solution: $e^y y' + xy' + y = 0$ (1)

$\Rightarrow \quad e^y y'' + e^y y'^2 + y' + xy'' + y' = 0$ (2)

For $x = 0$, $e^{y(0)} + 0.y(0) = e \quad \Rightarrow \quad y(0) = 1$

Putting $x = 0$ in (1) $e^y(0)\, y'^{(0)} + y(0) = 0 \quad \Rightarrow \quad y'(0) = -1/e$

Putting $x = 0$ in (2) $e^{y(0)}\, y''(0) + e^{y(0)}\, (y'(0))^2 + 2y'(0) = 0$

$$\Rightarrow \quad ey''(0) + e \,.\, \frac{1}{e^2} - \frac{2}{e} = 0$$

$\Rightarrow \quad y''(0) = 1/e^2$.

Example 64: Let $x = t^3 + 1$, $y = t^2 + t + 1$

Statement-1: $\frac{dy}{dx}$ at (1, 1) is 1

Statement-2: $\frac{dt}{dx}$ is not defined at $x = 1$.

Ans. (d)

Solution: $\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t + 1$

$$\frac{dy}{dx} = \frac{2t+1}{3t^2}. \text{ If } x = 1, \text{ then } t = 0$$

$\frac{dt}{dx}$ is not defined at $t = 0$ and also $\frac{dy}{dx}$ is not defined at $t = 0$.

Example 65: If $y = \log\cos\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)$

Statement-1: $y'(0) = 0$

Statement-2: $y'(x) = -\frac{e^x - e^{-x}}{1 + x^2}$

Ans. (c)

Solution: $y'(x)$

$$= -\frac{\sin\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)}{\cos\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)} \times \frac{1}{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} \times \frac{e^x + e^{-x}}{2}$$

$$= -\tan\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)\frac{2}{\left(e^x + e^{-x}\right)^2} \times \left(e^x + e^{-x}\right)$$

$$= -\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$y'(0) = 0$.

Example 66: Consider the function

$$f(x) = |x - 2| + |x - 5|, \ x \in \mathbf{R}$$

Statement 1. $f'(4) = 0$

Statement 2. f is continuous on [2, 5], differentiable on (2, 5) and $f(2) = f(5)$.

Ans. (b)

Solution: $f(x) = \begin{cases} 7 - 2x & if\ x \le 2 \\ 3 & if\ 2 \le x \le 5 \\ 2x - 7 & if\ x > 5 \end{cases}$

$f'(4) = 0$ as f is a constant function on (2, 5). Also f is continuous on [2, 5] and differentiable on (2, 5) and $f(2) = f(5) = 0$ but this is not correct explanation of statement-1.

Example 67: Statement 1 : $f(x) = (\sin \pi x)(x-2)^{1/3}$ is differentiable at $x = 2$.

Statement 2: Pointwise product of two differentiable functions in differentiable.

Ans. (b)

Solution:

$$f'(2+) = \lim_{h\to 0+} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0+} \frac{(\sin\pi(2+h))(2+h-2)^{1/3}}{h}$$

$$= \pi \lim_{h\to 0+} \frac{(\sin\pi h)(h)^{1/3}}{\pi h}$$

(h is small but positive)

$$= \pi.1.0 = 0$$

$$f'(2-) = \lim_{h\to 0-} \frac{\sin\pi(2+h)(2+h-2)^{1/3}}{h}$$

$$= \pi \lim_{h\to 0-} \frac{(-\sin\pi h)(h)^{1/3}}{\pi h}$$

(h is small and negative)

$$= -\pi \times 1 \times 0 = 0$$

Hence f is differentiable at $x = 2$. Statement 2 is also correct but since $(x-2)^{1/3}$ is not differentiable at $x = 2$, so statement 2 does not imply Statement 1.

Example 68: Suppose that f is a differentiable function and satisfies $f(x) + f(x-4) = 0$ for all $x \in \mathbf{R}$. If $f'(x) = u$ then $f'(x+800) = u$.

Statement 2: f satisfies $f(x) = f(x+8)$

Ans. (a)

Solution: Replacing x by $x+4$ in the given equation of statement 1, we have

$$f(x+4) + f(x) = 0 \qquad \ldots\text{(i)}$$

Again replacing x by $x+4$

$$f(x+8) + f(x+4) = 0$$

$\Rightarrow \quad f(x+8) - f(x) = 0$ using (i)

so $\quad f(x) = f(x+8)$, for all $x \in \mathbf{R}$

$$f'(x+800) = \lim_{h\to 0} \frac{f(x+800+h)-f(x+800)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Example 69: Suppose that a differentiable function f satisfies $f(x+y^5) = f(x) + f(y^5)$, $x, y \in \mathbf{R}$.

Statement 1: If $f'(9) = 8$ then $f'(-9) = 8$

Statement 2: f is an odd function.

Ans. (a)

Solution: Putting $x = y = 0$ in the given equation, we have $f(0) = f(0) + f(0) \Rightarrow f(0) = 0$

Putting $y = -x^{1/5}$, we have $0 = f(0) = f(x+y^5)$

$$= f(x) + f(y^5)$$

$$= f(x) + f(-x)$$

Hence f is an odd function.

$$f'(-9) = \lim_{h\to 0} \frac{f(-9+h)-f(-9)}{h}$$

$$= \lim_{h\to 0} \frac{-f(9-h)+f(9)}{h}$$

$$= \lim_{h\to 0} \frac{f(9-h)-f(9)}{-h} = f'(9).$$

Example 70: Let $f(x) = \log\left(x+\sqrt{x^2+1}\right) + x^3 + \sin^5 x$

Statement 1: $f'$is an even function

Statement 2: f is an odd function

Ans. (a)

Solution: $f(-x) = \log(-x+\sqrt{x^2+1}) + (-x^3) + (-\sin^5 x)$

$$= \log\left(\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right) + (-x^3) + (-\sin^5 x)$$

$$= \log\frac{1}{x+\sqrt{x^2+1}} - x^3 - \sin^5 x$$

$$= -f(x)$$

Hence f is an odd function. Morever f is a differentiable function so f' is even.

LEVEL 2

Straight Objective Type Questions

Example 71: If $f(x) = p\,|\sin x| + qe^{|x|} + r|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then

(a) $q + r = 0, p \in \mathbf{R}$ (b) $p + q = 0; r \in \mathbf{R}$
(c) $q = 0; r = 0; p \in \mathbf{R}$ (d) $r = 0; p = 0; q \in \mathbf{R}$

Ans. (b)

Solution: For $-\pi/2 < x \le 0$, $f(x) = -p\sin x + qe^{-x} - rx^3$, so

$$f'(0-) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0-}\left[-\frac{p\sin x}{x} - q\left(\frac{e^{-x}-1}{-x}\right) - rx^2\right]$$
$$= -p - q.$$

For $0 < x < \pi/2, f(x) = p\sin x + q\,e^x + rx^3$,

$$f'(0+) = \lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}$$
$$= \lim_{x\to 0+}\left[\frac{p\sin x}{x} + q\left(\frac{e^{x}-1}{x}\right) - rx^2\right]$$
$$= p + q.$$

For f to be differentiable at $x = 0$, we must have

$$p + q = -p - q \Rightarrow p + q = 0.$$

Example 72: Let $h(x) = \min\{x, x^2\}$ for $x \in \mathbf{R}$. Then which of the following is not correct

(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1$ for all $x > 1$
(d) h is not *a* differentiable function at atleast two points

Ans. (b)

Solution: $h(x) = \begin{cases} x, & x \ge 1 \\ x^2, & 0 \le x < 1 \\ x, & x < 0 \end{cases}$

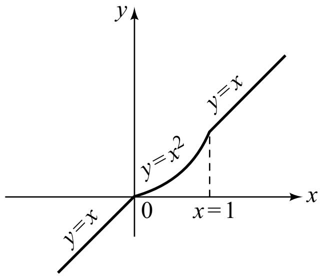


Fig. 10.3

From the graph it is clear that h is continuous. Also h is differentiable except possibly at $x = 0$ and 1.

$$h'(x) = \begin{cases} 1, & x > 1 \\ 2x, & 0 < x < 1 \\ 1 & x < 0 \end{cases}$$

For $x = 1$, $h'(1+) = \lim_{t\to 0+}\frac{h(1+t)-h(1)}{t}$

$$= \lim_{t\to 0+}\frac{1+t-1}{t} = 1$$

but $\quad h'(1-) = \lim_{t\to 0+}\frac{h(1-t)-h(1)}{t} = \lim_{t\to 0+}\frac{(1-t)^2-1}{t}$

$$= -2$$

So h is not differentiable at 1. Similarly $h'(0+) = 0$ but $h'(0-) = 1$

Example 73: Let $f(x) = \begin{cases} \dfrac{\sin|x^2-5x+6|}{x^2-5x+6} & x \ne 2, 3 \\ 1 & x = 2 \text{ or } 3 \end{cases}$

the set of all points where f is differentiable is

(a) $(-\infty, \infty)$ (b) $(-\infty, \infty) \sim \{2\}$
(c) $(-\infty, \infty) \sim \{3\}$ (d) $(-\infty, \infty) \sim \{2, 3\}$.

Ans. (d)

Solution: The function is clearly differentiable except possibly at $x = 2, 3$.

$$f'(2+) = \lim_{h\to 0+}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0+}\frac{\sin h(1-h) + h(1-h)}{h^2(-1+h)}$$
$$= -\lim_{h\to 0+}\left(\frac{\sin h(1-h)}{h(1-h)} + 1\right)\frac{1}{h}$$

The last limit doesn'*t* exist. If this limit then $\lim_{h\to 0+}\frac{1}{h}$ exist, which is not true.

Thus f is not differentiable at $x = 2$. Similarly f is not differentiable at $x = 3$. Hence the set of all points where f is differentiable is $(-\infty, \infty) \sim \{2, 3\}$.

Example 74: If $f(x) = x\cdot\frac{(a^{1/x} - a^{-1/x})}{a^{1/x} + a^{-1/x}}$, $x \ne 0$ ($a > 0$), $f(0) = 0$ then

(a) f is differentiable at $x = 0$
(b) f is not differentiable at $x = 0$
(c) f is not continuous at $x = 0$
(d) none of these.

Ans. (b)

Solution: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\frac{x\left(a^{1/x} - a^{-1/x}\right)}{a^{1/x} + a^{-1/x}}$

$$= \lim_{x\to 0+}\frac{x\left(1 - a^{-2/x}\right)}{1 + a^{-2/x}}$$

also $\quad \lim_{x\to 0-} f(x) = \lim_{x\to 0-}\frac{x\left(a^{2/x} - 1\right)}{a^{2/x} - 1} = 0.$

So f is continuous at $x = 0$.

$$f'(0+) = \lim_{x\to 0+}\frac{h\left(a^{1/h} - a^{-1/h}\right)}{h\left(a^{1/h} + a^{-1/h}\right)}$$
$$= \lim_{x\to 0+}\frac{1 - a^{-2/h}}{1 + a^{-2/h}} = 1$$

Similarly $f'(0-) = -1$. Hence, f is not differentiable at $x = 0$.

Example 75: Suppose that f is a differentiable function with the property that $f(x + y) = f(x) + f(y) + xy$ and $\lim_{h \to 0} \frac{1}{h} f(h) = 3$ then

(a) f is a linear function
(b) $f(x) = 3x + x^2$
(c) $f(x) = 3x + x^2/2$
(d) none of these.

Ans. (c)

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + xh - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} f(h) + x = 3 + x$$

Hence $f(x) = 3x + x^2/2 + C$. Putting $x = y = 0$ in the given equation, we have $f(0) = f(0 + 0) = f(0) + f(0) + 0 \Rightarrow f(0) = 0$. Thus $C = 0$ and $f(x) = 3x + x^2/2$.

Example 76: Let $f(x) = \begin{cases} x^2, & \text{if } x \le x_0 \\ ax + b, & x > x_0 \end{cases}$

If f is differentiable at x_0 then

(a) $a = x_0, b = -x_0$ (b) $a = 2x_0, b = -x_0^2$
(c) $a = 2x_0, b = x_0^2$ (d) none of these

Ans. (b)

Solution: Since f is differentiable so it is continuous also, therefore

$$x_0^2 = f(x_0) = \lim_{x \to x_0+} f(x) = ax_0 + b$$

Also $$\lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$= \lim_{h \to 0+} \frac{a(x_0 + h) + b - x_0^2}{h}$$

$$= \lim_{h \to 0+} \frac{x_0^2 + ah - x_0^2}{h} = a$$

(since $x_0^2 = ax_0 + b$)

Thus $$a = f'(x_0 -) = \lim_{h \to 0+} \frac{(x_0 + h)^2 - x_0^2}{h} = 2x_0$$

Hence $$x_0^2 = 2x_0^2 + b \quad \Rightarrow \quad b = -x_0^2$$

Example 77: Let $f(x)$ be *a* polynomial of degree two which is positive for all $x \in \mathbf{R}$. If $g(x) = f(x) + f'(x) + f''(x) + xf'''(x) + x^2 f^{IV}(x)$, then for any real x

(a) $g(x) < 0$ (b) $g(x) > 0$
(c) $g(x) = 0$ (d) $g(x) \ge 0$

Ans. (b)

Solution: Let $f(x) = ax^2 + bx + c$. As $f(x) > 0$ for all $x \in \mathbf{R}$, we must have, $a > 0$ and $b^2 - 4ac < 0$.

Now $$g(x) = ax^2 + bx + c + (2ax + b) + 2a + x.0 + x^2.0$$
$$= ax^2 + (b + 2a)x + b + c + 2a$$

Discriminant of $g(x) = (b + 2a)^2 - 4a(b + c + 2a)$
$$= -4a^2 + (b^2 - 4ac) < 0$$

Thus $g(x) > 0$ for all $x \in \mathbf{R}$.

Example 78: The left-hand derivative of $f(x) = [x] \sin \pi x$ at $x = k$, k is an integer is

(a) $(-1)^k (k - 1)\pi$ (b) $(-1)^{k-1} (k - 1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

Ans. (a)

Solution: Clearly $f(k) = 0$, so the left-hand derivative is equal to $\lim_{h \to 0-} \frac{f(k+h)}{h}$

$$= \lim_{h \to 0-} \frac{[k+h]\sin(k+h)\pi}{h}$$

$$= \lim_{h \to 0-} \frac{(k-1)\sin(k\pi + h\pi)}{h}$$

$$= \lim_{h \to 0-} \frac{(k-1)(-1)^k \sin h\pi}{h} \quad \text{(since } h < 0\text{)}$$

$$= (k - 1)(-1)^k \pi.$$

Example 79: If $y = \tan^{-1}\left(\frac{\log ex^{-2}}{\log ex^2}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$ then $\frac{d^2y}{dx^2}$ is

(a) 2 (b) 1
(c) 0 (d) x

Ans. (c)

Solution: $y = \tan^{-1}\left(\frac{\log e - 2\log x}{\log e + 2\log x}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$

$$= \tan^{-1}\left(\frac{1 - 2\log x}{1 + 2\log x}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$$

$$= \frac{\pi}{4} - \tan^{-1}(2\log x) + \tan^{-1} 3 + \tan^{-1}(2\log x)$$

$$= \frac{\pi}{4} + \tan^{-1} 3$$

Thus $$\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 0.$$

Example 80: If $f(x) = x^{1/x}$ then $f''(e)$ is equal to

(a) $e^{1/(e-3)}$ (b) $e^{1/e}$
(c) $e^{1/(e-2)}$ (d) none of these

Ans. (d)

◎ **Solution:** $\log f(x) = \frac{1}{x}\log x$. Differentiating both sides we have

$$\frac{f'(x)}{f(x)} = \frac{1}{x^2} - \frac{\log x}{x^2}$$

$$\Rightarrow \quad f'(x) = f(x)\,\frac{1-\log x}{x^2}$$

Differentiating again both the sides, we have

$$f''(x) = f'(x)\,\frac{1-\log x}{x^2} + f(x)\left[\frac{x^2(-1/x)-(1-\log x)\,2x}{x^4}\right]$$

thus $f''(e) = -\frac{f(e)}{e^3} = -e^{(1/e)-3}$.

◉ **Example 81:** If $y^2 = P(x)$ is a polynomial of degree 3, then $\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ is equal to

(a) $P(x) + P''(x)$ (b) $P(x)\,P''(x)$

(c) $P(x)\,P'''(x)$ (d) a constant.

Ans. (c)

◎ **Solution:** From $y^2 = P(x)$, we have $2yy_1 = P'(x)$ i.e. $2y_1 = P'(x)/y$

$$\Rightarrow \quad 2y_2 = \frac{yP''(x)-P'(x)y_1}{y^2} = \frac{yP''(x)-P'(x)P'(x)/2y}{y^2}$$

$$= \frac{2y^2P''(x)-(P'(x))^2}{2y^3}$$

$$\Rightarrow \quad 2y_2y^3 = \frac{1}{2}\,[2P(x)P''(x) - (P'(x))^2]$$

$$\Rightarrow 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = \frac{1}{2}\,[2\{P'(x)P''(x) + P(x)P'''(x)\} - 2P'(x)P''(x)]$$

$$= \frac{1}{2}\,2P(x)P'''(x) = P(x)P'''(x).$$

◉ **Example 82:** If $f(x) = (1+x)^n$, then the value of $f(0) + f'(0) + \frac{1}{2!}f''(0) + \cdots + \frac{1}{n!}f^n(0)$ is

(a) n (b) $2n$

(c) 2^{n-1} (d) none of these

Ans. (b)

◎ **Solution:** $f(0) = 1, f'(x) = n(1+x)^{n-1}$,

$f''(x) = n(n-1)(1+x)^{n-2} \cdots f^n(x) = n(n-1)\cdots 1\,(1+x)^{n-n}$.

So $f'(0) = n, f''(0) = n(n-1), \ldots f^n(0) = n!$.

$\therefore$ the given expression is equals to

$$1 + n + \frac{n(n-1)}{2!} + \cdots \frac{n!}{n!} = 2^n.$$

◉ **Example 83:** If $f(x) = x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \ldots + \frac{x^2}{(1+x^2)^n} + \ldots$ then at $x = 0$

(a) $f(x)$ has no limit

(b) $f(x)$ is discontinuous

(c) $f(x)$ is continuous but not differentiable

(d) $f(x)$ is differentiable

Ans. (b)

◎ **Solution:** For $x = 0, f(0) = 0$ and for $x \neq 0$,

$$f(x) = x^2\left[1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \ldots\right]$$

$$= x^2\,\frac{1}{1-1/(1+x^2)} = \frac{x^2(1+x^2)}{x^2} = 1 + x^2,$$

Thus, $\lim_{x\to 0} f(x) = 1$, Hence f is not continuous at $x = 0$, so f is also not differentiable at $x = 0$.

◉ **Example 84:** Let $f(x) = \begin{cases}\int_0^x \{1+|1-t|\}\,dt & \text{if } x > 2\\ 5x-7 & \text{if } x \le 2,\end{cases}$ then

(a) f is not continuous at $x = 2$

(b) f is continuous but not differentiable at $x = 2$

(c) f is differentiable everywhere

(d) $f'(2+)$ doesn't exist

Ans. (b)

◎ **Solution:** For $x > 2$,

$$\int_0^x \{1+|1-t|\}dt = \int_0^1 \{1+|1-t|\}dt + \int_1^x \{1+|1-t|\}dt$$

$$= \int_0^1 (2-t)dt + \int_1^x t\,dt = 1 + \frac{x^2}{2}.$$

Thus,. $$f(x) = \begin{cases}1 + x^2/2, & x > 2\\ 5x-7, & x \le 2\end{cases}$$

$$\lim_{x\to 2+} f(x) = 1 + 4/2 = 3 = f(2) = \lim_{x\to 2-} f(x)$$

$$f'(2+) = \lim_{h\to 0+}\frac{1+(1/2)(2+h)^2-3}{h}$$

$$= \lim_{h\to 0+}\frac{(1/2)h^2+2h}{h} = 2,$$

$$f'(2-) = \lim_{h\to 0-}\frac{5(2+h)-7-3}{h} = 5$$

Hence f is continuous but not differentiable at $x = 2$.

Example 85: Let $f(x) = \underset{k=1}{\pi}$ $(\cos(2k-1)x + i\sin(2k-1)x)$ then $(\text{Re}\, f(x))'' + i\,(\text{Im}\, f(x))''$ is equal to

(a) $n^2 f(x)$ (b) $-n^4 f(x)$

(c) $-n^2 f(x)$ (d) $n^4 f(x)$

Ans. (b)

Solution: $f(x) = \prod_{k=1}^{n} (\cos(2k-1)x + i\sin(2k-1)x)$

$$= \cos \sum_{k=1}^{n}(2k-1)\, x + i \sin \sum_{k=1}^{n}(2k-1)\, x$$

$= \cos n^2x + \text{i} \sin n^2x$ (Using De Movire's Theorem)

$\therefore$ $(\text{Re}\, f(x))'' = -n^4 \cos n^2x$ and

$\text{Im}\, f(x)'' = -n^4 \sin n^2x$. Thus

$(\text{Re}\, f(x))'' + i\,(\text{Im}\, f(x))'' = -n^4 [\cos n^2x + i \sin n^2x] = -n^4 f(x)$.

Example 86: Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f(x/y) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then which of the following may be true

(a) $f(x)$ is bounded

(b) $f(1/x) \to 0$ as $x \to \infty +$

(c) $x f(x) \to 1$ as $x \to 0+$

(d) $f'(x) = 1/x$

Ans. (d)

Solution: A function satisfying the given functional equation is $f(x) = \log x$. Clearly $f(x)$ is not bounded and $f(1/x) = -\log x \not\to 0$ as $x \to \infty$. Also $x f(x) = x \log x \not\to 1$ as $x \to 0$. But $f'(x) = 1/x$.

In fact, it can be shown that any continuous function satisfying the given functional equation is of the form $k \log x$, for some $k > 0$. Since $f(e) = 1$ so $k = 1$.

Example 87: If $x = \cos\theta$, $y = \sin^3\theta$, then $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}$ at $\theta = \pi/2$ is

(a) 1 (b) 2

(c) -2 (d) -3

Ans. (d)

Solution: $\frac{dx}{d\theta} = -\sin\theta$, $\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta \Rightarrow \frac{dy}{dx} = -3\sin\theta\cos\theta$. Differentiating again, we have $\frac{d^2y}{dx^2} = -3\cos 2\theta \frac{d\theta}{dx} = \frac{3\cos 2\theta}{\sin\theta}$.

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 9\sin^2\theta\cos^2\theta + \sin^3\theta\,\frac{3\cos 2\theta}{\sin\theta}$$

$$= 9\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos 2\theta$$

So the value of the expression on L.H.S. at $\theta = \pi/2$ is -3.

Example 88: If $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots$ + upto n terms, then $y'(0)$ is equal to

(a) $-1/(n^2+1)$ (b) $-n^2/(n^2+1)$

(c) $n^2/(n^2+1)$ (d) none of these

Ans. (b)

Solution: $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \ldots$ + upto n terms

$$= \tan^{-1}\frac{(x+1)-x}{1+x(x+1)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+1)(x+2)} + \ldots n \text{ terms}$$

$= \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \ldots$

$+ \tan^{-1}(x+n) - \tan^{-1}(x+(n-1))$

$= \tan^{-1}(x+n) - \tan^{-1}x$.

$$y'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

$$\Rightarrow \quad y'(0) = \frac{1}{1+n^2} - 1 = -\frac{n^2}{1+n^2}.$$

Example 89: If $f(x) = \sqrt{x+2\sqrt{2x-4}} + \sqrt{x-2\sqrt{2x-4}}$ then

(a) f is differentiable at all points of its domain except $x = 4$

(b) f is differentiable on $(2, \infty)$

(c) f is differentiable on $(-\infty, \infty)$

(d) $f'(x) = 0$ for all $x \in [2, 6)$

Ans. (a)

Solution: The domain of f is $[2, \infty)$. Put $t = \sqrt{2x-4}$

$$f(x) = \sqrt{t^2/2+2+2t} + \sqrt{t^2/2+2-2t}$$

$$= \frac{1}{\sqrt{2}}(t+2) + \frac{1}{\sqrt{2}}|t-2| = \begin{cases} \frac{1}{\sqrt{2}} \times 4 & \text{if } t < 2 \\ \sqrt{2}t & \text{if } t \geq 2 \end{cases}$$

$$= \begin{cases} 2\sqrt{2} & \text{if } x \in (2,4) \\ 2\sqrt{x-2} & \text{if } x \in (4,\infty) \end{cases}$$

Hence $f'(x) = \begin{cases} 0 & \text{if } x \in (2,4) \\ \frac{1}{\sqrt{x-2}} & \text{if } x \in (4,\infty) \end{cases}$

and $f'(4)$ does not exist.

Example 90: If $y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$, $x \in [0, \infty) \sim \{1\}$ then $\frac{dy}{dx}$ is equal to

(a) 1 (b) $\frac{x-1}{x+1}$

(c) 0 (d) $\frac{x+1}{x-1}$

Ans. (c)

Solution: $y = \cos^{-1}\frac{x-1}{x+1} + \sin^{-1}\frac{x-1}{x+1}$, since $-1 \le \frac{x-1}{x+1} \le 1$

$$\therefore \qquad y = \frac{\pi}{2}. \text{ So } \frac{dy}{dx} = 0$$

Example 91: The least value of n so that $y_n = y_{n+1}$ where $y = x^2 + e^x$ is

(a) 4 (b) 3

(c) 5 (d) 2

Ans. (b)

Solution: $y' = 2x + e^x$, $y'' = 2 + e^x$, $y''' = e^x$, $y^{iv} = e^x$. Thus $n = 3$.

Example 92: If $y = \cos^{-1}\frac{9-x^2}{9+x^2}$ then $y'(-1)$ is equal to

(a) $-3/5$ (b) $3/5$

(c) $2/7$ (d) $3/8$

Ans. (a)

Solution: $\cos^{-1}\frac{9-x^2}{9+x^2} = \cos^{-1}\frac{1-(x/3)^2}{1+(x/3)^2} = 2\tan^{-1}\frac{x}{3}$,

if $0 \le x < \infty$ and is equal to $-2\tan^{-1} x/3$ if $-\infty < x \le 0$. Hence

$$y'(x) = \begin{cases} \dfrac{2}{1+(x/3)^2}.\dfrac{1}{3} & \text{if } 0 < x < \infty \\ -\dfrac{2}{1+(x/3)^2}.\dfrac{1}{3} & \text{if } -\infty < x < 0 \end{cases}$$

Hence $y'(-1) = -3/5$.

Example 93: If $y = (x^2+1)^{\sin x}$, then $y'(0)$ is equal to

(a) 1/2 (b) e^2

(c) 0 (d) 3/2

Ans. (c)

Solution: $\log y = \sin x \log(x^2+1)$.

Differentiating we get $\frac{y'}{y} = (\sin x)\frac{2x}{x^2+1} + \cos x \log(x^2+1)$

so $y'(0) = y(0).0 = 0$.

Example 94: If $f(x) = x^n + 4$ then the value of $f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \ldots + \frac{f^n(1)}{n!}$ is

(a) 2^{n-1}

(b) $2^n + 4$

(c) $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$

(d) none of these

Ans. (b)

Solution: $f(1) = 5, f'(x) = nx^{n-1}$ so $f'(1) = n$

$f''(1) = n(n-1), \ldots, f^n(1) = 1.2 \ldots n$. Thus

$$f(1) + \frac{f'(1)}{1!} + \ldots + \frac{f^n(1)}{n!}$$

$$= 5 + \frac{n}{1} + \frac{n(n-1)}{2!} + \ldots + \frac{n!}{n!} = (1+1)^n + 4 = 2^n + 4.$$

Example 95: Let a function $y = y(x)$ be defined parametrically by $x = 2t - |t|$, $y = t^2 + t|t|$. Then $y'(x)$, $x > 0$

(a) 0 (b) $4x$

(c) $2x$ (d) does not exist

Ans. (b)

Solution: $x = 2t - |t| = \begin{cases} t, & t \ge 0 \\ 3t, & t < 0 \end{cases}$, so $t = \begin{cases} x, & x \ge 0 \\ x/3, & x < 0 \end{cases}$

Therefore, we can express $y = t^2 + t|t| = \begin{cases} 2t^2, & t \ge 0 \\ 0 & t < 0 \end{cases}$

$$= \begin{cases} 2x^2, & x \ge 0 \\ 0 & x < 0 \end{cases}. \text{ Hence } y'(x) = \begin{cases} 4x, & x > 0 \\ 0, & x < 0 \end{cases}$$

Note that we cannot find $\frac{dx}{dt}$ as the derivative does not exist at $t = 0$.

Example 96: $\frac{d^2x}{dy^2}$ equals

(a) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

(c) $\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

Ans (a)

Solution: $\frac{d^2x}{d^2y} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}$

$$= \frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$= (-1)\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1}$$

$$= -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}.$$

Example 97: Let $y = f(x) = \log(1+\sin x)^2, x \neq (2n+1)\frac{\pi}{2}$, $n \in \mathbf{I}$ then $f''(x) + 2e^{-y/2}$ is equal to

(a) 0 (b) $\frac{1}{1+\sin x}$

(c) $\frac{1}{(1+\sin x)^2}$ (d) $-\frac{1}{1+\sin x}$

Ans. (a)

Solution: $y = 2\log(1+\sin x)$

$$y' = \frac{2\cos x}{1+\sin x}$$

$$\Rightarrow \quad y'' = 2\frac{-(1+\sin x)\sin x - \cos^2 x}{(1+\sin x)^2}$$

$$= -2\frac{1+\sin x}{(1+\sin x)^2} = -\frac{2}{1+\sin x}$$

$$f''(x) + 2e^{-y/2} = f''(x) + 2e^{-\log(1+\sin x)}$$

$$= f''(x) + \frac{2}{1+\sin x} = 0.$$

Example 98: Let $f(x) = x + \cot x$ and f is the inverse of g, then $g'(x)$ is equal to

(a) $\frac{x}{x-g(x)}$ (b) $\frac{1}{(x-g(x))^2}$

(c) $-\frac{1}{(x-g(x))^2}$ (d) $\frac{x}{(x-g(x))^2}$

Ans. (c)

Solution: $f(g(x)) = x$

$\Rightarrow \quad g(x) + \cot g(x) = x$

Differentiating both sides, we get

$g'(x) - (\operatorname{cosec}^2 g(x))\, g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{1-\operatorname{cosec}^2 g(x)}$$

$$= \frac{1}{1-(1+\cot^2 g(x))} = -\frac{1}{(x-g(x))^2}$$

Example 99: If $f(x) = \begin{cases} xe^{-\left\{\frac{1}{|x|}+\frac{1}{x}\right\}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then f is

(a) discontinuous everywhere

(b) continuously differentiable at $x = 0$

(c) differentiable everywhere

(d) continuous but not differentiable

Ans. (d)

Solution: $f(x) = \begin{cases} x & x < 0 \\ xe^{-\frac{2}{x}} & x > 0 \\ 0 & x = 0 \end{cases}$

f is clearly a continuous function

$$f'(x) = \begin{cases} 1 & , x < 0 \\ e^{-\frac{2}{x}} + \frac{2}{x}e^{-\frac{2}{x}} & , x > 0 \end{cases}$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{he^{-\frac{2}{h}}}{h} = \lim_{h\to 0^+} e^{-\frac{2}{h}} = 0$$

f is not differentiable at $x = 0$.

Example 100: Let f and g be twice differentiable functions such that $\text{fog}(x) = x^3$ and $g(1) = 2$, $g'(1) = 1$, $g''(1) = 4$ then $f''(2)$ is equal to

(a) 0 (b) 2

(c) 4 (d) –6

Ans. (d)

Solution: Since $f(g(x)) = x^3$, differentiating both the sides, we get

$$f'(g(x))\, g'(x) = 3x^2$$

$$\Rightarrow \quad f''(g(x))\,(g'(x))^2 + f'(g(x))\, g''(x) = 6x$$

$$\Rightarrow \quad f''(g(x))\,(g'(x))^2 + \frac{3x^2}{g'(x)}\, g''(x) = 6x$$

Putting $x = 1$, we have

$$f''(2)\, 1^2 + \frac{3}{1}\, 4 = 6$$

$$\Rightarrow \quad f''(2) = -6.$$

EXERCISE

Concept-based Straight Objective Type Questions

1. If $f(t) = \frac{t^2-5t-1}{t^3}$ then $f'\left(\frac{1}{a}\right)$ is given by

(a) $3a^4 + 10a^3 + a^2$ (b) $3a^4 + 10a^3 - a^2$

(c) $a^4 + 5a^3 - a^2$ (d) $a^4 + 5a^2 - a$

2. If $y = \frac{1}{\sqrt[3]{2x-1}} + \frac{1}{\sqrt[4]{(x^2+2)^3}}$ then $y'(0)$ is given by

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $-\frac{2}{3}$ (d) $-\frac{1}{3}$

3. Let $f(x) = |\log x|$ then
   (a) $f'(1+) = -1$ (b) $f'(1+) = f'(1-)$
   (c) $f'(1-) = 1$ (d) $f'(1+) = 1$

4. For $f(x) = \sin^{-1}\frac{2x}{1+x^2}$, if $A = \{x \in \text{dom } f : f \text{ is not differentiable at } x\}$ then number of points in A is
   (a) 0 (b) 1
   (c) 2 (d) 3

5. Let $y = \sqrt{1-\sqrt{1-x^2}}$. The set of all points of non differentiability of y is
   (a) $\{0, 1\}$ (b) $\{1, -1\}$
   (c) $\{0, -1\}$ (d) $\{0, -1, 1\}$

6. If $y = \log_2 (\log_3(\log_5 x))$ then $y'(125)$ is equal to
   (a) $\frac{1}{375\log 2}$ (b) $\frac{1}{125\log 2 \log 3 \log 5}$
   (c) $\frac{1}{375\log 3 \log 3}$ (d) $\frac{1}{375\log 2 \log 3 \log 5}$

7. If $x = \frac{t+1}{t}$, $y = \frac{t-1}{t}$ then $\frac{d^2y}{dx^2}$ at $t = 1$
   (a) 0 (b) 1
   (c) -1 (d) $\frac{1}{\sqrt{2}}$

8. If $x = 3\cos t$, $y = 4\sin t$ then $\frac{dy}{dx}$ at $\left(3/\sqrt{2}, 2\sqrt{2}\right)$ is equal to
   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$
   (c) $-\frac{2}{3}$ (d) $-\frac{4}{3}$

9. If $x^4 + y^4 = x^2y^2$ then $\frac{dy}{dx}$ is given by
   (a) $\frac{x}{y}\cdot\frac{y^2-x^2}{2y^2-x^2}$ (b) $\frac{y}{x}\cdot\frac{y^2-2x^2}{y^2-x^2}$
   (c) $\frac{x}{y}\cdot\frac{y^2-2x^2}{2y^2-x^2}$ (d) $\frac{y}{x}\cdot\frac{y^2-2x^2}{2y^2-x^2}$

10. Let $f(x) = |x^2 - 4|x| + 3|$, then $A = \{x : f \text{ is not differentiable at } x\}$ is equal to
   (a) $\{-1, 1, -3, 3\}$ (b) $\{1, 3\}$
   (c) $\{1, -1\}$ (d) $\{3, -3\}$

LEVEL 1

Straight Objective Type Questions

11. The function
$$f(x) = \begin{cases} x^4 + x^2 - x + 2, & \text{for } x \le 1 \\ 3x^3 - x^2 + x, & \text{for } x > 1 \end{cases}$$
is
   (a) continuous everywhere
   (b) differentiable everywhere
   (c) differentiable at $x = 1$
   (d) such that f' exists everywhere but f'' is not continuous at $x = 1$.

12. If
$$f(x) = \begin{cases} x\sin(1/x), & \text{for } x \ne 0 \\ 0, & \text{for } x = 0 \end{cases}$$
then
   (a) f is a continuous function
   (b) $f'(0+)$ exists but $f'(0-)$ does not exist
   (c) $f'(0+) = f'(0-)$
   (d) $f'(0-)$ exists but $f'(0+)$ does not exist.

13. If $y = \log_{e^x}(x-2)^2$ for $x \ne 0, 2$, then $y'(3)$ is equal to
   (a) 1/3 (b) 2/3
   (c) 4/3 (d) none of these

14. The function f defined by
$$f(x) = \begin{cases} \frac{\sin x^2}{x}, & \text{for } x \ne 0 \\ 0, & \text{for } x = 0 \end{cases}$$
is
   (a) continuous and derivable at $x = 0$
   (b) neither continuous nor derivable at $x = 0$
   (c) continuous but not derivable at $x = 0$
   (d) none of these.

15. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$ is
   (a) 1/4 (b) 1/8
   (c) 1/2 (d) 1

16. Let $f(x) = x^n$, n being a non-negative integer. The value of n for which the equality $f'(a+b) = f'(a) + f'(b)$ is valid for all $a, b > 0$ is
   (a) 3 (b) 1
   (c) 2 (d) none of these

17. The function $f(x) = |x^3|$ is
    (a) differentiable everywhere
    (b) continuous but not differentiable at $x = 0$
    (c) not a continuous function
    (d) a function with range $(0, \infty)$
18. The set of all points of differentiability of the function $f(x) = e^{-|x|}$ is
    (a) $(0, \infty)$ (b) $(-\infty, \infty)$
    (c) $[0, \infty)$ (d) $(-\infty, \infty) \sim \{0\}$
19. If $x = \log t$ and $y = t^2 - 1$, then $y''(1)$ at $t = 1$ is
    (a) 2 (b) 4
    (c) 3 (d) none of these
20. For a differentiable function f on $\mathbf{R}$ there is an x_0 with
    $$f'(x_0) = \frac{1}{2}[f'(0) + f'(1)]$$
    (a) only if f is constant
    (b) only if f is increasing
    (c) if f is decreasing
    (d) if f is continuously differentiable
21. Let f be defined by $f(x) = |x - 2| + |x| + |x + 2|$ for $x \in \mathbf{R}$ then
    (a) $f'(-2+)$ doesn't exist
    (b) $f'(2-)$ doesn't exist
    (c) $f'(2+) = 3$
    (d) $f'(0+) = 2$
22. Let f be defined on $\mathbf{R}$ by $f(x) = x^4 \sin(1/x)$, if $x \neq 0$ and $f(0) = 0$ then
    (a) $f'(0)$ doesn't exist
    (b) $f'(2-)$ doesn't exist
    (c) f'' is not continous at $x = 0$
    (d) $f''(0)$ exist but f'' is not continuous at $x = 0$
23. If $x^y = e^{x-y}$ then
    (a) $\frac{dy}{dx}$ doesn't exist at $x = 1$
    (b) $\frac{dy}{dx} = 0$ when $x = 1$
    (c) $\frac{dy}{dx} = \frac{1}{2}$ when $x = e$
    (d) none of these
24. If f is one-one and satisfies $f'(x) = \sqrt{1-(f(x))^2}$ then $(f^{-1})'(x)$
    (a) is equal to $\frac{1}{\sqrt{1-x^2}}$
    (b) may not exist for every $x \in \mathbf{R}$
    (c) may not be known explicitly
    (d) is equal to $\sin^{-1}(f(x))$
25. The function $y = \sin^{-1} x$ satisfies
    (a) $(1 - x^2)\, y'' = xy''$ (b) $(1 - x^2)\, y'' = xy'$
    (c) $(1 - x^2)\, y'' = x^2 y'$ (d) $(1 - x^2)\, y' = 2xy'$
26. Let f be a one-one function satisfying $f'(x) = f(x)$ then $(f^{-1})''(x)$ is equal to
    (a) $-1/x^3$ (b) $-1/x^2$
    (c) $f(x)$ (d) $f^{-1}(x)$
27. If $y = \cos^{-1}(x^{-1})$ then $y'(-2)$ is equal to
    (a) $\frac{1}{2\sqrt{3}}$ (b) $-\frac{1}{2\sqrt{3}}$
    (c) $\frac{1}{2\sqrt{5}}$ (d) $-\frac{1}{2\sqrt{5}}$
28. If $x = \sin^{-1}(t^2 - 1)$, $y = \cos^{-1}(2t)$ then $\frac{dy}{dx}$ at $t = 0$ is
    (a) $-\sqrt{2}$ (b) $-\frac{1}{\sqrt{2}}$
    (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$
29. If $\tan^{-1} y - y + x = 0$ then $\frac{d^2y}{dx^2}$ is equal to
    (a) $\frac{-2(1+y^2)}{y^5}$ (b) $\frac{1+y^2}{y^5}$
    (c) $\frac{2(1+y^2)}{y^4}$ (d) $\frac{2(1+y^2)}{y^5}$
30. If $F(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$ then $F'(x)$ is equal to
    (a) $6x^3$ (b) $x^3 + 6x^2$
    (c) $3x$ (d) $6x^2$
31. If $y = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$ then $y'(x)$ is equal to
    (a) $\cos 2x$ (b) $-\cos 2x$
    (c) $2\cos^2 x$ (d) $\cos^3 x$
32. If $z(t) = \sqrt{t^3+1}$, then $z'(2)$ is equal to
    (a) 1/2 (b) -1
    (c) 2 (d) -1
33. If $y = \cos^{-1}\left(\frac{4\cos x - 5\sin x}{\sqrt{41}}\right)$, then $\frac{dy}{dx}$ is equal to
    (a) 0 (b) 1
    (c) -1 (d) none of these
34. If $y = \log\cos\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)$ then $y'(0)$ is equal to
    (a) $e + e^{-1}$ (b) $e - e^{-1}$
    (c) $\frac{e+e^{-1}}{2}$ (d) none of these

35. If the function $y = \log \frac{1}{1+x}$ satisfies the relation $xy' + 1 = f(y)$ then $f(y)$ is equal to
(a) y (b) $y^2 + 1$
(c) e^y (d) e^{-y}

36. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ satisfies the relation $(1 - x^2)y' - xy = k$ then the value of k is
(a) 1 (b) 0
(c) –1 (d) 2

37. If $x \sin y = \sin (y + a)$ and $\frac{dy}{dx} = \frac{A}{1+x^2-2x\cos a}$ then the value of A is
(a) 2 (b) $\cos a$
(c) $\sin a$ (d) none of these

38. If $f(x) = \cot^{-1} \frac{3x-x^3}{1-3x^2}$ and $g(x) = \sin^{-1} \frac{1-x^2}{1+x^2}$ then $\lim_{x\to t} \frac{f(x)-f(t)}{g(x)-g(t)}$ is
(a) $\frac{3}{2(1+t^2)}$ (b) $\frac{-5}{2(1+t^2)}$
(c) $\frac{5}{2}$ (d) $\frac{3}{2}$

39. Let f be a function defined for every x, such that $f'' = -f$, $f(0) = 0$, $f'(0) = 1$ then $f(x)$ is equal to
(a) $\tan x$ (b) $e^x - 1$
(c) $\sin x$ (d) $2 \sin x$

40. If $y = \sin^{-1} (\cos x)$ and $y'(x)$ is identically equal to $g(x)$ on $\mathbf{R} \sim \{kp, k \in \mathbf{I}\}$ then $g(x)$ is equal to
(a) $|\sin x|$ (b) $-\text{sgn}(\sin x)$
(c) $\text{sgn}(\sin x)$ (d) none of these

41. If $f(x) = \cos (x^2 - 4[x])$ for $0 < x < 1$, where $[x]$ denotes the greatest integer $\le$ x, then $f'\left(\sqrt{\pi}/2\right)$ is equal
(a) $-\sqrt{\frac{\pi}{2}}$ (b) $\sqrt{\frac{\pi}{2}}$
(c) 0 (d) $\frac{\sqrt{\pi}}{4}$

42. If $x = a \cos 2t$, $y = b \sin^2 t$ then $\frac{d^2y}{dx^2}$ is equal to
(a) $\frac{a}{b} \cos 2t$ (b) 0
(c) 1 (d) $\frac{b}{a} \sin 2t$

43. If $f(x) = \tan^{-1} \sqrt{(1+\sin x)/(1-\sin x)}$, $0 \le x < \pi/2$, then $f'(\pi/6)$ is
(a) – 1/4 (b) – 1/2
(c) 1/4 (d) 1/2

44. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = y = 1$ is
(a) 0 (b) – 1
(c) 1 (d) 2

45. If $t = \sin^{-1} 2^s$ then $\frac{ds}{dt}$ is equal to
(a) $\frac{\log 2}{\sqrt{1-t^2}}$ (b) $\frac{\sin t}{\log 2}$
(c) $\frac{\cot t}{\log 2}$ (d) none of these

46. Let (x, y) be any point on the unit circle with centre at the origin, then
(a) $yy'' - 2(y')^2 + 1 = 0$
(b) $yy'' + y'^2 + 1 = 0$
(c) $yy'' + y'^2 - 1 = 0$
(d) $yy'' + 2y'^2 + 1 = 0$

47. Let $f(x) = \log \sin x^2$, $0 < x < \pi/2$ then $f''\left(\sqrt{\pi}/2\right)$ is equal to
(a) 2π (b) $2(1 + \pi)$
(c) $2(1 - \pi)$ (d) $4(1 - \pi)$

48. Let $g(x)$ be the inverse of $f(x)$ and $f'(x) = \frac{1}{1+x^5}$ then $\frac{d^2}{dx^2}(g(x))$ is equal to
(a) $\frac{1}{1+(g(x))^5}$ (b) $5(g(x))^4 (1 + (g(x))^5$
(c) $\frac{g'(x)}{1+(g(x))^5}$ (d) $\frac{5(g'(x))^4}{1+(g(x))^5}$

49. The derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'\left(\sqrt{2}\right) = 4$, is:
(a) $1/\sqrt{2}$ (b) $\sqrt{2}$
(c) 1 (d) 0

50. If $y = \left(\log_{\cos x} \sin x\right) \left(\log_{nx} \cos x\right) + \sin^{-1} \frac{2x}{1+x^2}$, then $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$ is equal to:
(a) $\frac{8}{\left(4+\pi^2\right)}$ (b) 0
(c) $-\frac{8}{\left(4+\pi^2\right)}$ (d) 1

51. If $y = \sin^{-1}\left(\frac{5x + 12\sqrt{1-x^2}}{13}\right)$, then $\frac{dy}{dx}$ is equal to:

(a) $-\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{3}{\sqrt{1-x^2}}$ (d) $\frac{x}{\sqrt{1-x^2}}$

52. Let f: $(-1, 1) \to \mathbf{R}$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$

(a) 0 (b) -2

(c) 4 (d) -4

53. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(a) log 2 (b) $-$ log 2

(c) -1 (d) 1

54. Let $f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

Then which one of the following is true

(a) f is neither differentiable at $x = 0$ nor at $x = 1$

(b) f is differentiable at $x = 0$ and at $x = 1$

(c) f is differentiable at $x = 0$ but not at $x = 1$

(d) f is differentiable at $x = 1$ but not at $x = 0$.

Assertion-Reason Type Questions

55. Let $y = \frac{1}{3}\log\frac{x+1}{\sqrt{x^2 - x + 1}} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$

Statement-1: $\frac{dy}{dx}$ at $x = 0$ is 1

Statement-2: $\frac{dy}{dx} = \frac{2}{x^3 + 2}$

56. Let $y = \log\frac{1}{1+x}$

Statement-1: $y'(1) = -1/2$

Statement-2: $xy' + 1 = e^y$.

57. Suppose that $\cos(xy) = x$

Statement-1: $\frac{dy}{dx} = \frac{1 + y\sin(xy)}{x\sin(xy)}$

Statement-2: $\frac{dy}{dx} < 0$ for $x, y > 0$ such that $0 < xy < \pi$.

58. Suppose that $x = t - t^4$, $y = t^2 - t^3$

Statement-1: $\frac{dy}{dx}$ at (0, 0) is equal to $\frac{1}{3}$ or 0

Statement-2: $\frac{dy}{dx} = \frac{2t - 3t^2}{1 - 4t^3}$.

59. If $y = \sin(3\sin^{-1} x)$

Statement-1: $y''(0) = 0$

Statement-2: $y'(0) = 3$.

60. **Statement-1:** Let $f : \mathbf{R} \to \mathbf{R}$ be a real valued function satisfying $|f(x) - f(y)| \leq A|x - y|^{\alpha}$, for some constant A and $\alpha > 1$, for every $x, y \in \mathbf{R}$ then f is constant function.

Statement-2: $f'(x) = 0$ for all $x \in \mathbf{R}$ then f is a constant function.

61. Suppose that $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, $\pi/2 < x < \pi$

Statement-1: $\frac{d^2y}{dx^2} = 0$

Statement-2: y is a linear function of x.

62. Let f is a twice differentiable function satisfying $f'(x) = f(1 - x)$ for $x \in \mathbf{R}$. $f(\pi/2) = 1$

Statement-1: $f(x) = \left(-\frac{\cos 1}{1 + \sin 1}\right)\cos x + \sin x$

Statement-2: f satisfies $f''(x) - f(x) = 0$.

LEVEL 2

Straight Objective Type Questions

63. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \leq \pi/6 \\ ax + b & \text{if } \pi/6 < x \leq 1 \end{cases}$$

The values of a and b such that f and f' are continuous, are

(a) $a = 1, b = 1/\sqrt{2} + \pi/6$

(b) $a = 1/\sqrt{2}, b = 1/\sqrt{2}$

(c) $a = 1, b = \sqrt{3}/2 - \pi/6$

(d) none of these

64. Let
$$f(x) = \begin{cases} 1/|x| & \text{for } |x| \geq 1 \\ ax^2 + b & \text{for } |x| < 1 \end{cases}$$
The coefficients a and b so that f is continuous and differentiable at any point, are equal to
(a) $a = -1/2, b = 3/2$ (b) $a = 1/2, b = -3/2$
(c) $a = 1, b = -1$ (d) none of these.

65. For a real number y, let $[y]$ denote the greatest integer less than or equal to y. Then
$$f(x) = \frac{\tan(\pi[x - \pi])}{1 + [x]^2}$$
is
(a) discontinuous at some x
(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x
(c) $f'(x)$ exists for all x but the second derivative $f''(x)$ does not exist
(d) $f'(x)$ exists for all x.

66. If
$$f(x) = \begin{cases} (\cos x)^{1/\sin x} & \text{for } x \neq 0 \\ k & \text{for } x = 0 \end{cases}$$
the value of k, so that f is differentiable at $x = 0$ is
(a) 0 (b) 1
(c) 1/2 (d) none of these.

67. If $y = \sin^{-1}\left(\dfrac{\sin\alpha \sin x}{1 - \cos\alpha \sin x}\right)$, then $y'(0)$ is
(a) 1 (b) $2\tan\alpha$
(c) $(1/2)\tan\alpha$ (d) $\sin\alpha$.

68. The set of all points of differentiability of the function
$$f(x) = \frac{\sqrt{x+1} - 1}{\sqrt{x}}$$
for $x \neq 0$ and $f(0) = 0$ is
(a) $(-\infty, \infty)$ (b) $[0, \infty)$
(c) $(0, \infty)$ (d) $(-\infty, \infty) \sim \{0\}$.

69. If $x = \sin^{-1} t$ and $y = \log(1 - t^2)$; then $\left.\dfrac{d^2y}{dx^2}\right|_{t=1/2}$ is
(a) $-8/3$ (b) $8/3$
(c) $3/4$ (d) $-3/4$

70. If $y^2 + x^2 = R^2$ and $k = 1/R$, then k is equal to
(a) $\dfrac{|y''|}{\sqrt{1 + y'^2}}$ (b) $\dfrac{|y''|}{\sqrt{(1 + y'^2)^3}}$
(c) $\dfrac{2|y''|}{\sqrt{1 + y'^2}}$ (d) $\dfrac{|y''|}{2\sqrt{(1 + y'^2)^3}}$.

71. Let $f : \mathbf{R} \to \mathbf{R}$. Then f is differentiable on $\mathbf{R}$ if
(a) $|f(x) - f(y)| \leq k(x - y)$ for all $x, y \in \mathbf{R}$ and some $k > 0$
(b) $|f(x) - f(y)| \leq k|x - y|^{1/2}$ for all $x, y \in \mathbf{R}$ and some $k > 0$
(c) $|f(x) - f(y)| \leq k(x - y)^2$ for all $x, y \in \mathbf{R}$ and some $k > 0$
(d) f^2 is differentiable on $\mathbf{R}$

72. $f : \mathbf{R} \to \mathbf{R}$ be such that $|f(x) - f(y)| \leq |x - y|^3$ for all $x, y \in \mathbf{R}$ then the value of $f'(x)$ is
(a) $f(x)$
(b) constant possibly different from zero
(c) $(f(x))^2$
(d) 0

73. Let f be the function on $[0, 1]$ given by $f(x) = x\sin\dfrac{\pi}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is
(a) discontinuous but bounded
(b) differentiable
(c) continuous but not bounded
(d) continuous and bounded

74. Let $f(x) = \dfrac{|x|\left(3e^{1/|x|} + 4\right)}{2 - e^{1/|x|}}$, $x \neq 0$ and $f(0) = 0$ then
(a) f is not continuous
(b) f is continuous but not differentiable at $x = 0$
(c) $f'(0)$ exist
(d) $f'(0+) = 2$

75. Let $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x - 1), & x > 1 \end{cases}$ then
(a) f is continuous for only finitely many values of k
(b) f is discontinuous at $x = 1$
(c) f is differentiable only when $k = 2$
(d) there are infinitely many values of k for which f is differentiable

76. Let f be a function defined on $[0, \infty)$ by
$$f(x) = \begin{cases} 0 & x = 0 \\ (x + a/\sqrt{2})(1/\sqrt{2}) & 0 < x \leq a \\ (x/2 + a)(1/\sqrt{2}) & x \geq a \end{cases}$$
then
(a) $f'(x)$ is discontinuous at $x = 0, a$
(b) f is a continuous function
(c) f is discontinuous at $x = a$
(d) none of these

77. The number of times the function $y = (x^2 + 1)^{80}$ to be differentiated to result in a polynomial of degree 50 is
(a) 70 (b) 80
(c) 110 (d) 30

78. The derivative of the function $y = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$ doesn't exist for
(a) all values of x for which $|x| < 1$
(b) $x = -1, 1$

(c) all values of x for which $|x| > 1$
(d) none of these

79. If $0 < \alpha < 1$ and f is defined as

$$f(x) = \begin{cases} x^{\alpha} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{then}$$

(a) f is differentiable at $x = 0$
(b) f is not continuous
(c) f is not differentiable at $x = 0$
(d) f is bounded

80. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

then
(a) f is continuous on **R**
(b) f is differentiable on **R** ~ **Q**
(c) f is differentiable only at one point
(d) f is continuous at infinitely many points of **R**

81. A value of k such that $f(x) = x^k$ is $(n - 1)$ times differentiable at 0 but not n- times differentiable at 0 is

(a) $\frac{2n-3}{3}$ (b) $\frac{3n-2}{3}$
(c) $\frac{4n-3}{3}$ (d) none of these

82. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x| - 1) & \text{if } |x| > 1 \end{cases} \quad \text{is}$$

(a) **R** ~ {0} (b) **R** ~ {1}
(c) **R** ~ {– 1} (d) **R** ~ {– 1, 1}

83. The derivative of $\sin^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$ with respect to x^{2n} is

(a) $-\frac{1}{x^n + x^{3n}}$ (b) $\frac{1}{x^n + x^{3n}}$
(c) $\frac{1}{x^n + x^{2n}}$ (d) $-\frac{1}{x^n + x^{2n}}$

84. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$ and $\frac{dy}{dx} = f(x, y)\sqrt{\frac{1-y^6}{1-x^6}}$ then the value of $f(x, y)$ is

(a) y/x (b) y^2/x^2
(c) $2y^2/x^2$ (d) x^2/y^2

85. If $f(x) = x^2 + \tan x$ and f is inverse of g, then $g'(x)$ is equal to

(a) $\frac{1}{2 + [g(x) - x]^2}$ (b) $\frac{1}{2g(x) + (g(x) - x)^2}$
(c) $\frac{1}{2g(x) - (g(x) - x)^2}$
(d) $\frac{1}{2g(x) + 1 + (x - (g(x))^2)^2}$

86. Number of points of non-differentiability of $f(x) = \sin \pi(x - [x])$ in $(-\pi/2, \pi/2)$, where [.] denotes greatest integer function is
(a) 4 (b) 5
(c) 3 (d) 2

87. Suppose that f is differentiable function with the property $f(x + y) = f(x) + f(y) + x^2y^2$ and $\lim_{x \to 0} \frac{f(x)}{x} = 100$ then $f'(x)$ is equal to
(a) 100 (b) 20
(c) 30 (d) none of these

Previous Years' AIEEE/JEE Main Questions

1. $f(x)$ and $g(x)$ are two differential function on [0, 2] such that $f''(x) - g''(x) = 0$, $f'(1) = 2g'(1) = 4$, $f(2) = 3g(2) = 9$ then $f(x) - g(x)$ at $x = \frac{3}{2}$ is

(a) 0 (b) 2
(c) 10 (d) 5 **[2002]**

2. If $f(1) = 1, f'(1) = 2$ then $\lim_{x \to 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$ is

(a) 2 (b) 4
(c) 1 (d) $\frac{1}{2}$ **[2002]**

3. If $y = \left(x + \sqrt{1 + x^2}\right)^n$, then $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is

(a) n^2y (b) $-n^2y$
(c) $-y$ (d) $2x^2y$ **[2002]**

4. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \cdots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

(a) 2^{n-1} (b) 0
(c) 1 (d) 2^n **[2003]**

5. If $f(x) = \begin{cases} x\, e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then $f(x)$ is

(a) continuous for all x, but not differentially at $x = 0$
(b) neither differential nor continuous at $x = 0$
(c) discontinuous everywhere
(d) continuous as well as differentiable for all x. **[2003]**

6. Let $f(a) = g(a) = k$ and their nth derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if $\lim_{x \to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)} = 4$ then the value of k is

(a) 2 (b) 1
(c) 0 (d) 4 **[2003]**

7. If $x = e^{y + e^{y + \ldots \infty}}$, $x > 0$, then $\frac{dy}{dx}$ is

(a) $\frac{1-x}{x}$ (b) $\frac{1}{x}$
(c) $\frac{x}{1+x}$ (d) $\frac{1+x}{x}$ **[2004]**

8. If f is a real valued differentiable function satisfying $|f(x) - f(y)| \leq (x - y)^2$, $x, y \in \mathbf{R}$ and $f(0) = 0$, then $f(1)$ equals

(a) 2 (b) 1
(c) – 1 (d) 0 **[2005]**

9. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim_{h \to 0} \frac{1}{h} f(1+h) = 5$, then $f'(1)$ equals

(a) 5 (b) 6
(c) 3 (d) 4 **[2005]**

10. The set of points where $f(x) = \frac{x}{1+|x|}$ is differentiable is

(a) $(0, \infty)$
(b) $(-\infty, 0) \cup (0, \infty)$
(c) $(-\infty, -1) \cup (-1, \infty)$
(d) $(-\infty, \infty)$ **[2006]**

11. If $x^m y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ is

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$
(c) $\frac{x+y}{xy}$ (d) xy **[2006]**

12. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Then which of the following is true.

(a) $f(x) \geq 1$ for $x \in \mathbf{R}$
(b) $f(x)$ is not differentiable at $x = 1$
(c) $f(x)$ is differentiable everywhere
(d) $f(x)$ is not differentiable at $x = 0$ **[2007]**

13. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

Then which one of the following is true

(a) f is neither differentiable at $x = 0$ nor at $x = 1$
(b) f is differentiable at $x = 0$ and at $x = 1$
(c) f is differentiable at $x = 0$ but not at $x = 1$
(d) f is differentiable at $x = 1$ but not at $x = 0$ **[2008]**

14. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(a) log 2 (b) – log 2
(c) – 1 (d) 1 **[2009]**

15. Let $f : (-1, 1) \to \mathbf{R}$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$.

Then $g'(0) =$

(a) 0 (b) – 2
(c) 4 (d) – 4 **[2010]**

16. $\frac{d^2x}{dy^2}$ equals

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)^{-1}$
(c) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ **[2011]**

17. If function $f(x)$ is differentiable at $x = a$ than $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$ is

(a) $-a^2 f'(a)$ (b) $a^2 f'(a)$
(c) $2a f(a) - a^2 f'(a)$ (d) $2a f(a) + a^2 f'(a)$ **[2011]**

18. Consider the function , $f(x) = |x-2| + |x-5|$, $x \in \mathbf{R}$

Statement 1: $f'(4) = 0$

Statement 2: f is continuous is [2,5], differentiable in (2,5)and $f(2) = f(5)$ **[2012]**

19. Let f be a differentiable function such that $8f(x) + 6f\left(\frac{1}{x}\right) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = -1$ is

(a) $\frac{15}{14}$ (b) $-\frac{15}{14}$

(c) $-\frac{1}{14}$ (d) $\frac{1}{14}$ **[2013, online]**

20. For $a > 0$, $t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$.

Then $1+\left(\frac{dy}{dx}\right)^2$ equals

(a) $\frac{x^2}{y^2}$ (b) $\frac{y^2}{x^2}$

(c) $\frac{x^2+y^2}{y^2}$ (d) $\frac{x^2+y^2}{x^2}$ **[2013, online]**

21. If $y = \sec(\tan^{-1} x)$, then $\left.\frac{dy}{dx}\right|_{x=1}$ is equal to

(a) $\frac{1}{2}$ (b) 1

(c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$ **[2013]**

22. Let $f(x) = \frac{x^2 - x}{x^2 + 2x}$, $x \neq 0, -2$. Then $\frac{d}{dx}$ $(f^{-1}(x))$ (wherever defined) is equal to

(a) $\frac{-1}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$

(c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$ **[2013, online]**

23. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to

(a) $1 + x^5$ (b) $5x^4$

(c) $\frac{1}{1+(g(x))^5}$ (d) $1 + (g(x))^5$ **[2014]**

24. Let $f(x) = x|x|$, $g(x) = \sin x$ and $h(x) = go\,f(x)$. Then

(a) $h(x)$ is not differentiable at $x = 0$

(b) $h(x)$ is differentiable at $x = 0$, but $h'(x)$ is not continuous at $x = 0$

(c) $h'(x)$ continuous at $x = 0$ but it is not differentiable at $x = 0$

(d) $h'(x)$ is differentiable at $x = 0$

[2014, online]

25. If $f(x) = x^2 - x + 5$, $x > \frac{1}{2}$, and $g(x)$ is its inverse function, then $g'(7)$ equals

(a) $-\frac{1}{3}$ (b) $\frac{1}{13}$

(c) $\frac{1}{3}$ (d) $-\frac{1}{13}$ **[2014, online]**

26. Let $f : \mathbf{R} \to \mathbf{R}$ be function such that $|f(x)| \leq x^2$ for all $x \in \mathbf{R}$. Than at $x = 0$, f is

(a) continuous but not differentiable

(b) continuous as well as differentiable

(c) neither continuous nor differentiable

(d) differentiable but not continuous **[2014, online]**

27. If $y = e^{nx}$, then $\left(\frac{d^2y}{dx^2}\right)\frac{d^2x}{dy^2}$ is equal to

(a) ne^{nx} (b) ne^{-nx}

(c) 1 (d) $-ne^{-nx}$ **[2014]**

28. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx+2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is:

(a) 2 (b) $\frac{16}{5}$

(c) $\frac{10}{3}$ (d) 4 **[2015]**

29. For $x \in \mathbf{R}$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

(a) g is not differentiable at $x = 0$

(b) $g'(0) = \cos(\log 2)$

(c) $g'(0) = -\cos(\log 2)$

(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

[2016]

30. If $f(x)$ is differentiability function in the internal $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$, then $f\left(\frac{3}{2}\right)$ is equal to

(a) $\frac{23}{18}$ (b) $\frac{13}{6}$

(c) $\frac{25}{9}$ (d) $\frac{31}{8}$ **[2016, online]**

31. If the function $f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x+b) & 1 \leq x \leq 2 \end{cases}$ is differentiable at $x = 1$, then $\frac{a}{b}$ is equal to

(a) $\frac{\pi+2}{2}$ (b) $\frac{\pi-2}{2}$

(c) $\frac{-\pi-2}{2}$ (d) $-1 - \cos^{-1}(2)$

Previous Years' B-Architecture Entrance Examination Questions

1. Let $f(x) = \begin{cases} ax & , \quad x < 2 \\ ax^2 + bx + 3, & x \geq 2 \end{cases}$

   If f is differentiable for all x, then the value of (a, b) is equal to

   (a) $(1, 2)$ (b) $\left(\frac{3}{2}, \frac{9}{2}\right)$

   (c) $\left(\frac{3}{4}, -\frac{9}{2}\right)$ (d) $\left(\frac{3}{4}, -\frac{9}{4}\right)$ **[2007]**

2. Let $f : [0, \infty) \to [0, \infty)$ be given by

   $f(x) = \sum_{i=1}^{10} \frac{x^i}{i}$.

   If $f'(\cdot)$ denotes the derivative of $f(\cdot)$, then the value of $\lim_{x \to 1} \frac{f'(x) - 10}{x - 1}$ is

   (a) 35 (b) 40

   (c) 45 (d) 55 **[2008]**

3. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be given by

   $f(x) = x|x| + |x - 1| - |x - 2|^2 + |x - 3|^3$

   If A denotes the set of points where $f(\cdot)$ is not differentiable, then $A =$

   (a) $\{3\}$ (b) $\{1\}$

   (c) $\{0\}$ (d) $\{2\}$ **[2008]**

4. Let $f(\cdot)$ and $g(\cdot)$ be differentiable functions on $(-\infty, \infty)$ and let $f'(\cdot)$ and $g'(\cdot)$ denote derivatives of $f(\cdot)$ and $g(\cdot)$ respectively. If $f(0) = \frac{1}{2}$, $g(0) = \frac{1}{3}$, $f'(0) = 1$ and $g'(0) = 2$, then the value of $\lim_{x \to 0} \frac{2f(2x^2 + 3x) - 1}{3g(x) - 1}$ is

   (a) 1 (b) $\frac{1}{2}$

   (c) $\frac{1}{3}$ (d) $\frac{1}{4}$ **[2008]**

5. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be defined by

   $f(x) = \begin{cases} x^3, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$

   and let $f'(\cdot)$ denote the derivative of $f(\cdot)$

   Statement-1: $\lim_{x \to 0} f'(x) = f'(0)$

   Statement-2: $f'(x)$ exists, for each $x \in (-\infty, \infty)$ and $f'(x)$ is continuous at $x = 0$ **[2008]**

6. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be functions defined by $f(x) = \text{sgn}(\sin x)$, $g(x) = \sin(\text{sgn } x)$ where

   $\text{sgn } \alpha = \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$

   If $A = f'(\pi)$ and $B = g'(\pi)$, then

   (a) A does not exist and $B = 0$

   (c) $A = 0$ and $B = 0$

   (c) both A and B do not exist

   (d) $A = 0$ and B does not exist **[2010]**

7. Let $f(x) = 2\tan^{-1}\left(\frac{1+x}{1-x}\right) + \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \in \mathbf{R}$, $x \neq 1$

   Statement-1: $f''(x) = 0$

   Statement-2: The range of $f(x)$ is $\{\pi\}$ **[2010]**

8. If $f(x) = \begin{cases} 1 - x^2, & x \leq -1 \\ 2x + 2, & x > -1 \end{cases}$ then the derivative of $f(x)$ at $x = -1$ is

   (a) 2 (b) 0

   (c) $\frac{1}{2}$ (d) 3 **[2011]**

9. Amongst the following functions, a function that is differentiable at $x = 0$ is

   (a) $\cos(|x|) - |x|$ (c) $\cos(|x|) + |x|$

   (c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$ **[2012]**

10. Let f be a differentiable function such that $8f(x) + 6f\left(\frac{1}{x}\right) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$ then $\frac{dy}{dx}$ at $x = -1$ is

    (a) $\frac{15}{14}$ (c) $-\frac{15}{14}$

    (c) $-\frac{1}{14}$ (d) $\frac{1}{14}$ **[2013]**

11. If $f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 \\ a + bx^2 & \text{if } |x| \leq 2 \end{cases}$

    then $f(x)$ is differentiable at $x = -2$ for

    (a) $a = \frac{3}{4}$ and $b = -\frac{1}{16}$

    (b) $a = -\frac{1}{4}$ and $b = \frac{1}{16}$

(c) $a=\frac{1}{4}$ and $b=-\frac{1}{16}$

(d) $a=\frac{3}{4}$ and $b=\frac{1}{16}$ **[2014]**

12. Let $y(x) = (1 + x)(1 + x^2)(1 + x^4) \ldots(1 + x^{32})$. Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$ is

(a) $1-65\left(\frac{1}{2}\right)^{62}$ (b) $1-63\left(\frac{1}{2}\right)^{64}$

(c) $4-65\left(\frac{1}{2}\right)^{62}$ (d) $4-63\left(\frac{1}{2}\right)^{64}$ **[2015]**

13. If $y(x) = \begin{vmatrix} \sin x & \cos x & \sin x+\cos x+1 \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$, $x \in \mathbf{R}$, then $\frac{d^2y}{dx^2}+y$ is equal to

(a) 4 (b) –10

(c) 0 (d) 6 **[2016]**

Answers

Concept-based

1. (b)	**2.** (c)	**3.** (d)	**4.** (c)
5. (d)	**6.** (d)	**7.** (a)	**8.** (d)
9. (c)	**10.** (a)		

Level 1

11. (a)	**12.** (a)	**13.** (b)	**14.** (a)
15. (a)	**16.** (c)	**17.** (a)	**18.** (d)
19. (b)	**20.** (d)	**21.** (c)	**22.** (d)
23. (b)	**24.** (a)	**25.** (b)	**26.** (b)
27. (a)	**28.** (a)	**29.** (a)	**30.** (d)
31. (b)	**32.** (c)	**33.** (b)	**34.** (d)
35. (c)	**36.** (a)	**37.** (c)	**38.** (d)
39. (c)	**40.** (b)	**41.** (a)	**42.** (b)
43. (d)	**44.** (b)	**45.** (c)	**46.** (b)
47. (c)	**48.** (b)	**49.** (a)	**50.** (a)
51. (b)	**52.** (d)	**53.** (d)	**54.** (c)
55. (c)	**56.** (a)	**57.** (d)	**58.** (a)
59. (b)	**60.** (a)	**61.** (a)	**62.** (c)

Level 2

63. (c)	**64.** (a)	**65.** (d)	**66.** (b)
67. (d)	**68.** (c)	**69.** (a)	**70.** (b)
71. (c)	**72.** (d)	**73.** (d)	**74.** (b)
75. (c)	**76.** (a)	**77.** (c)	**78.** (b)
79. (c)	**80.** (c)	**81.** (b)	**82.** (d)
83. (a)	**84.** (d)	**85.** (d)	**86.** (c)
87. (a)			

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (a)	**3.** (a)	**4.** (b)
5. (a)	**6.** (d)	**7.** (a)	**8.** (d)
9. (a)	**10.** (d)	**11.** (b)	**12.** (c)
13. (a)	**14.** (d)	**15.** (d)	**16.** (a)
17. (c)	**18.** (b)	**19.** (c)	**20.** (d)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)
25. (c)	**26.** (b)	**27.** (d)	**28.** (a)
29. (b)	**30.** (c)	**31.** (a)	

Previous Years' B-Architecture Entrance Examination Questions

1. (d)	**2.** (c)	**3.** (b)	**4.** (a)
5. (a)	**6.** (a)	**7.** (a)	**8.** (a)
9. (d)	**10.** (d)	**11.** (a)	**12.** (c)
13. (d)			

Hints and Solutions

Concept-based

1. $f(t) = \frac{1}{t}-\frac{5}{t^2}-\frac{1}{t^3} \quad \Rightarrow \quad f'(t) = -\frac{1}{t^2}+\frac{10}{t^3}+\frac{3}{t^4}$

So $f'\left(\frac{1}{a}\right) = -a^2 + 10a^3 + 3a^4$

2. $y = (2x - 1)^{-1/3} + 5(x^2 + 2)^{-3/4}$

$$y'(x) = -\frac{2}{3}\frac{1}{(2x-1)^{4/3}}-\frac{15x}{2(x^2+2)^{7/4}}$$

$$\Rightarrow \quad y'(0) = -\frac{2}{3}$$

3. $f'(1+) = \lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0+}\frac{\log(1+h)-\log 1}{h}$

$$= \lim_{h\to 0+}\frac{1}{h}\log(1+h)=\lim_{h\to 0+}\log(1+h)^{1/h}$$

$$f'(1-) = \lim_{h\to 0-}\frac{f(1+h)}{h}=\lim_{h\to 0-}\frac{\log(1+h)}{h}$$

$= \lim_{h \to 0-} \log(1+h)^{-1/h(-1)} \qquad \begin{pmatrix} h \to 0-, \\ 1/h \to -\infty \end{pmatrix}$

$= \log e^{-1} = -1$

4. Since $\sin^{-1} x$ is not differentiable at $x = \pm 1$, so f is not differentiable at all x for which $\left|\dfrac{2x}{1+x^2}\right| = 1$

$\Rightarrow \quad 2|x| = 1 + x^2 \Rightarrow (|x| - 1)^2 = 0$

$\Rightarrow \quad |x| = 1$

Hence f is not differentiable at $x = 1, -1$.

5. The domain of definition of the function is $-1 \le x \le 1$

$$y' = \frac{1}{2\sqrt{1-\sqrt{1-x^2}}} \cdot \frac{-1}{2\sqrt{1-x^2}}(-2x) \text{ at } x \neq 0, \pm 1$$

As $x \to 1-$ or $x \to -1+$, we have $y' \to \infty$. At $x = 0$,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\sqrt{1-\sqrt{1-x^2}}}{x}$$

Since $(1 - x^2)^{1/2} - 1 = -\frac{1}{2}x^2 + 0(x^4)$, so

$$\lim_{x\to 0} \frac{\sqrt{1-\sqrt{1-x^2}}}{x} = \lim_{x\to 0} \frac{\sqrt{\frac{1}{2}x^2}}{x} = \begin{cases} \dfrac{1}{\sqrt{2}} & \text{as } x \to 0+ \\ -\dfrac{1}{\sqrt{2}} & \text{as } x \to 0- \end{cases}$$

6. $$y'(x) = \frac{1}{\log^2 \log_3 (\log_5 x)} \frac{d}{dx}(\log_3(\log_5 x))$$

$$= \frac{1}{\log_2 \log_3 (\log_5 x)} \frac{1}{\log^3 \log_5 x} \frac{d}{dx}(\log_5 x)$$

$$= \frac{1}{x(\log 2 \log 3 \log 5)\log_3(\log_5 x)\log_5 x}$$

$$y'(125) = y'(5^3) = \frac{1}{125(\log 2 \log 3 \log 5)3}$$

$$= \frac{1}{375 \log 2 \log 3 \log 5}$$

7. $x = 1 + \frac{1}{t} \Rightarrow \frac{dx}{dt} = -\frac{1}{t^2}$ and $\frac{dy}{dt} = \frac{1}{t^2}$

So $\frac{dy}{dx} = -1$ for all t. Hence $\frac{d^2y}{dx^2} = 0$

8. $\frac{dx}{dt} = -3 \sin t$, $\frac{dy}{dt} = 4 \cos t$ so that

$\frac{dy}{dx} = -\frac{4}{3} \cot t$. For $x = \frac{3}{\sqrt{2}}$,

we have $\cos t = \frac{1}{\sqrt{2}}$ and $y = 2\sqrt{2}$, $\sin t = \frac{1}{\sqrt{2}}$.

Hence $\cot t = 1$. So, $\frac{dy}{dx} = -\frac{4}{3}$.

9. $4x^3 + 4y^3 \frac{dy}{dx} = 2xy^2 + 2yx^2 \frac{dy}{dx}$

$\Rightarrow \quad (4y^3 - 2yx^2) \frac{dy}{dx} = 2xy^2 - 4x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x(y^2-2x^2)}{2y(2y^2-x^2)} = \frac{x}{y} \cdot \frac{y^2-2x^2}{2y^2-x^2}$$

10. $f(x) = |(|x| - 1)(|x| - 3)|$. Hence f is not differentiable at $|x| = 1$, $|x| = 3$ i.e. at $x = 1, -1, 3, -3$

Level 1

11. f being represented by polynomials is differentiable everywhere except possibly $x = 1$

$$f'(1+) = \lim_{h \to 0+} \frac{3(1+h)^3 - (1+h)^2 + (1+h) - 3}{h}$$

$$= \lim_{h \to 0} \frac{3[h^3 + 3h^2 + 3h] - (h^2 + 2h) + h}{h} = 8$$

$$f'(1-) = \lim_{h \to 0} \frac{(1+h)^4 + (1+h)^2 - (1+h) + 2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{[h^4 + 4h^3 + 6h^2 + 4h] + (h^2 + 2h) - h}{h} = 5$$

$\lim_{x \to 1+} f(x) = 3 = \lim_{x \to 1} f(x) = f(1)$.

So f is continuous everywhere.

12. f is continuous, $f'(0+) = \lim_{x \to 0+} \sin\frac{1}{x}$ which does not exist. Similarly $f'(0-)$ does not exist.

13. $y = \frac{2}{x} \log(x-2)$, $y'(x)$

$= 2\left[\frac{1}{x(x-2)} - \frac{1}{x^2}\log(x-2)\right]$, $y'(3) = \frac{2}{3}$.

14. f is continuous, since $\lim_{x \to 0} f(x) = 0 = f(0)$,

$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{\sin x^2}{x^2} = 1$.

15. Putting $x = \tan\theta$, $\frac{\sqrt{1+x^2}-1}{x} = \frac{\sec\theta - 1}{\tan\theta}$

$= \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$.

Thus $x = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$.

Putting $x = \sin\theta$, we obtain $v = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ $= 2\sin^{-1} x$

$$\frac{du}{dv} = \frac{\left(\frac{1}{2}\right)\frac{1}{1+x^2}}{\frac{2}{\sqrt{1-x^2}}} \quad \Rightarrow \quad \left.\frac{du}{dv}\right|_{x=0} = \frac{1}{4}.$$

16. $n(a + b)^{n-1} = n(a^n - 1 + b^n - 1)$ is valid if $n = 2$.

17. $f(x) = \begin{cases} x^3, & x \ge 0 \\ x^3, & x < 0 \end{cases}$, $f'(0+) = 0 = f'(0-)$

18. $f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$, $f'(0+) = -1$, $f'(0-) = 1$.

19. $\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2$, $y''(x) = \frac{d}{dt}(2t^2)\frac{dt}{dx} = 4t.\ t$

So $y''(1) = 4$.

20. For continuously differentiable function, apply intermediate value theorem to f' for the inequality

$f'(0) \le \frac{1}{2}(f'(0) + f'(1)) \le f'(1)$ if $f'(0) \le f'(1)$,

$f'(1) \le \frac{1}{2}(f'(0) + f'(1)) \le f'(0)$ if $f'(1) \le f'(0)$.

21. $f(x) = \begin{cases} -3x & x < -2 \\ -x+4 & -2 \le x < 0 \\ x+4 & 0 \le x \le 2 \\ 3x & x > 2 \end{cases}$

$f'(-2+) = -1$, $f'(2-) = 1$, $f'(0+) = 1$, $f'(2+) = 3$.

22. $f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), & x \ne 0 \\ 0 & x = 0 \end{cases}$

$$f''(x) = \begin{cases} 12x^2 \sin\left(\frac{1}{x}\right) - 4x\sin\frac{1}{x} + \sin\frac{1}{x} - 2x\cos\frac{1}{x}, & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Since $f''(0) = \lim\limits_{x \to 0} \dfrac{4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}}{x} = 0$.

Clearly $\lim\limits_{x \to 0} f''(x)$ doesn't exist.

23. $y \log x = x - y \Rightarrow \frac{y}{x} + (\log x)\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x(1+\log x)} \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 1 - y(1) = 0$.

24. Let $u = f^{-1}(x) \Rightarrow f(u) = x \Rightarrow f'(u)\frac{du}{dx} = 1$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-(f(u))^2}} = \frac{1}{\sqrt{1-x^2}}.$$

25. $y = \sin^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y'^2(1 - x^2) = 1$

Differentiating w.r.t. x, we get

$2y'y''(1 - x^2) - 2xy'^2 = 0$

$\Rightarrow y''(1 - x^2) = xy'$.

26. Let $u = f^{-1}(u) \Rightarrow f(u) = x \Rightarrow f'(u)\frac{du}{dx} = 1$

$$\Rightarrow f''(u)\left(\frac{du}{dx}\right)^2 + f'(u)\frac{d^2u}{dx^2} = 0$$

$$\Rightarrow f'(u)\left(\frac{1}{f(u)}\right)^2 + f'(u)\frac{d^2u}{dx^2} = 0$$

$$\Rightarrow \frac{d^2u}{dx^2} = -\frac{1}{(f(u))^2} = -\frac{1}{x^2}.$$

27. $y'(x) = \frac{1}{\sqrt{1-x^{-2}}}\ x^{-2}$ so $y'(2) = \frac{1}{2\sqrt{3}}$.

28. $$\frac{dy}{dx} = -\frac{2}{\sqrt{1-4t^2}} \Big/ \frac{2t}{\sqrt{1-(t^2-1)^2}}$$

$$= -\frac{\sqrt{2t^2-t^4}}{\sqrt{1-4t^2}.t} = -\frac{\sqrt{2-t^2}}{\sqrt{1-4t^2}}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = -\sqrt{2}$$

29. $$\frac{1}{1+y^2}\frac{dy}{dx} - \frac{dy}{dx} + 1 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = 1 + \frac{1}{y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{y^3}\frac{dy}{dx} = -\frac{2(1+y^2)}{y^5}.$$

30. $$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

$= 0 + 0 + 6x^2 = 6x^2$.

31. $$y = (\tan x \sin^2 x + \cos^2 x)\frac{1}{1+\tan x}$$

$$= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$$

$$= 1 - \frac{1}{2}\sin 2x$$

$$\frac{dy}{dx} = -\cos 2x.$$

32. $z'(t) = \frac{1}{2}\frac{3t^2}{(t^3+1)^{1/2}}$, $z'(2) = \frac{1}{2}.\frac{3.4}{3} = 2$.

33. $y = \cos^{-1}(\cos(\theta + x))$, $4 = r\cos\theta$, $5 = r\sin\theta$

$= \theta + x$ so $\dfrac{dy}{dx} = 1$.

34. $\dfrac{dy}{dx} = \dfrac{1}{\cos\left(\tan^{-1}\dfrac{e^x - e^{-x}}{2}\right)}\left(-\sin\left(\tan^{-1}\left(\dfrac{e^x - e^{-x}}{2}\right)\right)\right)\times$

$\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{e^x - e^{-x}}{2}\right)\right)$

$\left.\dfrac{dy}{dx}\right|_{x=0} = 0.$

35. $y = -\log(1+x)$ so $x\dfrac{dy}{dx} + 1 = -\dfrac{x}{1+x} + 1 = \dfrac{1}{1+x}$

$= e^y$.

36. $(1 - x^2)\,y^2 = (\sin^{-1} x)^2 \Rightarrow (1 - x^2)2yy' + (-2x)$

$y^2 = \dfrac{2\sin^{-1}x}{\sqrt{1-x^2}} = 2\text{y}$

$\Rightarrow (1 - x^2)\,y' - xy = 1$

37. $\dfrac{dy}{dx} = \dfrac{\sin^2 y}{\sin a} = \dfrac{\sin a \sin^2 y}{-\sin^2 y + \sin^2 a + \sin^2 y}$

$= \dfrac{\sin a \sin^2 y}{\sin(a+y)\sin(a-y) + \sin^2 y}$

$= \dfrac{\sin a \sin^2 y}{\sin(a+y)[\sin(a+y) - 2\sin y\cos a + \sin^2 y]}$

$= \dfrac{\sin a}{1 + \dfrac{\sin^2(a+y)}{\sin^2 y} - 2\cos a\dfrac{\sin(a+y)}{\sin y}}$

$= \dfrac{\sin a}{1 + x^2 - 2x\cos a}$.

38. $f(x) = \cot^{-1}\dfrac{3x - x^3}{1 - 3x^2} = \dfrac{\pi}{2} + 3\tan^{-1} x$

$f'(x) = \dfrac{3}{1+x^2}$ and $g(x) = \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= 2\tan^{-1} x$

so $g'(x) = \dfrac{2}{1+x^2}$. The required limit is equal to $\dfrac{f'(t)}{g'(t)}$

$= \dfrac{3}{2}$.

39. Verify the answers or consider $g(x) = \dfrac{f(x)}{\sin x}$ show that $g''(x) = -2\cot x\, g'(x) \Rightarrow g'(x) = c\,\mathrm{cosec}^2\, x$.

Thus $\sin x\, f'(x) - \cos x\, f(x) = C$ but $f(0) = 0$, $f'(0) = 1$ so const $= 0$. $g(x) =$ const $\Rightarrow f(x) = C \sin x$ but $f'(0) = 1$ so $C = 1$.

40. $y'(x) = -\dfrac{\sin x}{\sqrt{1-\cos^2 x}} = -\dfrac{\sin x}{|\sin x|} = -\operatorname{sgn}(\sin x)$.

41. $f(x) = \cos x^2$ as $[x] = 0$ for $0 < x < 1$ so $f'(x) = -2x\sin x^2$ hence $f'\left(\dfrac{\sqrt{\pi}}{2}\right) = -\dfrac{\sqrt{\pi}}{\sqrt{2}}$

42. $\dfrac{dy}{dx} = \dfrac{2b\sin t\cos t}{-2a\sin 2t} = -\dfrac{b}{a} \Rightarrow \dfrac{d^2y}{dx^2} = 0$.

43. $f(x) = \dfrac{\pi}{2} - \cot^{-1}\left(\cot\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)\right) = \dfrac{\pi}{4} + \dfrac{x}{2}$

Thus $f'(x) = \dfrac{1}{2}$.

44. $2^x\log 2 + 2^y\log 2\dfrac{dy}{dx} = 2^{x+y}\log 2\left(1 + \dfrac{dy}{dx}\right)$

Putting $x = y = 1$, we obtain

$(2 - 2^2)\dfrac{dy}{dx} = 2^2 - 2$

$\Rightarrow \dfrac{dy}{dx} = -1$

45. $2^s = \sin t \Rightarrow 2^s \log 2\dfrac{ds}{dt} = \cos t$

$\Rightarrow \dfrac{ds}{dt} = \dfrac{\cos t}{\sin t}\dfrac{1}{\log 2} = \dfrac{\cot t}{\log 2}$.

46. $x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow x + yy' = 0$

Differentiating again $1 + y'^2 + yy'' = 0$

47. $f'(x) = 2x\cot x^2$, $f''(x) = 2[\cot x^2 - 2x^2\,\mathrm{cosec}^2\, x^2]$

$f''\left(\sqrt{\pi}/2\right) = 2\left[\cot\pi/4 - \dfrac{2\pi}{4}\mathrm{cosec}^2\,\pi/4\right]$

$= 2(1 - \pi)$.

48. $f(g(x) = x \Rightarrow f'(g(x))\,g'(x) = 1$

$\Rightarrow g'(x) = \dfrac{1}{f'(g(x))} = 1 + (g(x))^5$

$\Rightarrow g''(x) = 5(g(x))^4\,g'(x) = 5(g(x))^4\,(1 + (g(x))^5)$

49. $u = f(\tan x)$, $v = g(\sec x)$

$\dfrac{du}{dx} = f'(\tan x)\sec^2 x$, $\dfrac{du}{dx} = g'(\sec x)\sec x\tan x$

$\therefore \dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$

$\Rightarrow \left.\dfrac{du}{dv}\right]_{x=\pi/4} = \dfrac{f'(1)(2)}{g'\left(\sqrt{2}\right)\left(\sqrt{2}\right)} = \dfrac{2\sqrt{2}}{4} = \dfrac{1}{\sqrt{2}}$

50. $y = (\log_{\cos x} \sin x)(\log_{nx} \cos x) + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$= \left(\frac{\log \sin x}{\log \cos x}\right)\left(\frac{\log \cos x}{\log nx}\right) + 2\tan^{-1} x$$

$$= \frac{\log \sin x}{\log(nx)} + 2\tan^{-1} x$$

$$\frac{dy}{dx} = \frac{\log(nx)(\cot x) - \frac{1}{x}\log \sin x}{(\log nx)^2} + \frac{2}{1+x^2}$$

$$\left.\frac{dy}{dx}\right]_{x=\pi/2} = 0 + \frac{2}{1+\pi^2/4} = \frac{8}{4+\pi^2}$$

51. Put $x = \sin\theta$ and $5/13 = \cos\alpha$, so that

$$y = \sin^{-1}[\sin(\theta+\alpha)] = \theta + \alpha$$
$$y = \sin^{-1} x + \cos^{-1}(5/13)$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

52. $g'(x) = 2[f(2f(x)+2)]\,[f'(2f(x)+2)\,(2f'(x))]$

$$g'(0) = 4[f(2f(0)+2)]\,[f'(2f(0)+2)\,(2f'(0))]$$
$$= 4f(0)f'(0)f'(0) = -4.$$

53. Put $x = 1$ in $x^{2x} - 2x^x \cot y - 1 = 0$, we have $1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \pi/2$.

Differenting the given expression,

$2x \,.\, x^{2x-1} + 2\log x \,.\, x^{2x} - 2[x.\, x^{x-1} + (\log x)\, x^x]\cot y + 2x^x (\operatorname{cosec}^2 y)\, y'(x) = 0$

Putting $x = 1$, $y = \pi/2$, we have $y'(1) = -1$

54. For $x \neq 1$, $f'(x) = \sin\dfrac{1}{(x-1)} - \dfrac{(x-1)}{(x-1)^2}\cos\dfrac{1}{x-1}$

$\therefore$ f is differentiable at $x = 0$

$f'(1)$ doesn't exist.

55. $\dfrac{dy}{dx} = \dfrac{1}{3}\left[\dfrac{1}{x+1} \times \dfrac{1}{2}\dfrac{3-3x}{x^2-x+1}\right]$

$$+\frac{1}{\sqrt{3}}\frac{2/\sqrt{3}}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2} = \frac{1}{x^3+1}$$

56. $y = -\log(1+x) \quad \Rightarrow \quad y' = -\dfrac{1}{1+x}$

$\Rightarrow \quad (1+x)y' = -1$

$xy' = -1 - y' = -1 + \dfrac{1}{1+x} = -1 + e^y$. Since $y(1) = -\log 2$, so putting $x = 1$ in $xy' + 1 = e^y$, we get $y'(1) + 1 = \dfrac{1}{2} \quad \Rightarrow \quad y'(1) = -\dfrac{1}{2}$.

57. $\dfrac{dy}{dx} = -\dfrac{1 + y\sin(xy)}{x\sin(xy)}$

58. $\dfrac{dy}{dx} = \dfrac{2t-3t^2}{1-4t^3}$ but $(x, y) = (0, 0)$ if and only if $t = 0$ or $t = 1$

59. $y'(x) = \cos(3\sin^{-1} x)\dfrac{3}{\sqrt{1-x^2}} \quad \Rightarrow \quad y'(0) = 3$

$(1-x^2)\,y'^2 = 9\cos^2(3\sin^{-1} x) = 9(1-y^2)$

$2(1-x^2)\,y''y' - 2xy'^2 = -18yy'$

$\Rightarrow \quad (1-x^2)y'' - xy' + 9y = 0$

Putting $x = 0$, $\quad y''(0) = -9y(0) = 0$.

60. $\left|\dfrac{f(x)-f(y)}{x-y}\right| \le A|x-y|^{\alpha-1}$

Since $\alpha - 1 > 0$ so $\lim\limits_{y\to x}|x-y|^{\alpha-1} = 0$.

Therefore $\lim\limits_{y\to x}\dfrac{f(x)-f(y)}{x-y} = 0$

$\Rightarrow f'(x) = 0$ for $x \in \mathbf{R}$.

$\Rightarrow f$ is a constant function

61. For $\pi/2 < x < \pi$, $\tan x < 0$, so

$$y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$$
$$= \tan^{-1}(\cot x) + \tan^{-1}(\cot x) + \pi$$
$$= 2\tan^{-1}(\cot x) + \pi$$
$$= -\cos^{-1}\left(\frac{1-\cot^2 x}{1+\cot^2 x}\right) + \pi$$
$$= -\cos^{-1}(\sin^2 x - \cos^2 x) + \pi$$
$$= -(\pi - \cos^{-1}(\cos 2x)) + \pi$$
$$= 2x$$

Hence $\dfrac{d^2y}{dx^2} = 0$ and y is a linear function of x.

62. Differentiating $f'(x) = f(1-x)$, we get

$$f''(x) = -f'(1-x) = -f(1-(1-x)) = -f(x)$$
$$\Rightarrow \quad f(x) = A\cos x + B\sin x$$
$$f(\pi/2) = 1 \quad \Rightarrow \quad B = 1$$

Also $f'(\pi/2) = f\left(1 - \dfrac{\pi}{2}\right) = A\cos\left(\dfrac{\pi}{2} - 1\right) + \sin\left(\dfrac{\pi}{2} - 1\right)$

$$= A\sin 1 + \cos 1$$

But $\quad f'(x) = -A\sin x + B\cos x$, so $f'\left(\dfrac{\pi}{2}\right) = -A$

$\Rightarrow \quad A(1 + \sin 1) + \cos 1 = 0 \quad$ i.e. $A = -\dfrac{\cos 1}{1+\sin 1}$

Hence $f(x) = -\dfrac{\cos 1}{1+\sin 1}\cos x + \sin x$.

Level 2

63. For f to be continuous $f(\pi/6) = \lim\limits_{x \to \pi/6+} f(x)$

$\sin \dfrac{\pi}{3} = a\dfrac{\pi}{6} + b$ i.e. $\dfrac{\sqrt{3}}{2} = a\dfrac{\pi}{6} + b$

$f'(\pi/6+) = \lim\limits_{h \to 0+} \dfrac{f(\pi/6+h) - f(\pi/6)}{h}$

$= \lim\limits_{h \to 0+} \dfrac{a(\pi/6+h) + b - \sin \pi/3}{h}$

$= \lim\limits_{h \to 0+} \dfrac{a(\pi/6+h) + b - (a\,\pi/6 + b)}{h}$

$= a.$

$f'(\pi/6-) = \lim\limits_{h \to 0-} \dfrac{f(\pi/6+h) - f(\pi/6)}{h}$

$= \lim\limits_{h \to 0-} \dfrac{\sin 2(\pi/6+h) - \sin \pi/3}{h}$

$= 2 \cos 2\dfrac{\pi}{6} = 2\dfrac{1}{2} = 1,$

so $a = 1$, $b = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$. For these values of a and b,

$f'(x) = \begin{cases} 2\cos 2x & 0 < x \le \pi/6 \\ 1 & \pi/6 < x \le 1 \end{cases}$ which is continuous.

64. $f(1) = 1 = \lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} ax^2 + b = a + b$

$f'(1+) = \lim\limits_{h \to 0+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0+} \dfrac{(1/(1+h)) - 1}{h}$

$= \lim\limits_{h \to 0+} \dfrac{1 - (1+h)}{h(1+h)} = -1$

$f'(1-) = \lim\limits_{h \to 0-} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0-} \dfrac{a(1+h)^2 + b - 1}{h}$

$= \lim\limits_{h \to 0-} \dfrac{a + b - 1 + ah^2 + 2ah}{h}$

$= 2a.$

so $a = -1/2$, $b = 3/2$

65. Since $[x - \pi]$ is an integer, so $\tan(\pi[x - \pi]) = 0$ for all x. Hence $f(x) = 0$ for all x, so $f'(x)$ exists for all x.

66. Since differentiability implies continuity so

$f(0) = \lim\limits_{x \to 0} f(x)$

$= \lim\limits_{x \to 0} (1 + (-1 + \cos x))^{\frac{1}{-1+\cos x} \times \frac{-2\sin^2 x/2}{2 \sin x/2 \cos x/2}}$

$= \lim\limits_{x \to 0} (1 + (-1 + \cos x))^{\frac{1}{-1+\cos x} \times -\tan x/2}$

$= e^0 = 1.$

67. $y(0) = \sin^{-1} 0 = 0$. Also

$\sin y = \sin \alpha \dfrac{\sin x}{1 - \cos\alpha \sin x}$

$\cos y \dfrac{dy}{dx}$

$= \sin \alpha \dfrac{(1 - \cos\alpha \sin x)\cos x + \sin^2 x \cos\alpha}{(1 - \cos\alpha \sin x)^2}$

At $x = 0$, $\left.\dfrac{dy}{dx}\right|_{x=0} = \sin \alpha.$

68. f is defined on $[0, \infty)$.

$\lim\limits_{h \to 0+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0+} \dfrac{\sqrt{h+1} - 1}{\sqrt{h}\, h}$

$= \lim\limits_{h \to 0+} \dfrac{1}{\sqrt{h}\,(\sqrt{h+1} + 1)}$

which does not exist. f is clearly differentiable on $(0, \infty)$.

69. $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{1-t^2}}$, $\dfrac{dy}{dt} = -\dfrac{2t}{1-t^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{-2t}{\sqrt{1-t^2}}$

$\dfrac{d^2y}{dx^2} = -2\dfrac{d}{dt}\left(\dfrac{t}{\sqrt{1-t^2}}\right)\dfrac{dt}{dx}$

$= -2\dfrac{\left(\sqrt{1-t^2} + \dfrac{t^2}{\sqrt{1-t^2}}\right) \times \sqrt{1-t^2}}{(1-t^2)}$

$= -2\dfrac{1}{1-t^2}$

$\Rightarrow \left.\dfrac{d^2y}{dx^2}\right|_{t=1/2} = -2 \cdot \dfrac{4}{3} = -\dfrac{8}{3}.$

70. $2yy' + 2x = 0 \Rightarrow y' = -x/y$

$y'^2 + yy'' + 1 = 0 \Rightarrow yy'' = -(1 + x^2/y^2)$

so $|y''| = \dfrac{x^2 + y^2}{y^3}$

$\dfrac{|y''|}{\sqrt{(1+y'^2)^3}} = \dfrac{x^2 + y^2}{y^3\sqrt{(1 + x^2/y^2)^3}} = \dfrac{x^2 + y^2}{(x^2 + y^2)^{3/2}}$

$= \dfrac{1}{\sqrt{x^2 + y^2}} = \dfrac{1}{R} = k.$

71. For (a), let $f(x) = |x|$ then $||x| - |y|| \le |x - y|$ for all $x, y \in \mathbf{R}$ so $|f(x) - f(y)| \le |x - y|$ but f is not differentiable at $x = 0$

For (b), let $h(x) = \begin{cases} x^{1/2}, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

Since for $x \in [0, 1]$ $|x^{1/2} - y^{1/2}| \le |x^{1/2} + y^{1/2}|$ so

$|x^{1/2} - y^{1/2}|^2 \le |x - y|$. Thus $|h(x) - h(y)| \le |x - y|^{1/2}$ but h is not differentiable.

If f satisfies $|f(x) - f(y)| \le k\,|x - y|^2$ then

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le |x - y|$$

$$\Rightarrow \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y| = 0$$

$\Rightarrow f$ *is* differentiable and $f'(x) = 0$

For (d) Let $f(x) = |x|$, $f^2(x) = x^2$ is differentiable on $\mathbf{R}$ but f is not differentiable.

72. $\left|\dfrac{f(x)-f(y)}{x-y}\right| \le |x - y|^2 \Rightarrow f'(x) = 0$

73. Since $\left|x \sin\dfrac{\pi}{x}\right| \le |x|$ so f is continuous and bounded on [0, 1]

$$\lim_{x\to 0}\frac{x\sin \pi/x - 0}{x - 0} = \lim_{x\to 0}\sin \pi/x$$ which does not exist so f is not differentiable.

74. $$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\frac{|x|\,(3e^{1/|x|}+4)}{2-e^{1/|x|}}$$

$$= \lim_{x\to 0+}\frac{x\,(3e^{1/|x|}+4)}{2-e^{1/x}} = \lim_{x\to 0+}\frac{x\,(3+4.e^{-1/x})}{2\,.\,e^{-1/x}-1}$$

$$= 0.\ \frac{3}{-1} = 0$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\frac{-x\,(3e^{-1/x}+4)}{2-e^{-1/x}}$$

$$= -\lim_{x\to 0-}\frac{x\,(3+4e^{-1/x})}{2\,e^{-1/x}-1} = 0$$

Thus f is continuous at $x = 0$

$$\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+}\frac{x\,(3e^{1/x}+4)}{x\left(2-e^{1/x}\right)}$$

$$= \lim_{x\to 0+}\frac{(3+4e^{-1/x})}{2\,e^{-1/x}-1} = -3$$

$$\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-}\frac{-(3+4e^{1/x})}{\left(2\,e^{1/x}-1\right)} = 3.$$

75. $\lim_{x\to 1+} f(x) = k \lim_{x\to 1+}(x - 1) = 0$

$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(x^2 - 1) = 0$

f is continuous for any value of k

$$\lim_{h\to 0+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0+}\frac{k(1+h-1)}{h} = k.$$

$$\lim_{h\to 0-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0-}\frac{(1+h)^2-1)}{h} = 2$$

f is differentiable only when $k = 2$.

76. $$\lim_{x\to a-} f(x) = \lim_{x\to a-}\left(1/\sqrt{2}\right)\left(x+a/\sqrt{2}\right)$$

$$= \frac{1}{\sqrt{2}}\left(a+a/\sqrt{2}\right)$$

$$= \frac{\sqrt{2}+1}{2}a$$

$$\lim_{x\to a+} f(x) = \lim_{x\to a+}\left(1/\sqrt{2}\right)\left(\frac{x}{2}+a\right) = \frac{3a}{2\sqrt{2}}$$

f is clearly not conttnuous at $x = 0$,

so f' is not defined at 0 and a, hence not continuous.

77. Given function is a polynomial of degree 160. Differentiation one-time reduces a polynomial by one degree. Thus to obtain polynomial of degree 50, we has to differentiate 110 times.

78. The derivative of $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ doesn't exist for $\left|\dfrac{2x}{1+x^2}\right| = 1$. So $2\,|x| = x^2 + 1$. For $x \ge 0$,

$(x - 1)^2 = 0 \Rightarrow x = 1$, if $x < 0$, then $(x + 1)^2 = 0$

so $x = -1$.

79. $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x^{\alpha} = 0$, f is continuous at $x = 0$

$$\lim_{h\to 0+}\frac{f(h)-f(0)}{h-0} = \lim_{h\to 0+}\frac{h^{\alpha}}{h} = \lim_{h\to 0+}\frac{1}{h^{1-\alpha}}$$

which does not exist.

80. $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = 0$ when x approach through rational or irrational so f is differentiable at

$x = 0$. If $a \neq 0$ then

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ does not exist. Morever f is continuous at $x = 0$ only.

81. $f(x) = x^k$ is $(n - 1)$ times differentiable at $x = 0$ if $k - (n - 1) \geq 0$ and not n-times differentiable if $k - n < 0$ so $n - 1 \leq k < n$, $k = \frac{3n-2}{3}$ satisfy this inequality.

82. We have

$$f(x) = \begin{cases} (-\tfrac{1}{2})(x+1) & \text{if} \quad x < -1 \\ \tan^{-1} x & \text{if} \quad - \leq x \leq 1 \\ (\tfrac{1}{2})(x-1) & \text{if} \quad x > 1 \end{cases}$$

Since $f(-1) = \frac{\pi}{4}$, $f(1) = \frac{\pi}{4}$ and $\lim_{x \to 1-} f(x) = -1$, $\lim_{x \to 1+} f(x) = 0$ so f is not continuous at $x = -1, 1$ and hence not differentiable at $-1, 1$. Also

$$f'(x) = \begin{cases} -\tfrac{1}{2} & \text{if} \quad x < -1 \\ \frac{1}{1+x^2} & \text{if} \quad -1 < x < 1 \\ \tfrac{1}{2} & \text{if} \quad x > 1 \end{cases}$$

Thus the domain of f' is $\mathbf{R} \sim \{-1, 1\}$

83. Let $y = \sin^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$ and $u = x^{2n}$, then

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$= \frac{1}{\sqrt{1 - \left(\frac{1-x^{2n}}{1+x^{2n}}\right)^2}} \frac{d}{dx}\left(\frac{1-x^{2n}}{1+x^{2n}}\right) \frac{1}{2\,n\,x^{2n-1}}$$

$$= \frac{1+x^{2n}}{\sqrt{4x^{2n}}} \frac{-(1+x^{2n})2nx^{2n-1} - (1-x^{2n})2n\,x^{2n-1}}{(1+x^{2n})^2}$$

$$\times \frac{1}{2n\,x^{2n-1}}$$

$$= \frac{1}{2\,x^n} \times \frac{-4\,n\,x^{2n-1}}{(1+x^{2n})} \frac{1}{2n\,x^{2n-1}}$$

$$= -\frac{1}{x^n\,(1+x^{2n})} = -\frac{1}{x^n + x^{3n}}.$$

84. Put $x^3 = \cos\theta$, $y^3 = \cos\phi$ so that given equation reduces to $\sin\theta + \sin\phi = a(\cos\theta - \cos\phi)$

$$\Rightarrow 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = -a\,2\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2}$$

$$\Rightarrow \cos\frac{\theta-\phi}{2} = -a\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \theta - \phi = 2\tan^{-1}(-1/a)$$

$$\Rightarrow \frac{d\theta}{dx} - \frac{d\phi}{dx} = 0$$

$$\frac{d}{dx}(\cos^{-1} x^3) - \frac{d}{dx}(\cos^{-1} y^3) = 0$$

$$\Rightarrow -\frac{3x^2}{\sqrt{1-x^6}} + \frac{3y^2}{\sqrt{1-y^6}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

85. $f(g(x)) = x \Rightarrow f'(g(x))\,g'(x) = 1$

Also $f'(x) = 2x + \sec^2 x = 2x + 1 + \tan^2 x$

$= 2x + 1 + (f(x) - x^2)^2$

$f'(g(x)) = 2g(x) + 1 + (f(g(x)) - (g(x)^2)^2$

$= 2\,g(x) + 1 + (x - (g(x))^2)^2$

Thus $g'(x) = \frac{1}{f'(g(x))}$

$$= \frac{1}{2\,g(x) + 1 + (x - (g(x))^2)^2}$$

86. $[x]$ is not continuous at integral points.

The number of integers lying in $(-\pi/2, \pi/2)$ is 3.

87. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) + x^2h^2 - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h} + x^2 \lim_{h \to 0} h = 10.$$

Previous Years' AIEEE/JEE Main Questions

1. $f''(x) - g''(x) = 0 \Rightarrow f'(x) - g'(x) =$ constant. Putting $x = 1$, we get $f'(1) - g'(1) = C \Rightarrow C = 2g'(1) - g'(1) = g'(1) = 2$.

So $f'(x) - g'(x) = 2 \Rightarrow f(x) - g(x) = 2x + C'$. Putting $x = 2$ we have $f(2) - g(2) = 4 + C' \Rightarrow 4 + C' = 3g(2) - g(2) = 2g(2) = 6 \Rightarrow C' = 6 - 4 = 2$. Hence $f(x) - g(x) = 2x + 2$. At $x = \dfrac{3}{2}$

$$f\left(\frac{3}{2}\right) - g\left(\frac{3}{2}\right) = 2 \cdot \frac{3}{2} + 2 = 5.$$

2. $\lim\limits_{x\to 1} \dfrac{\sqrt{f(x)}-1}{\sqrt{x}-1} = \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1} \times \dfrac{\sqrt{x}+1}{\sqrt{f(x)}+1}$

$= f'(1) \times \dfrac{2}{\sqrt{f(1)}+1} = 2 \times \dfrac{2}{1+1} = 2$

3. $\dfrac{dy}{dx} = n\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\dfrac{x}{\sqrt{1+x^2}}\right) = \dfrac{ny}{\sqrt{1+x^2}}$

$(1 + x^2)\left(\dfrac{dy}{dx}\right)^2 = n^2y^2$

$\Rightarrow (1 + x^2)\ 2\dfrac{dy}{dx}\dfrac{d^2y}{dx} + 2x\left(\dfrac{dy}{dx}\right)^2 = 2n^2y\dfrac{dy}{dx}$

$\Rightarrow (1 + x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = n^2y.$

4. $f'(x) = nx^{n-1} \Rightarrow f'(1) = n$

Also $f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$

$\Rightarrow f^k(1) = n(n-1) \dots (n-k+1)$

So $f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''}{2!} \cdots + \dfrac{(-1)^n f^n(1)}{n!}$

$= 1 - \dfrac{n}{1!} + \dfrac{n(n-1)}{2!} + \cdots + \dfrac{(-1)^n n!}{n!}$

$= (1-1)^n = 0.$

5. $f(x) = \begin{cases} x & x<0 \\ xe^{-2/x} & x>0 \\ 0 & x=0 \end{cases}$

$\lim\limits_{x\to 0-} f(x) = 0$ and $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} xe^{-2/x} = 0$

$\lim\limits_{x\to 0+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0+} \dfrac{xe^{-2/x}}{x} = \lim\limits_{x\to 0+} e^{-2/x} = 0$

$\lim\limits_{x\to 0-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0-} \dfrac{x-0}{x} = 1.$

So f is not differentiable at $x = 0$ but is continuous for all x.

6. $\lim\limits_{x\to a} \dfrac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}$

$= k\lim\limits_{x\to a} \dfrac{g(x)-1-f(x)+1}{g(x)-f(x)}$

$= k$, so $k = 4$

7. We have $x = e^{y+x}$

$\Rightarrow \log x = y + x \Rightarrow \dfrac{1}{x} = \dfrac{dy}{dx} + 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

8. For $x \neq y$,

$\left|\dfrac{f(x)-f(y)}{x-y}\right| \le |x-y|$

$\Rightarrow \left|\dfrac{f(x+h)-f(x)}{h}\right| \le |h|$

Taking limit as $h \to 0$, we get

$|f'(x)| \le 0$

But $|f'(x)| \ge 0$

$\therefore\ |f'(x)| = 0 \Rightarrow f'(x) = 0\ \forall\ x$

Thus, $f(x)$ is a constant function.

Now, $f(0) = 0 \Rightarrow f(1) = 0$

9. As f is differentiable at $x = 1$, f is continuous at $x = 1$. Therefore,

$f(1) = \lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0} h\left\{\dfrac{f(1+h)}{h}\right\}$

$= \left(\lim\limits_{h\to 0} h\right)\left(\lim\limits_{h\to 0} \dfrac{f(1+h)}{h}\right) = (0)\ (5) = 0$

Now,

$f^1(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \dfrac{f(1+h)}{h} = 5$

10. We have

$$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{if} \quad x<0 \\ \dfrac{x}{1+x} & \text{if} \quad x \ge 0 \end{cases}$$

The function f is differentiable at all points except possibly at $x = 0$.

We have

$$Lf'(0) = \lim_{x\to 0-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{1}{1-x} = 1$$

and $Rf'(0) = \displaystyle\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0+}\frac{1}{1+x} = 1$$

As $Lf'(0) = Rf'(0) = 1$, we get f is differentiable at $x = 0$. Thus f is differentiable for all $x \in (-\infty, \infty)$.

11. $m \log x + n \log y = (m + n) \log (x + y)$

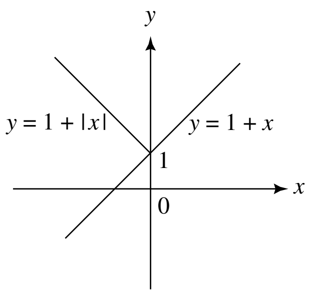
$$\Rightarrow \frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y}-\frac{m}{y}$$

$$\Rightarrow \frac{nx-my}{y(x+y)}\frac{dy}{dx} = \frac{nx-my}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}.$$

12. $f(x) = \min\{x + 1, |x| + 1\} = x + 1$, so is differentiable everywhere.



13. For $x \neq 1$,

$$f'(x) = \sin\left(\frac{1}{x-1}\right)-\frac{1}{x-1}\cos\left(\frac{1}{x-1}\right)$$

$\therefore$ f is differentiable at $x = 0$

$\displaystyle\lim_{x\to 1}\frac{f(x)-f(0)}{x-1} = \lim_{x\to 1}\sin\frac{1}{x-1}$ which does not exist.

$\therefore$ f is differentiable at $x = 0$ but not at $x = 1$.

14. Putting $x = 1$ in $x^{2x} - 2x^x \cot y - 1 = 0$,

we get $1 - 2 \cot y - 1 = 0$

$\Rightarrow \cot y = 0 \Rightarrow y = \pi/2$.

Using the formula,

$\dfrac{d}{dx}\left[f(x)^{g(x)}\right] = g(x)\, f(x)^{g(x)-1} f'(x) + (g'(x) \log f(x))\, f(x)^{g(x)}$, we have

$2x.x^{2x-1}(1) + (2 \log x)x^{2x} - 2[x.x^{x-1}(1) + (\log x)x^x] \cot y + 2x^x (\operatorname{cosec}^2 y)\, y'(x) = 0$

Putting $x = 1$, $y = \pi/2$, we get

$2 + 0 - 2(1)(0) + 2 \operatorname{cosec}^2(\pi/2)\, y'(1) = 0$

$\Rightarrow y'(1) = -1.$

15. $g'(x) = 2[f(2f(x) + 2)]f'(2f(x) + 2)\,(2f'(x))$

$\Rightarrow g'(0) = 2[f(2f(0) + 2)]f'(2f(0) + 2)\,(2f'(0))$

$= 4(f(0))\, f'(0)^2 = 4(-1)(1)^2 = -4$

16. $$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \frac{d}{dy}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$= \frac{-1}{(dy/dx)^2}\left(\frac{d^2y}{dx^2}\right)\frac{dx}{dy}$$

$$= \frac{d^2y/dx^2}{(dy/dx)^3}$$

17. $$\lim_{x\to a}\frac{x^2f(a)-a^2f(x)}{x-a}$$

$$= \lim_{x\to a}\frac{(x^2-a^2)f(a)+a^2(f(a)-f(x))}{x-a}$$

$$= \lim_{x\to a}\left[(x+a)f(a)-a^2\frac{f(x)-f(a)}{x-a}\right]$$

$= 2a\, f(a) - a^2 f'(a)$

18. We have

$$f(x) = \begin{cases}(2-x)+(5-x) & \text{if } x\le 2\\ (x-2)+(5-x) & \text{if } 2<x\le 5\\ (x-2)+(x-5) & \text{if } x>5\end{cases}$$

$$= \begin{cases}7-2x & \text{if } x\le 2\\ 3 & \text{if } 2<x\le 5\\ 2x-7 & \text{if } x>5\end{cases}$$

Note that $f'(4) = 0$ as f is a constant function in the interval $(2, 5)$.

Thus, the statement-1 is true.

Statement-2 is also true. But it is not the correct explanation for the statement-1.

19. $8f(x) + 6f\left(\frac{1}{x}\right) - x = 5.$ (i)

Replace x by $\frac{1}{x}$, we have $8f\left(\frac{1}{x}\right)+6f(x)-\frac{1}{x} = 5$ (ii)

Solving (i) and (ii) for $f(x)$ and $f(1/x)$, we have

$14f(x) = 5 - \frac{3}{x} + 4x$

$\Rightarrow 14f'(x) = \frac{3}{x^2}+4$

So $f(-1) = \frac{1}{14}\ [5 + 3 - 4] = \frac{2}{7}$

and $f'(-1) = \frac{1}{14}[3 + 4] = \frac{1}{2}$

Differentiating $y = x^2f(x)$, we have $\frac{dy}{dx} = 2xf(x) + x^2f'(x)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-1} = -2f(-1) + f'(-1) = -\frac{4}{7}+\frac{1}{2} = -\frac{1}{14}.$

20. $\frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-\frac{1}{2}} \cdot a^{\sin^{-1}t}\frac{1}{\sqrt{1-t^2}}\ \log a$

$= \frac{1}{2}\frac{\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}}{\sqrt{1-t^2}}\ \log a = \frac{1}{2}\frac{x}{\sqrt{1-t^2}}\log a$

$\frac{dy}{dt} = -\frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-\frac{1}{2}} \cdot a^{\cos^{-1}t}\cdot\frac{1}{\sqrt{1-t^2}}\ \log a$

$= -\frac{1}{2}\frac{y}{\sqrt{1-t^2}}\ \log a$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ so $1+\left(\frac{dy}{dx}\right)^2 = \frac{x^2+y^2}{x^2}.$

21. $y = \sec(\tan^{-1}x) = \sqrt{1+(\tan(\tan^{-1}x))^2}$

$\Rightarrow y = \sqrt{1+x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$

$\therefore \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$

22. Let $y = f(x) = \frac{x^2-x}{x^2+2x} = 1-\frac{3x}{x(x+2)} = 1-\frac{3}{x+2}$

$\Rightarrow x = -\frac{3}{y-1}-2$ so $f^{-1}(x) = -\frac{3}{x-1}-2$

$\frac{d}{dx}(f^{-1}(x)) = \frac{3}{(x-1)^2}.$

23. $g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

$\Rightarrow g'(f(x)) = 1 + x^5$

$\Rightarrow g'(x) = 1 + (f^{-1}(x))^5 = 1 + (g(x))^5.$

24. $h(x) = (gof)(x) = g(f(x)) = \sin(x|x|)$

$= \begin{cases} -\sin(x^2); & \text{if } x<0 \\ \sin(x^2); & \text{if } x\ge 0\end{cases}$

We have

$h'(x) = \begin{cases} -2x\cos(x^2); & \text{if } x<0 \\ 2x\cos(x^2); & \text{if } x>0\end{cases}$

Also, $Lh'(0) = \lim\limits_{x\to 0-}\frac{h(x)-h(0)}{x-0}$

$= \lim\limits_{x\to 0-}\frac{-\sin(x^2)}{x} = 0$

and $Rh'(0) = 0$

Thus, $h'(x) = \begin{cases} -2\cos(x^2) & \text{if } x<0 \\ 2x\cos(x^2) & \text{if } x\ge 0\end{cases}$

Note that $h'(x)$ is continuous at all points except possibly at $x = 0$.

Now, $\lim\limits_{x\to 0-} h'(x) = \lim\limits_{x\to 0-}[-2x\cos(x^2)] = 0 = h'(0)$

and $\lim\limits_{x\to 0+} h'(x) = \lim\limits_{x\to 0+}[2x\cos(x^2)] = 0 = h'(0)$

This shows that h' is continuous at $x = 0$.

Also, h' is differentiable at all points except possibly at $x = 0$.

Now, $Lh''(0) = \lim\limits_{x\to 0-}\frac{h'(x)-h'(0)}{x-0} = -2$

and $Rh''(0) = 2$

This shows that h' is differentiable at all points except at $x = 0$.

25. We have $g(f(x)) = x$

$\Rightarrow g'(f(x))\, f'(x) = 1$

Now, $f(x) = 7 \Rightarrow x^2 - x + 5 = 7$

$\Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2$ as $x > 1/2$.

Thus, $g'(7)\, f'(2) = 1$

$\Rightarrow g'(7) = \dfrac{1}{2(2)-1} = \dfrac{1}{3}$.

26. For $x \in \mathbf{R}$ $0 \le |f(x)| \le x^2$

$\Rightarrow \lim\limits_{x\to 0} 0 \le \lim\limits_{x\to 0} |f(x)| \le \lim\limits_{x\to 0}(x^2)$

$\Rightarrow \lim\limits_{x\to 0} |f(x)| = 0 \Rightarrow \left|\lim\limits_{x\to 0} f(x)\right| = 0$

$\Rightarrow \lim\limits_{x\to 0} f(x) = 0.$

Also, $0 \le |f(0)| \le 0^2 \Rightarrow f(0) = 0$

$\therefore \lim\limits_{x\to 0} f(x) = 0 = f(0)$

$\Rightarrow f$ is continuous at $x = 0$.

Also, for $x \ne 0$,

$$0 \le \left|\frac{f(x)-f(0)}{x}\right| \le \left|\frac{x^2-0}{x}\right| = |x|$$

$\therefore \lim\limits_{x\to 0}\left|\dfrac{f(x)-f(0)}{x-0}\right| = 0$

$\Rightarrow f'(0) = 0$

27. $y = e^{nx} \Rightarrow$ In $y = nx$

Now, $\dfrac{d^2y}{dx^2} = n^2e^{nx} = n^2y$

and $\dfrac{d^2x}{dy^2} = \dfrac{1}{ny^2}$

$\therefore \left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{d^2x}{dy^2}\right) = \dfrac{n}{y} = -ne^{-nx}$

28. As g is differentiable at $x = 3$, g is continuous at $x = 3$.

$\therefore \lim\limits_{x\to 3-} g(x) = \lim\limits_{x\to 3+} g(x) = g(3)$

$= k\sqrt{3+1} = 3m + 2 = k\sqrt{3+1}$

$\Rightarrow 2k = 3m + 2$

Also, as g is differentiable at $x = 3$,

$$\lim_{x\to 3-}\frac{g(x)-g(3)}{x-3} = \lim_{x\to 3+}\frac{g(x)-g(3)}{x-3}$$

$\Rightarrow \dfrac{k}{2(2)} = m \Rightarrow k = 4m.$

$\therefore m = 2/5$ and $k = 8/5$

$\Rightarrow k + m = 2$

29. $f(x) = |\log 2 - \sin x|,\ x \in \mathbf{R}$

Let $a = \sin^{-1}(\log 2)$

For $-a < x < a$, $f(x) = \log 2 - \sin x$

$\Rightarrow g(x) = f(f(x)) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

$\Rightarrow g'(x) = (-\cos(\log 2 - \sin x))\,(-\cos x)$

$\therefore g'(0) = (-\cos(\log 2\,))\,(-1) = \cos(\log 2)$

30. $\lim\limits_{t\to x}\dfrac{t^2 f(x) - x^2 f(t)}{t-x} = 1$

$\Rightarrow \lim\limits_{t\to x}\dfrac{(t^2-x^2)f(x) - x^2(f(t)-f(x))}{t-x} = 1$

$\Rightarrow \lim\limits_{t\to x}\left[(t+x)f(x) - x^2\dfrac{f(t)-f(x)}{t-x}\right] = 1$

$\Rightarrow 2xf(x) - x^2f'(x) = 1$

$\Rightarrow x^2f'(x) - 2xf(x) = -1$

$\Rightarrow f'(x) - \dfrac{2}{x}f(x) = \dfrac{1}{x^2}$

$\Rightarrow \dfrac{1}{x^2}f'(x) - \dfrac{2}{x^3}f(x) = \dfrac{1}{x^4}$

$\Rightarrow \dfrac{d}{dx}\left[\dfrac{1}{x^2}f(x)\right] = \dfrac{1}{x^4}$

$\Rightarrow \dfrac{1}{x^2}f(x) = \int x^{-4}dx = -\dfrac{1}{3}x^{-3} + C$

$\Rightarrow f(x) = -\dfrac{1}{3x} + Cx^2$

Now, $1 = f(1) = -\dfrac{1}{3} + C \Rightarrow C = \dfrac{4}{3}$

Thus, $f(x) = -\dfrac{1}{3x} + \dfrac{4}{3}x^2$

$\therefore\ f\left(\frac{3}{2}\right) = -\frac{2}{9}+\frac{4}{3}\cdot\frac{9}{4} = \frac{25}{9}$

31. As f is continuous at $x = 1$,

$\lim_{x\to1-} f(x) = \lim_{x\to1+} f(x)$

$\Rightarrow -1 = a + \cos^{-1}(1 + b)$

Also, as f is differentiable at $x = 1$, (1)

$$\lim_{x\to1-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow -1 = \frac{-1}{\sqrt{1-(1+b)^2}}$$

$\Rightarrow 1 - (1 + b)^2 = 1 \Rightarrow 1 + b = 0$ (2)

From (1) and (2)

$-1 = a + \cos^{-1}(0) = a+\frac{\pi}{2}$

$\Rightarrow a = -1 - \pi/2$

Thus, $\frac{a}{b} = \frac{2+\pi}{2}$

Previous Years' B-Architecture Entrance Examination Questions

1. $\lim_{x\to2-} f(x) = \lim_{x\to2-} ax = 2a$ and $\lim_{x\to2+} f(x) = \lim_{x\to2+} (ax^2 + bx + 3) = 4a + 2b + 3$

Since f is differentiable so continuous, hence

$2a + 2b + 3 = 0$ (i)

Also $\lim_{h\to0+}\frac{f(2+h)-f(2)}{h}$

$$= \lim_{h\to0+}\frac{a(2+h)^2+b(2+h)+3-(4a+2b+3)}{h}$$

$$= \lim_{h\to0+}\frac{ah^2+4ah+bh}{h}$$

$= 4a + b$

$$\lim_{h\to0-}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to0-}\frac{a(2+h)-(4a+2b+3)}{h}$$

$= a$ (using (i))

So $a = 4a + b \Rightarrow -3a = b$. Putting this in (i),

we get $a = \frac{3}{4}$, $b = -\frac{9}{4}$.

2. $f'(x) = \sum_{i=1}^{10} x^{i-1} \Rightarrow f'(1) = 10$ and $f''(x) = \sum_{i=2}^{10}(i-1)x^{i-2}$.

Therefore,

$$\lim_{x\to1}\frac{f'(x)-10}{x-1} = \lim_{x\to1}\frac{f'(x)-f'(1)}{x-1} = f''(1) = \sum_{i=2}^{10}(i-1)$$

$= 1 + \ldots + 9 = \frac{9\times10}{2} = 45.$

3. $h(x) = x|x| = \begin{cases} x^2, & x\ge0 \\ -x^2, & x<0 \end{cases}$ is a differentiable function

$u(x) = |x - 2|^2 = (x - 2)^2$ is a differentiable function

$v(x) = |x - 3|^3 = \begin{cases} (x-3)^3, & x\ge3 \\ -(x-3)^3, & x<3 \end{cases}$ is a differentiable function

$w(x) = |x - 1|$ is not differentiable at $x = 1$.

f is differentiable everywhere except at $x = 1$.

Hence $A = \{1\}$

4. $\lim_{x\to0}\frac{2f(2x^2+3x)-1}{3g(x)-1}\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to0}\frac{2(4x+3)f'(2x^2+3x)}{3g'(x)}$$

$$= \frac{2(0+3)f'(0)}{3g'(0)} = \frac{2.3.1}{3.2} = 1$$

5. $f'(x) = \begin{cases} 3x^2, & x>0 \\ 0, & x<0 \end{cases}$ and $\lim_{x\to0+}\frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to0+}\frac{x^3}{x} = 0.$$

$$\lim_{x\to0-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0-}\frac{0-0}{x} = 0.$$

$f'(x)$ exists for all x and is continuous so

$\lim_{x\to0} f'(x) = f'(0)$

6. $f(x) = \begin{cases} 1 & \text{if} \quad \sin x > 0 \\ -1 & \text{if} \quad \sin x < 0 \\ 0 & \text{if} \quad \sin x = 0 \end{cases}$ so

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in (0, \pi) \\ -1 & \text{if} \quad x \in (\pi, 2\pi) \\ 0 & \text{if} \quad x = \pi \end{cases}$$

$\lim\limits_{h \to 0+} \dfrac{f(\pi + h) - f(\pi)}{h} = \lim\limits_{h \to 0+} \dfrac{-1}{h}$ which doesn't exist

So A doesnot exist.

$g(x) = \begin{cases} \sin 1, & x > 0 \\ \sin(-1), & x < 0 \\ 0 & x = 0 \end{cases}$ so $\lim\limits_{h \to 0+} \dfrac{g(\pi + h) - g(\pi)}{h}$

$$\lim_{h \to 0+} \frac{\sin 1 - \sin 1}{h} = 0.$$

and $\lim\limits_{h \to 0-} \dfrac{g(\pi + h) - g(\pi)}{h} = \lim\limits_{x \to 0-} \dfrac{\sin 1 - \sin 1}{h} = 0$.

Thus $B = 0$.

7. $u = 2\tan^{-1} \dfrac{1+x}{1-x} = 2\left(\dfrac{\pi}{4} + \tan^{-1} x\right)$ and

$= \dfrac{\pi}{2} - \cos^{-1} \dfrac{1-x^2}{1+x^2}$

Hence $f(x) = u + v = \pi$ for all x

$f'(x) = 0 \Rightarrow f''(x) = 0$.

8. $\lim\limits_{h \to 0-} \dfrac{f(-1+h) - f(-1)}{h} = \lim\limits_{h \to 0-} \dfrac{1 - (-1+h)^2 - 0}{h}$

$= \lim\limits_{h \to 0-} \dfrac{1 - (1 + h^2 - 2h)}{h} = 2$

$\lim\limits_{h \to 0+} \dfrac{f(-1+h) - f(-1)}{h} = \lim\limits_{h \to 0+} \dfrac{2(-1+h) + 2 - 0}{h}$

$= 2$.

So $f'(-1) = 2$.

9. $\cos |x| - |x| = \cos x - |x|$ which is not differentiable at $x = 0$.

$\cos |x| + |x| = \cos x + |x|$ which is not differentiable at $x = 0$

$u(x) = \sin(|x|) + |x| = \begin{cases} \sin x + x & \text{if } x \geq 0 \\ -(\sin x + x) & \text{if } x < 0 \end{cases}$

$\lim\limits_{h \to 0+} \dfrac{u(h) - u(0)}{h} = \lim\limits_{h \to 0+} \dfrac{\sin h + h}{h} = 2$

$\lim\limits_{h \to 0-} \dfrac{u(h) - u(0)}{h} = -\lim\limits_{h \to 0-} \dfrac{\sin h + h}{h} = -2$

So u is not differentiable at $x = 0$

Let $v(x) = \sin(|x|) - |x| = \begin{cases} \sin x - x, & \text{if } x \geq 0 \\ -\sin x + x, & \text{if } x < 0 \end{cases}$

$\lim\limits_{h \to 0+} \dfrac{v(h) - v(0)}{h} = \lim\limits_{h \to 0+} \dfrac{\sin h - h}{h} = 0$

$\lim\limits_{h \to 0-} \dfrac{v(h) - v(0)}{h} = \lim\limits_{h \to 0-} \dfrac{-\sin h + h}{h} = 0$

v is a differentiable function.

10. Same as Q.19 in AIEEE/JEE Question.

11. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x > 2 \\ -\dfrac{1}{x} & \text{if} \quad x < -2 \\ a + bx^2 & \text{if} \quad -2 \leq x \leq 2 \end{cases}$

$\lim\limits_{x \to -2-} f(x) = \lim\limits_{x \to -2-} -\dfrac{1}{x} = \dfrac{1}{2}$

$\lim\limits_{x \to -2+} f(x) = \lim\limits_{x \to -2+} (a + bx^2) = a + 4b$

Since f is differentiable so continuous, hence $a + 4b = \dfrac{1}{2}$

$\lim\limits_{h \to 0+} \dfrac{f(-2+h) - f(-2)}{h}$

$= \lim\limits_{h \to 0+} \dfrac{a + b(-2+h)^2 - (a + 4b)}{h}$

$= \lim\limits_{h \to 0+} \dfrac{bh^2 - 4bh}{h} = -4b$

$\lim\limits_{h \to 0-} \dfrac{f(-2+h) - f(-2)}{h} = \lim\limits_{h \to 0-} \dfrac{-\dfrac{1}{-2+h} - (a + 4b)}{h}$

$= \lim\limits_{h \to 0-} \dfrac{-\dfrac{1}{-2+h} - \dfrac{1}{2}}{h} = -\lim\limits_{h \to 0-} \dfrac{2 - 2 + h}{2h(-2+h)}$

$= -\lim\limits_{h \to 0-} \dfrac{1}{2(-2+h)}$

$= \frac{1}{4}.$

So $b = \frac{-1}{16}$ and $a = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}.$

12. For $0 < x < 1$

$(1 - x)\ y(x) = (1 - x)\ (1 + x)\ (1 + x^2)\ \ldots\ (1 + x^{32})$

$= (1 - x^2)\ (1 + x^2)\ \ldots\ (1 + x^{32})$

$= (1 - x^4)\ (1 + x^4)\ \ldots\ (1 + x^{32})$

$= \ldots = 1 - x^{64}$

$\Rightarrow y(x) = \frac{1-x^{64}}{1-x}$

$$y'(x) = \frac{(1-x)(-64x^{64})-(-1)(1-x^{64})}{(1-x)^2}$$

$$\Rightarrow y'\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)(-64)\left(\frac{1}{2}\right)^{63}+1-\left(\frac{1}{2}\right)^{64}}{\left(1-\frac{1}{2}\right)^2}$$

$$= 4\left[1-65\left(\frac{1}{2}\right)^{64}\right] = 4-65\left(\frac{1}{2}\right)^{62}$$

13. Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$y(x) = \begin{vmatrix} \sin x & \cos x - \sin x & \cos x + 1 \\ 23 & -6 & -10 \\ 1 & 0 & 0 \end{vmatrix}$$

$= -10(\cos x - \sin x) + 6 \cos x + 6$

$= 10 \sin x - 4 \cos x + 6$

$\frac{dy}{dx} = 10 \cos x + 4 \sin x$

$\frac{d^2 y}{dx^2} = -10 \sin x + 4 \cos x = -y + 6$

$\Rightarrow \frac{d^2 y}{dx^2} + y = 6.$