UNIT 2

Conditional Identities

Conditional identities are the equalities of two expressions subject to certain given conditions.

ILLUSTRATIVE EXAMPLES

Example 1. If $a + b + c = 0$, then the value of

$$\frac{a^2+b^2+c^2}{bc+ca+ab} \text{ is ....}$$

Sol: $(a+b+c)^2 = 0$

$\Rightarrow a^2+b^2+c^2+2(ab+bc+ca) = 0$

$a^2+b^2+c^2 = -2(ab+bc+ca)$

$$\Rightarrow \quad \frac{a^2+b^2+c^2}{bc+ca+ab} = -2$$

Ans: -2

Example 2. If $a + b + c = 0$, what is the value of

$$\frac{8abc}{a^3+b^3+c^3}?$$

Sol: If $a+b+c=0$, the $a+b=-c$

or $(a+b)^3 = (-c)^3$

$\Rightarrow a^3+b^3+3ab(a+b) = -c^3$

$\Rightarrow a^3+b^3+3ab(-c)+c^3 = 0$

or $a^3+b^3+c^3 = 3abc$

$$\therefore \quad \frac{8abc}{a^3+b^3+c^3} = \frac{8abc}{3abc} = \frac{8}{3} = 2\frac{2}{3}$$

Ans: $2\frac{2}{3}$

[Important: If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$]

Example 3. What is the value of the expression

$$\frac{(x-y)^3+(y-z)^3+(z-x)^3}{(x-y)(y-z)(z-x)}?$$

Sol: If $x-y=a$, $y-z=b$, $z-x=c$, then $a+b+c=0$ and thus $a^3+b^3+c^3=3abc$

or $\frac{a^3+b^3+c^3}{abc} = 3$. Hence the value of the given expression is 3.

Ans: 3

Example 4. If $a + b + c = 0$ what is the value of

$$\frac{a^2+b^2+ab}{b^2+c^2+bc} + \frac{c^2+ca+a^2}{b^2+c^2+bc}?$$

Sol: $a+b+c=0$

$$a+b=-c \text{ or } c=-(a+b)$$

$$b^2+c^2+bc = b^2+(a+b)^2-b(a+b)$$

$$= b^2+a^2+2ab+b^2-ab-b^2$$

$$= b^2+a^2+ab$$

$\therefore \quad \dfrac{b^2+a^2+ab}{b^2+c^2+bc}=1$; Also $\dfrac{c^2+a^2+ca}{b^2+c^2+bc}=1$

$\therefore$ Given expression is equal to 2.

Alternative Solution:

Since $a+b+c=0 \Rightarrow a=1, b=-1, c=0$

put these value in given expression to get equal to 2.

Ans: 2

Example 5. If $x=b+c, y=c+a, z=a+b,$ then find the value of $\dfrac{x^2+y^2+z^2-yz-zx-xy}{a^2+b^2+c^2-bc-ca-ab}$

Sol: $x-y=(b+c)-(c+a)=b-a$

Similarly, $y-z=c-b$ and $z-x=a-c$

Now $x^2+y^2+z^2-yz-zx-xy$

$$=\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]$$

$$=\frac{1}{2}[(b-a)^2+(c-b)^2+(a-c)^2]$$

$$=\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$=a^2+b^2+c^2-bc-ca-ab$$

$\therefore$ the value of the given expression of equal to 1.

Ans: 1

Example 6. If $a+b+c=0$, find the value of

$$\frac{a^4+b^4+c^4}{a^2b^2+c^2(a^2+b^2)}$$

Sol: $a+b+c=0$ means $a+b=-c$

or $a^2+b^2+2ab=c^2$

or $a^2+b^2-c^2=2ab$

squaring $a^4+b^4-c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $\dfrac{a^4+b^4+c^4}{a^2b^2+c^2(a^2+b^2)}=2$

Ans: 2

Example 7. If $x+y+z=0$, then what is the value of

$$\frac{1}{x^2+y^2-z^2}+\frac{1}{y^2+z^2-x^2}+\frac{1}{z^2+x^2-y^2}?$$

Sol: $x+y=-z$ or $x^2+y^2+2yx=z^2$

$\therefore \quad x^2+y^2-z^2=-2yz$

Similarly, $y^2+z^2-z^2=-2yz$ and $z^2+x^2-y^2=-2zx$

Thus the given expression is equal to

$$=-\frac{1}{2}\left[\frac{1}{yx}+\frac{1}{yz}+\frac{1}{xy}\right]$$

$$=-\frac{1}{2}\left(\frac{x+y+z}{xyz}\right)=-\frac{1}{2}\times\frac{0}{xyz}=0$$

Ans: 0

Example 9. If $a=x(y-z)$, $b=y(z-x)$ and $c=z(x-y)$

What is the value of $\dfrac{xyz}{abc}\left(\dfrac{a^3}{x^3}+\dfrac{b^3}{y^3}+\dfrac{c^3}{z^3}\right)$?

Sol: $\dfrac{a}{x}=-y-z, \dfrac{b}{y}=z-x$ and $\dfrac{c}{z}=x-y.$

Now $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=y-z+z-x+x-y$

$$=0$$

$\therefore \quad \dfrac{a^3}{x^3}+\dfrac{b^3}{y^3}+\dfrac{c^3}{z^3}=3\dfrac{a}{x}\cdot\dfrac{b}{y}\cdot\dfrac{c}{z}=\dfrac{3abc}{xyz}$

$\therefore$ The given expression is equal to

$$\frac{xyz}{abc}\times\frac{3\cdot abc}{xyz}=3$$

Ans: 3

Example 10. It $\dfrac{a}{b+c}=x, \dfrac{b}{c+a}=y, \dfrac{c}{a+b}=z$, what is the value of $\dfrac{1}{1+x}+\dfrac{1}{1+y}+\dfrac{1}{1+z}$?

Sol: $1+x=1+\dfrac{a}{b+c}=\dfrac{a+b+c}{b+c}$ so $\dfrac{1}{1+x}=\dfrac{b+c}{a+b+c}$

etc.

$\therefore$ Given expression $= \frac{b+c}{a+b+c} + \frac{c+a}{a+b+c}$

$+ \frac{a+b}{a+b+c}$

$= \frac{2(a+b+c)}{(a+b+c)} = 2.$

Ans: 2

Example 11. If $x + y = 2z$, what is the value of $\frac{x}{x-z} + \frac{z}{y-z}$?

Sol: $x + y = 2z$ measn $x - z = y$. ...(i)

$\therefore \frac{x}{x-z} + \frac{z}{y-z} = \frac{x}{z-y} + \frac{z}{y-z}$

$\Rightarrow \frac{-x+z}{y-z} = \frac{z-x}{z-x}$ from (i)

$= 1$

Ans: 1

Example 12. If $x = a^2 - bc, y = b^2 - ca, z = c^2 - ab$ then what it the value of $\frac{ax+by+cz}{(a+b+c)(x+y+z)}$

Sol: $ax + by + cz = a(a^2 - bc) + b(b^2 - ca) + c(c^2 - ab)$

$= a^3 + b^3 + c^3 - 3abc.$

Also $(a+b+c)(x+y+z) = (a+b+c)(a^2 - bc + b^2 - ca + c^2 - ab)$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc$

$\therefore$ the value of the expression = 1

Ans: 1

MULTIPLE CHOICE QUESTIONS

Tick (✓) the correct choice amongst the following:

1. If $a+b+c=0$, then $(a^3+b^3+c^3) \div abc$ is equal to

(a) 1 (b) 2
(c) 3 (d) 9

2. If $a + b + c = 0$, then $\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}$ is equal to

(a) 4 (b) 2
(c) 1 (d) 16

3. If $a + b + c = 0$, then $\frac{1}{b^2+c^2-a^2} + \frac{1}{c^2+a^2-b^2} + \frac{1}{a^2+b^2-c^2}$ is equal to

(a) 3 (b) 6
(c) 1 (d) 0

4. If $x+y+z=0$, then $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$ is equal to

(a) 3 (b) 27
(c) 1 (d) -3

5. If $a+b+2c=0$, then the value of $a^3+b^3+8c^3$ is equal to

(a) $3abc$ (b) $4abc$
(c) abc (d) $6abc$

6. The value of $\frac{(x-y)^3+(y-z)^3+(z-x)^3}{(x-y)(y-z)(z-x)}$ is equal to

(a) 1 (b) 2
(c) 3 (d) -1

7. If $x = a(b-c)$; $y = b(c-a)$; $z = c(a-b)$, then $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + \left(\frac{z}{c}\right)^3$ is equal to

(a) $\frac{xyz}{abc}$ (b) $\frac{1}{3}\frac{xyz}{abc}$

(c) $3\frac{xyz}{abc}$ (d) $\frac{3(x+y+z)}{(a\,b\,c)}$

8. If $a+b+c=0$ then $\dfrac{a^2+b^2+c^2}{ab+bc+ca}$ is equal to

(a) 2 (b) 1
(c) –1 (d) 3

9. If $a+b+c=0$ then the value of $\dfrac{(a+b)(b+c)(c+a)}{abc}$ is

(a) 1 (b) –1
(c) –3 (d) 3

10. If $a+b+c=0$ then the value of $\dfrac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{abc}$ is

(a) 3 (b) –3
(c) –1/3 (d) 1/3

11. If $a+b+c=0$ then the value of $(a+b+c)^3-(a^3-b^3-c^3$ is equal to

(a) $2abc$ (b) abc
(c) $-3abc$ (d) $-abc$

12. If $ab+bc+ca=0$, then the value of $\dfrac{1}{a^2-bc}+\dfrac{1}{b^2-ca}+\dfrac{1}{c^2-ab}$ is equal to

(a) 3 (b) $3abc$
(c) abc (d) 0

13. If $\dfrac{x}{y+z}=a;\ \dfrac{y}{z+x}=b;\ \dfrac{z}{x+y}=c;$ then $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}$ is equal to

(a) $a+b+c$ (b) 3
(c) 2 (d) 1

14. If $a+b=2c$; then $\dfrac{a}{a-c}+\dfrac{c}{b-c}$ is equal to

(a) 0 (b) –1
(c) 1 (d) 2

15. $\dfrac{a^2-b^2-2bc-c^2}{a^2+b^2+2ab-c^2}$ is equal to

(a) $\dfrac{a+b+c}{a-b+c}$ (b) $\dfrac{a-b-c}{a+b-c}$

(c) $\dfrac{a-b-c}{a-b+c}$ (d) $\dfrac{a-b+c}{a+b+c}$

16. If $a=\dfrac{x+y}{z}$, $b=\dfrac{y+z}{x}$ and $c=\dfrac{z+x}{y}$, then the value of $\dfrac{1}{a+1}\div\dfrac{1}{b+1}+\dfrac{1}{c+1}$ is equal to

(a) 3 (b) 2
(c) 1 (d) –3

17. If $a^2=b+c;\ b^2=c+a;\ c^2=a+b$, then the value of $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}$ is equal to

(a) 4 (b) 3
(c) 2 (d) 1

18. If $x=a+b;\ y=b+c;\ z=c+a$, then the value of $\dfrac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$ is equal to

(a) 3 (b) 2
(c) 1 (d) 1/3

19. If $x=a^2-bc,\ y=b^2-ca,\ z=c^2+ab$, then the value of $\dfrac{(a+b+c)(x+y+z)}{ax+by+cz}$ is equal to

(a) 3 (b) 2
(c) 1 (d) 0

20. If $a+b+c)^2=3(ab+bc+ca)$, then which one of the following is true?

(a) $a\neq b\neq c$ (b) $a>b>c$
(c) $a<b<c$ (d) $a=b=c$

ANSWERS

1. (c)	2. (b)	3. (d)	4. (a)	5. (d)
6. (c)	7. (c)	8. (d)	9. (b)	10. (b)
11. (c)	12. (d)	13. (c)	14. (c)	15. (b)
16. (c)	17. (d)	18. (b)	19. (c)	20. (d)

SOLUTIONS

[For Some Selected Problems]

1. $a+b+c=0$

$\Rightarrow$ $a+b=-c$, cub both sides

$(a+b)^3=(-c)^3$

$\Rightarrow$ $a^3+b^3+c^3+3abc\,(-c)=0$

$\Rightarrow$ $a^3+b^3+c^3=3abc.$

2. $a+b=-c$ or $a^2+b^2+2ab=c^2$

$\therefore$ $a^2+b^2-c^2=-2ab$, squaring again both sides

$a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2=-4a^2b^2$

$\Rightarrow$ $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

$$\Rightarrow \frac{a^4+b^4+c^4}{(a^2b^2+b^2c^2+c^2a^2)}=2.$$

3. $a+b+c=0$

$\Rightarrow$ $b+c-a$; squaring both sides

$\therefore$ $b^2+c^2+2bc=a^2$

or $b^2+c^2-a^2=-2bc$

Similarly $c^2+a^2-b^2=-2ca$

and $a^2+b^2-c^2=-2ab$

$$\therefore \text{ Givein expression } = -\frac{1}{2}\left[\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right]$$

$$=-\frac{1}{2}\left(\frac{a+b+c}{abc}\right)$$

$$=-\frac{1}{2abc}\times 0=0$$

$[\because (a+b+c)=0.]$

4. $$\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=\frac{x^3+y^3+z^3}{xyz}$$

$$\Rightarrow \frac{3xyz}{xyz}=3 \quad (\because x^3+y^3+z^3=3xyz)$$

6. $a=(x-y)$; $b=(y-z)$ and $c=(z-x)$

$$\text{G.E}=\frac{a^3+b^3+c^3}{abc}=\frac{3abc}{abc}=3.$$

7. $$\frac{x}{a}=b-c\,;\ \frac{y}{b}=c-a;\ \frac{z}{c}=a-b.$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$

$$\therefore \left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3=3\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\left(\frac{y}{b}\right)$$

8. $a+b+c=0$; squaring both side

$a^2+b^2+c^2+2(ab+bc+ca)=0$

$$\Rightarrow \frac{a^2+b^2+c^2}{(ab+bc+ca)}=-2$$

9. $a+b=-c$ ect.

$$\therefore \text{ G. E}=\frac{(-c)(-a)(-b)}{abc}=-1.$$

10. $$\text{G.E}=\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{abc}$$

$$=\frac{a^2(-a)+b^2(-b)+c^2(-c)}{abc}$$

$$=\frac{(a^3+b^3+c^3)}{abc}=-\frac{3abc}{abc}=-3$$

11. $(a+b+c)^3-(a^3+b^3+c^3)$

$=-(a^3+b^3+c^3)$

$=-(3abc)=-3abc.$

12. $a^2-bc=a^2+ab+ac=a(a+b+c)$ etc.

$$\therefore \text{ G.E}=\frac{1}{a(a+b+c)}+\left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]$$

$=0 \because (ab+bc+ca)=0$

13. $\dfrac{x}{y+z} = a \quad \Rightarrow 1 + \dfrac{x}{y+z} = a+1$

$\therefore \dfrac{1}{1+a} = \dfrac{y+z}{x+y+z}$ etc.

$\therefore$ GE $= \dfrac{1}{x+y+z}[y+z+x+x+y]$

$= \dfrac{2(x+y+z)}{(x+y+z)}$

$= 2.$

14. $a+b = 2c \quad \Rightarrow \quad a-c = a-b.$

$\dfrac{a}{a-c} + \dfrac{c}{b-c} = \dfrac{a}{c-b} + \dfrac{c}{b-c}$

$= \dfrac{-a+c}{(b-c)} = \dfrac{c-a}{b-c} = \dfrac{b-c}{b-c} = 1$

15. GE $= \dfrac{a^2-(b+c)^2}{(a+b)^2-c^2}$

$= \dfrac{(a+b+c)(a-b-c)}{(a+b+c)(a+b-c)}$

$= \dfrac{a-b-c}{a+b-c}$

16. $a = \dfrac{x+y}{z} \quad \Rightarrow \quad a+1 = \dfrac{x+y+z}{z}$

$\therefore \dfrac{1}{a+1} = \dfrac{z}{x+y+z}$ etc.

$\therefore$ G.E $= 6\sqrt{3} = 1.$

17. $\dfrac{1}{a+1} = \dfrac{a}{a^2+a} = \dfrac{a}{b+c+a}$ etc.

$\therefore$ G.E $= \dfrac{a+b+c}{a+b+c} = 1.$

18. $x^3+y^3+z^3-3xyz$

$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$= (x+y+z)\dfrac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]$ ...(i)

$x+y+z = 2(a+b+c)$ and

$x-y = a-c;\ y-z = b-a$ and $z-x = c-b$

$\therefore$ (i)can be writtn as $2(a+b+c).\dfrac{1}{2}[(a-c)^2$

$+(b-a)^2+(c-b)^2]$

$= 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= 2(a^3+b^3+c^3-3abc)$

$\therefore$ GE $\Rightarrow \dfrac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc} = 2.$

19. $(a+b+c)(x+y+z)$

$= (a+b+c)[a^2+b^2+c^2-bc-ca-ab]$

$= a^3+b^3+c^3-3abc$ ...(i)

Also $ax+by+cz = a(a^2-bc)+b(b^2-ca)$

$+c(c^2-ab)$

$= a^3+b^3+c^3-3abc)$ ....(ii)

$\therefore$ GE $= 1.$

20. $(a+b+c)^2 = 3(ab+bc+ca)$

$\Rightarrow a^2+b^2+c^2+2ab+2bc+2ca$

$-3ab-3bc-3ca = 0$

$\Rightarrow a^2+b^2+c^2-ab-bc-ca = 0$

$\Rightarrow \dfrac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2] = 0$

Which is true only if $a = b = c$