Differential Equations

- An equation is called a differential equation, if it involves variables as well as derivatives of dependent variable with respect to independent variable.

  For example:

  $$x\frac{d^4y}{dx^4} + y\left(\frac{d^2y}{dx^2}\right)^3 - 2x^2y\frac{dy}{dx} + 3 = 0$$ is a differential equation.

  Sometimes, we may write $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ etc. as $y' y'', y''', y''''$ etc. respectively. Also, note that we cannot say that $\tan(y') + x = 0$ is a differential equation.

- Order of a differential equation is defined as the order of the highest order derivative of dependent variable with respect to independent variable involved in the given differential equation.

  For example: The highest order derivative present in the differential equation $x^3y^5y'''' - 3x^2y'' + xyy' - 5 = 0$ is y''''. Therefore, the order of this differential equation is 4.

- Degree of a differential equation is the highest power of the highest order derivative in it.

  For example: The degree of the differential equation $(y'')^2 - 2x(y'')^5 - xy(y'')^2 + y' = 0$ is 2, since the highest power of the highest order derivative, y'', is 2.

  Note: The degree of the differential equation $\frac{d^2y}{dx^2} + \cos\left(\frac{dy}{dx}\right) = 0$ is not defined since it is not a polynomial equation in $\frac{dy}{dx}$. However, its order is 2.

- If a differential equation is defined, then its order and degree are always positive integers.

- A function that satisfies the given differential equation is called a solution of a given differential equation.

Example: Verify whether $y = \sin x + \cos x - 5$ is a solution of the differential equation $y'' + y' = 0$ or not.

Solution:
We have, $y = \sin x + \cos x - 5$
$\because y' = \cos x - \sin x$
$y' = -\sin x - \cos x = -(\sin x + \cos x)$
$y'' = -(\cos x - \sin x) = -y'$
$\Rightarrow y''' + y' = 0$
Therefore, $y = \sin x + \cos x - 5$ is a solution of the differential equation $y'' + y' = 0$

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- To form a differential equation from a given function, we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

Example: Form the differential equation, representing the family of circle $(x - a)^2 + (y - b)^2 = r^2$, where a and b are arbitrary constants.

Solution:
We have $(x - a)^2 + (y - b)^2 = r^2$ (1)
Differentiating with respect to x, we obtain
$2(x - a) + 2(y - b)\frac{dy}{dx} = 0$ ...(2)

Again differentiating with respect to *x*, we obtain

$$2+2\left[\left(\frac{dy}{dx}\right)^2+(y-b)\frac{d^2y}{dx^2}\right]=0$$

$$\Rightarrow (y-b)\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow y-b=-\left[\frac{1+(y')^2}{y''}\right]$$

Substituting $y-b=-\frac{[1+(y')^2]}{y''}$ in equation (2), we obtain

$$2(x-a)-2\frac{[1+(y')^2]}{y''}\times y'=0$$

$$\Rightarrow x-a=\frac{y'+(y')^3}{y''}$$

Substituting the values of $(x-a)$ and $(y-b)$ in equation (1), we obtain

$$\left[\frac{y'+(y')^3}{y''}\right]^2+\left[-\frac{1+(y')^2}{y''}\right]^2=r^2$$

$$\Rightarrow (y')^2\left[1+(y')^2\right]^2+\left[1+(y')^2\right]^2=r^2(y'')^2$$

$$\Rightarrow \left[1+(y')^2\right]^2\left[1+(y')^2\right]=r^2(y'')^2$$

$$\Rightarrow \left[1+(y')^2\right]^3-r^2(y'')^2=0$$

This is the required differential equation of the given circle.

- The three methods of solving first order, first degree differential equations are given as follows:
  - **Variable separable method:** This method is used to solve such an equation in which variables can be separated completely i.e., terms containing *y* should remain with *dy* and terms containing *x* should remain with *dx*.

Example: Solve the differential equation: $x(1+y^2)dx+y(4+x^2)dy=0$

Solution:

$$x(1+y^2)dx + y(4+x^2)dy = 0$$

$$\Rightarrow \frac{x}{4+x^2}dx + \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{1}{2}\cdot\left(\frac{2x}{4+x^2}\right)dx + \frac{1}{2}\cdot\left(\frac{2y}{1+y^2}\right)dy = 0$$

$$\Rightarrow \int\frac{2x}{4+x^2}dx = -\int\frac{2y}{1+y^2}dy$$

$$\Rightarrow \log(4+x^2) = -\log(1+y^2) + \log C$$

$$\Rightarrow \log(4+x^2)(1+y^2) = \log C$$

$$\Rightarrow (4+x^2)(1+y^2) = C$$

This is the required solution of the given differential equation.

- **Homogeneous differential equation:**

A differential equation which can be expressed as $\frac{dy}{dx} = f(x,y)$ or $\frac{dx}{dy} = g(x,y)$ **,** where $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogenous differential equation. To solve such an equation, we have to substitute $y = vx$ in the given differential equation and then solve it by variable separable method.

Example: Solve the differential equation: $xydy - (x^2 - 3y^2)dx = 0$

Solution:

$xydy - (x^2 - 3y^2)dx = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 3y^2}{xy} = F(x, y)$, … (1)

Now,

$$F(\lambda x, \lambda y) = \frac{\lambda^2 x^2 - 3\lambda^2 y^2}{\lambda^2 xy} = \frac{x^2 - 3y^2}{xy}$$

$$= \lambda^{\circ} f(x, y)$$

F is homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{x^2 - 3v^2x^2}{vx^2} = \frac{1-3v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v} - v = \frac{1-4v^2}{v}$$

$$\Rightarrow \int \frac{v}{1-4v^2} dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{8}\int \frac{-8v}{1-4v^2} dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{8}\log(1-4v^2) = \log(x) - \log C_1$$

$$\Rightarrow \log\left[x(1-4v^2)^{\frac{1}{8}}\right] = \log C_1$$

$$\Rightarrow x(1-4v^2)^{\frac{1}{8}} = C_1$$

$$\Rightarrow x^8(1-4v^2) = C_1^8 = C(\text{say})$$

$$\Rightarrow x^8\left(1 - 4\times\frac{y^2}{x^2}\right) = C$$

$$\Rightarrow x^6(x^2 - 4y^2) = C$$

This is the required solution of the given differential equation.

- **Linear differential equation:**
  - A differential equation which can be expressed in the form of $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only, is called a first order linear differential equation.

In this case, we find integrating factor (I.F.) by using the formula:

I.F. $= e^{\int P\,dx}$
Then, the solution of the differential equation is given by,
$y(\text{I.F.}) = \int (Q \times \text{I.F.})\,dx + C$

- A linear differential equation can also be of the form $\frac{dx}{dy} + P_1 x = Q$, where P_1 and Q_1 are constants or functions of *y* only.

In this case, I.F. $= e^{\int P_1 dy}$
And the solution of the differential equation is given by,
$x(\text{I.F.}) = \int (Q_1 \times \text{I.F.})\,dy + C$

Example: Find the solution of the differential equation $\sin y\,dx = \cos y(\sin y - x)dy$, satisfying the condition that *x* = 5 when $y = \frac{\pi}{2}$.

Solution:
We have,
$\sin y\,dx = \cos y(\sin y - x)dy$

$\Rightarrow \frac{dx}{dy} = \frac{\cos y(\sin y - x)}{\sin y} = \cos y - x\cot y$

$\Rightarrow \frac{dx}{dy} + x\cot y = \cos y$

This is a linear differential equation of the form $\frac{dx}{dy} + P_1 x = Q$ where $P_1 = \cot y$ and $Q_1 = \cos y$

Now, I.F = $= e^{\int P_1 dy} = e^{\int \cot y\,dy} = e^{\int \log \sin y\,dy} = \sin y$

Therefore, the general solution of the given differential equation is

$$x \times \sin y = \int \cos y \times \sin y \, dy + C$$

$$\Rightarrow x \sin y = \frac{1}{2} \int \sin 2y \, dy + C$$

$$\Rightarrow x \sin y = -\frac{1}{4} \cos 2y + C$$

Substituting $y = \frac{\pi}{2}$ and $x = 5$ in this equation, we obtain

$$5 \sin\left(\frac{\pi}{2}\right) = -\frac{1}{4} \cos\left(2 \times \frac{\pi}{2}\right) + C$$

$$5 \sin\left(\frac{\pi}{2}\right) = -\frac{1}{4} \cos \pi + C$$

$$5 \times 1 = -\frac{1}{4} \times (-1) + C$$

$$\Rightarrow C = \frac{19}{4}$$

Therefore, the required solution is

$$x \sin y = -\frac{1}{4} \cos 2y + \frac{19}{4}$$

$$\Rightarrow x \sin y + \frac{1}{4} \cos 2y = \frac{19}{4}$$

$$\Rightarrow 4x \sin y + \cos 2y = 19$$