Mind Map-7

INDEFINITE INTEGRAL

1. Standard Integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{1}{x} dx = \log|x| + C$

(iii) $\int e^x dx = e^x + C$

(iv) $\int a^x dx = \frac{a^x}{\log a} + C$

(v) $\int \sin x\, dx = -\cos x + C$

(vi) $\int \cos x\, dx = \sin x + C$

(vii) $\int \sec^2 x\, dx = \tan x + C$

(viii) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(ix) $\int \sec x \tan x\, dx = \sec x + C$

(x) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

(xi) $\int \cot x\, dx = \log|\sin x| + C$

(xii) $\int \tan x\, dx = \log|\sec x| + C$

(xiii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(xiv) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C$

(xv) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xviii) $\int -\frac{1}{a^2 + x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(xix) $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xx) $\int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$

2. Methods of Integration

When integration cannot be reduced into some standard form, then integration is performed using following methods :

(i) Integration by Substitution
(ii) Integration using Partial Fractions
(iii) Integration by Parts

3. Integration by Substitution

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method by which we change the variable of integration to some other variable is known as the method of substitution.

Consider $I = \int f(x)dx$

Put $x = g(t)$, so $\frac{dx}{dt} = g'(t)$

i.e., $dx = g'(t)\, dt$

Thus, $I = \int f(x)dx = \int f(g(t))g'(t)dt$

Some Important Substitutions are:

Function	Substitutions
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

4. Integration Using Partial Fractions

Consider a rational function of the form $\frac{P(x)}{Q(x)}$

where $P(x)$ & $Q(x)$ are polynomials in x & $Q(x) \neq 0$. If degree of $P(x)$ is greater than the degree of $Q(x)$, then we may divide $P(x)$ by $Q(x)$ such that

$$\frac{P(x)}{Q(x)} = T(x) + \frac{R(x)}{Q(x)}$$

where, $T(x)$ is a polynomial in x & degree of $R(x)$ is less than the degree of $Q(x)$.
$T(x)$ being a polynomial can be easily integrated.

$\frac{R(x)}{Q(x)}$ can be integrated by expressing $\frac{R(x)}{Q(x)}$ as the sum of partial fractions of the following types:

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$

(ii) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ cannot be factorised further.

5. Integration by Parts

$$\int u.v dx = u\int v dx - \int\left[\frac{du}{dx}\cdot\int v dx\right]dx$$

Here, u is the first function & v is the second function.
Selection of first function : For applying integration by parts, we choose the first function as the function which comes first in the word **ILATE**, where

I stands for the inverse trigonometric function ($\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ etc)
L stands for the logarithmic function
A stands for the algebraic functions
T stands for the trigonometric functions
E stands for the exponential functions

6. Integrals of Some Special Functions

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

7. Two Standard Forms of an Integral

(i) $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

(ii) $\int [xf'(x) + f(x)]dx = xf(x) + C$

8. Some Special Types of Integrals

(i) $\sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

(ii) $\sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

(iii) $\int\sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(iv) Integrals of the types $\int\frac{dx}{ax^2+bx+c}$ or $\int\frac{dx}{\sqrt{ax^2+bx+c}}$

can be transformed into standard form by expressing

$$ax^2 + bx + c = a\left[x^2 + \frac{bx}{a} + \frac{c}{a}\right]$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

(v) Integrals of the types $\int\frac{px+q}{ax^2+bx+c}dx$

or $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

can be transformed into standard form by expressing $px + q = A\frac{d}{dx}(ax^2+bx+c) + B$

$= A(2ax + b) + B$ where A & B can be determined by comparing coefficients on both sides.

DEFINITE INTEGRAL

1. Definite Integral

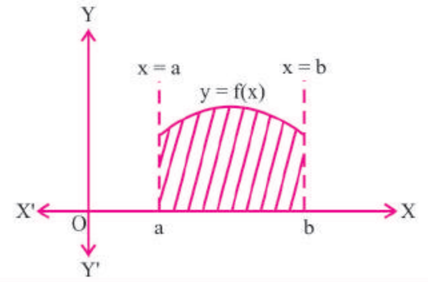
The definite integral of f(x) between the limits a to b i.e., in the interval [a, b] is denoted by

$\int_a^b f(x)dx$ and is defined as follows:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where, $\int f(x)dx = F(x)$

The definite integral $\int_a^b f(x)dx$ is also defined as the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis



2. Definite Integral as the Limit of a Sum

$$\int_a^b f(x)dx = \lim_{h\to 0}[f(a) + f(a+h) + ---- + f(a+(n-1)h)]$$

or

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + ---- + f(a+(n-1)h]$$

where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

The above expression is known as the definite integral as the limit of a sum.

3. Fundamental Theorem of Calculus

Theorem 1 : Let f be a continuous function on the closed interval [a, b] and let A(x) be area function. Then $A'(x) = f(x), \forall x \in [a, b]$

Theorem 2 : Let f be a continuous function defined on the closed interval [a, b] & F be the anti-derivative of f.

Then $\int_a^b f(x)dx = [F(x + c)]_a^b = F(b) - F(a)$

This is called the definite integral of f over the range [a, b], where a & b are called the limits of integration, a being the lower limit & b the upper limit.

4. Evaluations of Definite Integrals by Substitution

Consider a definite integral of the following form

$$\int_a^b f(g(x))g'(x)dx$$

To evaluate this integral we proceed as following

Step 1 : Substitute

Step 2 : Find the limits of integration in new system of variable, i.e. the lower limit is g(a) and the upper limit is g(b), and the integral is now $\int_{g(a)}^{g(b)} f(t)dt$

Step 3 : Evaluate the integral so obtained by usual method.

5. Properties of Definite Integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$, in particular $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$

(iv) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

(vi) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if f(x) is an even function i.e., $f(-x) = f(x)$

$\int_{-a}^a f(x)dx = 0$, if f(x) is an odd function i.e., $f(-x) = -f(x)$

(vii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$

(viii) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a - x) = f(x)$

$= 0$, if $f(2a - x) = -f(x)$