KEY CONCEPTS (DIF-FERENTIABILITY)

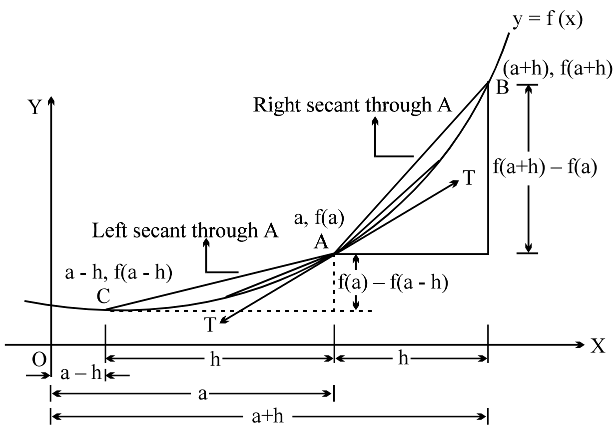
THINGS TO REMEMBER :

1. Right hand & Left hand Derivatives ; By definition

$: f'(a) = \underset{h\to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$ if it exist

(i) The right hand derivative of f' at $x = a$ denoted by $f'(a^+)$ is defined by : $f'(a^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$,

provided the limit exists & is finite.



(ii) The left hand derivative : of f at x = a denoted by $f'(a^+)$ is defined by :

$f'(a^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$,

Provided the limit exists & is finite.

We also write $f'(a^+) = f'_+(a)$ & $f'(a^-) = f'_-(a)$.

* This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.

(iii) Derivability & Continuity :

(a) If $f'(a)$ exists then f(x) is derivable at x= a $\Rightarrow$ f(x) is continuous at x = a.

(b) If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \underset{h\to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$ exists.

Also

$$f(x+h)-f(x) = \frac{f(x+h)-f(x)}{h}.h \; [h \neq 0]$$

Therefore :

$$\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)] = \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}.h = f'(x).0 = 0$$

Therefore $\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)] = 0 \Rightarrow \underset{h\to 0}{\text{Limit}}$ f(x+h) = f(x) $\Rightarrow$ f is continuous at x.

Note : If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true :

" IF f IS CONTINUOUS AT x , THEN f IS DERIVABLE AT x " IS NOT TRUE.

e.g. the functions $f(x) = |x|$ & $g(x) = x \sin\frac{1}{x}$; $x \neq 0$ & $g(0) = 0$ are continuous at x = 0 but not derivable at x = 0.

NOTE CAREFULLY :

(a) Let $f'_+(a) = p$ & $f'_-(a) = q$ where p & q are finite then :

(i) p = q $\Rightarrow$ f is derivable at x = a $\Rightarrow$ f is continuous at x = a.

(ii) p $\neq$ q $\Rightarrow$ f is not derivable at x = a.

It is very important to note that f may be still continuous at x = a.

In short, for a function f :

Differentiability $\Rightarrow$ Continuity ;

Continuity $\not\Rightarrow$ derivability ;

Non derivibality $\not\Rightarrow$ discontinuous ;

But discontinuity $\Rightarrow$ Non derivability

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

3. **DERIVABILITY OVER AN INTERVAL :**

f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &

(ii) for any point c such that a < c < b, $f'(c+)$ & $f'(c-)$ exist & are equal.

NOTE :

1. If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) − g(x) , f(x).g(x) will also be derivable at x = a & if g (a) $\neq$ 0 then the function f(x)/g(x) will also be derivable at x = a.

2. If f(x) is differentiable at x = a & g(x) is not differentiable at x = a, then the product function F(x) = f(x)· g(x) can still be differentiable at x = a e.g. f(x) = x & g(x) = $|x|$.

3. If f(x) & g(x) both are not differentiable at x = a then the product function ;

F(x) = f(x)· g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$.

4. If f(x) & g(x) both are non-deri. at x = a then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

5. If f(x) is derivable at x = a $\not\Rightarrow$ f′(x) is continuous at x = a.

e.g. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

6. **A surprising result :** Suppose that the function f (x) and g (x) defined in the interval (x_1, x_2) containing the point x_0, and if f is differentiable at x = x_0 with f $(x_0) = 0$ together with g is continuous as x = x_0 then the function F (x) = f (x) · g (x) is differentiable at x = x_0

e.g. F (x) = sinx · $x^{2/3}$ is differentiable at x = 0.

1. If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then indicate the correct alternative(s)
(A) f(x) is continuous but not differentiable at x = 0
(B) f(x) is differentiable at x = 0
(C) f(x) is not differentiable at x = 0
(D) None of these

2. If $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then f(x) is
(A) continuous as well differentiable at x = 0
(B) continuous but not differentiable at x = 0
(C) neither differentiable at x = 0 not continuous at x = 0
(D) None of these

3. The function f(x) is defined as follows
$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ x^3 - x + 1 & \text{if } x > 1 \end{cases}$$ then f(x) is
(A) derivable & cont. at x = 0
(B) derivable at x = 1 but not cont. at x = 1
(C) neither derivable nor cont. at x = 1
(D) not derivable at x = 0 but cont. at x = 1

4. If $f(x) = \begin{cases} \sqrt{x}\left(1+\sin\dfrac{1}{x}\right) & , x > 0 \\ -\sqrt{-x}\left(1+\sin\dfrac{1}{x}\right) & , x < 0 \\ 0 & , x = 0 \end{cases}$, then f(x) is
(A) continuous as well diff. at x = 0
(B) continuous at x = 0, but not diff. at = 0
(C) neither continuous at x = 0 nor diff. at x = 0
(D) None of these

5. Let f(x) be defined in [–2, 2] by
$$f(x) = \begin{cases} \max(\sqrt{4-x^2}, \sqrt{1+x^2}), & -2 \le x \le 0 \\ \min(\sqrt{4-x^2}, \sqrt{1+x^2}), & 0 < x \le 2 \end{cases}$$ then f(x)
(A) is continuous at all points
(B) is not continuous at more than one point
(C) is not differentiable only at one point
(D) is not differentiable at more than one point.

6. The function $f(x) = \sin^{-1}(\cos x)$ is
(A) discontinuous at x = 0
(B) continuous at x = 0
(C) differentiable at x = 0
(D) None of these

7. If $f(x) = [x]^2 + \sqrt{\{x\}^2}$, then
(where, [*] and { * } denote the greatest integer and fractional part functions respectively)
(A) f(x) is continuous at all integral points
(B) f(x) is continuous and differentiable at x = 0
(C) f(x) is discontinuous $\forall\ x \in I - \{1\}$
(D) f(x) is differentiable $\forall\ x \in I$

8. If $f(x) = p|\sin x| + q \cdot e^{|x|} + r|x|^3$ and f(x) is differentiable at x = 0, then
(A) p = q = r = 0 (B) p = 0, q = 0, $r \in R$
(C) q = 0, r = 0, $p \in R$ (D) p + q = 0, $r \in R$

9. If f(x) is differentiable everywhere, then
(A) | f | is differentiable everywhere
(B) $|f|^2$ is differentiable everywhere
(C) f | f | is not differentiable at some point
(D) f + | f | is differentiable everywhere

10. Let f(x + y) = f(x) f(y) all x and y. Suppose that f(3) = 3 and f'(0) = 11 then f'(3) is given by
(A) 22 (B) 44
(C) 28 (D) 33

11. If $f: R \to R$ be a differentiable function, such that $f(x+2y) = f(x) + f(2y) + 4xy\ \forall\ x, y \in R$, then
(A) f'(1) = f'(0) + 1
(B) f'(1) = f'(0) – 1
(C) f'(0) = f'(1) + 2
(D) f'(0) = f'(1) – 2

12. Let $f: R \to R$ be a function such that
$f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)}{3}$, f(0) = 0 and f'(0) = 3, then
(A) $\dfrac{f(x)}{x}$ is differentiable in R
(B) f(x) is continuous but not differentiable in R
(C) f(x) is continuous in R
(D) f(x) is bounded in R

13. If a differentiable function f satisfies $f\left(\frac{x+y}{3}\right) = \frac{4-2(f(x)+f(y))}{3}$ $\forall$ x, r $\in$ R, find f(x)

(A) 1/7 (B) 2/7
(C) 8/7 (D) 4/7

14. The functions defined by $f(x) = \max \{x^2, (x-1)^2, 2x(1-x)\}, 0 \le x \le 1$

(A) is differentiable for all x
(B) is differentiable for all x except at one point
(C) is differentiable for all x except at two points
(D) is not differentiable at more than two points

15. If f is an even function such that $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ has some finite non-zero value, then

(A) f is continuous and derivable at x = 0
(B) f is continuous but not derivable at x = 0
(C) f may be discontinuous at x = 0
(D) None of these

16. Let $f : R \to R$ be a function defined by f(x) = Min {x + 1, |x| + 1}. Then which of the following is true ?

(A) $f(x) \ge 1$ for all $x \in R$
(B) f(x) is not differentiable at x = 1
(C) f(x) is differentiable everywhere
(D) f(x) is not differentiable at x = 0

17. If $f(x) = \begin{cases} \frac{x^2-1}{x^2+1}, & 0 < x \le 2 \\ \frac{1}{4}(x^3 - x^2), & 2 < x \le 3 \\ \frac{9}{4}(|x-4| + |2-x|), & 3 < x < 4 \end{cases}$, then

(A) f(x) is differentiable at x = 2 & x = 3
(B) f(x) is non–differentiable at x = 2 & x = 3
(C) f(x) is differentiable at x = 3 but not at x = 2
(D) f(x) is differentiable at x = 2 but not at x = 3.

18. A function f defined as f(x) = x[x] for $-1 \le x \le 3$ where [x] defines the greatest integer $\le$ x is

(A) conti. at all points in the domain of but nonderivable at a finite number of points
(B) discontinuous at all points & hence non-derivable at all points in the domain of f
(C) discont. at a finite number of points but not derivable at all points i the domain of f
(D) discont. & also non-derivable at a finite number of points of f.

19. Function $f(x) = \frac{x}{1+|x|}$ is differentiable in the set-

(A) $(-\infty, \infty)$ (B) $(-\infty, 0)$
(C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$

20. If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$ then

(where { * } denotes the fractional part function)

(A) 'f' is cont. & diff. at x = 0
(B) 'f' is cont. but not diff. at x = 0
(C) 'f' is cont. & diff. at x = 2
(D) None of these

21. Let $f : R \to R$ is a differentiable function satisfying the condition
$f(x + y^5) = f(x) + (f(y))^5$ $\forall$ x, y $\in$ R.
If f '(0) > 0 then the value of $\left[\frac{f(20)}{2}\right]$ is
(where [*] denotes greatest integer function)

(A) 9 (B) 10
(C) 11 (D) 12

22. For what triplets of real number (a, b, c) with a $\ne$ 0 the function $f(x) = \begin{cases} x & x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$ is differentiable for all real x ?

(A) {(a, 1 – 2a, a)| a ε R, a ≠ 0}
(B) {(a, 1 – 2a, c)| a, c ε R, a ≠ 0}
(C) {(a, b) | a, b, c ε R, a + b + c = 1}
(D) {(a, 1 – 2a, 0)| a ε R, a ≠ 0}

23. Let $f(x) = [n + p\sin x], x \in (0, \pi), n \in I$ and p is a prime number. Then number of points where f(x) is not differentiable is
(where [*] denotes greatest integer function)

(A) p – 1 (B) p + 1
(C) 2p + 1 (D) 2p – 1

24. Let $f : R \to R$ be a function defined by f(x) = Min {x + 1, |x| + 1}. Then which of the following is true **[AIEEE 2007]**

(A) $f(x) \ge 1$ for all $x \in R$
(B) f(x) is not differentiable at x = 1
(C) f(x) is differentiable everywhere
(D) f(x) is not differentiable at x = 0

25. Let $f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1} & \text{if} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$

Then which one of the following is true ?

[AIEEE 2008]

(A) f is differentiable at x = 0 and at x = 1
(B) f is differentiable at x = 0 but not at x = 1
(C) f is differentiable at x = 1 but not at x = 0
(D) f is neither differentiable at x = 0 nor at x = 1

26. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement – 1 : gof is differentiable at x = 0 and its derivative is continuous at that point.

[AIEEE 2009]

Statement – 2 : gof is twice differentiable at x = 0.

(A) Statement -1 is true, Statement -2 is true; Statement -2 is a correct explanation for Statement -1
(B) Statement -1 is true, Statement -2 is true; Statement -2 is not a correct explanation for Statement -1.
(C) Statement -1 is true, Statement -2 is false.
(D) Statement -1 is false, Statement -2 is ture.

27. If the function $g(x) = \begin{cases} k\sqrt{x+1} & , \quad 0 \le x \le 3 \\ mx+2 & , \quad 3 < x \le 5 \end{cases}$ is differentiable, then the value of k + m is :

[JEE MAIN 2015]

(A) $\frac{10}{3}$ (B) 4
(C) 2 (D) $\frac{16}{5}$

28. If $f(x) = \begin{cases} x\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), x \neq 0 \\ 0 \quad , x = 0 \end{cases}$ then at x = 0, f(x) is -

(A) differentiable (B) not differentiable
(C) $f'(0^+) = -1$ (D) $f'(0^-) = 1$

29. If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$ where {x} denotes the fractional part function, then -

(A) 'f' is continuous & differentiable at x = 0
(B) 'f' is continuous but not differentiable at x = 0
(C) 'f' is continuous & differentiable at x = 2
(D) none of these

30. If $f(x) = \begin{cases} \frac{x-1}{2x^2 - 7x + 5}, x \neq 1 \\ \frac{-1}{3} \quad , x = 1 \end{cases}$, then f'(1) equals -

(A) $\frac{2}{9}$ (B) $-\frac{2}{9}$
(C) 0 (D) does not exist

31. Let $f(x) = \begin{cases} 4x^2 + 2[x]x & \text{if} \quad -\frac{1}{2} \le x < 0 \\ ax^2 - bx & \text{if} \quad 0 \le x < \frac{1}{2} \end{cases}$ where [x] denotes the greatest integer function. Then -

(A) f(x) is continuous and differentiable in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ for all a, provided b = 2
(B) f(x) is continuous and differentiable in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ if f(a) = 4, b = 2
(C) f(x) is continuous and differentiable in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ if a = 4 and b = 0
(D) for no choice of a and b, f(x) is differentiable in $\left(-\frac{1}{2}, \frac{1}{2}\right)$

32. A function f defined as f(x) = x[x] for $-1 \le x \le 3$ where [x] defines the greatest integer $\le x$ is -

(A) continuous at all points in the domain of f but non-derivable at a finite number of points
(B) discontinuous at all points & hence non-derivable at all points in the domain of f
(C) discontinuous at a finite number of points but not derivable at all points in the domain of f
(D) discontinuous & also non-derivable at a finite number of points of f

33. Consider $f(x) = \begin{cases} \left[\dfrac{2(\sin x - \sin^3 x) + |\sin x - \sin^3 x|}{2(\sin x - \sin^3 x) - |\sin x - \sin^3 x|}\right], & x \neq \dfrac{\pi}{2} \\ 3 & x = \dfrac{\pi}{2} \end{cases}$ for $x \in (0,\pi)$;

where [ ] denotes the greatest integer function, then -

(A) f is continuous & differentiable at $x = \pi/2$

(B) f is continuous but not differentiable at $x = \pi/2$

(C) f is neither continuous not differentiable at $x = \pi/2$

(D) none of these

34. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \le (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then $f(1)$ equals

(A) 1 (B) 2

(C) 0 (D) –1

35. Let $f(x) = \begin{cases} g(x).\cos\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ where g(x) is an even function differentiable at x = 0, passing through the origin. Then $f'(0)$

(A) is equal to 1 (B) is equal to 0

(C) is equal to 2 (D) does not exist

DIFFERENTIABILITY

1.	B	**2.**	B	**3.**	D	**4.**	B	**5.**	D	**6.**	B	**7.**	C
8.	D	**9.**	B	**10.**	D	**11.**	D	**12.**	C	**13.**	D	**14.**	C
15.	B	**16.**	C	**17.**	B	**18.**	D	**19.**	A	**20.**	D	**21.**	B
22.	A	**23.**	D	**24.**	C	**25.**	B	**26.**	C	**27.**	C	**28.**	B
29.	D	**30.**	B	**31.**	A	**32.**	D	**33.**	A	**34.**	C	**35.**	B