

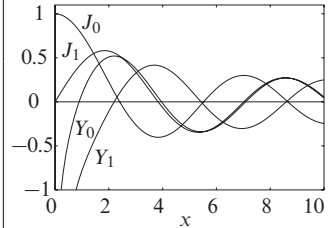
2.9 Special functions and polynomials

Gamma function

Definition	$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad [\Re(z) > 0]$	(2.407)
Relations	$n! = \Gamma(n+1) = n\Gamma(n) \quad (n=0,1,2,\dots)$	(2.408)
	$\Gamma(1/2) = \pi^{1/2}$	(2.409)
	$\binom{z}{w} = \frac{z!}{w!(z-w)!} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$	(2.410)
Stirling's formulas (for $ z , n \gg 1$)	$\Gamma(z) \simeq e^{-z} z^{z-(1/2)} (2\pi)^{1/2} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \dots\right)$	(2.411)
	$n! \simeq n^{n+(1/2)} e^{-n} (2\pi)^{1/2}$	(2.412)
	$\ln(n!) \simeq n \ln n - n$	(2.413)

Bessel functions

Series expansion	$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k! \Gamma(\nu+k+1)}$	(2.414)	$J_\nu(x)$ Bessel function of the first kind
	$Y_\nu(x) = \frac{J_\nu(x) \cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}$	(2.415)	$Y_\nu(x)$ Bessel function of the second kind
Approximations	$J_\nu(x) \simeq \begin{cases} \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu & (0 \leq x \ll \nu) \\ \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) & (x \gg \nu) \end{cases}$		(2.416)
	$Y_\nu(x) \simeq \begin{cases} \frac{-\Gamma(\nu)}{\pi} \left(\frac{x}{2}\right)^{-\nu} & (0 < x \ll \nu) \\ \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) & (x \gg \nu) \end{cases}$		(2.417)
Modified Bessel functions	$I_\nu(x) = (-i)^\nu J_\nu(ix)$	(2.418)	$I_\nu(x)$ modified Bessel function of the first kind
	$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} [J_\nu(ix) + iY_\nu(ix)]$	(2.419)	$K_\nu(x)$ modified Bessel function of the second kind
Spherical Bessel function	$j_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{\nu+\frac{1}{2}}(x)$	(2.420)	$j_\nu(x)$ spherical Bessel function of the first kind [similarly for $y_\nu(x)$]



Legendre polynomials^a

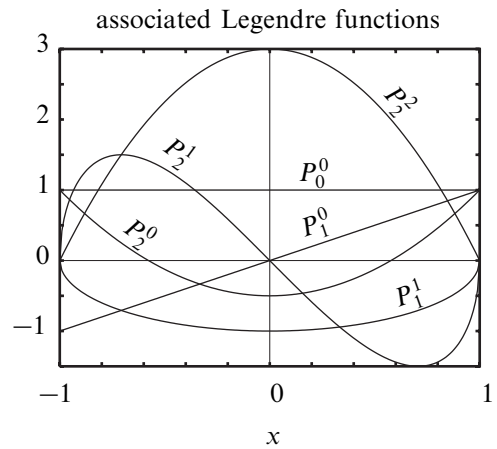
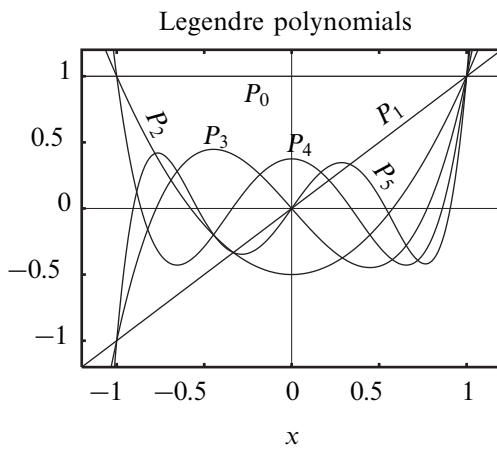
Legendre equation	$(1-x^2)\frac{d^2P_l(x)}{dx^2} - 2x\frac{dP_l(x)}{dx} + l(l+1)P_l(x) = 0 \quad (2.421)$	P_l Legendre polynomials l order ($l \geq 0$)
Rodrigues' formula	$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad (2.422)$	
Recurrence relation	$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x) \quad (2.423)$	
Orthogonality	$\int_{-1}^1 P_l(x)P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'} \quad (2.424)$	$\delta_{ll'}$ Kronecker delta
Explicit form	$P_l(x) = 2^{-l} \sum_{m=0}^{l/2} (-1)^m \binom{l}{m} \binom{2l-2m}{l} x^{l-2m} \quad (2.425)$	$\binom{l}{m}$ binomial coefficients
Expansion of plane wave	$\exp(\mathbf{i}kz) = \exp(\mathbf{i}kr \cos \theta) \quad (2.426)$	k wavenumber z propagation axis $z = r \cos \theta$
	$= \sum_{l=0}^{\infty} (2l+1) \mathbf{i}^l j_l(kr) P_l(\cos \theta) \quad (2.427)$	j_l spherical Bessel function of the first kind (order l)
$P_0(x) = 1$ $P_1(x) = x$	$P_2(x) = (3x^2 - 1)/2$ $P_3(x) = (5x^3 - 3x)/2$	$P_4(x) = (35x^4 - 30x^2 + 3)/8$ $P_5(x) = (63x^5 - 70x^3 + 15x)/8$

^aOf the first kind.

Associated Legendre functions^a

Associated Legendre equation	$\frac{d}{dx} \left[(1-x^2) \frac{dP_l^m(x)}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] P_l^m(x) = 0 \quad (2.428)$	P_l^m associated Legendre functions
From Legendre polynomials	$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad 0 \leq m \leq l \quad (2.429)$ $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \quad (2.430)$	P_l Legendre polynomials
Recurrence relations	$P_{m+1}^m(x) = x(2m+1)P_m^m(x) \quad (2.431)$	$!!$ $5!! = 5 \cdot 3 \cdot 1$ etc.
	$P_m^m(x) = (-1)^m (2m-1)!! (1-x^2)^{m/2} \quad (2.432)$	
	$(l-m+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x) - (l+m)P_{l-1}^m(x) \quad (2.433)$	
Orthogonality	$\int_{-1}^1 P_l^m(x)P_{l'}^m(x) dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'} \quad (2.434)$	$\delta_{ll'}$ Kronecker delta
$P_0^0(x) = 1$ $P_2^0(x) = (3x^2 - 1)/2$	$P_1^0(x) = x$ $P_2^1(x) = -3x(1-x^2)^{1/2}$	$P_1^1(x) = -(1-x^2)^{1/2}$ $P_2^2(x) = 3(1-x^2)$

^aOf the first kind. $P_l^m(x)$ can be defined with a $(-1)^m$ factor in Equation (2.429) as well as Equation (2.430).



Spherical harmonics

Differential equation	$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_l^m + l(l+1) Y_l^m = 0 \quad (2.435)$	Y_l^m spherical harmonics
Definition ^a	$Y_l^m(\theta, \phi) = (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi} \quad (2.436)$	P_l^m associated Legendre functions
Orthogonality	$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_l^{m*}(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{mm'} \delta_{ll'} \quad (2.437)$	Y^* complex conjugate $\delta_{ll'}$ Kronecker delta
Laplace series	$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi) \quad (2.438)$ where $a_{lm} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_l^{m*}(\theta, \phi) f(\theta, \phi) \sin \theta \, d\theta \, d\phi \quad (2.439)$	f continuous function
Solution to Laplace equation	if $\nabla^2 \psi(r, \theta, \phi) = 0$, then $\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_l^m(\theta, \phi) \cdot [a_{lm} r^l + b_{lm} r^{-(l+1)}] \quad (2.440)$	ψ continuous function a, b constants
$Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$ $Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \quad Y_2^0(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$ $Y_2^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi} \quad Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$ $Y_3^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{7}{4\pi}} (5 \cos^2 \theta - 3) \cos \theta \quad Y_3^{\pm 1}(\theta, \phi) = \mp \frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i\phi}$ $\begin{matrix} \pm 2 \\ \end{matrix} \quad Y_3^{\pm 2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{105}{4\pi}} \sin^2 \theta \cos \theta e^{\pm 2i\phi} \quad \begin{matrix} \pm 3 \\ \end{matrix} \quad Y_3^{\pm 3}(\theta, \phi) = \mp \frac{1}{8} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{\pm 3i\phi}$		

Delta functions

Kronecker delta	$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (2.441)$	δ_{ij} Kronecker delta $i, j, k, \dots$ indices (= 1, 2 or 3)
	$\delta_{ii} = 3 \quad (2.442)$	
Three-dimensional Levi-Civita symbol (permutation tensor) ^a	$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$	ϵ_{ijk} Levi-Civita symbol (see also page 25)
	$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \quad (2.443)$	
	$\text{all other } \epsilon_{ijk} = 0$	
	$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \quad (2.444)$	
	$\delta_{ij}\epsilon_{ijk} = 0 \quad (2.445)$	
	$\epsilon_{ilm}\epsilon_{jlm} = 2\delta_{ij} \quad (2.446)$	
Dirac delta function	$\epsilon_{ijk}\epsilon_{ijk} = 6 \quad (2.447)$	
	$\int_a^b \delta(x) dx = \begin{cases} 1 & \text{if } a < 0 < b \\ 0 & \text{otherwise} \end{cases} \quad (2.448)$	$\delta(x)$ Dirac delta function $f(x)$ smooth function of x a, b constants
	$\int_a^b f(x)\delta(x-x_0) dx = f(x_0) \quad (2.449)$	
	$\delta(x-x_0)f(x) = \delta(x-x_0)f(x_0) \quad (2.450)$	
	$\delta(-x) = \delta(x) \quad (2.451)$	
	$\delta(ax) = a ^{-1}\delta(x) \quad (a \neq 0) \quad (2.452)$	
	$\delta(x) \simeq n\pi^{-1/2}e^{-n^2x^2} \quad (n \gg 1) \quad (2.453)$	

^aThe general symbol $\epsilon_{ijk\dots}$ is defined to be +1 for even permutations of the suffices, -1 for odd permutations, and 0 if a suffix is repeated. The sequence (1, 2, 3, ..., n) is taken to be even. Swapping adjacent suffices an odd (or even) number of times gives an odd (or even) permutation.