CHAPTER 7

BINOMIAL THEOREM

7.1 INTRODUCTION

We have dealt with expansions of $(x + a)^2$ while dealing with quadratic equations. Herein we will study expansions of the form $(x + a)^n$. Any power of binomial expression $\{(a + x)^2\}$ can be expanded in the form of a series, which is obtained the by process of continuous multiplication as shown here: $(a + x)^2 = (a + x)(a + x) = a^2 + ax + ax + x^2 = a^2 + 2ax + x^2$ which can be explained as the terms of expansion are obtained when any one of two terms 'a' or 'x' are selected from each factor and finally they are multiplied together.

7.2 BINOMIAL

Any algebraic expression containing two terms is called 'binomial expression'. [Bi (two) + Nomial (terms)] is an expression containing sum of two different terms.

7.2.1 Binomial Expansion (Natural Index)

Binomial expansion is a polynomial equivalent of powers of a given binomial expression. The expressions for $(a + x)^n$ has been obtained as, $(a + x)^n = nC_0\, a^n x^0 + nC_1\, a^{n-1} x^1 + {}^nC_2\, a^{n-2} x^2 + \ldots\ldots + {}^nC_r\, a^{n-r} x^r + \ldots\ldots + {}^nC_n\, a^0 x^n$

- Where n is a positive integer which is given by $(a+x)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} x^r$ and $(a-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r a^{n-r} x^r$.
- $(1+x)^n = \sum_{r=0}^{n} x^r . {}^nC_r \quad (1-x)^n = \sum_{r=0}^{n} (-1)^r .\ {}^nC_r\ x^r$; where $n \in I^+$ is known as index of binomial and ${}^nC_r$ is binomial coefficient).
- ${}^nC_r$ are known as **binomial coefficients**.
- n is called **index** of binomial.
- The binomial expansion is homogenous in 'a' and 'x'. i.e., **sum of powers of a and x in each term remains constant** and this constant is equal to index of binomial.
- Number of **distinct terms** in the expansion is equal to (n + 1).

- The equidistant binomial coefficients from beginning and end are equal.
- The number of terms in the expansion $\{(a + x)^n + (a - x)^n\}$ will be n/2+1, when n is even $\frac{n+1}{2}$ and when n is odd.
- The number of terms in the above expansion $\{(a + x)^n - (a - x)^n\}$ will be n/2, when n is even and $\frac{n+1}{2}$ when n is odd.

7.3 GENERAL TERM

A general term is known as representative term of binomial and it is $(r + 1)^{th}$ term of the expansion and is given by $T_{r+1} = {}^nC_r\, a^{n-r} x^r$ in expansion of $(a + x)^n$.

7.3.1 r^{th} Term from Beginning

The term ${}^nC_r\, x^{n-r} y^r$ is the $(r + 1)^{th}$ term from beginning in the expansion of $(x + y)^n$. It is usually called the **general term** and it is denoted by T_{r+1}. i.e., $T_{r+1} = {}^nC_r\, x^{n-r} y^r$.

7.3.2 k^{th} Term from End

k^{th} term from end in the expansion of $(x + y)^n = (n - k + 2)^{th}$ term from beginning.

7.4 MIDDLE TERM

The middle term depends upon the value of n.

Case I: If n is even: Then total number of terms in the expansion of $(x + y)^n$ is n+1 (odd). So, there is only one middle term i.e., (n/2 +1)th term is the middle term i.e., $T_{n/2+1} = {}^nC_{n/2}\, x^{n/2} y^{n/2}$.

Case II: If n is odd: Then total number of terms in the expansion of $(x + y)^n$ is n+1 (even). So there are two middle terms i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th are two middle terms. They are given by $C_{\frac{n-1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$ and ${}^nC_{\frac{n+1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}}$.

7.5 NUMBER OF TERMS IN EXPANSIONS

- $(a + x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \ldots + {}^nC_{n-1} a^1 x^{n-1} + {}^nC_n a^0 x^n = \sum_{r=0}^{n} {}^nC_r a^{n-r}.x^r$.
- $(a - x)^n = {}^nC_0 a^n x^0 - {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \ldots + {}^nC_{n-1} a(-x)^{n-1} + {}^nC_n a^0 (-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r a^{n-r}.x^r$.
- $(a + x)^n + (a - x)^n = 2\sum_{r=0}^{m} {}^nC_{2r} a^{n-2r} x^{2r}$; where $\begin{cases} n = 2m & \text{if n is even} \\ n-1 = 2m & \text{if n is odd} \end{cases}$.

$\Rightarrow$ Number of terms: $m+1=\begin{cases}\dfrac{n+2}{2} & \text{if n is even}\\ \dfrac{n+1}{2} & \text{if n is odd}\end{cases}$.

- $(a + x)^n - (a - x)^n = 2\sum_{r=0}^{m} {}^nC_{2r+1}a^{n-2r-1}x^{2r+1}$; where $\begin{cases} n=2m+1 & \text{if n is odd}\\ n-1=2m+1 & \text{if n is even}\end{cases}$.

$\Rightarrow$ Number of terms : $m+1=\begin{cases}\dfrac{n}{2} & \text{if n is even}\\ \dfrac{n+1}{2} & \text{if n is odd}\end{cases}$.

7.6 GREATEST TERM

If T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms in the expansion of $(1 + x)^n$, then $\dfrac{T_{r+1}}{T_r}=\dfrac{{}^nC_rx^r}{{}^nC_{r-1}x^{r-1}}=\dfrac{n-r+1}{r}x$.

Let numerically, T_{r+1} be the greatest term in the above expansion. Then $T_{r+1} \geq T_r$ or $\dfrac{T_{r+1}}{T_r}\geq 1$.

$\therefore$ $\dfrac{n-r+1}{r}|x|\geq 1$ to find the value of r, i.e., $r\leq\dfrac{(n+1)}{(1+|x|)}|x|$.

Now substituting values of n and x in (i), we get $r \leq m + f$ or $r \leq m$. where m is a positive integer and f is a fraction such that $0 < f < 1$. In the first case, T_{m+1} is the greatest term, while in the second case, T_m and T_{m+1} are the greatest terms and both are equal.

7.6.1 To Find the Greatest Term in the Expansion of $(1 + x)^n$

- Calculate m = ...
- If m is integer, then T_m and T_{m+1} are equal and both are greatest term.
- If m is not integer, then $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integral part.

Note:

To find the greatest term in the expansion of $(x+y)^n$ since $(x+y)^n = x^n(1+ y/x)^n$ and then find the greatest term in $(1+y/x)^n$.

7.7 GREATEST COEFFICIENT

To determine the greatest coefficient in the binomial expansion of $(1 + x)^n$, consider the following: $\left(\dfrac{T_{r+1}}{T_r}\right)=\dfrac{C_r}{C_{r-1}}=\dfrac{n-r+1}{r}=\dfrac{n+1}{r}-1$. Now, the (r + 1)th binomial coefficient will be greater than the r^{th} binomial coefficient, when $\dfrac{n+1}{r}-1>1$.

$\Rightarrow \frac{n+1}{2} > r$ ......(i)

But r must be an integer, and therefore, when n is even, the greatest binomial coefficient is given by the greatest value of r, consistent with (i) i.e., . r = n/2 and hence the greatest binomial coefficient is ${}^nC_{n/2}$.

- If n is even, then greatest coefficient = ${}^nC_{n/2}$.
- If n is odd, then greatest coefficients are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$.

7.8 PROPERTIES OF BINOMIAL COEFFICIENT

The binomial coefficient for general term of the expansion $(a + x)^n$ is given as ${}^nC_r$ which states the number of ways, the term $a^{n-r} x^r$ occurs in the expansion.

7.8.1 Properties of ${}^nC_r$

It is defined as number of selections of r objects out of n different objects and is given by, ${}^nC_r = \frac{n!}{r!(n-r)!}$, when n > r, (= 0 if n < r).

- ${}^nC_r$ is always an integer. Product of r consecutive integers is always divisible by r!.

$\therefore \ {}^nC_r = \frac{n(n-1)(n-2)(n-3).....(n-r+1)}{r!} \in I$ (Clearly, the numerator is completely divisible by r!)

- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_x = {}^nC_y \quad \Rightarrow \quad x = y$ or $x + y = n$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_r = \frac{n}{r}\left({}^{n-1}C_{r-1}\right) = \frac{n}{r}\left(\frac{n-1}{r-1}\right){}^{n-2}C_{r-2} = \ldots\ldots\ldots\ldots$
- ${}^nC_r = {}^{n+1}C_{r+1} \cdot \left(\frac{r+1}{n+1}\right) = \frac{(r+1)(r+2)}{(n+1)(n+2)}\ {}^{n+2}C_{r+2}$
- $r \cdot {}^nC_r = n\ {}^{n-1}C_{r-1}$ and $\frac{{}^nC_r}{r+1} = \left(\frac{{}^{n+1}C_{r+1}}{n+1}\right)$

7.9 PROPERTIES OF COEFFICIENTS

Properties of binomial expression are derived from:

- $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots. + {}^nC_n x^n$ ...... (i)
- $(1-x)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_r x^r = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots\ldots + (-1)^n\ {}^nC_n x^n$ ..... (ii)
- $\sum_{r=0}^{n} {}^nC_r x^r = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$

- $\sum_{r=0}^{n}(-1)^r\ {}^nC_r = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots.. + (-1)^n\ {}^nC_n = 0$
- The sum of the binomial coefficients of the odd terms in the expansion of$(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.
- $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots. = 2^{n-1}$
- $\sum_{r=0}^{n} r\ {}^nC_r = 1C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n\ 2^{n-1}$
- $\sum_{r=0}^{n} r^2\ {}^nC_r = 1^2 C_1 + 2^2 C_2 + 3^2 C_3 + \ldots.. + n^2 C_n$
- $\sum \frac{C_r}{r+1} = \frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots\ldots + \frac{C_n}{n+1}$
- $\sum_{r=1}^{n} r.\frac{{}^nC_r}{{}^nC_{r-1}} = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

7.10 MULTINOMIAL THEOREM

- The general term in the multinomial expansion is $\frac{n!}{r_1!r_2!\ldots..r_k} x_1^{r_1}.x_2^{r_2}\ldots\ldots..x_k^{r_k}$
- The total number of terms in the multinomial expansion = number of non -negative integral solutions of the equation $r_1 + r_2 + \ldots. + r_k = n = {}^{n+k-1}C_n$ or ${}^{n+k-1}C_{k-1}$
- Coefficient of $x_1^{r1}.\ x_2^{r2} \ldots..x_5^{r5}$ in the expansion of $a_1x_1 + a_2x_2 + \ldots + a_kx_k =$
- Greatest coefficient in the expansion of $(x_1 + x_2 + \ldots.. + xk)^n$ where q is the quotient and r the remainder when n is divided by k.
- The number of terms in the expansion of $(x + y + z)^n$, where n is a positive integer, is 1/2 (n + 1) (n + 2).
- Sum of all the coefficients is obtained by putting all the variables x_i equal to 1 and it is equal to n^m.

7.11 TIPS AND TRICKS

1. $(x + y)^n$ = sum of odd terms + sum of even terms.
2. In the expansion of $(x + y)^n$, $n \in N \frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right)\frac{y}{x}$
3. The coefficient of x^{n-1} in the expansion of $(x + 1)(x + 2) \ldots\ldots(x + n) = \frac{n(n+1)}{2}$.
4. The coefficient of x^{n-1} in the expansion of $(x + 1)(x - 2) \ldots\ldots(x - n) = \frac{-n(n+1)}{2}$.
5. Greatest term in $(x + y)^n = x^n$. Greatest terms in $\left(1+\frac{y}{x}\right)^n$.
6. The number of terms in the expansion of $(x_1 + x_2 + \ldots. + x_n)\ n = {}^{n+r-1}C_{r-1}$
7. If the coefficients of the rth, (r + 1) and (r + 2) th terms in the expansion of $(1 + x)^n$ are in H.P. then $n + (n - 2r)^2 = 0$.
8. If the coefficients of the rth (r + 1) th and (r + 2) th terms in the expansion of $(1 + x)^n$ are in A.P. then $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.