10. PARABOLA

1. INTRODUCTION TO CONIC SECTIONS

1.1 Geometrical Interpretation

Conic section, or conic is the locus of a point which moves in a plane such that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the Focus.

(b) The fixed straight line is called the Directrix.

(c) The constant ratio is called the Eccentricity denoted by e.

(d) The line passing through the focus and perpendicular to the directrix is called the Axis.

(e) The point of intersection of a conic with its axis is called the Vertex.

Sections on right circular cone by different planes

(a) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure.

(b) Section of a right circular cone by a plane parallel to its base is a circle.

(c) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola.

(d) Section of a right circular cone by a plane neither parallel to a side of the cone nor perpendicular to the axis of the cone is an ellipse.

(e) Section of a right circular cone by a plane parallel to the axis of the cone is an ellipse or a hyperbola.

3D View:

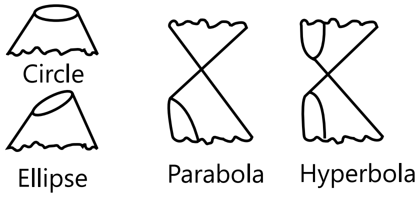


Figure 10.1

1.2 Conic Section as a Locus of a Point

If a point moves in a plane such that its distances from a fixed point and from a fixed line always bear a constant ratio 'e' then the locus of the point is a conic section of the eccentricity e (focus-directrix property). The fixed point is the focus and the fixed line is the directrix.

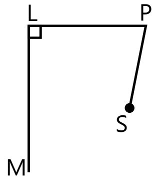


Figure 10.2

(a) If e > 1, it is a hyperbola.

(b) If e = 1, it is a parabola.

(c) If e < 1, it is an ellipse.

(d) If e = 0, it is a circle.

(e) If the focus is (α, β) and the directrix is ax + by + c = 0 then the equation of the conic section whose eccentricity =e, is $(x-\alpha)^2 + (y-\beta)^2 = e^2 . \frac{(ax+by+c)^2}{a^2+b^2}$

1.3 General Equation of Conic

The general equation of a conic with focus (p, q) and directrix lx + my + n = 0 is:

$(l^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2(lx + my + n)^2$

This equation, when simplified, can be written in the form $ax^2 + 2gx + 2hxy + by^2 + 2fy + c = 0$.

This general equation represents a pair of straight lines if it is degenerate, i.e. $\Delta = 0$,

where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. Further, this equation represents

(a) A pair of parallel straight lines, if $\Delta = 0$ and $h^2 = ab$

(b) A pair of perpendicular straight lines, if $\Delta = 0$ and a + b = 0

(c) A point, if $\Delta = 0$ and $h^2 < ab$

The general equation given above, represents a conic section if it is non-degenerate, i.e. $\Delta \neq 0$, also this equation represents

(a) A circle, if $\Delta \neq 0$, a = b and h = 0

(b) A parabola, if $\Delta \neq 0$ and $h^2 = ab$

(c) An ellipse, if $\Delta \neq 0$ and $h^2 < ab$

(d) A hyperbola, if $\Delta \neq 0$ and $h^2 > ab$

CONCEPTS

Always use a geometrical approach in the coordinate geometry problem.

Vaibhav Gupta (JEE 2009, AIR 54)

2. GENERAL EQUATION OF PARABOLA

Definition: A parabola is the locus of a point which moves such that its distance from a fixed point is equal to its distance from a fixed straight line.

Standard equation of the parabola:

The standard form of a parabola is taken with the origin as its vertex and the focus lying on the x-axis.

Let F be the focus and ZM be the directrix of the parabola.

Let P be a point on the parabola.

Join FP and from P drawn PM $\perp$ to ZM. $\therefore$ FP = PM

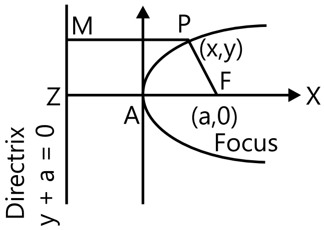


Figure 10.3

To find the co-ordinates of focus and equation of directrix: From F draw FZ ⊥ on ZM. Bisect FZ in A, i.e. FA = AZ. Then A lies on the parabola. Let FZ = 2a then AF = AZ = a.

Take A as the origin, AF as the x-axis and AY as the y-axis.

Then the co-ordinates of F are (a, 0) and the equation of the directrix is x = –a, of x + a = 0.

Let the co-ordinates of P be (x, y). We know that for a parabola FP = PM.

$$\sqrt{(x-a)^2+(y-0)^2} = x + a$$

$\Rightarrow \quad x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2 \quad \Rightarrow \quad y^2 = 4ax$

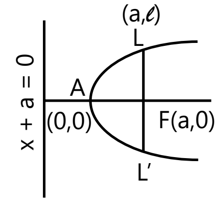


Figure 10.4

Which is the standard equation of a parabola.

Note: Focus F is (a, 0); **Directrix** is x + a = 0; **Vertex** A is (0, 0); **Axis** AF of the parabola is y = 0.

Latus rectum LL' is 4a as calculated below:

Let the co-ordinates of L be (a, l). The point L (a, l) lies on the parabola $y^2 = 4ax$.

$\therefore \quad l^2 = 4a.a \quad$ or $\quad l = 2a$.

Latus rectum LSL' = $2l$ = 2(2a) = 4a = 4.FA

Illustration 1: Find the equation of the parabola whose focus is (1, 1) and the tangent at the vertex is x + y = 1. Also find its latus rectum. **(JEE MAIN)**

Sol: In order to get the equation of a parabola, we need to find the equation of a directrix. Using the equation of the tangent at the vertex and the focus we can find the directrix and hence the equation of the parabola.

The directrix is parallel to the tangent at the vertex V.

$\therefore$ The directrix will be of the form $x + y = \lambda$ ...(i)

Now, V is the foot of the perpendicular from S(1, 1) to the line x + y = 1.

Let $V = (\alpha, \beta)$. Then $\alpha + \beta = 1$ ...(ii)

and $\frac{\beta-1}{\alpha-1}.(-1) = -1$, i.e., $\alpha = \beta$ ...(iii)

Solving (ii), (iii) we get $\alpha = \beta = \frac{1}{2}$. So $V = \left(\frac{1}{2}, \frac{1}{2}\right)$

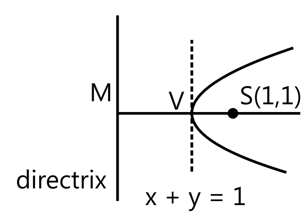


Figure 10.5

Let $M = (x_1, y_1)$. As MV = VS, V is the middle point of MS.

$\therefore \quad \frac{x_1+1}{2} = \frac{1}{2}, \frac{y_1+1}{2} = \frac{1}{2}$

$\therefore \quad x_1 = 0, y_1 = 0$. So M = (0, 0)

As M is on the directrix, (0, 0) satisfies (i). Hence, $\lambda = 0$

$\therefore$ the equation of the directrix is x + y = 0.

Using focus-directrix property, the equation of the parabola is $(x-1)^2 + (y-1)^2 = \left(\frac{x+y}{\sqrt{1^2+1^2}}\right)^2$

$\Rightarrow 2[(x-1)^2 + (y-1)^2] = (x+y)^2$

$\Rightarrow \quad 2(x^2 + y^2 - 2x - 2y + 2) = x^2 + y^2 + 2xy$

$\Rightarrow \quad x^2 + y^2 - 2xy - 4x - 4y + 4 = 0$

$\Rightarrow \quad (x-y)^2 = 4(x + y - 1)$

Length of latus rectum $= 4 \times VS = 4\sqrt{\left(\frac{1}{2}-1\right)^2+\left(\frac{1}{2}-1\right)^2} = 4 . \frac{1}{\sqrt{2}} = 2\sqrt{2}$

Illustration 2: Find the equation of the parabola whose focus is (–1, –2) and equation of the directrix is x – 2y + 3 = 0. **(JEE MAIN)**

Sol: Use the standard definition of a parabola.

Let P(x, y) be any point on the parabola whose focus is S(–1, –2) and the directrix x – 2y + 3 = 0

By definition, SP = PM

$\Rightarrow \quad (x + 1)^2 + (y + 2)^2 = \left(\dfrac{x - 2y + 3}{\sqrt{1 + 4}}\right)^2$

$\Rightarrow \quad 5[(x + 1)^2 + (y + 2)^2] = (x - 2y + 3)^2$

$\Rightarrow \quad 5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$

$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

Illustration 3: Find the equation of the parabola whose focus is the point (4, 0) and whose directrix is x = –4. Also, find the length of the latus rectum. **(JEE MAIN)**

Sol: Same as the previous question. Refer Fig. 10.6

In this illustration, the focus is (4, 0) and directrix is x + 4 = 0

Let P(x, y) be any moving point then draw ZZ′ ⊥ PM from P to the directrix.

$\dfrac{FP}{PM} = 1; FP = PM \quad \Rightarrow \quad FP^2 = (PM)^2$

$(x - 4)^2 + (y - 0)^2 = \left(\dfrac{x + 4}{\sqrt{1}}\right)^2$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 = x^2 + 8x + 16$

$\Rightarrow \quad y^2 = 16x$

Length of latus rectum = coefficient of x = 16.

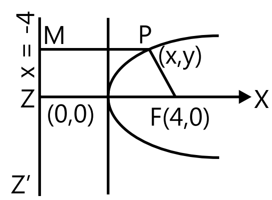
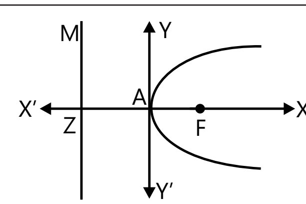
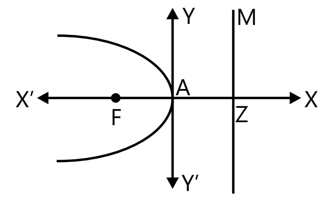
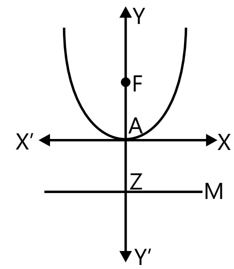
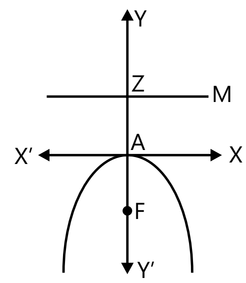


Figure 10.6

3. STANDARD FORMS OF PARABOLA

Right handed parabola: The equation of this type of parabola is of the form $y^2 = 4ax$, a > 0. See Fig. 10.7	 $y^2 = 4ax$ **Figure 10.7**
Left handed parabola: The equation of this type of parabola is of the form $y^2 = -4ax$, a > 0. See Fig. 10.8	 $y^2 = -4ax$ **Figure 10.8**

Upward parabola: The equation of this type of parabola is of the form $x^2 = 4ay$, $a > 0$. See Fig. 10.9	 $x^2 = 4ay$ **Figure 10.9**
Downward parabola: The equation of this type of parabola is of the form $x^2 = -4ay$, $a > 0$. See Fig. 10.10	 $x^2 = -4ay$ **Figure 10.10**

Equation of the parabola Properties	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex (Co-ordinates)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus (Co-ordinates)	(a, 0)	(–a, 0)	(0, a)	(0, –a)
Latus rectum (length)	4a	4a	4a	4a
Axis (Equation)	y = 0	y = 0	x = 0	x = 0
Directrix (Equation)	x = –a	x = a	y = –a	y = a
Symmetry (about)	x-axis	x-axis	y-axis	y-axis

CONCEPTS

Don't get confused between $x^2 = 4ay$ and $y^2 = 4ax$.

Almost every condition is different for both parabolas.

Nivvedan (JEE 2009, AIR 113)

Illustration 4: Find the vertex, the axis, the focus, the directrix, and latus rectum of the parabola, $4y^2 + 12x - 20y + 67 = 0$. **(JEE MAIN)**

Sol: Represent the given equation in the standard form and then compare it with the standard forms to get vertex, axis, focus, directrix and latus rectum to get the answers.

The given equation is

$$4y^2 + 12x - 20y + 67 = 0 \quad \Rightarrow \quad y^2 + 3x - 5y + 67/4 = 0$$

$$\Rightarrow \quad y^2 - 5y = -3x - 67/4 \quad \Rightarrow \quad y^2 - 5y + \left(\frac{5}{2}\right)^2 = -3x - \frac{67}{4} + \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4} \quad \Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right) \quad \text{... (i)}$$

Let $x = X - 7/2,\ y = Y + 5/2$ ... (ii)

Using these relations, equation (i) reduces to $Y^2 = -3X$ ... (iii)

This is of the form $Y^2 = -4aX$. On comparing, we get $a = 3/4$.

Vertex: The coordinates of the vertex of $Y^2 = -3X$ are $(X = 0, Y = 0)$ So, the coordinates of the vertex before transformation are $\left(-\frac{7}{2}, \frac{5}{2}\right)$.

Axis: The equation of the axis of the parabola $Y^2 = -3X$ is $Y = 0$. So, the equation of the axis before transformation is $y = 5/2$.

Focus: The coordinates of the focus are $(X = -3/4, Y = 0)$. So, the coordinates of the focus are $\left(-\frac{17}{4}, \frac{5}{2}\right)$.

Directrix: The equation of the directrix is $X = +a = +3/4$. So, the equation of the directrix before transformation is $x = -11/4$

Latus rectum: The length of the latus rectum of the given parabola is $4a = 3$.

Illustration 5: Prove that $9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$ represents a parabola. Also find its focus and directrix. **(JEE MAIN)**

Sol: A general equation of a conic represents a parabola if $\Delta \neq 0$ and $h^2 = ab$. In order to get the focus and the directrix, convert the given equation into the standard form and compare with the standard form.

Here $h^2 - ab = (-12)^2 - 9(16) = 144 - 144 = 0$. Also, $\Delta = \begin{vmatrix} 9 & -12 & -10 \\ -12 & 16 & \frac{-15}{2} \\ -10 & \frac{-15}{2} & -60 \end{vmatrix} \neq 0$

$\therefore$ The given equation represents a parabola. Now, the equation is $(3x - 4y)^2 = 5(4x + 3y + 12)$.

Clearly, the lines $3x - 4y = 0$ and $4x + 3y + 12 = 0$ are perpendicular to each other. So, let

$$\frac{3x - 4y}{\sqrt{3^2 + (-4)^2}} = Y, \quad \frac{4x + 3y + 12}{\sqrt{4^2 + 3^2}} = X \quad \text{... (i)}$$

The equation of the parabola becomes $Y^2 = X = 4.\frac{1}{4}X$

$\therefore$ Here $a = \frac{1}{4}$ in the standard equation.

$\therefore$ The focus $= (a, 0)_{X, Y} = \left(\frac{1}{4}, 0\right)_{X, Y}$

If, $X = \frac{1}{4}$, $Y = 0$, then from the equations of transformation in (i), we get

$$\frac{3x - 4y}{5} = 0, \ \frac{4x + 3y + 12}{5} = \frac{1}{4} \quad \Rightarrow \quad 3x - 4y = 0,\ 4x + 3y = \frac{-43}{4}$$

$\Rightarrow \quad y = \frac{3x}{4}, y = \frac{1}{3}\left(-\frac{43}{4} - 4x\right) \qquad \therefore \quad \frac{3x}{4} = \frac{1}{3}\left(-\frac{43}{4} - 4x\right)$

$\Rightarrow \quad 9x = -43 - 16x\,;\ \therefore\ x = \frac{-43}{25} \qquad$ and $y = \frac{3}{4}x = \frac{3}{4}.\frac{-43}{25} = \frac{-129}{100}$

$\therefore \quad \text{Focus} = \left(\frac{-43}{25}, \frac{-129}{100}\right)$

The equation of the directrix is $\qquad X + a = 0$, i.e., $X + \frac{1}{4} = 0$

or $\quad \frac{4x+3y+12}{5} + \frac{1}{4} = 0 \qquad$ or $\quad 4x + 3y = -\frac{5}{4} - 12 = -\frac{53}{4}$

$\therefore$ The equation of the directrix is $4x + 3y + \frac{53}{4} = 0$.

Illustration 6: Find the equation of the parabola whose latus rectum is 4 units, the axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$. **(JEE ADVANCED)**

Sol: The square of the distance of a point from the directrix is equal to the product of latus rectum and the distance of the point from the axis.

Let P(x, y) be any point on the parabola and let PM and PN be perpendiculars from P to the axis and to the tangent at the vertex respectively. Then

$PM^2 = (\text{Latus rectum})(PN) \quad \Rightarrow \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right) \Rightarrow (3x + 4y - 4)^2 = 20(4x - 3y + 7)$

This is the equation of the required parabola.

Note: In the above examples, we have learnt how to find the vertex, the focus, the axis, the directrix etc. of parabolas reducible to one of the various forms given. If the equation of a parabola is quadratic in both x and y, then to find its vertex, focus, axis, etc., we follow the following algorithm.

Step I: Obtain the equation of the parabola and express it in the form $(ax + by + c)^2 = (\text{Constant})(bx - ay + c')$

It should be noted here that $ax + by + c = 0$ and $bx - ay + c' = 0$ represent perpendicular lines.

Step II: Divide both sides by $\sqrt{a^2+b^2}$ to obtain $\left(\frac{ax+by+c}{\sqrt{a^2+b^2}}\right)^2 = (\text{Constant})\left(\frac{bx-ay+c'}{\sqrt{a^2+b^2}}\right)$

Step III: Now substitute $\frac{ax+by+c}{\sqrt{a^2+b^2}} = Y$ and $\frac{bx-ay+c'}{\sqrt{a^2+b^2}} = X$ in step II to obtain $Y^2 = (\text{Constant})\,X$.

Step IV: Compare the equation obtained in step III with $Y^2 = 4ax$ to obtain various elements like vertex, focus, axis, etc., and use the transformation in step III to obtain the corresponding elements of the given parabola.

Illustration 7: Find the equation of the parabola whose axis is parallel to the y-axis and which passes through the points (0, 4), (1, 9) and (–2, 6). Also find its latus rectum. **(JEE ADVANCED)**

Sol: Consider a standard equation of a parabola with the vertex at (α, β) such that the axis is parallel to the Y-axis. Substitute the points given and find the. (see Fig. 10.11) equation

As the axis is parallel to the y-axis, it will be of the form $x - \alpha = 0$ and the tangent to the vertex (which is perpendicular to the axis) will be of the form $y - \beta = 0$ for some β.

Hence, the equation of the parabola will be of the form

$$(x - \alpha)^2 = 4a(y - \beta) \qquad \text{... (i)}$$

where α, β, a are the unknown constants and 4a being the latus rectum.

(i) passes through (0, 4), (1, 9) and (–2, 6). So

$(0 - \alpha)^2 = 4a(4 - \beta)$,

i.e., $a^2 = 4a(4 - \beta)$ ... (ii)

and $(1 - \alpha)^2 = 4a(9 - \beta)$

i.e, $1 - 2\alpha + a^2 = 4a(9 - \beta)$ ... (iii)

and $(-2 - \alpha)^2 = 4a(6 - \beta)$,

i.e, $4 + 4\alpha + a^2 = 4a(6 - \beta)$ ... (iv)

(iii) – (ii) $\Rightarrow$ $1 - 2\alpha = 20a$ ... (v)

(iv) – (iii) $\Rightarrow$ $3 + 6\alpha = -12a$

$\therefore \quad \dfrac{1-2\alpha}{3+6\alpha} = \dfrac{20a}{-12a} \quad \Rightarrow \quad -3(1 - 2\alpha) = 5(3 + 6\alpha)$ Z

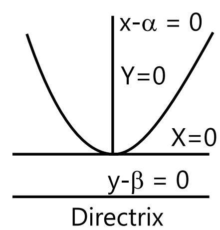


Figure 10.11

$\Rightarrow \quad 24\alpha = -18; \qquad \therefore \qquad \alpha = -\dfrac{3}{4}$

$\therefore \quad$ (v) $\Rightarrow \quad 1 + \dfrac{3}{2} = 20a; \qquad \therefore \qquad a = \dfrac{5}{40} = \dfrac{1}{8}$

$\therefore \quad$ (ii) $\Rightarrow \quad \left(-\dfrac{3}{4}\right)^2 = 4.\,\dfrac{1}{8}\,(4 - \beta)$

$\Rightarrow \quad \dfrac{9}{16} = \dfrac{1}{2}(4 - \beta)$, i.e., $4 - \beta = \dfrac{9}{8} \quad \Rightarrow \quad \beta = \dfrac{23}{8}$

$\therefore \quad$ From (i), the equation of the parabola is $\left(x+\dfrac{3}{4}\right)^2 = 4.\,\dfrac{1}{8}.\left(y-\dfrac{23}{8}\right)$

$\Rightarrow \quad x^2 + \dfrac{3}{2}x + \dfrac{9}{16} = \dfrac{1}{2}y - \dfrac{23}{16} \qquad \Rightarrow \quad x^2 + \dfrac{3}{2}x - \dfrac{1}{2}y + 2 = 0$

$\therefore \quad 2x^2 + 3x - y + 4 = 0$ and its latus rectum $= 4a = 4.\,\dfrac{1}{8} = \dfrac{1}{2}$

4. PARAMETRIC EQUATIONS OF A PARABOLA

For the standard equation of the parabola $y^2 = 4ax$ we write the parametric equations as $x = at^2$ and $y = 2at$. Thus, the parametric coordinates of a point on the parabola are $(at^2, 2at)$. Unlike the rest of conics, there is no physical significance for the parameter t.

5. A POINT AND A PARABOLA

The point (x_1, y_1) lies inside, on, or outside $y^2 = 4ax$ according to $y_1^2 - 4ax_1$ <, =, or > 0.

Illustration 8: Find the set of values of α in the interval $[\pi/2, 3\pi/2]$ for which the point $(\sin\alpha, \cos\alpha)$ does not lie outside the parabola $2y^2 + x - 2 = 0$. **(JEE MAIN)**

Sol. Use the concept of the Position of a point w.r.t. a parabola. If the point $(\sin\alpha, \cos\alpha)$ lies inside or on the parabola $2y^2 + x - 2 = 0$.

$2\cos^2\alpha + \sin\alpha - 2 \le 0 \quad \Rightarrow \quad 2 - 2\sin^2\alpha + \sin\alpha - 2 \le 0$

$\Rightarrow \quad \sin\alpha(2\sin\alpha - 1) \ge 0 \quad \Rightarrow \quad \sin\alpha \le 0$ or, $2\sin\alpha - 1 \ge 0$

$\Rightarrow \quad \alpha \in [\pi, 3\pi/2]$ or, $\alpha \in [\pi/2, 5\pi/6] \quad \Rightarrow \quad \alpha \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$

6. CHORD

6.1 Equation af a Chord

Let P $(at_1^2, 2at_1)$ and Q$(at_2^2, 2at_2)$ be two points on the parabola $y^2 = 4ax$. Then, the equation of the chord PQ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

Note: (a) If the chord joining points t_1 and t_2 on the parabola $y^2 = 4ax$ passes through the focus then $t_1 t_2 = -1$.

(b) If one end of a focal chord of the parabola $y^2 = 4ax$ is P$(at^2, 2at)$, then the coordinates of the other end is $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

Length of a focal chord

(a) Let P$(at^2, 2at)$ be one end of a focal chord PQ of the parabola $y^2 = 4ax$. Then, the length of the focal chord with ends as P and Q is $a(t + 1/t)^2$

(b) We know that, $\left|t+\frac{1}{t}\right| \geq 2$ for all $t \neq 0$. $\therefore$ $a\left(t+\frac{1}{t}\right)^2 \geq 4a$ $\Rightarrow$ PQ $\geq$ 4a.

Thus, the length of the smallest focal chord of the parabola is 4a which is the length of its latus rectum.For this reason, the latus rectum of a parabola is the smallest focal chord.

(c) The semi-latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

(d) The circle described on any focal chord of a parabola as a diameter of that circle, also touches the directrix.

(e) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ at two points that can be real, coincident or imaginary according to a >,=,< cm $\Rightarrow$ condition of tangency is, $c = a/m$.

(f) Length of the chord intercepted by the parabola on the line $y = mx + c$ is: $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$

CONCEPTS

The length of the chord joining two points t_1 and t_2 on the parabola $y^2 = 4ax$ is

$a(t_1 - t_2)\sqrt{(t_1+t_2)^2+4}$

Nitin Chandrol (JEE 2012, AIR 134)

Illustration 9: A quadrilateral ABCD is inscribed in $y^2 = 4ax$ and 3 of its sides AB, BC, CD pass through fixed points $(\alpha, 0)$, $(\beta, 0)$ and $(\gamma, 0)$, then show that the 4th side also passes through a fixed point. Also find this fixed point.

Sol: Consider four parametric coordinates and form equations according to the given conditions.

Let t_1, t_2, t_3 and t_4 be the parametric coordinates of A, B, C and D respectively.

Equation of AB is $y(t_1 + t_2) = 2x + 2at_1t_2$. Given that this line passes through $(\alpha, 0)$.

$\Rightarrow \quad t_1t_2 = -\frac{\alpha}{a}$

Similarly, $t_2t_3 = -\frac{\beta}{a}$, $t_3t_4 = -\frac{\gamma}{a}$ Accordingly, $t_1t_4 = \frac{t_1t_2\,t_3t_4}{t_2t_3} = \frac{-\frac{\alpha}{a}.\frac{\gamma}{a}}{\frac{\beta}{a}} = \frac{-\alpha\gamma}{\beta a}$

From the above result we can say that AD always passes through a point $\left(\frac{\alpha\gamma}{\beta}, 0\right)$

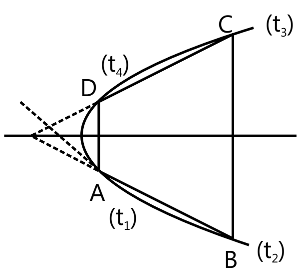


Figure 10.12

Illustration 10: Find the relation between the line y = x + 1 and the parabola $y^2 = 4x$. **(JEE MAIN)**

Sol: Solve the two given equations and based on the intersection we can find the relation between the two.

Solving the line and parabola, we get $(x + 1)^2 = 4x \Rightarrow (x - 1)^2 = 0$. Therefore y = x + 1 is a tangent to the parabola.

Illustration 11: Through the vertex O of a parabola $y^2 = 4x$, the chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ intersects the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. **(JEE MAIN)**

Sol: Use parametric coordinates for P (t_1) and Q (t_2) to find the relation between them and obtain the equation of chord PQ. Let the coordinates of P and Q be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$ respectively. Then,

m_1 = Slope of OP = $\frac{2}{t_1}$; m_2 = Slope of OQ = $\frac{2}{t_2}$

Since OP is perpendicular to OQ. Therefore, $m_1m_2 = -1$

$\Rightarrow \frac{2}{t_1} \times \frac{2}{t_2} = -1 \qquad \Rightarrow t_1t_2 = -4$

The equation of chord PQ is $y(t_1 + t_2) = 2x + 2t_1t_2$

$\Rightarrow y(t_1 + t_2) = 2x - 8$

Clearly, it passes through (4, 0) for all values of t_1 and t_2. Thus, PQ cuts x-axis at a fixed point (4, 0) for all position of point P.

Let R(h, k) be the mid-point of PQ. Then, $2h = t_1^2 + t_2^2$ and $k = t_1 + t_2$ ... (ii)

$\therefore (t_1 + t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2 \qquad \Rightarrow k^2 = 2h - 8$ [Using (i) and (ii)]

Hence, the locus is $y^2 = 2x - 8$. (This is also a parabola)

Illustration 12: Find the locus of the centre of the circle described on any focal chord of a parabola $y^2 = 4ax$ as diameter. **(JEE ADVANCED)**

Sol: Use the parametric form.

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord PQ of the parabola $y^2 = 4ax$. Then, $t_1t_2 = -1$.

Let (h, k) be the coordinates of the centre of the circle described on PQ as the diameter. Then

$h = a/2\,(t_1^2 + t_2^2)$ and $k = a(t_1 + t_2)$

$\Rightarrow 2h/a = t_1^2 + t_2^2$ and $(k/a)^2 = (t_1 + t_2)^2$

$\Rightarrow 2h/a = t_1^2 + t_2^2$ and $k^2/a^2 = t_1^2 + t_2^2 + 2t_1t_2$

$\Rightarrow k^2/a^2 = 2h/a - 2$ $[\because t_1t_2 = -1]$

$\Rightarrow k^2 = 2a(h - a)$

Hence, the locus of (h, k) is $y^2 = 2a(x - a)$

Illustration 13: A triangle ABC of area Δ is inscribed in the parabola $y^2 = 4ax$ such that the vertex A lies at the vertex of the parabola and side BC is a focal chord. Prove that the difference of the distances of B and C from the axis of the parabola is $2\Delta/a$. **(JEE ADVANCED)**

Sol: Use the parametric form for the points B and C and proceed according to the conditions given in the question.

Let the coordinates of B and C be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively.

Since BC is a focal chord of the parabola $y^2 = 4ax$. Therefore,

Since, Δ = Area of ΔABC

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix} \Rightarrow \Delta = |a^2t_1t_2(t_1 - t_2)| \Rightarrow \Delta = |-a^2(t_1 - t_2)| \quad [\because t_1t_2 = -1]$$

$\Rightarrow \quad \Delta = a^2|t_2 - t_1|$

We have, BL = $2at_1$ and CM = $2at_2$ $\quad \therefore \quad |BL - CM| = |2at_1 - 2at_2|$

$2a|t_1 - t_2| = 2a \times \Delta/a^2 = 2\Delta/a$

Illustration 14: Let PQ be a variable focal chord of the parabola $y^2 = 4ax$ whose vertex is A. Prove that the locus of the centroid of ΔAPQ is a parabola whose latus rectum is 4a/3. **(JEE ADVANCED)**

Sol: Take two general points on the focal chord of the parabola and use the formula for a centroid to obtain a relation between the ordinate and the abscissae of the centroid.

Let the coordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively. Since PQ is a focal chord.

Therefore $t_1t_2 = -1$ ... (i)

Let (h, k) be the coordinates of the centroid of ΔAPQ. Then,

$h = \frac{at_1^2 + at_2^2}{3}$ and $k = \frac{2at_1 + 2at_2}{3}$ $\quad \Rightarrow \quad \frac{3h}{a} = t_1^2 + t_2^2$ and $3k/2a = t_1 + t_2$ ... (ii)

Now $\quad (t_1 + t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2 \quad \Rightarrow \quad \left(\frac{3k}{2a}\right)^2 = \frac{3h}{a} - 2 \quad \Rightarrow \quad \frac{9k^2}{4a^2} = \frac{3h - 2a}{a}$

$\Rightarrow \quad k^2 = \frac{4a}{9}(3h - 2a) \quad \Rightarrow \quad k^2 = \frac{4a}{3}\left(h - \frac{2a}{3}\right)$

Hence, the locus of (h, k) is $\quad y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

Clearly, it represents a parabola whose latus rectum is $\frac{4a}{3}$

Illustration 15: A variable chord through the focus of the parabola $y^2 = 4ax$ intersects the curve at P and Q. The straight line joining P to the vertex cuts the line joining Q to the point (–a, 0) at R. Show that the locus of R is $y^2 + 8x^2 + 4ax = 0$. **(JEE ADVANCED)**

Sol: Consider two points on the parabola and obtain the equation of the straight line passing through P and Q. Then obtain the locus of the intersection point.

Let the coordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively. It is given that the chord PQ passes through the focus S(–a, 0). Therefore $t_1t_2 = -1$.

Equation of OP is $\quad y = \frac{2}{t_1}x$ ... (i)

Equation of OQ is $\quad y - 0 = \frac{2at_2 - 0}{at_2^2 + a}(x + a) \quad \Rightarrow \quad y = \frac{2t_2}{t_2^2 + 1}(x + a)$ ... (ii)

Let R(h, k) be the point of intersection of OP and OQ. Then, $k = \frac{2}{t_1}h$ and $k = \frac{2t_2}{t_2^2 + 1}(h + a)$

$\Rightarrow \quad k = -2ht_2$ and $k = \frac{2t_2}{t_2^2 + 1}(h + a) \quad [\because t_1t_2 = -1]$

$\Rightarrow \quad k = \frac{-k/h}{1 + \frac{k^2}{4h^2}}(h + a) \quad \Rightarrow \quad k^2 + 4h^2 = -4h^2 - 4ah \quad \Rightarrow \quad k^2 + 8h^2 + 4ah = 0$

Hence, the locus of (h, k) is obtained by replacing (h, k) with (x, y) in the above equation.

Illustration 16: AP is any chord of the parabola $y^2 = 4ax$ passing through the vertex A. PQ is a chord perpendicular to AP. Find the locus of the mid-point of PQ. **(JEE ADVANCED)**

Sol: Use the parametric form to find the equation of the line PQ and then the locus of the mid-point of PQ.

Let the coordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively. Then

m_1 = Slope of AP = $\frac{2}{t_1}$ $\qquad$ m_2 = Slope of PQ = $\frac{2}{t_1+t_2}$

Since AT ⊥ PQ. Therefore, $m_1m_2 = -1$

$$\Rightarrow \frac{4}{t_1(t_1+t_2)} = -1 \Rightarrow t_1(t_1+t_2) = -4 \qquad \text{... (i)}$$

Let R(h, k) be the mid-point of PQ. Then, $2h = a(t_1^2 + t_2^2)$ ... (ii)

$k = a(t_1 + t_2)$ ... (iii)

From (i) and (iii), we have $\frac{kt_1}{a} = -4 \quad \Rightarrow \quad t_1 = \frac{-4a}{k}$

From (i), we have $t_1^2 + t_1t_2 = -4 \quad \Rightarrow \quad t_1t_2 = -t_1^2 - 4$

$$\Rightarrow t_1t_2 = -\frac{16a^2}{k^2} - 4 \qquad \left[\because t_1 = -\frac{4a}{k}\right]$$

$$\Rightarrow t_1t_2 = -\frac{4}{k^2}(4a^2+k^2) \qquad \text{... (iv)}$$

Now $(t_1 + t_2)^2 = (t_1^2 + t_2^2) + 2t_1t_2 \quad \Rightarrow \quad \frac{k^2}{a^2} = \frac{2h}{a} - \frac{8}{k^2}(4a^2 + k^2)$ [Using (2), (3) and (4)]

$\Rightarrow \quad k^4 = 2k^2ha - 8a^2(4a^2 + k^2)$ $\qquad$ Hence, the locus of (h, k) is

$y^4 = 2y^2xa - 8a^2(4a^2 + y^2)$ or, $\qquad y^4 + 8a^2y^2 - 2axy^2 + 32a^4 = 0$

Illustration 17: Find a point K on axis of $y^2 = 4ax$ which has the property that if chord PQ of the parabola is drawn through it then $\frac{1}{PK^2}+\frac{1}{KQ^2}$ is same for all positions of the chord.

Sol: Consider a point of the axis of the parabola and use the distance form of a straight line to find the relation between the parameters. The next step is to prove that $\frac{1}{PK^2}+\frac{1}{KQ^2}$ remains unchanged.

$\frac{1}{PK^2}+\frac{1}{KQ^2}$ = constant $\qquad \frac{x-b}{\cos\theta} = \frac{y-0}{\sin\theta} = r \qquad (r\sin\theta)^2 = 4a(r\cos\theta + b)$

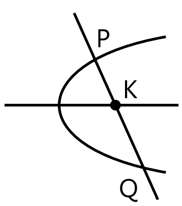


Figure 10.13

$$\Rightarrow r^2\sin^2\theta - 4a\, r\cos\theta - 4ab = 0$$

$$\Rightarrow \frac{r_1^2+r_2^2}{r_1^2\, r_2^2} = \frac{(r_1+r_2)^2 - 2r_1r_2}{r_1^2\, r_2^2}$$

$$\Rightarrow \frac{\left[\left((4a\cos\theta)/(\sin^2\theta)\right)^2 - 2\left((-4ab)/(\sin^2\theta)\right)\right]}{16a^2b^2/\sin^4\theta} = \frac{16a^2\cos^2\theta + 8ab\sin^2\theta}{16a^2b^2}$$

$f(\theta) = \frac{8a(2a\cos^2\theta + b\sin^2\theta)}{16a^2b^2}$; $\qquad f'(\theta) = \sin2\theta(b - 2a) = 0 \qquad b = 2a$

Equation of the chord bisected at a given point: The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$ $\quad$ or $\quad T = S_1$

where $\quad S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

6.2 Diameter of a Parabola

The locus of the middle points of a system of parallel chords is called a diameter and in the case of a parabola this diameter is shown to be a straight line which is parallel to the axis of the parabola.

The equation of the diameter bisecting the chords of the parabola $y^2 = 4ax$, with slope m, is $y = 2a/m$.

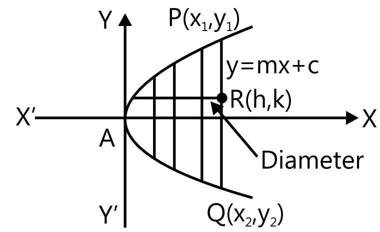


Figure 10.14

CONCEPTS

1. The area of a triangle formed inside the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ where y_1, y_2, y_3 are the ordinates of vertices of the triangle.
2. If the vertex and the focus of a parabola are on the x-axis and at distance a and a′ from the origin respectively, then the equation of the parabola is $y^2 = 4(a' - a)(x - a)$

Vijay Senapathi (JEE 2009, AIR 71)

Illustration 18: A ray of light is coming along the line y = b, (b > 0) from the positive direction of the x-axis and strikes a concave mirror whose intersection with the x-y plane is the parabola $y^2 = 4ax$, (a > 0). Find the equation of the reflected ray and show that it passes through the focus of the parabola. **(JEE ADVANCED)**

Sol: In this question, we need to use the concept of angle between two lines. Use this concept to find the equation of the reflected ray and to show that the focus lies on the reflected ray.

Let P be the point of incidence. Then P is the intersection of the line y = b and the parabola $y^2 = 4ax$. $\therefore P = \left(\frac{b^2}{4a}, b\right)$

$\therefore$ The equation of the tangent PT at P is $y.b = 2a\left(x + \frac{b^2}{2}\right)$ or $by = 2ax + \frac{b^2}{2}$ ... (i)

'm' of (i) is $\frac{2a}{b}$. So, $\tan\theta = \frac{2a}{b}$

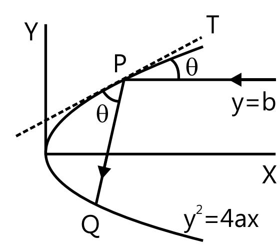


Figure 10.15

Let the slope of the reflected ray PQ be m. $\therefore \tan\theta = \left|\frac{m - (2a/b)}{1 + m.(2a/b)}\right|$

or $\frac{2a}{b} = \left|\frac{m - (2a/b)}{1 + m.(2a/b)}\right|$ $\therefore \frac{m - (2a/b)}{1 + m.(2a/b)} = \pm\frac{2a}{b}$

or $m - \frac{2a}{b} = \pm\left(\frac{2a}{b} + m.\frac{4a^2}{b^2}\right)$ $\therefore m - \frac{2a}{b} = \frac{2a}{b} + m.\frac{4a^2}{b^2}$

and $m - \frac{2a}{b} = -\frac{2a}{b} - m.\frac{4a^2}{b^2}$ $\therefore m\left(1 - \frac{4a^2}{b^2}\right) = \frac{4a}{b}$

and $m\left(1 + \frac{4a^2}{b^2}\right) = 0.$ But $m \neq 0$ $\therefore m\left(1 - \frac{4a^2}{b^2}\right) = \frac{4a}{b} \Rightarrow m = \frac{4ab}{b^2 - 4a^2}$

$\therefore$ The equation of the reflected ray PQ is $y - b = \frac{4ab}{b^2 - 4a^2}\left(x - \frac{b^2}{a^2}\right)$

or $(b^2 - 4a^2)y - b(b^2 - 4a^2) = 4abx - b^3$ or $(b^2 - 4a^2)y - 4abx + 4a^2b = 0$

This will pass through the focus (a, 0) if $(b^2 - 4a^2)0 - 4ab \,.\, a + 4a^2b = 0$, which is true.

$\therefore$ The reflected ray passes through the focus.

7. TANGENT

The equation of tangent at (x_1, y_1) to any conic section can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$ and without changing the constant (if any) in the equation of the curve.

7.1 Equation of Tangent to Parabola (Standard Form)

The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $yy_1 = 2a(x + x_1)$. The equation of tangents to all standard forms of parabola at point (x_1, y_1) are given below for ready reference.

Equation of parabola	Equation of tangent
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

7.2 Parametric Form

The equation of the tangent to the parabola $y^2 = 4ax$ at a point $(at^2, 2at)$ is given by $ty = x + at^2$. The parametric equations of tangents to all standard forms of parabola are as given below:

Equation of parabola	Point of contact	Equation of the tangent
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

7.3 Slope Form of a Tangent

The equation of the tangent with slope m to the parabola $y^2 = 4ax$ is $y = mx + a/m$

The equation of tangents to various standard forms of the parabola in terms of the slope of the tangent are as follows:

Equation of parabola	Equation of the tangent	Condition of tangency	Point of contact
$y^2 = 4ax$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
$y^2 = -4ax$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$	$\left(-\frac{a}{m^2}, \frac{-2a}{m}\right)$
$x^2 = 4ay$	$x = my + \frac{a}{m}$	$c = \frac{a}{m}$	$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$
$x^2 = -4ay$	$x = my - \frac{a}{m}$	$c = -\frac{a}{m}$	$\left(\frac{-2a}{m}, \frac{-a}{m^2}\right)$

7.4 Point of Intersection of Tangents

The x-coordinate of the point of intersection of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is the G.M. of the x-coordinates of P and Q and the y-coordinate is the A.M. of the y-coordinates of P and Q that is, the tangents at the points $(at_1^2, 2at_1)$ and $(at^2, 2at_2)$ intersect at $(at_1t_2, a(t_1 + t_2))$

7.5 Director Circle

The locus of the point of intersection of perpendicular tangents to a conic is known as its director circle. The director circle of a parabola is its directrix.

7.6 Pair of Tangents

We see that two tangents can be drawn from a external point to a parabola. The two tangents are real and distinct or coincident or imaginary according to whether the given point lies outside, on or inside the parabola and the combined equation of the pair of tangents drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$ is $(y^2 - 4ax)(y_1^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2$ or, $SS_1 = T^2$

where $S \equiv y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T \equiv yy_1 - 2a(x + x_1)$.

7.7 Chord of Contact

The chord joining the points of contact of two tangents drawn from an external point P to a parabola is known as the chord of contact of tangents drawn from P.

The chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

7.8 Common tangents to two conics

In this section, we shall discuss some problems on finding the common tangents to two conics. The following algorithm may be used to find the common tangents to two given conics.

Algorithm

Step I: Observe the equations of the two conics.

Step II: Identify the conic whose equation is either in standard form or it is reducible to standard form.

Step III: Write the equation of the tangent in slope form to the conic obtained in step II.

Step IV: Apply the condition that the tangent obtained in step III also touches the second conic and find the value of m (m is the slope).

Step V: Substitute the value(s) of m obtained in step IV in the equation written in step III. The equation obtained is the required tangent(s).

7.9 Important Results

(a) The tangent at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

(b) The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point.

(c) The portion of the tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.

(d) The perpendicular drawn from the focus on any tangent to parabola, intersects on the tangent at the vertex.

(e) The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

(f) The circumcircle of the triangle formed by any three points on a parabola passes through the focus of the parabola.

(g) The tangent at any point on a parabola is equally inclined to the focal distance of the point and axis of the parabola.

(h) If SZ is perpendicular to the tangent at a point P of a parabola, then Z lies on the tangent at the vertex and SZ^2 = AS.SP, where A is the vertex of the parabola.

(i) The image of focus w.r.t. any tangent to a parabola lies on its directrix.

(j) The length of the subtangent at any point on a parabola is equal to twice the abscissa of that point.

(k) If the tangents to the parabola $y^2 = 4ax$ at the points P and Q intersect at T, then TP and TQ subtend equal angles at the focus.

CONCEPTS

- The locus of the point of the intersection of the tangent at P and the perpendicular from the focus to this tangent is the tangent at the vertex of the parabola.
- $y = mx - am^2$ is a tangent to the parabola $x^2 = 4ay$ for all values of m and its point of contact is $(2am, am^2)$.

  The formula given in the previous table for the parabola $x^2 = 4ay$ is different as in that case, the slope of the tangent is 1/m. Don't confuse between the two formulae
- Angle between the tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_1)$ on the parabola $y^2 = 4ax$ is $\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1 t_2}\right|$

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 19: Prove that the straight line y = mx + c touches the parabola $y^2 = 4a(x + a)$ if c = ma + a/m. **(JEE MAIN)**

Sol: Apply the tangency condition for standard form.

Equation of the tangent of slope m to the parabola $y^2 = 4a(x + a)$ is $y = m(x + a) + a/m$

$\Rightarrow \quad y = mx + a\left(m + \frac{1}{m}\right)$ but the given tangent is y = mx + c $\quad \therefore \quad c = am + a/m$

Illustration 20: A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line y = 3x + 5. Find its equation and its point of contact. **(JEE MAIN)**

Sol: Find the slopes of the lines making an angle of 45°. Then use the standard equation of the tangent to get the answer.

Slope values of the required tangents are $m = \frac{3 \pm 1}{1 \mp 3} \quad \Rightarrow \quad m_1 = -2, m_2 = 1/2$

$\because$ Equation of the tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + a/m$

$\therefore$ Tangents are $y = -2x - 1$ at $\left(\frac{1}{2}, -2\right)$ and $2y = x + 8$ at $(8, 8)$.

Illustration 21: Find the equation of the common tangents of the parabola $y^2 = 4ax$ & $x^2 = 4by$. **(JEE MAIN)**

Sol: Use the standard slope form of the equation of the tangent to find the slope and hence, the equation of the common tangent.

Equation of tangent to $y^2 = 4ax$ is $y = mx + a/m$ ... (i)

Equation of tangent to $x^2 = 4by$ is $x = m_1y + b/m_1 \Rightarrow y = \frac{1}{m_1}x - \frac{b}{(m_1)^2}$ ... (ii)

For common tangent, (i) & (ii) must represent the same line,

$$\therefore \quad \frac{1}{m_1} = m \text{ and } \frac{a}{m} = -\frac{b}{m_1^2} \quad \Rightarrow \quad \frac{a}{m} = -m^2 b \quad \Rightarrow \quad m = \left(-\frac{a}{b}\right)^{1/3}$$

$\therefore$ Equation of the common tangent is $y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{b}{a}\right)^{1/3}$

Illustration 22: A chord of the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Find the locus of the point of intersection of tangents at its extremities. **(JEE MAIN)**

Sol: Find the relation between the parametric coordinates of the point subtending right angle at the vertex. Then, use this relation to find the locus of the intersection of the tangents.

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be two points on the parabola $y^2 = 4ax$ such that the chord PQ subtends a right angle at the vertex O(0, 0). Then, Slope of OP × Slope of OQ = –1

$$\Rightarrow \quad \frac{2}{t_1} \times \frac{2}{t_2} = -1 \quad \Rightarrow \quad t_1t_2 = -4 \qquad ....(i)$$

Let R(h, k) be the point of intersection of tangents at P and Q. Then,

$h = at_1t_2$ and $k = a(t_1 + t_2) \Rightarrow h = -4a$

Hence, the locus of R(h, k) is $x = -4a$.

Illustration 23: The inclinations θ and ϕ of two tangents to the parabola $y^2 = 4ax$ with the axis are given by $\tan\theta = 1/m$ and $\tan\phi = m/2$. Show that, as m varies, the point of intersection of the tangents traces a line parallel to the directrix of the parabola. **(JEE ADVANCED)**

Sol: Use the parametric form to find the relation between the two points and hence, find the locus of the intersection point.

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be two points on the parabola $y^2 = 4ax$ such that the tangents at P and Q are inclined at angles θ and ϕ with the axis of the parabola. Then

$\tan\theta$ = Slope of the tangent at P

and, $\tan\phi$ = Slope of the tangent at Q

$\Rightarrow \tan\theta = 1/t_1$ and $\tan\phi = 1/t_2$

$\Rightarrow 1/m = 1/t_1$ and $m/2 = 1/t_2$

$$\Rightarrow \quad 1/m \times m/2 = \frac{1}{t_1t_2} \quad \Rightarrow \quad t_1t_2 = 2 \qquad ... (i)$$

Let R(h, k) be the point of intersection of tangents at P and Q. Then

$h = at_1t_2$ and $k = a(t_1 + t_2) \quad \Rightarrow \quad h = 2a$ [Using (1)]

Hence, the locus of R(h, k) is $x = 2a$, which is a line parallel to the directrix of the parabola.

Illustration 24: Tangents PQ and PR are drawn to a parabola $y^2 = 4ax$. If p_1, p_2, p_3 be the perpendiculars from P, Q and R to any tangent of the parabola, prove that p_1 is the geometric mean of p_2 and p_3. **(JEE ADVANCED)**

Sol: Consider two points on the parabola and find the point of intersection of the tangents at these points. Then, consider a tangent in the general form and find p_1, p_2 and p_3.

Let the coordinates of Q and R be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively. Then, the equations of the tangents at Q and R are $t_1y = x + at_1^2$, $t_2y = x + at_2^2$ respectively.

The coordinates of P are $(at_1t_2, a(t_1 + t_2))$. Let $ty = x + at^2$ ... (i)

be any tangent to the parabola $y^2 = 4ax$

Then, p_1 = Length of the perpendicular from P $(at_1t_2, a(t_1 + t_2))$ on (1)

$$= \left|\frac{at_1t_2 - a(t_1 + t_2)t + at^2}{\sqrt{t^2+1}}\right| \qquad = a\left|\frac{(t-t_1)(t-t_2)}{\sqrt{t^2+1}}\right|$$

p_2 = Length of the perpendicular from $Q(at_1^2, 2at_1)$ on (1)

$$\Rightarrow \quad p_2 = \left|\frac{2att_1 - at_1^2 - at^2}{\sqrt{t^2+1}}\right| = \frac{a(t-t_1)^2}{\sqrt{t^2+1}}$$ and, p_3 = Length of the perpendicular from $R(at_2^2, 2at_2)$ on (i)

$$p_3 = \left|\frac{2att_2 - at_2^2 - at^2}{\sqrt{t^2+1}}\right| = \frac{a(t-t_2)^2}{\sqrt{t^2+1}}$$ Clearly, $p_1^2 = p_2p_3$.

Hence, p_1 is the geometric mean of p_2 and p_3.

Illustration 25: Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x + a)$ and the other touches the parabola $y^2 = 4b(x + b)$. Prove that the point of intersection of the lines is $x + a + b = 0$.

(JEE ADVANCED)

Sol: Consider the two equations of the tangent in the slope form. Find the intersection of the two tangents and prove the above result.

The equation of the tangent of slope m_1 to $y^2 = 4a(x + a)$ is

$y = m_1(x + a) + a/m_1$ ... (i)

The equation of the tangent to the slope m_2 to the parabola $y^2 = 4b(x + b)$ is

$y = m_2(x + b) + b/m_2$ ... (ii)

It is given that (1) and (2) are perpendicular to each other. Therefore $m_2 = -1/m_1$

Putting $m_2 = -1/m_1$ in (1), we get $y = -1/m_1(x + b) - bm_1$ ... (iii)

The x-coordinate of the point of intersection of (1) and (3) is obtained by subtracting (3) from (1),

we get $0 = \left(m_1 + \frac{1}{m_1}\right)x + a\left(m_1 + \frac{1}{m_1}\right) + b\left(m_1 + \frac{1}{m_1}\right)$

$\Rightarrow \quad 0 = (x + a + b)m_1\left(m_1 + \frac{1}{m_1}\right) \qquad \Rightarrow \quad x + a + b = 0 \qquad \Rightarrow \quad x = -(a + b).$

Clearly, the point $(-(a + b), y)$ lies on the line $x + a + b = 0$ for all values of y. Thus, the point of intersection of (i) and (ii) lies on the line $x + a + b = 0$.

Illustration 26: Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(JEE ADVANCED)

Sol: Consider three points on a parabola and find the intersection of the tangents at these points. Then, find the equation of the circle passing through these three points and prove that it passes through the focus.

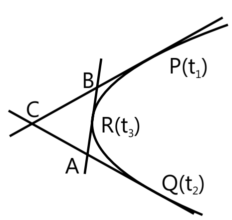


Figure 10.17

Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ and $R(at_3^2, 2at_3)$ be the three points on the parabola $y^2 = 4ax$.

The equations of tangents at P and Q are

$yt_1 = x + at_1^2$; $yt_2 = x + at_2^2$

The point of intersection C of the tangents at P and Q is $\{at_1t_2, a(t_1 + t_2)\}$.

Similarly, other points of intersection of tangents are

$B = \{(at_3t_1, a(t_3 + t_1)\}, A = \{(at_2t_3, a(t_2 + t_3)\}$

Let the equation of the circumcircle of the ΔABC be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

(i) will pass through the focus (a, 0) if

$a^2 + 2ga + c = 0$ ... (ii)

A, B, C are points on (i). So

$a^2t_1^2t_2^2 + a^2(t_1 + t_2)^2 + 2g.at_1t_2 + 2f.a(t_1 + t_2) + c = 0$ ... (iii)

$a^2t_2^2t_3^2 + a^2(t_2 + t_3)^2 + 2g.at_2t_3 + 2f.a(t_2 + t_3) + c = 0$ ... (iv)

$a^2t_3^2t_1^2 + a^2(t_3 + t_1)^2 + 2g.at_3t_1 + 2f.a(t_3 + t_1) + c = 0$ ... (v)

(iii) – (iv) $\Rightarrow$ $a^2t_2^2(t_1^2 - t_3^2) + a^2(t_1 - t_3)(t_1 + 2t_2 + t_3) + 2g.at_2(t_1 - t_3) + 2f.a(t_1 - t_3) = 0$

or $a^2t_2^2(t_1 + t_3) + a^2(t_1 + 2t_2 + t_3) + 2gat_2 + 2fa = 0$ ... (vi)

Similarly, (iv) – (v)

$a^2t_3^2(t_2 + t_1) + a^2(t_1 + t_2 + 2t_3) + 2gat_3 + 2fa = 0$ ... (vii)

Also, (vi) – (vii) $\Rightarrow$ $a^2[t_2^2(t_1 + t_3) - t_3^2(t_2 + t_1) + t_2 - t_3] + 2ga(t_2 - t_3) = 0$

or $a[t_2t_3(t_2 - t_3) + t_1(t_2^2 - t_3^2) + (t_2 - t_3)] + 2g(t_2 - t_3) = 0$

or $a[t_2t_3 + t_1t_2 + t_1t_3 + 1] + 2g = 0$

$\therefore$ $2g = -a(1 + t_1t_2 + t_2t_3 + t_3t_1)$.

From $t_3 \times$ (vi) – $t_2 \times$ (vii), we get $2f = -a(t_1 + t_2 + t_3 - t_1t_2t_3)$.

Substituting the values of 2g, 2f in (iii), we get $c = a^2(t_1t_2 + t_2t_3 + t_3t_1)$

$\therefore$ the equation of the circumcircle is $x^2 + y^2 - a(1 + St_1t_2)x - a(St_1 - t_1t_2t_3)y + a^2St_1t_2 = 0$.

It passes through (a, 0) because $a^2 - a^2(1 + St_1t_2) + a^2 St_1t_2 = 0$. Hence proved.

8. POLE AND POLAR

Let P be a point lying within or outside a given parabola. Suppose any straight line drawn through P intersects the parabola at Q and R. Then, the locus of the point of intersection of the tangents to the parabola at Q and R is called the polar of the given point P with respect to the parabola, and the point P is called the pole of the polar.

The polar of a point (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

CONCEPTS

- The chord of contact and the polar of any point on the directrix always pass through the focus.
- The pole of a focal chord lies on the directrix and the locus of the poles of the focal is a directrix.
- The polars of all points on the directrix always pass through a fixed point and this fixed point is the focus.
- The polar of the focus is the directrix and the pole of the directrix is the focus.

Anurag Saraf (JEE 2009, AIR 226)

Illustration 27: The general equation to a system of parallel chords of the parabola $y^2 = \frac{25}{7}x$ is $4x - y + k = 0$. What is the equation of the corresponding diameter? **(JEE MAIN)**

Sol: Solve the equation of the line and the parabola. Then use the definition of the diameter to find the answer.

Let PQ be a chord of the system whose equation is $4x - y + k = 0$ ... (i)

Where k is a parameter.

Let $M(\alpha, \beta)$ be the middle point of PQ. The locus of M is the required diameter.

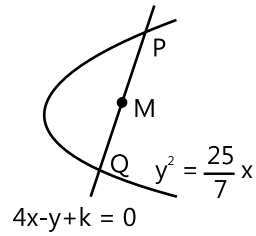


Figure 10.17

The equation of the parabola is $y^2 = \frac{25}{7}x$ ... (ii)

Solving (i) and (ii), $4 \times \frac{7y^2}{25} - y + k = 0$ or $28y^2 - 25y + 25k = 0$.

Let its roots be y_1, y_2. Then $\beta = \frac{y_1 + y_2}{2} = \frac{25}{2 \times 28} = \frac{25}{56}$ $\therefore$ The equation of the locus of $M(\alpha, \beta)$ is $y = \frac{25}{56}$

Illustration 28: Find the locus of the middle points of the normal chords of the parabola $y^2 = 4ax$. **(JEE ADVANCED)**

Sol: The locus of the middle points of the normal chords is nothing but the diameter corresponding to the normal chords. Using this, we can easily find the answer.

Let PQ be a normal chord to the parabola $y^2 = 4ax$, which is normal at $P(at_1^2, 2at_1)$

Let the chord extend to intersect the parabola again at $Q(at_2^2, 2at_2)$

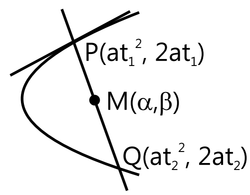


Figure 10.18

The equation of the normal at $P(at_1^2, 2at_1)$ is

$$y + t_1x = 2at_1 + at_1^3 \quad \text{... (i)}$$

$Q(at_2^2, 2at_2)$ satisfies it. So

$$2at_2 + t_1 . at_2^2 = 2at_1 + at_1^3$$

or $2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$. As $t_1 \neq t_2$, we get

$$2 + t_1(t_2 + t_1) = 0 \quad \text{... (ii)}$$

Let $M(\alpha, \beta)$ be the middle point of the normal chord. Then

$$\alpha = \frac{at_1^2 + at_2^2}{2} = \frac{a}{2}(t_1^2 + t_2^2) \quad \text{... (iii)}$$

$$\beta = \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2) \quad \text{... (iv)}$$

(4) $\Rightarrow$ $t_1 + t_2 = \frac{\beta}{a}$ (2) $\Rightarrow$ $2 + t_1 . \frac{\beta}{a} = 0$ $\therefore$ $t_1 = \frac{-2a}{\beta}$

$\therefore$ (2) $\Rightarrow$ $2 + \frac{-2a}{\beta}\left(t_2 - \frac{2a}{\beta}\right) = 0$ or $\frac{\beta}{a} - \left(t_2 - \frac{2a}{\beta}\right) = 0$ $\therefore$ $t_2 = \frac{\beta}{a} + \frac{2a}{\beta}$

Substituting t_1, t_2 in (3) we get

$$\alpha = \frac{a}{2}\left[\left(\frac{-2a}{\beta}\right)^2 + \left(\frac{\beta}{a} + \frac{2a}{\beta}\right)^2\right] = \frac{a}{2}\left[\frac{4a^2}{\beta^2} + \frac{\beta^2}{a^2} + \frac{4a^2}{\beta^2} + 4\right]$$

$\therefore$ The equation of the required locus is $x = \frac{a}{2}\left[\frac{8a^2}{y^2} + \frac{y^2}{a^2} + 4\right]$ or $x = \frac{4a^3}{y^2} + \frac{y^2}{2a} + 2a$

Illustration 29: Show that the locus of the poles of the tangents to the parabola $y^2 = 4ax$ with respect to the parabola $y^2 = 4bx$ is the parabola $ay^2 = 4b^2x$. **(JEE ADVANCED)**

Sol: Consider the general equation of the tangent to parabola. Taking a point as the pole find the polar w.r.t. $y^2 = 4bx$. Compare the two equations and prove the above result.

Any tangent to the parabola $y^2 = 4ax$ is $ty = x + at^2$ ....(i)

Let (α, β) be the pole of (1) with respect to the parabola $y^2 = 4bx$.

Then (1) is the polar of (α, β) with respect to $y^2 = 4bx$ $\therefore$ (1) and $y\beta = 2b(x + \alpha)$ are identical.

So, comparing these,

$$\frac{t}{\beta} = \frac{1}{2b} = \frac{at^2}{2b\alpha}; \qquad \therefore \quad t = \frac{\beta}{2b}, t^2 = \frac{\alpha}{a}$$

$$\therefore \quad \left(\frac{\beta}{2b}\right)^2 = \frac{\alpha}{a} \qquad \text{or} \qquad ab^2 = 4b^2\alpha$$

$\therefore$ The equation of the required locus of the poles is $ay^2 = 4b^2x$.

9. NORMAL

9.1 Equation of Normal

Point Form: The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

The equation of the normals to all standard forms of parabola at (x_1, y_1) are given below for ready reference.

Equation of the parabola	Equation of the normal
$y^2 = 4ax$	$y - y_1 = -\frac{y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$x - x_1 = -\frac{x_1}{2a}(y - y_1)$
$x^2 = -4ay$	$x - x_1 = \frac{x_1}{2a}(y - y_1)$

SLOPE FORM: The equations of normals to various standard form of the parabola in terms of the slope of the normal are as given below.

Equation of the parabola	Equation of the normal	(Feet of the normal)
$y^2 = 4ax$	$y = mx - 2am - am^3$	$(am^2, -2am)$
$y^2 = -4ax$	$y = mx + 2am + am^3$	$(-am^2, 2am)$
$x^2 = 4ay$	$x = my - 2am - am^3$	$(-2am, am^2)$
$x^2 = -4ay$	$x = my + 2am + am^3$	$(2am, -am^2)$

Also $y = mx + c$ is normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$.

Parametric Form: The equation of the normal to the parabola $y^2=4ax$ at point $(at^2, 2at)$ is given by $y+tx=2at+at^3$. The equation of normals to all standard forms of parabola in terms of parameter 't' are listed below for ready reference.

Equation of the parabola	Parametric coordinates	Equation of the normal
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

The point of intersection of normals at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R\{2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$

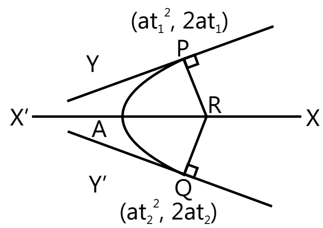


Figure 10.19

Illustration 30: If the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have a common normal other than x-axis then prove that $\frac{b}{a-c} > 2$. **(JEE MAIN)**

Sol: Consider the slope of the normal to be m.

Write the equation of two normals w.r.t. the two parabolas and solve them to prove the above inequality.

Normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$

Normal to $y^2 = 4c(x - b)$ is $y = m(x - b) - 2cm - cm^3$

$\Rightarrow 2am + am^3 = bm + 2cm + cm^3 \quad \Rightarrow \quad m^2(a - c) + 2(a - c) = b$

$\Rightarrow m^2 = \frac{b}{a-c} - 2 \quad \Rightarrow \frac{b}{a-c} > 2.$

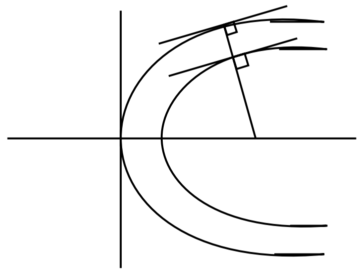


Figure 10.20

9.2 Co-normal Points

We see that in general, three normals can be drawn from a point to a parabola. We shall also study the relations between their slopes and conditions so that the three normals are distinct.

Co-normal Points: The points on the parabola at which the normals pass through a common point are called co-normal points. The co-normal points are also called the feet of the normals.

Let P(h, k) be a point and $y^2 = 4ax$ be a parabola. The equation of any normal to the parabola $y^2 = 4ax$ is ;

$y = mx - 2am - am^3$

If it passes through the point P(h, k) then

$$k = mh - 2am - am^3 \quad \Rightarrow \quad am^3 + m(2a - h) + k = 0 \qquad ....(i)$$

This is a cubic equation in m. So, it gives three values of m, say (m_1, m_2 and m_3). Corresponding to each value of m there is a normal passing through the point P(h, k).

Let, A, B, C be the feet of the normals. Then, AP, BP and CP are three normals passing through point P. Let m_1, m_2 and m_3 respectively, be their slopes. Then, their equations are

$$y = m_1x - 2am_1 - am_1^3,\ y = m_2x - 2am_2 - am_2^3,\ y = m_3x - 2am_3 - am_3^3$$

The coordinates of A, B and C are

$A(am_1^2, 2ma_1)$, $B(am_2^2, -2am_2)$ and $C(am_3^2, -am_3)$.

Since, m_1, m_2, m_3 are the roots of the equation (i). Therefore, we have

$$m_1 + m_2 + m_3 = 0 \quad \text{....(ii)}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a} \quad \text{....(iii)}$$

and, $m_1m_2m_3 = \frac{-k}{a}$ ....(iv)

We have the following results related to co-normal points and the slopes of the normals at co-normal points.

Note:

(a) The algebraic sum of the slopes of the normals at co-normal point is zero.

(b) The sum of the ordinates of the co-normal points is zero.

(c) The centroid of the triangle formed by the co-normal points on a parabola lies on its axis.

9.3 Useful Results

(a) If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

(b) The tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.

(c) If the normals at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ meet on the parabola, then $t_1.t_2 = 2$.

(d) If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of the ordinates of P and Q is $8a^2$.

(e) If the normal chord at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.

(f) The normal chord of a parabola at a point whose ordinate is equal to the abscissa subtends a right angle at the focus.

(g) The normal at any point of a parabola is equally inclined to the focal distance of the point and to the axis of the parabola.

(h) The sub-normal of a point on a parabola is always constant and equal to semi-latus rectum of the parabola.

(i) The normal at any point P of a parabola bisects the external angle between the focal distance of the point and the perpendicular on the directrix from the point P.

Remark: It follows from this property, that is if there is a concave parabolic mirror whose intersection with xy-plane is the parabola $y^2 = 4ax$, then all rays of light coming from the positive direction of x-axis and parallel to the axis of the parabola, after reflection, will pass through the focus of the parabola.

CONCEPTS

If a circle intersects a parabola at four points, then the sum of their ordinates is zero.

Anand K (JEE 2009, AIR 47)

Illustration 31: If a chord which is normal to $y^2 = 4ax$ at one end subtends a right angle at the vertex then find the angle at which it is inclined to the axis. **(JEE MAIN)**

Sol: Let m_1 and m_2 be the slopes of the line joining the vertex and the two ends of the chord. Using the relation between m_1 and m_2 find the slope of the chord and hence the angle it makes with the positive direction of X-axis.

$m_1 = \frac{2at_1}{at_1^2} = \frac{2}{t_1}$ The point at which the normal intersects the parabola is

$t_2 = -t_1 - \frac{2}{t_1} \Rightarrow m_2 = \frac{2}{t_1 - \frac{2}{t_1}} = \frac{2t_1}{-t_1^2 - 2}$

$m_1m_2 = -1 \Rightarrow \frac{4}{t_1^2 + 2} = 1 \Rightarrow t_1^2 + 2 = 4$

$t_1 = \pm\sqrt{2} \Rightarrow m_3 = \frac{2}{t_1 + t_2} = -t_1$

$\Rightarrow \tan\theta = \pm\sqrt{2} \Rightarrow \theta = \tan^{-1}(\pm\sqrt{2})$

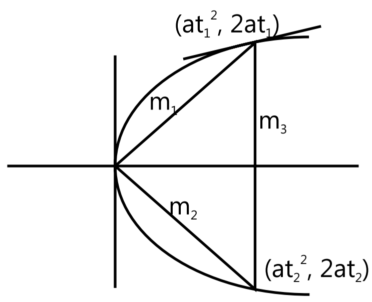


Figure 10.21

Illustration 32: Find the equation of the normal to the parabola $y^2 = 4x$, which is (i) parallel to the line $y = 2x - 5$, (ii) perpendicular to the line $2x + 6y + 5 = 0$. **(JEE MAIN)**

Sol: Use the slope form of the normal to get the two equations accordingly.

The equation of the normal to the parabola,

$y^2 = 4ax$ at $(am^2, -2am)$ is $y = mx - 2am - am^3$

Where m is the slope of the normal. Here, a = 1. So, the equation of the normal at $(m^2, -2m)$ is

$y = mx - 2m - m^3$ ...(i)

(a) If the normal is parallel to the line $y = 2x - 5$. Then, m = Slope of the line $y = 2x - 5$ is 2

Putting the value of m in (i), we obtain $y = 2x - 12$

as the equation of the required normal at (4, –4).

(b) If the normal in (i) is perpendicular to the line $2x + 6y + 5 = 0$. Then, m = 3

hence equation of the normal is $y = 3x - 33$

Illustration 33: Show that the distance between a tangent to the parabola $y^2 = 4ax$ and the parallel normal is $a\sec^2\theta \operatorname{cosec}\theta$, where θ is the inclination of the either of them with the axis of the parabola. **(JEE MAIN)**

Sol: In the equation of the normal, use the concept of distance between two parallel lines to prove that the distance between the tangent and the normal is $a\sec^2\theta \operatorname{cosec}\theta$.

Let m be the slope of the tangent or parallel normal to the parabola $y^2 = 4ax$. Then, $m = \tan\theta$.

The equations of the tangent and normal of slope m to the parabola $y^2 = 4ax$ are

$y = mx + a/m$ and $y = mx - 2am - am^3$

The distance between these two parallel lines is given by $d = \left|\frac{(a/m) + 2am + am^3}{\sqrt{1+m^2}}\right|$

Using: $d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} \Rightarrow d = \left|\frac{a + 2am^2 + am^4}{m\sqrt{1+m^2}}\right| \Rightarrow \frac{a(1+m^2)^2}{m\sqrt{1+m^2}} \Rightarrow d = \frac{a(1+m^2)^{3/2}}{m}$

$\Rightarrow d = a(1 + \tan^2\theta)^{3/2}\cot\theta$ [$\because$ m = tan q] $\Rightarrow d = a\sec^2\theta \operatorname{cosec}\theta$.

Illustration 34: Find the values of θ for which the line $y = x\cos\theta + 4\cos^3\theta - 14\cos\theta - 1$ is a normal to the parabola $y^2 = 16x$. **(JEE ADVANCED)**

Sol: Compare the given equation with the standard equation of the normal and obtain the value of θ.

The slope of the given line is m = $\cos\theta$.

We know that the line y = mx + c is a normal to the parabola $y^2 = 4ax$, if $c = -2am - am^3$. Therefore, the given line will be a normal to the parabola $y^2 = 16x$, if

$4\cos^3\theta - 14\cos\theta - 1 = -18\cos\theta - 4\cos^3\theta$; $\Rightarrow$ $8\cos^3\theta - 6\cos\theta = 1$

$\Rightarrow$ $2(4\cos^3\theta - 3\cos\theta) = 1$ $\Rightarrow$ $2\cos 3\theta = 1$

$\Rightarrow$ $\cos 3\theta = \frac{1}{2}$ $\Rightarrow$ $\cos 3\theta = \cos \pi/3 = 1$

$\Rightarrow$ $3\theta = 2n\pi \pm \pi/3, n \in Z$ $\Rightarrow$ $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}, n \in Z$

Illustration 35: Prove that three normals can be drawn from the point (c, 0) to the parabola $y^2 = x$ if $c > \frac{1}{2}$ and then one of the normals is always the axis of the parabola. Also find c for which the other two normals will be perpendicular to each other. **(JEE ADVANCED)**

Sol: The standard equation of the normal of a parabola is a cubic equation in 'm' (slope). Find the condition for the cubic equation to have three real roots. Once we have the slope of the three normals, we can easily find the condition for the two normals, other than the axis of the parabola, to be perpendicular to each other.

Let $(t^2, 1)$ be a foot of one of the normals to the parabola $y^2 = x$ from the point (c, 0).

Now, the equation of the normal to $y^2 = x$ at (t^2, t) is

$$y - t = \frac{-1}{\left(\frac{dy}{dx}\right)_{t^2,t}}.(x - t^2) \Rightarrow \quad y - t = \frac{-1}{(1/2t)}.(x - t^2) \quad \left\{\because y^2 = x \Rightarrow 2y\frac{dy}{dx} = 1 ; \therefore \frac{dy}{dx} = \frac{1}{2y}\right\}$$

$\Rightarrow$ $y - t = -2t(x - t^2)$ $\Rightarrow$ $y + 2tx = t + 2t^3$ ....(i)

It passes through (c, 0) if $0 + 2tc = t + 2t^3$

$\Rightarrow$ $2t^3 + t(1 - 2c) = 0$ $\Rightarrow$ $t[2t^2 - (2c - 1)] = 0$ $\therefore t = 0, \pm\sqrt{\frac{2c-1}{2}}$

Three normals can be drawn if t has three real distinct values.

So, $2c - 1 > 0$, i.e., $c > \frac{1}{2}$.

The foot of one of the normals is (t^2, t) where t = 0, i.e., the foot is (0, 0).

From (i), the corresponding normal is y = 0, i.e., the x-axis which is the axis of the parabola.

For the other two normals $t = \pm\sqrt{\frac{2c-1}{2}}$.

From (i), 'm' of a normal = –2t $\therefore$ 'm' of the other two normals are $-2 . \sqrt{\frac{2c-1}{2}}, 2 . \sqrt{\frac{2c-1}{2}}$.

They are perpendicular if $-2 . \sqrt{\frac{2c-1}{2}} \times 2 . \sqrt{\frac{2c-1}{2}} = -1$

$\Rightarrow$ $-2(2c - 1) = -1 \Rightarrow$ $c = 3/4$

10. SUBTANGENT AND SUBNORMAL

Let the parabola be $y^2 = 4ax$. Let the tangent and normal at $P(x_1, y_1)$ meet the axis of parabola at T and G respectively, and the tangent at $P(x_1, y_1)$ makes an angle Ψ with the positive direction of x-axis.

A(0, 0) is the vertex of the parabola and PN = y. Then,

(a) Length of tangent = PT = PN cosec Ψ = y_1 cosec Ψ

(b) Length of normal = PG = PN cosec(90 – Ψ) = y_1 sec Ψ

(c) Length of subtangent = TN = PN cot Ψ = y_1 cot Ψ

(d) Length of subnormal = NG = PN cot(90° – Ψ) = y_1 tan Ψ

Where, $\tan\Psi = \frac{2a}{y_1} = m$, {Slope of tangent at P(x, y)]

Note: **(a)** Length of the tangent at $(at^2, 2at) = 2at\ \text{cosec}\ \Psi = 2at\sqrt{(1+\cot^2\Psi)} = 2at\sqrt{1+t^2}$

(b) Length of the normal at $(at^2, 2at) = 2at\sec\Psi = 2at\sqrt{(1+\tan^2\Psi)} = 2a\sqrt{t^2+t^2\tan^2\Psi} = 2a\sqrt{(t^2+1)}$

(c) Length of subtangent at $(at^2, 2at) = 2at\cot\Psi = 2at^2$

(d) Length of subnormal at $(at^2, 2at) = 2at\tan\Psi = 2a$.

CONCEPTS

Two parabolas are said to be equal when their latus rectums are equal.

The sub tangent at any point on the parabola is twice the abscissa or proportional to square of the ordinate of the point.

GV Abhinav (JEE 2012, AIR 329)

PROBLEM-SOLVING TACTICS

A Working Rule to find the equation of a parabola when focus & directrix are given:

Step 1: Find the distance between focus and general point P(x, y) by the distance formula.

Step 2: Find the perpendicular distance from the point P(x, y) to the given directrix.

(The perpendicular distance from a point $P(x_1, y_1)$ to the line ax + by + c = 0 is $\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$)

Step 3: Equate the distances calculated in step 1 and step 2. On simplification we get the required equation of the parabola.

A Working Rule to find the equation of a parabola when the vertex and the focus are given:

Step 1: Find the slope of the axis formed by joining the focus and the vertex by the formula $\frac{y_2-y_1}{x_2-x_1}$

Step 2: Find the slope of the directrix by the formula $m_1.m_2 = -1$; where m_1 is the slope of the axis of the parabola and m_2 is the slope of the directrix.

Step 3: Find a point on the directrix as the vertex, which is the middle point between the focus and the point on the directrix, by means of the mid-point formula.

Step 4: Write the equation of the directrix, using the slope point formula.

Step 5: The focus and the directrix are now known so we can find the equation of the parabola by the method given above.

FORMULAE SHEET

1. **Definition:** A parabola is the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed straight line.

For e.g. if the focus is (α, β) and the directrix is $ax + by + c = 0$ then the equation of the parabola is

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{(ax+by+c)^2}{a^2+b^2}$$

2. The general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $\Delta \neq 0$ and $h^2 = ab$.

3.

Equation of the parabola Properties	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex (Co-ordinates)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus (Co-ordinates)	(a, 0)	(–a, 0)	(0, a)	(0, –a)
Latus rectum (length)	4a	4a	4a	4a
Axis (Equation)	y =0	y =0	x = 0	x = 0
Directrix (Equation)	x = –a	x = a	y = –a	y = a
Symmetry (about)	x-axis	x-axis	y-axis	y-axis

4. The equation of the chord joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

5. If the equation of the chord joining points t_1 and t_2 on the parabola $y^2 = 4ax$ passes through the focus then $t_1t_2 = -1$.

In other words, if one end of a focal chord of the parabola $y^2 = 4ax$ is $P(at^2, 2at)$ then the co-ordinates of the other end is $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

6. The length of the focal chord passing through $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ is $a(t + 1/t)^2$.

7. The length of the chord intercepted by the parabola on the line $y = mx + c$ is $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

8. The length of the chord joining two points 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1+t_2)^2+4}$.

9. The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $yy_1 = 2a(x + x_1)$

10. Parametric Form

Equation of the parabola	Point of contact	Equation of the tangent
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

11. Slope form

Equation of the parabola	Equation of the tangent	Condition of tangency	Point of contact
$y^2 = 4ax$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
$y^2 = -4ax$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$	$\left(\frac{-a}{m^2}, \frac{-2a}{m}\right)$
$x^2 = 4ay$	$x = mx + \frac{a}{m}$	$c = \frac{a}{m}$	$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$
$x^2 = -4ay$	$x = mx - \frac{a}{m}$	$c = -\frac{a}{m}$	$\left(\frac{-2a}{m}, \frac{-a}{m^2}\right)$

12. The point of intersection of tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is given by $(at_1t_2, a(t_1 + t_2))$

13. If SZ be perpendicular to the tangent at a point P of a parabola, then Z lies on the tangent at the vertex and $SZ^2 = AS \times SP$, where A is the vertex of the parabola.

14. Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$

15. Equation of normal in different forms

Equation of the parabola	Equation of the normal
$y^2 = 4ax$	$y - y_1 = -\frac{y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$x - x_1 = -\frac{x_1}{2a}(y - y_1)$
$x^2 = -4ay$	$x - x_1 = -\frac{x_1}{2a}(y - y_1)$

Equation of the parabola	Parametric coordinates	Equation of the normal
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Equation of the parabola	Equation of the normal	(Feet of the normal)
$y^2 = 4ax$	$y = mx - 2am - am^3$	$(am^2, -2am)$
$y^2 = -4ax$	$y = mx + 2am + am^3$	$(-am^2, 2am)$
$x^2 = 4ay$	$x = my - 2am - am^3$	$(-2am, am^2)$
$x^2 = -4ay$	$x = my + 2am + am^3$	$(2am, -am^2)$

16. The point of intersection of normals at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $R[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$

17. **(i)** The algebraic sum of the slopes of the normals at the co-normal point is zero.

(ii) The centroid of a triangle formed by the co-normal points on a parabola lies on its axis.

18. If the normal at the point P $(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $(at_2^2, 2at_2)$. Then $t_2 = -t_1 - \frac{2}{t_1}$.

19. If the normal drawn at the point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ intersect at a third point on the parabola then $t_1.t_2 = 2$.

20. If the normal chord at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.

21. The chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

22. The combined equation of the pair of tangents drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$ is $SS_1 = T^2$ where, $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.

23. The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 \quad \text{or,} \quad T = S_1.$$

24. The polar of a point (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

25. The equation of the diameter of the parabola $y^2 = 4ax$ bisecting chords of slope m is $y = 2a/m$.

26. A circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

27. If the tangents at P and Q meet at T, then

(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP \times SQ$, and

(iii) Δ SPT and Δ STQ are similar.

28. Tangents and normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ and $(3a, 0)$.

29. The semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola, i.e. $2a = \frac{2bc}{b+c}$ or, $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

30. The orthocentre of any triangle formed by tangents at any three points $P(t_1)$, $Q(t_2)$ and $R(t_3)$ on a parabola $y^2 = 4ax$ lies on the directrix and has the coordinates $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

31. If a normal drawn to a parabola passes through a point P(h, k), then $k = mh - 2am - am^3$, i.e,
$am^3 + m(2a - h) + k = 0$,

$\Rightarrow m_1 + m_2 + m_3 = 0$; $\quad m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; and $m_1m_2m_3 = -\frac{k}{a}$.

32. The equation of a circle circumscribing the triangle formed by three co-normal points and which passes through the vertex of the parabola is $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$.

33. The area of a triangle formed inside the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ where y_1, y_2, y_3 are the ordinates of the vertices of the triangle.

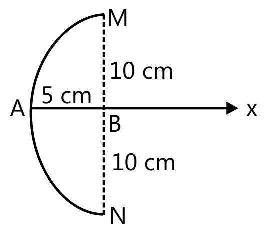
34. If the vertex and the focus of a parabola are on the x-axis and at a distance a and b from the origin respectively then the equation of the parabola is $y^2 = 4(b - a)(x - a)$.

Solved Examples

JEE Main/Boards

Example 1: If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

Sol: Refer to Fig. 10.22. If we consider the origin to be the vertex of the parabola. Then we know that the point (5 , 10) will lie on the parabola. Using this we can solve the question easily.



Let 'MAN' be the parabolic reflector such that MN is its diameter and AB is its depth. It is given that AB = 5 cm and MN = 20 cm

$\therefore$ MB = BN = 10 cm

Taking the equation of the reflector as

$y^2 = 4ax$ ... (i)

Co-ordinates of point M are (5, 10) and lies on (i). Therefore,

$(10)^2 = 4(a)(5) \Rightarrow a = 5$

Thus, the equation of the reflector is

$y^2 = 20x$

Its focus is at (5, 0), i.e., at point B.

Hence, the focus is at the mid-point of the given diameter.

Example 2: The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is-

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$

Sol: Rewrite the given equation in the standard form and compare with the equation of directrix.

The given equation can be written as

$(y + 2)^2 = -4x + 2 = -4(x - 1/2)$

Which is of the form $Y^2 = 4aX$

Where $Y = y + 2$, $X = x - 1/2$, $a = -1$

The directrix of the parabola

$Y^2 = 4ax$ is $X = -a$

$\Rightarrow x - 1/2 = -(-1) \Rightarrow x = 3/2$

is the equation of the directrix of the given parabola.

Example 3: If the focus of a parabola divides a focal chord of the parabola in segments of length 3 and 2, the length of the latus rectum of the parabola is-

(A) 3/2 (B) 6/5 (C) 12/5 (D) 24/5

Sol: Let $y^2 = 4ax$ be the equation of the parabola, then the focus is S(a, 0). Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be vertices of a focal chord of the parabola, then $t_1t_2 = -1$. Let SP = 3, SQ = 2

$SP = \sqrt{a^2(1-t_1^2)+4a^2t_1^2} = a(1 + t_1^2) = 3$ ..(i)

and $SQ = a\left(1+\frac{1}{t_1^2}\right) = 2$ ..(ii)

From (i) and (ii), we get $t_1^2 = 3/2$ and $a = 6/5$. Hence, the length of the latus rectum = 24/5.

Example 4: The tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q and R, the coordinates of the mid-point of QR are-

(A) $(x_1 - a, y_1 + b)$ (B) (x_1, y_1)

(C) $(x_1 + b, y_1 + a)$ (D) $(x_1 - b, y_1 - b)$

Sol: Consider a mid point of the chord and find the equation w.r.t. $y^2 = 4a(x + b)$. Compare this equation with the equation of the tangent to $y^2 = 4ax$ and get the coordinates of the mid point.

Equation of the tangent at $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

or $2ax - y_1y + 2ax_1 = 0$ ...(i)

If M(h, k) is the mid-point of QR, then the equation of QR, a chord of the parabola $y^2 = 4a(x + b)$ in terms of its mid-point is

$ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$ (Using T = S')

or $2ax - ky + k^2 - 2ah = 0$ ...(ii)

Since (i) and (ii) represent the same line, we have

$$\frac{2a}{2a} = \frac{y_1}{k} = \frac{2ax_1}{k^2 - 2ah}$$

$\Rightarrow k = y_1$ and $k^2 - 2ah = 2ax_1$

$\Rightarrow \quad y_1^2 - 2ah = 2ax_1 \Rightarrow 4ax_1 - 2ax_1 = 2ah$

(As $P(x_1, y_1)$ lies on the parabola $y^2 = 4ax$)

$\Rightarrow h = x_1$ so that $h = x_1$, $k = y_1$ is the mid point of QR.

Example 5: P is a point on the parabola whose ordinate equals its abscissa. A normal is drawn to the parabola at P to meet it again at Q. If S is the focus of the parabola then the product of the slopes of SP and SQ is-

(A) – 1 (B) 1/2 (C) 1 (D) 2

Sol: Proceed according to the given condition. Clearly, the point with the same abscissa and the ordinate is the point (4a , 4a).

Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$, then $at^2 = 2at \Rightarrow t = 2$ and thus the coordinates of P are (4a, 4a).

Equation of the normal at P is $y = -tx + 2at + at^3$

$\Rightarrow \quad y = -2x + 4a + 8a$

$\Rightarrow \quad 2x + y = 12a \qquad \text{...(i)}$

Which meets the parabola $y^2 = 4ax$ at points given by

$y^2 = 2a(12a - y) \Rightarrow y^2 + 2ay - 24a^2 = 0$

$\Rightarrow \quad y = 4a$ or $y = -6a$

$y = 4a$ corresponds to the point P

and $y = -6a \Rightarrow x = 9a$ from (i)

So that the coordinates of Q are (9a, –6a). Since the coordinates of the focus S are (a,0), slope of SP = 4/3 and slope of SQ = –6/8. Product of the slopes = –1.

Example 6: The common tangents to the circle $x^2 + y^2 = a^2/2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola.

(A) $x^2 = 4ay$ (B) $x^2 = -4ay$

(C) $y^2 = -4ax$ (D) $y^2 = 4a(x + a)$

Sol: In this case, first we need to find the two common tangents and then find the point of intersection. Start with the standard equation of the tangent to a parabola and apply the condition of tangency on the circle to get the slope of the tangents and proceed to find the point of intersection.

The equation of a tangent to the parabola $y^2 = 4ax$ is $y = mx + a/m$.

If it touches the circle $x^2 + y^2 = a^2/2$

$$\frac{a}{m} = \left(\frac{a}{2}\right)\sqrt{1+m^2} \Rightarrow 2 = m^2(1 + m^2)$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0 \Rightarrow (m^2 - 1)(m^2 + 2) = 0$

$\Rightarrow \quad m^2 = 1 \Rightarrow m = \pm 1$

Hence, the common tangents are $y = x + a$ and $y = -x - a$, which intersect at the point (–a, 0) Which is the focus of the parabola $y^2 = -4ax$.

Example 7: The locus of the vertices of the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a \text{ is-}$$

(A) xy = 64/105 (B) xy = 105/64

(C) $xy = \frac{3}{4}$ (D) xy = 35/16

Sol: Convert the given equation to the standard form. The equation of the parabola can be written as

$$\frac{y}{a} = \left(\frac{ax}{\sqrt{3}} + \frac{\sqrt{3}}{4}\right)^2 - \frac{3}{16} - 2$$

or $$\left(x + \frac{3}{4a}\right)^2 = \frac{a^2}{3a}\left(y + \frac{35}{16}a\right)$$

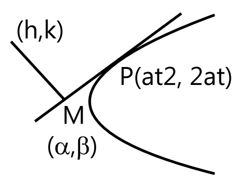
Vertex is $x = -3/4a$, $y = -35a/16$

Locus of the vertex is xy = 105/64.

Example 8: Find the locus of the foot of the perpendicular drawn from a fixed point to any tangent to a parabola.

Sol: Take a fixed point and use it to find the foot of the perpendicular on a general equation of a tangent.

Let the parabola be $y^2 = 4ax$ and the fixed point be (h, k)



The tangent at any point $P(at^2, 2at)$ is

$ty = x + at^2 \qquad \text{... (i)}$

Let $M(\alpha, \beta)$ be the foot of the perpendicular to the tangent (i) from the point (h, k)

Using perpendicularly, $\frac{\beta - k}{\alpha - h}.\frac{1}{t} = -1 \qquad \text{... (ii)}$

As $M(\alpha, \beta)$ is on (i), $t\beta = \alpha + at^2 \qquad \text{... (iii)}$

We have to eliminate t from (ii) and (iii)

From (ii), $t = -\frac{\beta - k}{\alpha - h}$. Putting in (iii),

$$\beta\left(-\frac{\beta-k}{\alpha-h}\right) = \alpha + a.\left(\frac{\beta-k}{\alpha-h}\right)^2$$

or $\quad -\beta(\beta - k)(\alpha - h) = \alpha(\alpha - h)^2 + a(\beta - k)^2$

$\therefore$ The equation of the locus of the foot M is

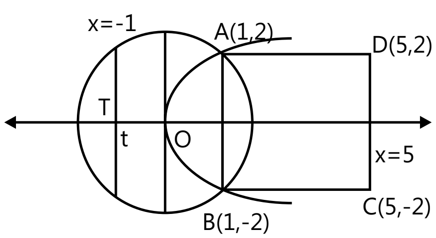
$x(x - h)^2 + y(x - h)(y - k) + a(y - k)^2 = 0.$

Example 9: Tangents to the parabola at the extremities of a common chord AB of the circle $x^2 + y^2 = 5$ and the parabola $y^2 = 4x$ intersect at the point T. A square ABCD is constructed on this chord lying inside the parabola, then $[(TC)^2 + (TD)^2]^2$ is equal to ?

Sol: Find the point of intersection of the circle and the parabola. Then get the equation of the chord and the point of intersection of the tangents at the end of the chord. In the last step use simple geometry to find $[(TC)^2 + (TD)^2]^2$.

The points of intersection of the circle and the parabola are A(1, 2), B(1, –2)

The equation of the common chord is x = 1, which is the latus rectum of the parabola.



$\therefore$ Tangents at the extremities of AB intersect on the directrix x = –1.

Coordinates of T are (–1, 0)

Since the length of AB = 4, the sides of the square ABCD are of length 4, and the coordinates of C are

(–5, 2) and of D are (5, 2).

$(TC)^2 = (TD)^2 = (5 + 1)^2 + 4 = 40.$

$\Rightarrow \quad [(TC)^2 + (TD)^2]^2 = 80^2 = 6400.$

JEE Advanced/Boards

Example 1: If the normal chord at a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex, then a value of t is-

(A) 4 (B) $\sqrt{3}$ (C) $\sqrt{2}$ (D) 1

Sol: Use the concept of homogenization of a conic and a straight line and then apply the condition of the sum of the co-efficients of x^2 and y^2 equal to zero.

The equation of the normal of a parabola

$y^2 = 4ax$ is $y = -tx + 2at + at^3$ ...(i)

The joint equation of the lines joining the vertex (origin) to the points of intersection of the parabola and the line (i) is

$$y^2 = 4ax\left[\frac{y+tx}{2at+at^3}\right]$$

$\Rightarrow \quad (2t + t^3)y^2 = 4x(y + tx)$

$\Rightarrow \quad 4t\,x^2 - (2t + t^3)y^2 + 4xy = 0$

Since these lines are at right angles co-efficient of x^2 + coefficient of $y^2 = 0$

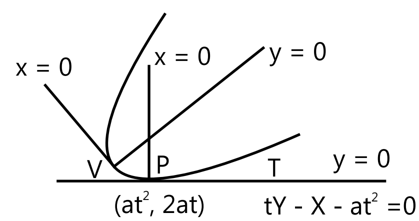
$\Rightarrow \quad 4t - 2t - t^3 = 0 \Rightarrow t^2 = 2$

For t = 0, the normal line is y = 0, i.e. the axis of the parabola which passes through the vertex (0, 0).

Example 2: A parabola is drawn touching the x-axis at the origin and having its vertex at a given distance k from the x-axis. Prove that the axis of the parabola is a tangent to the parabola $x^2 + 8k(y - 2k) = 0$.

Sol: Use the relation between the tangent at the vertex and the axis of the parabola to prove it.

Let the equation of the parabola be $Y^2 = 4ax$.



Any tangent to it at the point $(at^2, 2at)$ is

$Yt = X + at^2$ ... (i)

The normal at the point $(at^2, 2at)$ is

$Y + tX = 2at + at^3$ ... (ii)

Take the equations of transformation

$$\frac{tY - X - at^2}{\sqrt{1+t^2}} = y \qquad \text{... (iii)}$$

and $$\frac{Y + tX - 2at - at^3}{\sqrt{1+t^2}} = x \qquad \text{... (iv)}$$

$\therefore$ in x, y coordinates P = (0, 0) and PT is the x-axis which is the tangent to the parabola at the origin.

Now, (3) $\Rightarrow \quad tY - X - at^2 = y\sqrt{1+t^2}$ ... (v)

(4) $\Rightarrow \quad Y + tX - 2at - at^3 = x\sqrt{1+t^2}$ ... (vi)

$\therefore \quad (5) \times t + (6) \quad \Rightarrow \quad (t^2 + 1)Y - 2at^3 - 2at$

$= yt\sqrt{1+t^2} + x\sqrt{1+t^2}$

$\therefore$ The axis of the parabola (Y = 0) becomes

$-2at^3 - 2at = (yt + x)\sqrt{1+t^2}$

or $\quad yt + x = \dfrac{-2at(1+t^2)}{\sqrt{1+t^2}} = -2at\sqrt{1+t^2}$ ... (vii)

The distance of the vertex V(0, 0) in the X, Y coordinates from PT

$= \dfrac{-at^2}{\sqrt{1+t^2}} = k$ (from the equation)

$\therefore$ From (vii), the equation of the axis of the parabola in x, y coordinates becomes

$yt + x = -2at\left(\dfrac{-at^2}{k}\right)$

or $\quad yt + x - \dfrac{2a^2t^3}{k} = 0$ ... (viii)

The given parabola is $x^2 = -8k(y - 2k)$ ... (ix)

Solving (viii) and (ix), we get

$x^2 = -8k.\dfrac{1}{t}\left(-x + \dfrac{2a^2t^3}{k}\right) + 16k^2$

or $\quad x^2 = 8\dfrac{k}{t}x - 16a^2t^2 + 16k^2$

or $\quad tx^2 - 8kx + 16t(a^2t^2 - k^2) = 0$

Here, $\quad D = 64k^2 - 64t^2(a^2t^2 - k^2)$

$= 64[k^2 - a^2t^4 + t^2k^2]$

$= 64\left[(1+t^2).\dfrac{a^2t^4}{1+t^2} - a^2t^4\right] = 0$

$\therefore$ The axis, given by (8) touches the given parabola.

Note: If we take $k = \dfrac{at^2}{\sqrt{1+t^2}}$, the points of intersection of the axis and the given parabola will be imaginary.

Example 3: If the line x – 1 = 0 is the directrix of the parabola $y^2 - kx + 8 = 0, k \neq 0$ and the parabola intersects the circle $x^2 + y^2 = 4$ in two real distinct points, then the value of k is-

(A) – 4 (B) – 8 (C) 4 (D) None

Sol: Represent the parabola in the standard form. Compare the equation of the directrix with the given equation and form a quadratic in k. Solve the quadratic for two real roots to get the desired value of k.

The equation of the parabola can be written as

$y^2 = k(x - 8/k)$ which is of the form $Y^2 = 4AX$

Where $\quad Y = y, X = x - 8/k$ and $A = k/4$

Equation of the directrix is

$X = -A \Rightarrow x - 8/k = -k/4$

Which represents the given line x – 1 = 0

If $\dfrac{8}{k} - \dfrac{k}{4} = 1$

$\Rightarrow k^2 + 4k - 32 = 0 \Rightarrow k = -8$ or 4

For k = 4, the parabola is $y^2 = 4(x - 2)$ whose vertex is (2, 0) and touches the circle $x^2 + y^2 = 4$ at the vertex. Therefore $k \neq 4$.

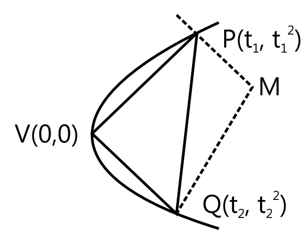
For k = –8, the parabola is $y^2 = -8(x + 1)$ which intersects the circle $x^2 + y^2 = 4$ at two real distinct points.

Example 4: A variable chord PQ of the parabola $y = x^2$ subtends a right angle at the vertex. Find the locus of points of intersection of the normals at P and Q.

Sol: Take two points on the parabola and find the relation between the parametric coordinates. Use this relation to find the locus.

The vertex V of the parabola is (0, 0) and any point on $y = x^2$ has the coordinates (t, t^2).

So let us take $P = (t_1, t_1^2)$, $Q(t_1, t_2^2)$ and ÐPVQ = 90º



As 'm' of VP $= \dfrac{t_1^2 - 0}{t_1 - 0} = t_1$

and 'm' of VQ $= \dfrac{t_2^2 - 0}{t_2 - 0} = t_2$,

$VP \perp VQ \Rightarrow t_1 . t_2 = -1$ ... (i)

The equation of the normal to a curve at (x_1, y_1) is

$y - y_1 = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{x_1, y_1}}.(x - x_1)$

The normals at P & Q intersect at

$M(x,y) = \left(2a\left(t_1^2 + t_2^2 + t_1t_2\right) - at_1t_2\left(t_1 + t_2\right)\right)$

From (1) $t_1t_2 = -1$

$\therefore y = a\left(t_1 + t_2\right)$; $\quad x = 2a(t_1^2 + t_2^2 - 1)$

$$y^2 = a^2\left(t_1^2 + t_2^2 - 2\right)$$

$$\Rightarrow \frac{y^2}{a^2} + 1 = \frac{x}{2a} \Rightarrow 2y^2 + 2a^2 = xa$$

Here, a=1/4

$\therefore$ Locus is $16y^2 = 2x - 1$

Example 5: A parabola is drawn to pass through A and B, the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; the axes of reference being AB and a perpendicular diameter, prove that the locus of the focus of the parabola is $\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1$.

Sol: Consider a circle with its centre at the origin. Let the two points A and B lie on the X-axis. Write the equation of the tangent in standard form and apply the focus-directrix property to prove the given statement.

Let A = (–a, 0) and B = (a, 0)

The centre of the circle = (0, 0)

The equation of the concentric circle will be

$x^2 + y^2 = b^2$

Any tangent to $x^2 + y^2 = b^2$ is

$y = mx + b\sqrt{1+m^2}$

Which is the directrix of the parabola.

Let (α, β) be the focus.

Then by focus-directrix property, the equation of the parabola will be

$$(x - \alpha)^2 + (y - \beta)^2 = \left(\frac{y - mx - b\sqrt{1+m^2}}{\sqrt{1+m^2}}\right)^2$$

It passes through A (–a, 0), B (a, 0); so

$$(a + \alpha)^2 + b^2 = \left(\frac{ma - b\sqrt{1+m^2}}{\sqrt{1+m^2}}\right)^2$$

$$= \frac{m^2a^2 + b^2(1+m^2) - 2abm\sqrt{1+m^2}}{1+m^2} \quad \text{... (i)}$$

$$(a - \alpha)^2 + b^2 = \left(\frac{-ma - b\sqrt{1+m^2}}{\sqrt{1+m^2}}\right)^2$$

$$= \frac{m^2a^2 + b^2(1+m^2) + 2abm\sqrt{1+m^2}}{1+m^2} \quad \text{... (ii)}$$

$$\text{or } a^2 + b^2 = \frac{m^2}{1+m^2}a^2 + b^2 \quad \text{... (iii)}$$

$$\text{(ii) – (i)} \quad \Rightarrow \quad -4a\alpha = \frac{4abm\sqrt{1+m^2}}{1+m^2}$$

$$\text{or} \quad a^2 = \frac{b^2m^2}{1+m^2} \quad \therefore \quad \frac{m^2}{1+m^2} = \frac{\alpha^2}{b^2} \quad \text{... (iv)}$$

Putting in (iii) from (iv)

$$a^2 + a^2 + b^2 = \frac{a^2\alpha^2}{b^2} + b^2$$

$$\therefore \quad \left(1 - \frac{a^2}{b^2}\right)a^2 + b^2 = b^2 - a^2$$

$$\text{or} \quad \frac{b^2 - a^2}{b^2}a^2 + b^2 = b^2 - a^2 \quad \therefore \quad \frac{\alpha^2}{b^2} + \frac{\beta^2}{b^2 - a^2} = 1$$

$\therefore$ The equation of the locus of the focus (α, β) is

$$\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1.$$

Example 6: Let (x_r, y_r) ; r = 1,2,3,4 be the points of the intersection of the parabola $y^2 = 4ax$ and the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

Prove that $y_1 + y_2 + y_3 + y_4 = 0$.

Sol: Solve the equation of the circle and the parabola. Then use the theory of equations to prove $y_1 + y_2 + y_3 + y_4 = 0$.

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

$y^2 = 4ax$ ... (ii)

Solving (i) and (ii), we get the coordinates of points of intersection

From (ii), $x = \frac{y^2}{4a}$ putting in (i),

$$\left(\frac{y^2}{4a}\right)^2 + y^2 + 2g.\frac{y^2}{4a} + 2fy + c = 0$$

$$\text{or} \quad \frac{1}{(4a)^2}y^4 + \left(1 + \frac{g}{2a}\right)y^2 + 2fy + c = 0$$

It has four roots.

Its roots are y_1, y_2, y_3 and y_4.

$$\text{Now, sum of roots} = -\frac{\text{coefficient of } y^3}{\text{coefficient of } y^4}$$

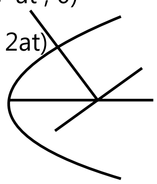
$$\therefore \quad y_1 + y_2 + y_3 + y_4 = -\frac{0}{1/(4a)^2} = 0.$$

Example 7: From the point, where any normal to the parabola $y^2 = 4ax$ meets the axis, a line perpendicular to the normal is drawn. Prove that this line always touches the parabola $y^2 + 4a(x - 2a) = 0$.

Sol: Get the equation of the line perpendicular to the normal, passing through the intersection of the normal and the axis. Use the theory of equation

The normal at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is $y + tx = 2at + at^3$.

It cuts the axis $y = 0$ of the parabola at $(2a + at^2, 0)$.



∴ The equation of the line through this point drawn perpendicular to the normals is

$$y - 0 = \frac{1}{t}(x - \overline{2a + at^2})$$

{∵ 'm' of normal = –t}

or $ty = x - 2a - at^2$ ... (i)

We have to prove that (1) touches the parabola

$y^2 + 4a(x - 2a) = 0$ ... (ii)

Solving (i) and (ii), $y^2 + 4a(ty + at^2) = 0$

or $y^2 + 4aty + (2at)^2 = 0$ or $(y + 2at)^2 = 0$

∴ $y = -2at, -2at$

∴ (i) cuts (ii) at coincident points, i.e., (i) touches (ii).

Example 8: Consider a parabola $y^2 = 4ax$, the length of focal chord is ℓ and the length of the perpendicular from the vertex to the chord is p then-

(A) ℓ.p is constant (B) $\ell.p^2$ is constant

(C) ℓ^2.p is constant (D) None of these

Sol: A quantity is constant if it does not depend on the parameter. Represent ℓ and p in terms of the parameter and look for the quantity in which the parameter gets eliminated.

Let $P(at^2, 2at)$ and $Q(a/t^2, -2a/t)$ be a focal chord of the parabola (as $t_1t_2 = -1$)

The length of PQ $= \ell = \sqrt{(at^2 - a/t^2)^2 + (2at + 2a/t)^2}$

$= a\sqrt{(t^2 - 1/t^2)^2 + 4(t + 1/t)^2}$

$= a(t + 1/t)\sqrt{(t - 1/t)^2 + 4} = a(t + 1/t)^2$

The length of the perpendicular from the vertex (0,0) on the line PQ whose equation is

$y\left(t - \frac{1}{t}\right) = 2(x - a)$ is given by

$$p = \frac{2a}{\sqrt{(t - 1/t)^2 + 2^2}} = \frac{2a}{(t + 1/t)}$$

So that $\ell.p^2 = \frac{4a^2}{(t + 1/t)^2} \times a(t + 1/t)^2 = 4a^3$,

which is constant.

Paragraph for Example No. 9 to 11

C: $y = x^2 - 3$, D: $y = kx^2$, L_1: $x = a$, L_2: $x = 1$. $(a \neq 0)$

Example 9: If the parabolas C & D intersect at a point A on the line L_1, then the equation of the tangent line L at A to the parabola D is-

(A) $2(a^3 - 3)x - ay + a^3 - 3a = 0$

(B) $2(a^3 - 3)x - ay + a^3 - 3a = 0$

(C) $(a^3 - 3)x - 2ay - 2a^3 + 6a = 0$

(D) None of these

Sol: In this case we need to calculate the point of intersection of C and D and then find the equation of the tangent to the parabola $y = kx^2$.

C and D intersect at the points for which $x^2 - 3 = kx^2$.

But x = a(given)

$\Rightarrow k = \frac{a^2 - 3}{a^2}$.

So the coordinates of A are $(a, a^2 - 3)$

The equation of the tangent L at A to D: $y = kx^2$ is

$$\frac{1}{2}(y + a^2 - 3) = \frac{a^2 - 3}{a^2}xa$$

$\Rightarrow 2(a^2 - 3)x - ay - a^3 + 3a = 0$ (L)

Example 10: If the line L meets the parabola C at a point B on the line L_2, other than A then a is equal to-

(A) –3 (B) –2 (C) 2 (D) 3

Sol: Proceed further from the previous solution.

The line L meets the parabola

C: $y = x^2 - 3$ at the points for which

$$x^2 - 3 = \frac{2(a^2 - 3)}{a}x - a^2 + 3 \Rightarrow (x - a)(ax + 6 - a^2) = 0.$$

But $x = 1$ & $x \neq a$.

$$\Rightarrow x = \frac{a^2 - 6}{a} = 1 \quad \Rightarrow a^2 - a - 6 = 0$$

Example 11: If a > 0, the angle subtended by the chord AB at the vertex of the parabola C is-

(A) $\tan^{-1}(5/7)$ (B) $\tan^{-1}(1/2)$

(C) $\tan^{-1}(2)$ (D) $\tan^{-1}(1/8)$

Sol: Calculate the point of intersection of C and D depending the value of 'a' and hence find the angle.

If a > 0, then a = 3. The coordinates of A and B are (3, 6) and (1, –2) respectively, and the equation of C: is

$y = x^2 - 3$ or $x^2 = y + 3$

The coordinates of the vertex O of the parabola C are (0, –3).

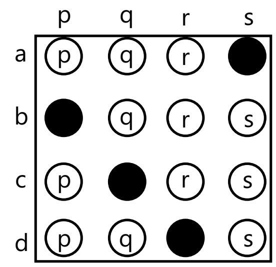
Slope of OA = 3, slope of OB = 1

∴ The angle between OA and OB is

$\tan^{-1}\dfrac{3-1}{1+3} = \tan^{-1}(1/2)$.

Example 12: Let $y^2 = 4ax$ be the equation of a parabola, then

(A) $yy_1 = 2a(x + x_1)$	(p) Equation of thenormal at (x_1,y_1)
(B) $xy_1=2a(y_1-y)+x_1y_1$	(q) Equation of the focal chord through (x_1,y_1)
(C) $xy_1=y(x_1-a)+ay_1$	(r) Equation of the through (x_1,y_1) and the point of intersection of axis with the direct
(D) $(x+a)y_1=(x_1+a)y$	(s) Equation of the tangent at (x_1,y_1)



Sol: Use the standard results and simple transformations to match the given option.

For the parabola $y^2 = 4ax$,

The equation of the tangent at (x_1, y_1) is

$yy_1 = 2a(x + x_1)$

The equation of the normal at (x_1, y_1) is

$y - y_1 = -\dfrac{y_1}{2a}(x - x_1)$

$\Rightarrow \quad xy_1 = 2a(y_1 - y) + x_1y_1$

Next, the equation of the focal chord through (x_1, y_1) and (a, 0) is

$y = \dfrac{y_1 - 0}{x_1 - a}(x - a)$

$\Rightarrow \quad xy_1 = y(x_1 - a) + ay_1$

Lastly, the equation of the line joining (–a, 0), the point of intersection of the axis y = 0 and the directrix x + a = 0 with (x_1, y_1) is

$(x + a)y_1 = (x_1 + a)y$

JEE Main/Boards

Exercise 1

Q.1 Find the equation of the parabola whose focus is (1, –1) and vertex is (2, 1).

Q.2 Find focus, vertex, directrix and axis of the parabola $4x^2 + y - 3x = 0$.

Q.3 The focal distance of a point on the parabola $y^2 = 12x$ is four units. Find the abscissa of this point.

Q.4 A double ordinate of parabola $y^2 = 4ax$ is of length 8a. Prove that the lines joining vertex to the end points of this chord are at right angles.

Q.5 Show that $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if $ln = am^2$.

Q.6 If x + y + 1 = 0 touches the parabola $y^2 = lx$, then show that $\lambda = 4$.

Q.7 Find the equation of the tangent, to the parabola $y^2 = 8x$, which makes an angle of 45° with the line y = 3x+5.

Q.8 Find equation of the tangent and the normal to the parabola $y^2 = 4x$ at the point (4, –4).

Q.9 Find the equation and the point of contact of the tangents to $y^2 = 6x$ drawn from the point (10, –8).

Q.10 Find the equation of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$.

Q.11 Find the point where normal to the parabola $y^2 = x$ at $\left(\frac{1}{4}, \frac{1}{2}\right)$ cuts it again.

Q.12 Find shortest distance between $y^2 = 4x$ and $x^2 + y^2 - 24y + 128 = 0$.

Q.13 AB is a chord of the parabola $y^2 = 4ax$ with the end A at the vertex of the given parabola. BC is drawn perpendiculars to AB meeting the axis of the parabola at C. Find the projection of BC on this axis.

Q.14 M is the foot of the perpendicular from a point P on the parabola $y^2 = (x - 3)$ to its directrix and S is the focus of the parabola, if SPM is an equilateral triangle, find the length of each side of the triangle.

Q.15 PQ is a double ordinate of a parabola $y^2 = 4ax$. If the locus of its points of trisection is another parabola length of whose latus rectum is k times the length of the latus rectum of the given parabola, then find the value of k.

Q.16 Find the equation of the parabola, the extremities of whose latus rectum are (1, 2) and (1, –4).

Q.17 Prove that the normal chord to a parabola at the point whose ordinate is equal to the abscissa subtends a right angle at the focus.

Q.18 If from the vertex of the parabola $y^2 = 4ax$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be drawn, prove that the locus of the vertex of the farther angle of the rectangle is the parabola $y^2 = 4a(x - 8a)$.

Q.19 Prove that the locus of the middle points of all chords of the parabola $y^2 = 4ax$ which are drawn through the vertex is the parabola $y^2 = 2ax$.

Q.20 Show that the locus of the middle point of all chords of the parabola $y^2 = 4ax$ passing through a fixed point (h, k) is $y^2 - ky = 2a(x - h)$.

Q.21 Prove that the area of the triangle formed by the tangents at points t_1 and t_2 on the parabola $y^2 = 4ax$ with the chord joining these two points is $\frac{a^2}{2}|t_1 - t_2|^3$.

Q.22 Show that the portion of the tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.

Q.23 If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and the tangent at the vertex A in Y and rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.

Q.24 Two equal parabolas have the same vertex and their axes are at right angles. Prove that they cut again at an angle $\tan^{-1}\frac{3}{4}$.

Q.25 Find the locus of the point of intersection of the tangents to the parabola $y^2 = 4ax$ which include an angle α.

Q.26 Find the set of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$ from which all the three normals drawn to the parabola are real and distinct.

Q.27 Show that the locus of points such that two of the three normals to the parabola $y^2 = 4ax$ from them coincide is $27ay^2 = 4(x - 2a)^3$.

Q.28 If a circle passes through the feet of normals drawn from a point to the parabola $y^2 = 4ax$, Prove that the circle also passes through origin.

Q.29 The middle point of a variable chord of the parabola $y^2 = 4ax$ lies on the line $y = mx + c$. Show that it always touches the parabola $\left(y + \frac{2a}{m}\right)^2 = 8a\left(x + \frac{c}{m}\right)$.

Exercise 2

Single Correct Choice Type

Q.1 The length of the chord intercepted by the parabola $y^2 = 4x$ on the straight line $x + y = 1$ is-

(A) 4 (B) $4\sqrt{2}$ (C) 8 (D) $8\sqrt{2}$

Q.2 A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is-

(A) $x^2 - 6x - 8y + 25 = 0$ (B) $x^2 - 8x - 6y + 25 = 0$

(C) $x^2 - 6x + 8y - 25 = 0$ (D) $x^2 + 6x - 8y - 25 = 0$

Q.3 The curve describes parametrically by $x = t^2 - 2t + 2$, $y = t^2 + 2t + 2$ represents-

(A) Straight line (B) Pair of straight line

(C) Circle (D) Parabola

Q.4 If $y = 2x - 3$ is a tangent to the parabola $y^2 = 4a\left(x - \frac{1}{3}\right)$, then 'a' is equal to-

(A) 1 (B) –1 (C) $\frac{14}{3}$ (D) $\frac{-14}{3}$

Q.5 Two tangents to the parabola $y^2=4a$ make angles a_1 and a_2 with the x-axis. The locus of their point of intersection if $\frac{\cot\alpha_1}{\cot\alpha_2} = 2$ is-

(A) $2y^2 = 9ax$ (B) $4y^2 = 9ax$

(C) $y^2 = 9ax$ (D) None

Q.6 Through the vertex 'O' of the parabola $y^2 = 4ax$, variable chords OP and OQ are drawn at right angles. If the variable chord PQ intersects the axis of x at R, then distance OR-

(A) Varies with different positions of P and Q

(B) Equals the semi latus rectum of the parabola

(C) Equals latus rectum of the parabola

(D) Equals double the latus rectum of the parabola

Q.7 A point P moves such that the difference between its distances from the origin and from the axis of 'x' is always a constant c. The locus-

(A) A straight line having equal intercepts C on the axis

(B) A circle having its centre at $\left(0, -\frac{c}{2}\right)$ & passing through $\left(c\sqrt{2}, \frac{c}{2}\right)$

(C) A parabola with its vertex at $\left(0, -\frac{c}{2}\right)$ & passing through $\left(c\sqrt{2}, \frac{c}{2}\right)$

(D) None of these

Q.8 Tangents are drawn from the point (–1, 2) on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line x = 2, is-

(A) 6 (B) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) None

Q.9 Which one of the following equations represented parametric-cally represents equation to a parabolic profile ?

(A) $x = 3\cos t$; $y = 4\sin t$

(B) $x^2 - 2 = -\cos t$; $y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$

(D) $x = \sqrt{1-\sin t}$; $y = \sin^2 + \cos\frac{1}{2}$

Q.10 From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If q_1 and q_2 are the inclinations of these tangents with the axis of x such that, $q_1 + q_2 = \frac{\pi}{4}$, then the locus of P is-

(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$

(C) $x - y - 1 = 0$ (D) $x + y + 1 = 0$

Q.11 From the point (4, 6) a pair of tangent lines are drawn to the parabola, $y^2 = 8x$. The area of the triangle formed by these pair of tangent lines and the chord of contact of the point (4, 6) is-

(A) 8 (B) 4 (C) 2 (D) None

Q.12 Let PSQ be the focal chord of the parabola, $y^2 = 8x$. If the length of SP = 6 then, ℓ(SQ) is equal to-

(A) 3 (B) 4 (C) 6 (D) None

Q.13 The line $4x - 7y + 10 = 0$ intersects the parabola, $y^2 = 4x$ at the points A and B. The coordinates of the point of intersection of the tangents drawn at the points A and B are-

(A) $\left(\frac{7}{2}, \frac{5}{2}\right)$ (B) $\left(-\frac{5}{2}, \frac{7}{2}\right)$ (C) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (D) $\left(-\frac{7}{2}, \frac{5}{2}\right)$

Q.14 A line passing through the point (21,30) and normal to the curve $y = 2\sqrt{x}$ can have the slope-

(A) 2 (B) 3 (C) –2 (D) –5

Q.15 If the chord of contact of tangents from a point P to the parabola $y^2=4ax$ touches the parabola $x^2=4by$, the locus of P is-

(A) Circle (B) Parabola

(C) Ellipse (D) Hyperbola

Q.16 If M is the foot of the perpendicular from a point P of a parabola $y^2=4ax$ its directrix and SPM is an equilateral triangle, where S is the focus, the SP is equal to-

(A) a (B) 2a (C) 3a (D) 4a

Q.17 The latus rectum of a parabola whose focal chord is PSQ such that SP = 3 and SQ = 2 is given by-

(A) 24/5 (B) 12/5 (C) 6/5 (D) None

Q.18 The normal chord of to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to the abscissa, then angle subtended by normal chord at the focus is-

(A) $\frac{\pi}{4}$ (B) $\tan^{-1}\sqrt{2}$ (C) $\tan^{-1} 2$ (D) $\frac{\pi}{2}$

Q.19 P is any point on the parabola, $y^2=4ax$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. The angle subtended by MD at the focus is-

(A) $\pi/4$ (B) $\pi/3$ (C) $5\pi/12$ (D) $\pi/2$

Q.20 A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha, 0)$, $(\beta, 0)$ both to right of the origin. A circle also passes through these two points. The length of a tangent from the origin to the circle is-

(A) $\sqrt{\frac{bc}{a}}$ (B) ac^2 (C) $\frac{b}{a}$ (D) $\sqrt{\frac{c}{a}}$

Q.21 TP and TQ are tangents to the parabola, $y^2 = 4ax$ at P and Q. If the chord PQ passes through the fixed point (–a, b) then the locus of T is-

(A) $ay = 2b(x - b)$ (B) $bx = 2a(y - a)$

(C) $by = 2a(x - a)$ (D) $ax = 2b(y - b)$

Q.22 The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modulus of the difference of the ordinates of the point Q and R is-

(A) $\frac{A}{2a}$ (B) $\frac{A}{a}$ (C) $\frac{2A}{a}$ (D) $\frac{4A}{a}$

Previous Years' Questions

Q.1 If $x + y = k$ is normal to $y^2 = 12x$, then k is- ***(2000)***

(A) 3 (B) 9 (C) –9 (D) –3

Q.2 If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of k is- ***(2000)***

(A) $\frac{1}{8}$ (B) 8 (C) 4 (D) $\frac{1}{4}$

Q.3 The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is- ***(2000)***

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$

Q.4 The locus of the midpoint of the line segment joining the focus to a moving point on the parabola $y^2=4ax$ is another parabola with directrix- ***(2000)***

(A) $x = -a$ (B) $x = -\frac{a}{2}$ (C) $x = 0$ (D) $x = \frac{a}{2}$

Q.5 The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is- ***(2000)***

(A) $3y = 9x + 2$ (B) $y = 2x + 1$

(C) $2y = x + 8$ (D) $y = x + 2$

Q.6 Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1: 3. Then, the locus of P is- ***(2011)***

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

Q.7 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are- ***(2008)***

(A) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

Q.8 Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by- ***(2011)***

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$

(C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

Q.9 The point of intersection of the tangents at the ends of the latus rectum of parabola $y^2 = 4x$ is........ ***(1994)***

Q.10 Find the shortest distance of the point (0, c) from the parabola $y = x^2$ where $0 \le c \le 5$. ***(1982)***

Q.11 Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2). ***(1984)***

Q.12 Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. ***(1994)***

Q.13 From a point 'A' common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola. ***(1996)***

Q.14 The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45°. Show that the locus of the point P is a hyperbola. ***(1998)***

Q.15 At any point P on the parabola $y^2 - 2y - 4x + 5 = 0$ a tangent is drawn which meets the directrix at Q. Find the locus of point R, which divides QP externally in the ratio $\frac{1}{2}:1$. ***(2004)***

Q.16 The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is ***(2014)***

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $\frac{1}{8}$ (D) $\frac{2}{3}$

JEE Advanced/Boards

Exercise 1

Q.1 Find the equations of the tangents to the parabola $y^2 = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Also find the coordinates of their points of contact.

Q.2 Find the equations of the tangents of the parabola $y^2 = 12x$, which pass through the point (2, 5).

Q.3 Show that the locus of points of intersection of two tangents to $y^2=4ax$ and which are inclined at an angle α is $(y^2 - 4ax)\cos^2\alpha = (x + a)^2\sin^2\alpha$.

Q.4 Two straight lines one being a tangent to $y^2 = ax$ and the other to $x^2 = 4by$ are right angles. Find the locus of their point of intersection.

Q.5 Prove that the locus of the middle point of portion of a normal to $y^2=4ax$ intercepted between the curve and the axis is another parabola. Find the vertex and the latus rectum of the second parabola.

Q.6 Show that the locus of a point, such that two of the three normals drawn from it to the parabola $y^2 = 4ax$ are perpendicular is $y^2 = a(x - 3a)$.

Q.7 A variable chord t_1t_2 of the parabola $y^2 = 4ax$. Show that its passes through a fixed point. Also find the co-ordinates of the fixed point.

Q.8 Through the vertex O of the parabola $y^2 = 4ax$, a perpendicular is drawn to any tangent meeting it at P and the parabola at Q. Show that OP.OQ = constant.

Q.9 If the normal at P(18, 12) to the parabola $y^2 = 8x$ cuts its again at Q, show that $9PQ = 80\sqrt{10}$.

Q.10 O is the vertex of the parabola $y^2=4ax$ and L is the upper end of the latus rectum. If LH is drawn perpendicular to OL meeting OX in H, prove that the length of the double ordinate through H is $4a\sqrt{5}$.

Q.11 The normal at a point P to the parabola $y^2 = 4ax$ meets its axis at G. Q is another point on the parabola such that QG is perpendicular to the axis of the parabola. Prove that $QG^2 - PG^2$ = constant.

Q.12 Find the condition on 'a' and 'b' so that the two tangents drawn to the parabola $y^2 = 4ax$ from a point are normals to the parabola $x^2 = 4by$.

Q.13 Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is $y^2(2x + a) = a(3x + a)^2$.

Q.14 Prove that, the normal to $y^2 = 12x$ at (3, 6) meets the parabola again in (27, –18) and circle on this normal chord as diameter is $x^2 + y^2 - 30x + 12y - 27 = 0$.

Q.15 Show that, the normals at the points (4a, 4a) and at the upper end of the latus rectum of the parabola $y^2 = 4ax$ intersect on the same parabola.

Q.16 If from the vertex of a parabola a pair of chords be drawn at right angles to one another, & with these chords as adjacent sides a rectangle be constructed, then find the locus of the outer corner of the rectangle.

Q.17 Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that one of the normals is given by $y = x - 3$ and find the equations of the others.

Q.18 A circle is described whose centre is the vertex and whose diameter is three-quarters of the latus rectum of a parabola $y^2 = 4ax$. Prove that the common chord of the circle and parabola bisects the distance between the vertex and the focus.

Q.19 TP and TQ are tangents to the parabola and the normals at P and Q meet at a point R on the curve. Prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y^2 = a(x - a)$.

Q.20 Find the equation of the circle which passes through the focus of the parabola $x^2 = 4y$ and touches it at the point (6, 9).

Q.21 P and Q are the point of contact of the tangents drawn from the point T to the parabola $y^2 = 4ax$. If PQ be the normal to the parabola at P, prove that TP is bisected by the directrix.

Q.22 Prove that the locus of the middle points of the normal chords of the parabola $y^2 = 4ax$ is

$$\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a.$$

Q.23 Two tangents to the parabola $y^2=8x$ meet the tangent at its vertex in the point P and Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y^2=8(x + 2)$.

Q.24 Two perpendicular straight lines through the focus of the parabola $y^2=4x$ meet its tangents to the parabola parallel to the perpendicular lines intersect in the mid point of TT′.

Q.25 A variable chord PQ of the parabola $y^2 = 4x$ is drawn parallel to the line y=x. If the parameters of the points P and Q on the parabola are p and q respectively, show, that p + q = 2. Also show that the locus of the point of intersection of the normals at P and Q is $2x - y = 12$.

Q.26 Show that the circle through three points the normals at which to the parabola $y^2 = 4ax$ are concurrent at the point (h, k) is $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$.

Q.27 Find the condition such that the chord t_1t_2 of the parabola $y^2 = 4ax$ passes through the point (a, 3a). Find the locus of intersection of t tangents at t_1 and t_2 under this condition.

Q.28 A variable tangent to the parabola $y^2 = 4ax$ meets the circle $x^2 + y^2 = r^2$ at P and Q. Prove that the locus of the mid point of PQ is $x(x^2 + y^2) + ay^2 = 0$.

Q.29 A variable chord PQ of a parabola $y^2=4ax$, subtends a right angle at the vertex. Show that it always passes through a fixed point. Also show that the locus of the point of intersection of the tangents at P and Q is a straight line. Find the locus of the mid point of PQ.

Exercise 2

Single Correct Choice Type

Q.1 The tangent at P to a parabola $y^2=4ax$ meets the directrix at U and the latus rectum at V then SUV (where S is the focus).

(A) Must be a right triangle

(B) Must be an equilateral triangle

(C) Must be an isosceles triangle

(D) Must be a right isosceles triangle

Q.2 If the distances of two points P and Q from the focus of a parabola $y^2=4ax$ are 4 and 9, then the distance of the point of intersection of tangents at P and Q from the focus is-

(A) 8 (B) 6 (C) 5 (D) 13

Q.3 The chord of contact of the pair of tangents drawn from each point on the line 2x + y = 4 to the parabola $y^2=-4x$ passes through a fixed point.

(A) (–2, 1) (B) (–2, –1)

(C) (1/2, 1/4) (D) (–1/2, –1/4)

Q.4 The locus of the foot of the perpendiculars drawn from the vertex on a variable tangent to the parabola $y^2 = ax$ is-

(A) $x(x^2 + y^2) + ay^2 = 0$ (B) $y(x^2 + y^2) + ax^2 = 0$

(C) $x(x^2 - y^2) + ay^2 = 0$ (D) None of these

Q.5 Locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$ is-

(A) $2x - 1 = 0$ (B) $2x + 3 = 0$

(C) $2y + 3 = 0$ (D) $2x + 5 = 0$

Q.6 If the tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q and R, then the mid point of QR is-

(A) $(x_1 + b, y_1 + b)$ (B) $(x_1 - b, y_1 - b)$

(C) (x_1, y_1) (D) $(x_1 + b, y_1)$

Q.7 The point (s) on the parabola $y^2 = 4x$ which are closest to the circle, $x^2 + y^2 - 24y + 128 = 0$ is/are-

(A) (0, 0) (B) $(2, 2\sqrt{2})$ (C) (4, 4) (D) None

Q.8 Length of the focal chord of the parabola $y^2 = 4ax$ at a distance p from the vertex is-

(A) $\frac{2a^2}{p}$ (B) $\frac{a^3}{p^2}$ (C) $\frac{4a^3}{p^2}$ (D) $\frac{p^2}{a^2}$

Q.9 If two normals to a parabola $y^2=4ax$ intersect at right angles then the chord joining their feet passes through a fixed point whose co-ordinates are-

(A) (–2a, 0) (B) (a, 0) (C) (2a, 0) (D) None

Q.10 Locus of a point p if the three normals drawn from it to the parabola $y^2 = 4ax$ are such that two of them make complementary angles with the axis of the parabola is-

(A) $y^2 = a(x + a)$ (B) $y^2 = 2a(x - a)$

(C) $y^2 = a(x - 2a)$ (D) $y^2 = a(x - a)$

Q.11 A tangent to the parabola $x^2 + 4ay = 0$ cuts the parabola $x^2 = 4by$ at A and B the locus of the mid point of AB is-

(A) $(a + 2b)x^2 = 4b^2y$ (B) $(b + 2a)x^2 = 4b^2y$

(C) $(a + 2b)y^2 = 4b^2x$ (D) $(b + 2x)x^2 = 4a^2y$

Q.12 The circle drawn on the latus rectum of the parabola $4y^2 + 25 = 4(y + 4x)$ as diameter cuts the axis of the parabola at the points-

(A) $\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{9}{2},\frac{1}{2}\right)$ (B) $\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{9}{2}\right)$

(C) $\left(\frac{1}{2},\frac{1}{2}\right)$, (0, 0) (D) $\left(\frac{1}{2},\frac{7}{2}\right),\left(\frac{1}{2},\frac{9}{2}\right)$

Q.13 The distance between a tangent to the parabola $y^2 = 4Ax(A > 0)$ and the parallel normal with gradient 1 is-

(A) 4A (B) $2\sqrt{2}$ A (C) 2A (D) $\sqrt{2}$ A

Q.14 The equation to the directrix of a parabola if the two extremities of its latus rectum are (2, 4) and (6, 4) and the parabola passes through the point (8, 1) is-

(A) $y - 5 = 0$ (B) $y - 6 = 0$ (C) $y - 1 = 0$ (D) $y - 2 = 0$

Q.15 A variable circle is drawn to touch the line $3x - 4y = 10$ and also the circle $x^2 + y^2 = 4$ externally then the locus of its centre is-

(A) Straight line

(B) Circle

(C) Pair of real, distinct straight lines

(D) Parabola

Q.16 The tangent and normal at P(t), for all real positive t, to the parabola $y^2=4ax$ meet the axis of the parabola in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through the points P, T and G is-

(A) $\cot^{-1}t$ (B) $\cot^{-1}t^2$ (C) $\tan^{-1}t$ (D) $\tan^{-1}t^2$

Q.17 P is a point on the parabola $y^2=4x$ where abscissa and ordinate are equal. Equation of a circle passing through the focus and touching the parabola at P is-

(A) $x^2 + y^2 - 13x + 2y + 12 = 0$

(B) $x^2 + y^2 - 13x - 18y + 12 = 0$

(C) $x^2 + y^2 + 13x - 2y - 14 = 0$

(D) None of these

Q.18 A circle is described whose centre is the vertex and whose diameter is three – quarters of the latus rectum of the parabola $y^2 = 4ax$. If PQ is the common chord of the circle and the parabola and L_1L_2 is the latus rectum, then the area of the trapezium PL_1L_2Q is-

(A) $3\sqrt{2}\,a^2$ (B) $2\sqrt{2}\,a^2$ (C) $4a^2$ (D) $\left(\frac{2+\sqrt{2}}{2}\right)a^2$

Multiple Correct Choice Type

Q.19 Let P, Q and R are three co-normal points on the parabola $y^2 = 4ax$. Then the correct statement(s) is/are-

(A) Algebraic sum of the slopes of the normals at P, Q and R vanishes

(B) Algebraic sum of the ordinates of the points P, Q and R vanishes

(C) Centroid of the triangle PQR lies on the axis of the parabola

(D) Circle circumscribing the triangle PQR passes through the vertex of the parabola

Q.20 Let A be the vertex and L the length of the latus rectum of the parabola, $y^2 - 2y - 4x - 7 = 0$. The equation of the parabola with A as vertex, 2L, the length of the latus rectum and the axis at right angles to the of the given curve is-

(A) $x^2 + 4x + 8y - 4 = 0$ (B) $x^2 + 4x - 8y + 12 = 0$

(C) $x^2 + 4x + 8y + 12 = 0$ (D) $x^2 + 8x - 4y + 8 = 0$

Q.21 Two parabolas have the same focus. If their directrices are the x-axis and the y-axis respectively, then the slope of their common chord is-

(A) 1 (B) –1 (C) 4/3 (D) 3/4

Q.22 Equation of common tangent to the circle, $x^2 + y^2 = 50$ and the parabola, $y^2 = 40x$ can be-

(A) $x + y - 10 = 0$ (B) $x - y + 10 = 0$

(C) $1 + y + 10 = 0$ (D) $x - y - 10 = 0$

Q.23 The equation $y^2 + 3 = 2(2x + y)$ represents a parabola with the vertex at-

(A) $\left(\frac{1}{2},1\right)$ and axis parallel to x-axis

(B) $\left(1,\frac{1}{2}\right)$ and axis parallel to x-axis

(C) $\left(\frac{1}{2},1\right)$ and focus at $\left(\frac{3}{2},1\right)$

(D) $\left(\frac{1}{2},1\right)$ and axis parallel to y-axis

Q.24 Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally then-

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$

(C) $a > 0, b > 0$ (D) $a < 0, b < 0$

Q.25 P is a point on the parabola $y^2 = 4ax$ $(a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular P on the directrix. If a circle is described on MD as a diameter then it intersects the x-axis at a point whose co-ordinates are-

(A) (–3a, 0) (B) (–a, 0) (C) (–2a, 0) (D) (a, 0)

Previous Years' Questions

Q.1 The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represent ***(1999)***

(A) A pair of straight lines (B) An ellipse

(C) A parabola (D) A hyperbola

Q.2 The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is- ***(2001)***

(A) $\sqrt{3}y = 3x + 1$ (B) $\sqrt{3}y = -(x + 3)$

(C) $\sqrt{3}y = x + 3$ (D) $\sqrt{3}y = -(3x + 1)$

Q.3 The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible values of the slope of this chord, are- ***(2003)***

(A) {–1, 1} (B) {–2, 2} (C) {–2, 1/2} (D) {2, –1/2}

Q.4 Axis of a parabola is y = x and vertex and focus are at a distance $\sqrt{2}$ and $2\sqrt{2}$ respectively from the origin. Then equation of the parabola is- ***(2006)***

(A) $(x - y)^2 = 8(x + y - 2)$ (B) $(x + y)^2 = 2(x + y - 2)$

(C) $(x - y)^2 = 4(x + y - 2)$ (D) $(x + y)^2 = 2(x - y + 2)$

Q.5 Normals at P, Q, R are drawn to $y^2=4x$ which intersect at (3, 0). Then

Column I	Column II
(A) Area of ΔPQR	(p) 2
(B) Radius of circumcircle of ΔPQR	(q) $\frac{5}{2}$
(C) Centroid of ΔPQR	(r) $\left(\frac{5}{2},0\right)$
(D) Circumcentre of ΔPQR	(s) $\left(\frac{2}{3},0\right)$

Q.6 Equation of common tangent of $y=x^2$ and $y=-x^2 + 4x - 4$ is- ***(2006)***

(A) $y = 4(x - 1)$ (B) $y = 0$

(C) $y = -4(x - 1)$ (D) $y = -30x-50$

Q.7 The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose- ***(2009)***

(A) Vertex is $\left(\frac{2a}{3},0\right)$ (B) Directrix is x = 0

(C) Latus rectum is $\frac{2a}{3}$ (D) Focus is (a, 0)

Q.8 Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be- ***(2010)***

(A) $-\frac{1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $-\frac{2}{r}$

Q.9 Consider the parabola $y^2 = 8x$. Let D_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2},2\right)$ on the parabola and D_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is ...... ***(2011)***

Q.10 Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, 0). Show that h > 2. ***(1981)***

Q.11 Three normals are drawn from the point (c, 0) to the curve $y^2 = x$. Show that c must be greater than $\frac{1}{2}$. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other. ***(1991)***

Q.12 Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4ax$ internally in the ratio 1: 2 is a parabola. Find the vertex of this parabola. ***(1995)***

Q.13 Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C, taken in pairs, intersect at points P, Q and R. Determine the ratio of the areas of the triangle ABC and PQR. ***(1996)***

Q.14 Let C_1 and C_2 be, respectively, the parabolas $x^2 = y - 1$ and $y^2 = x - 1$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively, with respect to the line y = x. Prove that P_1 lies on C_2, Q_1 lies on C_1 and $PQ \geq \min\{PP_1, QQ_1\}$. Hence or otherwise, determine points P_0 and Q_0 on the parabolas C_1 and C_2 respectively such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) with P on C_1 and Q on C_2. ***(2000)***

Q.15 Normals are drawn from the point P with slopes m_1, m_2, m_3 to the parabola $y^2 = 4x$. If locus of P with $m_1m_2 = \alpha$ is a part of the parabola itself, then find α. ***(2003)***

Q.16 Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by ***(2011)***

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$

(C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

Q.17 Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1 : 3. Then the locus of P is ***(2011)***

(A) $y^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

Q.18 Let the straight line x = b divide the area enclosed by $y = (1-x)^2$, $y = 0$ and x = 0 into two parts $R_1(0 \leq x \leq b)$ and $R_2(b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals ***(2011)***

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Q.19 Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus and the point $P\left(\frac{1}{2},2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is ***(2011)***

Paragraph (Questions 20 and 21)

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line y = 2x + a, a > 0.

Q.20 Length of chord PQ is

(A) 7a (B) 5a (C) 2a (D) 3a

Q.21 If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

Q.22 A line L : y = mx + 3 meets y-axis at E(0, 3) and the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y -axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum.

Match List I with List II and select the correct answer using the code given below the lists: ***(2013)***

List I		List II	
(i)	m =	(p)	$\frac{1}{2}$
(ii)	Maximum area of ΔEFG is	(q)	4
(iii)	$y_0 =$	(r)	2
(iv)	$y_1 =$	(s)	1

(A) i $\to$ s, ii $\to$ p, iii $\to$ q, iv $\to$ iii

(B) i $\to$ r, ii $\to$ s, iii $\to$ p, iv $\to$ ii

(C) i $\to$ p, ii $\to$ r, iii $\to$ q, iv $\to$ iv

(D) i $\to$ p, ii $\to$ r, iii $\to$ s, iv $\to$ ii

Paragraph (Questions 23 and 24)

Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at)$, Q $(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a, 0).

Q.23 The value of r is ***(2014)***

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

Q.24 If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is ***(2014)***

(A) $\frac{\left(t^2-1\right)^2}{2t^3}$ (B) $\frac{a\left(t^2-1\right)^2}{2t^3}$

(C) $\frac{a\left(t^2-1\right)^2}{t^3}$ (D) $\frac{a\left(t^2+2\right)^2}{t^3}$

Q.25 Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 > 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent P_2 which passes through $(f_1, 0)$, The m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m^2}+m_2^2\right)$ is ***(2015)***

Questions

JEE Main/Boards

Exercise 1

Q.12 Q.15 Q.18 Q.19
Q.23 Q.27 Q.29

Exercise 2

Q.1 Q.10 Q.16 Q.21

Previous Years' Questions

Q.4 Q.7 Q.12 Q.15

JEE Advanced/Boards

Exercise 1

Q.7 Q.13 Q.18 Q.21
Q.25 Q.29

Exercise 2

Q.1 Q.7 Q.9 Q.15
Q.19 Q.21 Q.22 Q.25

Previous Years' Questions

Q.3 Q.6 Q.7 Q.9
Q.11 Q.14

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$

Q.2 $\left(\frac{3}{8},\frac{1}{2}\right), \left(\frac{3}{8},\frac{9}{16}\right)$, $8y-5 = 0, 8x-3=0$

Q.3 1

Q.7 $y + 2x + 1 = 0, 2y = x + 8$

Q.8 $x + 2y + 4 = 0, y - 2x + 12 = 0$

Q.9 $x + 2y + 6 = 0$ at $(6, -6)$ $3x + 10y + 50 = 0$ at $\left(\frac{50}{3},-10\right)$

Q.10 $2x + 3y + 36 = 0$

Q.11 $\left(\frac{9}{4},-\frac{3}{2}\right)$

Q.12 $4(\sqrt{5}-1)$

Q.13 $y\ (y/x) = y^2/x = 4ax/x = 4a$

Q.14 $2 \times 4 = 8$

Q.15 $k = 1/9$

Q.16 $(y + 1)^2 = -3(2x - 5)$; $(y + 1)^2 = 3(2x + 1)$

Q.25 $(x + a)^2 \tan^2\alpha = y^2 - 4ax$

Q.26 $\{(x, 1) ; x > 3\}$

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** A	**Q.3** D	**Q.4** D	**Q.5** A	**Q.6** C
Q.7 C	**Q.8** B	**Q.9** B	**Q.10** C	**Q.11** C	**Q.12** A
Q.13 C	**Q.14** D	**Q.15** D	**Q.16** D	**Q.17** A	**Q.18** D
Q.19 D	**Q.20** D	**Q.21** C	**Q.22** C		

Previous Years' Questions

Q.1 B	**Q.2** C	**Q.3** D	**Q.4** C	**Q.5** D	**Q.6** C
Q.7 B,C	**Q.8** A,B,D	**Q.9** $(-1, 0)$	**Q.10** $\sqrt{c-\frac{1}{4}}$, $\frac{1}{2} \le c \le 5$		**Q.11** $x + y = 3$
Q.12 $y^2 = 2(x - 4)$	**Q.13** $\frac{15a^2}{4}$	**Q.15** $(x + 1)(y - 1)^2 + 4 = 0$		**Q.16** A	

JEE Advanced/Boards

Exercise 1

Q.1 $2x - y + 2 = 0, (1, 4)$; $x + 2y + 16 = 0, (16, -16)$

Q.2 $3x - 2y + 4 = 0$; $x - y + 3 = 0$

Q.4 $(ax + by)(x^2 + y^2) + (ay - bx)^2 = 0$

Q.5 $(a, 0)$; a

Q.7 $[a(t_0^2 + 4), -2at_0]$

Q.12 $a^2 > 8b^2$

Q.16 $y^2 = 4a(x - 8a)$

Q.17 $y = -4x + 72, y = 3x - 33$

Q.20 $x^2 + y^2 + 18x - 28y + 27 = 0$

Q.27 $2 - 3(t_1 + t_2) + 2t_1t_2 = 0$; $2x - 3y + 2a = 0$

Q.28 $x(x^2 + y^2) + ay^2 = 0$

Q.29 $(4a, 0)$; $x + 4a = 0$; $y^2 = 2a(x - 4a)$

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** B **Q.3** A **Q.4** D **Q.5** D **Q.6** C

Q.7 C **Q.8** C **Q.9** B **Q.10** D **Q.11** A **Q.12** A

Q.13 B **Q.14** B **Q.15** D **Q.16** C **Q.17** A **Q.18** D

Multiple Correct Choice Type

Q.19 A, B, C, D **Q.20** A, B **Q.21** A, B **Q.22** B, C **Q.23** A, C **Q.24** A, D

Q.25 A, D

Previous Years' Questions

Q.1 C **Q.2** C **Q.3** A **Q.4** A **Q.5** $A \to p$; $B \to q$; $C \to s$; $D \to r$

Q.6 A,B **Q.7** A,D **Q.8** C,D **Q.9** 2 **Q.10** $h > 2$ **Q.11** $c = 3/4$

Q.12 $\left(\frac{2}{9}, \frac{8}{9}\right)$ **Q.13** 2 **Q.14** $P_0\left(\frac{1}{2}, \frac{5}{4}\right)$, $Q_0\left(\frac{5}{4}, \frac{1}{2}\right)$ **Q.15** $\alpha = 2$ **Q.16** A,B,D

Q.17 C **Q.18** B **Q.19** 2 **Q.20** B **Q.21** D **Q.22** A

Q.23 B **Q.24** D **Q.25** 4

Solutions

JEE Main/Boards

Exercise 1

Sol 1: Vertex is the midpoint of point of intersection of directrix (let say M(a, b)) & the focus and axis of parabola.

$\therefore (2, 1) = \left(\frac{a+1}{2}, \frac{b-1}{2}\right)$

$\therefore M(a, b) = (3, 3)$

Also directrix $\perp$ to axis.

$\therefore$ Slope of directrix $= -\frac{1}{2}$

The equation of directrix is $(y - 3) = -\frac{1}{2}(x - 3)$

$\therefore x + 2y - 9 = 0$

Let (x, y) be the point on parabola

$\therefore \frac{x+2y-9}{\sqrt{5}} = \sqrt{(x-1)^2 + (y+1)^2}$

$\therefore (x + 2y - 9)^2 = 5[(x - 1)^2 + (y + 1)^2]$

$\therefore x^2 + 4y^2 + 81 + 4xy - 18x - 36y$

$= 5x^2 + 5y^2 - 10x + 10y + 10$

$\therefore 4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$

Sol 2: $4x^2 - 3x + \frac{9}{16} = -y + \frac{9}{16}$

$\therefore \left(2x - \frac{3}{4}\right)^2 = \frac{\left(-y + \frac{9}{16}\right)}{1}$

$\therefore \left(x-\frac{3}{8}\right)^2 = \frac{1}{4}\left(-y+\frac{9}{16}\right)$

Let Y = -y + $\frac{9}{16}$ & X = x $\frac{3}{8}$

$\therefore X^2 = -\frac{Y}{4}$

Comparing to $X^2 = -4aY \Rightarrow a = \frac{1}{16}$

Vertex is (X = 0, Y = 0)

∴ Coordinates of vertex in original Cartesian system is $\left(\frac{3}{8}, \frac{9}{16}\right)$

Focus = $\left(0, -\frac{1}{16}\right) \Rightarrow$ actual focus = $\left(\frac{3}{8}, \frac{1}{2}\right)$

and foot of directrix = (0, a) = $\left(0, \frac{1}{16}\right)$

∴ Actual foot = $\left(\frac{3}{8}, \frac{5}{8}\right)$

equation of directrix is $y = \frac{5}{8}$ & axis is $x = \frac{3}{8}$.

Sol 3: $y^2 = 12x \therefore a = 3$

Focal distance = distance from directrix

$\therefore x + a = 4 \quad \Rightarrow x = 1$

Sol 4: Length of double ordinate = 8a

∴ The ends of ordinate are (x, 4a) & (x – 4a). Substituting in $y^2 = 4ax$ we get x = 4a

∴ A(4a, 4a) & B(4a, – 4a) are the ends of ordinates

$m_{OA} = 1$ & $m_{OB} = -1$

∴ These points subtend 90° at origin.

Sol 5: Equation of line is $y = -\frac{l}{m}x - \frac{n}{m}$.

but equation of tangent to a parabola with slope s is $y = sx + \frac{a}{s}$

$\therefore s = -\frac{l}{m}$ & $\frac{a}{s} = -\frac{n}{m} \Rightarrow am^2 = ln$

Sol 6: $x + y + 1 = 0$ & $y^2 = lx$

From above result $ln = am^2$

$\therefore 1 \times 1 = \frac{\lambda}{4} \times (-1)^2 \qquad \Rightarrow \lambda = 4$

Sol 7: Let slope of line be m

$\therefore \tan \pm 45^0 = \frac{m-3}{1+3m}$

$1 = \frac{(m-3)^2}{(3m+1)^2}$

$\therefore 8m^2 + 12m - 8 = 0$

$2m^2 + 3m - 2 = 0 \Rightarrow (2m - 1)(m + 2) = 0$

$\therefore m = -2$ or $m = \frac{1}{2}$

∴ Equation of tangent is $y = mx + \frac{a}{m}$ (a = 2)

$\therefore y = -2x + \frac{2}{-2}$ or $y = \frac{1}{2}x + \frac{2}{\frac{1}{2}}$

$\Rightarrow 2x + y + 1 = 0$ or $x - 2y + 8 = 0$

Sol 8: $y^2 = 4x \quad (x_1, y_1) = (4, -4)$

Equation of tangent is $yy_1 = 2(x + x_1)$

$\therefore -4y = 2x + 8$

Equation of normal = $(y - y_1) = -\frac{y_1}{2a}(x - x_1)$

$\Rightarrow y + 4 = 2(x - 4)$

$\therefore 2x - y - 12 = 0$

Sol 9: Let tangents be $y = mx + \frac{6}{4m}$

Now (10 – 8) lies on it $-8 = 10m + \frac{3}{2m}$

$20m^2 + 16m + 3 = 0$

$\Rightarrow 20m^2 + 10m + 6m + 3 = 0$

$\Rightarrow (10m + 3)(2m + 1) = 0$

$\therefore m = -\frac{3}{10}$ or $m = -\frac{1}{2}$

∴ Equation is $y = -\frac{3}{10}x - 5$ or $3x + 10y + 50 = 0$

& $y = -\frac{1}{2}x - 3$ or $x + 2y + 6 = 0$

∴ Points of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

∴ For line 3x + 10y + 50 point of contact

$= \left(\frac{3}{2\times 9}\times 100, \frac{3}{-3/10}\right) = \left(\frac{50}{3}, -10\right)$

and for line 2y + x + 6 = 0 point of contact

$$= \left(\frac{3}{2 \times 1} \times 4, \frac{3}{-\frac{1}{2}} \right) = (6, -6)$$

Sol 10: Tangent to parabola $y^2 = 32x$ is

$y = mx + \frac{8}{m}$ & tangent to parabola

$x^2 = 108y$ is $x = \frac{y}{m} + 27m$

$\therefore \frac{8}{m} = -27m^2$

$\therefore m^3 = -\frac{8}{27} \Rightarrow m = -\frac{2}{3}$

$\therefore$ Common tangent is $y = -\frac{2}{3}x - 12$

$\therefore 2x + 3y + 36 = 0$

Sol 11: For the parabola $a = \frac{1}{4}$

$\therefore$ Point $= \left(\frac{1}{4}, \frac{1}{2}\right)$

$\therefore 2at = \frac{1}{2} \Rightarrow t = 1$

$\therefore t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_2 = -1 - 2 = -3$

$\therefore$ Point, when the normal again cuts the parabola

is $\left(\frac{1}{4}t_2^2, \frac{1}{2}t_2\right) = \left(\frac{9}{4}, \frac{-3}{2}\right)$

Sol 12: The circle lies outside the parabola the shortest distance is when normal to parabola is normal to circle, i. e. , it passes through center of circle

The equation of normal in parametric form is

$y = -tx + 2at + at^3 \Rightarrow y = -tx + 2t + t^3$

$\Rightarrow$ It passes through (0, 12)

$\therefore t^3 + 2t - 12 = 0$

$(t - 2)(t^2 + 2t + 6) = 0$

t = 2 is only positive point

$\therefore$ Point is (4, 4) & shortest distance

$= \sqrt{4^2 + (4 - 12)^2} - r = \sqrt{80} - 4 = 4(\sqrt{5} - 1)$

Sol 13: Let $B = (at_1^2, 2at_1)$

A = (0, 0)

& let C be (h, 0)

we have to find $|h - at_1^2|$

Since $BC \perp AB$

$\therefore \frac{2at_1}{at_1^2 - h} \times \frac{2at_1}{at_1^2} = -1$

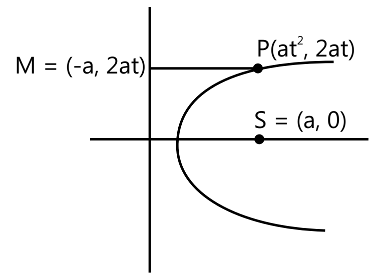
$\therefore 4a = h - at_1^2$

$\therefore$ Projection of BC on x-axis = 4a

Sol 14: Changing the coordinate system to y = 4 & x = x + 3 won't change the dimensions of parabola

$\therefore$ Let parabola be $y^2 = 8x$

$\therefore P = (at^2, 2at)$



M = (–a, 2at) S = (a, 0)

Since PM = SP always

$\therefore (2a)^2 + (2at)^2 = (a + at^2)^2$

$\Rightarrow 4 + 4t^2 = t^4 + 2t^2 + 1$

$\therefore t^4 - 2t^2 - 3 = 0 \Rightarrow t^2 = 3$

$\therefore$ Side of triangle $= a + at^2 = 2 + 2 \times 3 = 8$

Sol 15: $y^2 = 4ax$

$P = (at^2, 2at)$ & $Q = (at^2, -2at)$

Let M = point of trisection = (x, y)

$\therefore (x, y) = \left(\frac{2at^2 + at^2}{3}, \frac{2 \times (2at) - 2at}{3}\right)$

$\therefore (x, y) = \left(at^2, \frac{2at}{3}\right)$

$\therefore x = at^2$ & $y = \frac{2at}{3}$

$\therefore x = a \times \left(\frac{3y}{2a}\right)^2$

$\therefore y^2 = \frac{4ax}{9}$

Length of latus rectum of $y^2 = 4ax$ is 4a. Length of latus rectum of $y^2 = \frac{4ax}{9}$ is $\frac{4a}{9}$; $\therefore k = \frac{1}{9}$

Sol 16: Focus = mid point of ends of latus rectum = (1, – 1)

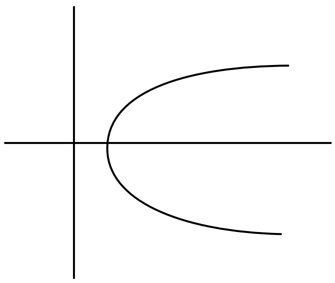
$a = \frac{6}{4} = \frac{3}{2}$

Since latus rectum is ⊥ to x-axis

∴ Only co-ordinates are shifted.

vertex = $\left(1 \pm \frac{3}{2}, -1\right) \Rightarrow \left(-\frac{1}{2}, -1\right)$ or $\left(\frac{5}{2}, -1\right)$

When vertex lies to left of latus rectum equation parabola is



∴ Equation of parabola is

$(y + 1)^2 = 6\left(x + \frac{1}{2}\right) \Rightarrow (y + 1)^2 = 3(2x + 1)$

or $(y + 1)^2 = -6\left(x - \frac{5}{2}\right) \Rightarrow (y + 1)^2 = -3(x - 5)$

Sol 18: Let AP be the chord with A = (0, 0) & P $(at^2, 2at)$. Let Q = $\left(at_2^2, 2at_2\right)$

Since AQ ⊥ AP

$\therefore \frac{2at_2}{at_2^2} \times \frac{2at}{at^2} = -1 \Rightarrow t_2 = -\frac{4}{t}$

$\therefore Q = \left(\frac{16a}{t^2}, -\frac{8a}{t}\right)$

Let R be the required point = (h, k)

∴ Since OPQR is a rectangle

∴ Midpoint of OR = Mid-point of P & Q

$\therefore \left(\frac{h}{2}, \frac{k}{2}\right) = \left(\frac{at^2 + \frac{16a}{t^2}}{2}, \frac{2at - \frac{8a}{t}}{2}\right)$

$\therefore h = a\left(t^2 + \frac{16}{t^2}\right); \quad k = 2a\left(t - \frac{4}{t}\right)$

$\therefore \left(\frac{k}{2a}\right)^2 + 8 = \frac{h}{a} \Rightarrow k^2 = (h - 8a) \times 4a$

∴ Locus of R is $y^2 = 4a(x - 8a)$

Sol 19: Let $P(at^2, 2at)$ be a point on parabola

∴ The middle point of segment OP is

$(h, k) = \left(\frac{at^2}{2}, at\right)$

$\therefore h = \frac{a}{2}\left(\frac{k}{a}\right)^2$

∴ Locus of M is $y^2 = 2ax$

Sol 20: Let $A\left(at_1^2, 2at_1\right)$ & $B(at_2^2, 2at_2)$ be two points, then mid point

$(x, y) = \left(\frac{a(t_1^2 + t_2^2)}{2}, \frac{2a(t_1 + t_2)}{2}\right)$

$\therefore (t_1^2 + t_2^2) = \frac{2x}{a}$ & $t_1 + t_2 = \frac{y}{a}$

The chord passing through A, B is

$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$

Since (h, k) satisfies this

$\therefore (t_1 + t_2)k - 2at_1t_2 = 2h$

$\therefore \frac{yk}{a} - a((t_1 + t_2)^2 - (t_1^2 + t_2^2)) = 2h$

$\Rightarrow \frac{yk}{a} - a\left(\frac{y^2}{a^2} - \frac{2x}{a}\right) = 2h$

$\therefore \frac{y(y - k)}{a} = 2(x - h)$

$\therefore y^2 - ky = 2a(x - h)$

Sol 21: Since point of intersection of

$P = (at_1^2, 2at_1)$ & $Q = (at_2^2, 2at_2^2)$ is $[at_1t_2 a(t_1 + t_2)]$

Area of PQR

$$APQR = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_1t_2 & a(t_1 + t_2) & 1 \end{vmatrix}$$

$$\therefore A = \frac{1}{4}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ 2at_1t_2 & 2a(t_1 + t_2) & 2 \end{vmatrix}$$

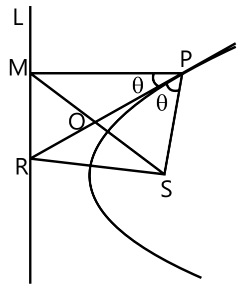
$R_3 \rightarrow R_3 - R_1 - R_2$

$$A = \frac{1}{4}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ a(t_1 - t_2)^2 & 0 & 0 \end{vmatrix}$$

$\therefore A = \frac{1}{4} 2a|(t_1 - t_2)| \times a(t_1 - t_2)^2$

$\therefore A = \frac{a^2}{2} |t_1 - t_2|^3$

Sol 22: Consider the parabola with focus S & L as the directrix & P is a point



SP = PM & tangent bisects PM & SP

$\therefore$ PO $\perp$ SM

$\therefore \angle PSM = 90 - \theta. \angle PMR = 90^o$

$\therefore \angle SMR = 90 - \angle PMS = \theta$

Now in ΔMOR & ΔSOR

MO = SO & OR is common &

$\angle MOR = \angle SOR = 90^o$

$\therefore$ By RHS $\Delta MOR = \Delta SOR$

$\therefore \angle ROS = \angle RNO = \theta$

$\therefore$ Angle subtended by the position of tangent Between P and R = $\theta + (90 - \theta) = 90^o$

Sol 23: Equation of tangent to parabola is

$y = mx + \frac{a}{m}$

Tangent at A is x = 0

$T = \left(-\frac{a}{m^2}, 0\right)$

$A = (0, 0)$ & $Y = \left(0, \frac{a}{m}\right)$

Coordinations of G are $\left(\frac{-a}{m^2}, \frac{a}{m}\right)$

$\therefore x = -\frac{a}{m^2}; y = \frac{a}{m}$

$\therefore x = -\frac{a}{\left(\frac{a}{y}\right)^2}$

$\therefore y^2 + ax = 0$ is the locus of point G

Sol 24: Let the parabolas be $y^2 = 4ax$ & $x^2 = 4ay$

Let P be their point of intersection

$\Rightarrow x^4 = (4a)^2 \times (4ax)$

$\therefore x = 4a$ & $y = \frac{(4a)^2}{4a} = 4a$

$\therefore P = (4a, 4a)$

The equation of tangents for $y^2 = 4ax$ at (4a, 4a) is

$4ay = 2a (x + 4a)$

$\therefore$ Slope $= \frac{1}{2}$

and equation of tangent to $x^2 = 4ay$ at (4a, 4a) is

$4ax = 2a (y + 4a)$

$\therefore$ Slope = 2

$\therefore$ Angle between parabolas $= \tan\theta = \left(\frac{2-\frac{1}{2}}{1+2\times\frac{1}{2}}\right) = \frac{3}{4}$

Sol 25: Let tangents be $y = mx + \frac{a}{m}$

It passes through h, k

$\therefore k = mh + \frac{a}{m}$

$\Rightarrow m^2h - km + a = 0$

$m_1m_2 = \frac{a}{h}$ and $m_1 + m_2 = \frac{k}{h}$

Angle between tangents = a

$\therefore \tan\alpha = \frac{m_1 - m_2}{1+m_1m_2}$

$\therefore \tan^2\alpha = \frac{\left(\frac{k}{h}\right)^2 - 4\times\frac{a}{h}}{\left(1+\frac{a}{h}\right)^2} \Rightarrow \tan^2\alpha = \frac{k^2 - 4ha}{(h+a)^2}$

$\therefore$ Locus is $y^2 = \tan^2\alpha(x + a)^2 + 4ax$

Sol 26: Let (h, 0) be point on axis of parabola

Equation of normal is $y = mx - 2am - am^3$

(h, 0) passes through it

$\therefore am^3 + 2am - mh = 0$

$m(am^2 + 2a - h) = 0$

For m to be real and distinct $-\frac{(2a-h)}{a} > 0$

$\therefore \frac{(h-2a)}{a} > 0$

The parabola is $(y-1)^2 = 4(x-1)$

$\therefore$ a = 1 y = Y + 1 & x = X + 1

h – 2a > 0 & y – 1 = 0

x > (2a + 1) & y = 1

$\therefore$ The points which satisfy are (x, 1) where x > 3

Sol 27: The equation of normal to parabola is

$y = mx - 2am - am^3$

$\therefore$ (h, k) satisfies it

$\therefore$ Let $f(m) = am^3 + (2a-h)m + k$

Two of the 3 tangents coincide

$\therefore$ f(m) has two equal root

$\therefore$ f′(m) at m = p is 0 & f(p) = 0

$f'(m) = 3am^2 + (2a-h)$

$\therefore$ f′(m) is O at $m^2 = \frac{(h-2a)}{3a}$

Let m = p

$f(m) = m(am^2 + (2a-h)) + k$

$\therefore\ f(p) = p\left(\frac{h-2a}{3} + 2a - h\right) + k = 0$

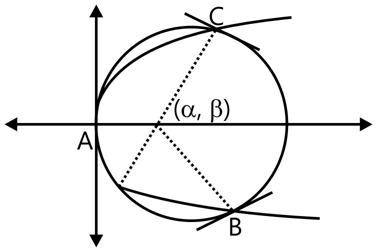
$\therefore\ k = 2\frac{(h-2a)}{3}p$

$9k^2 = 4(h-2a)^2 \times p^2 = 4(h-2a)^2 \times \frac{h-2a}{3a}$

$\therefore\ 4(h-2a)^3 = 27ak^2$

$\therefore$ For two tangents to be coincide locus of P is $27ay^2 = 4(h-2a)^3$

Sol 28: Let $A(am_1^2, -2am_1)$, $B(am_2^2, 2am_2)$ and $C(am_3^2, -2am_3)$ be 3 points on parabola $y^2 = 4ax$



Since point of intersection of normals is (α, β) then

$am^3 \times (2a-\alpha)m + \beta = 0$

$m_1 + m_2 + m_3 = 0$

Let the equation of circle through A, B, C be

$x^2 + y^2 + 2gx + 2fy + c = 0$

If the points $(am^2, -2am)$ lies on circle then

$(am^2)^2 + (-2am)^2 + 2g(am^2) + 2t(-2am) + c = 0$

This is a biquadratic equation in m. Hence these are four values of m, say m_1, m_2, m_3 & m_4 such that circle passes through these points

$m_1 + m_2 + m_3 + m_4 = 0$

$0 + m_4 = 0$; $m_4 = 0$

$\therefore$ (0, 0) lies on the circle

Sol 29: Let (h, k) be the middle point

$\therefore$ Equation of chord is

$yk - 2a(x+h) = k^2 - 4ah$

Now let $y = Y - \frac{2a}{m}$ & $x = X - \frac{c}{m}$

$\therefore\ \left(Y - \frac{2a}{m}\right)k - 2a\left(X - \frac{c}{m} + h\right) = k^2 - 4ah$

$Y = \frac{2a}{k}X + \left(\frac{2ak}{m} + 2ah + k^2 - 4ah - \frac{2ac}{m}\right)/k$

& parabola is $Y^2 = 8aX$

Line touches the parabola if mc = a

$\therefore\ mc = \frac{2a}{k}\left(\frac{2a}{m} + 2ah + k^2 - 4ah - \frac{2ac}{m}\right)/k$

$= \frac{2a}{k}\left(\frac{2ak}{m} - 2ah + k^2 - \frac{2ac}{m}\right)/k$

$= \frac{2a}{k}\left(\frac{2ak}{m} - 2ah + k^2 - \frac{2a}{m}(k - mh)\right)/k$

$= \frac{2a}{k}\left(\frac{2ak}{m} - 2ah + k^2 - \frac{2ak}{m} + 2ah\right)/k = 2a$

$\therefore$ It is tangent to parabola

Exercise 2

Single Correct Choice Type

Sol 1: (C) x = 1 – y

$\therefore\ y^2 = 4 - 4y$

$\therefore\ y^2 + 4y - 4 = 0$

$(y_2 - y_1)^2 = (4)^2 + 4 \times 4 = 32 \Rightarrow |y_2 - y_1| = 4\sqrt{2}$

$|x_2 - x_1| = (1 - y_2) - (1 - y_1) = |y_2 - y_1|$

$\therefore$ Length of chord $= \sqrt{2} \times |y_2 - y_1| = 8$

Sol 2: (A) $y^2 - 12x - 4y + 4 = 0$

$(y-2)^2 = 12x$

Let x = X and y = Y + 2

$\therefore\ Y^2 = 12X$

Focus of parabola is (3, 0)

∴ Focus in original coordinate system is (3, 2)

∴ Vertex of new parabola is (3, 2)&focus =(3, 4)

∴ a = 2

Let x = X + 3 & y = Y + 2

$\therefore$ Equation is $X^2 = 8Y \Rightarrow (x-3)^2 = 8(y-2)$

$\therefore\ x^2 - 6x - 8y + 25 = 0$ is the equation of parabola

Sol 3: (D) $x = t^2 - 2t + 2,\ y = t^2 + 2t + 2$

$x + y - 4 = 2t^2$

$y - x = 4t$

∴ (x + y – 4, y – x) lies on a parabola

⇒ (x + y, y – x) lies on a parabola by rotating axis (x, y) lies on parabola

Sol 4: (D) $y^2 = 4a\left(x - \frac{1}{3}\right)$

Let y = Y and x = X + $\frac{1}{3}$

$\therefore$ Equation of parabola is $Y^2 = 4aX$

$\therefore$ equation of line is $Y = 2X - \frac{7}{3}$

$c = \frac{a}{m} \Rightarrow -\frac{7}{3} = \frac{a}{2}$

$\therefore\ a = -\frac{14}{3}$

Sol 5: (A) $\cot a_1 = 2\cot a_2 \therefore\ \tan a_1 = \frac{\tan\alpha_2}{2}$

let $P = (at_1^2, 2at_1)$ & $Q = (at_2^2, 2at_2)$

T = point of intersection = $(at_1t_2, a(t_1 + t_2)) = (x, y)$

Slope of P = $\frac{1}{t_1}$ and slope of Q = $\frac{1}{t_2}$

$\Rightarrow \frac{1}{t_1} = \frac{1}{2t_2}$

$\therefore\ t_1 = 2t_2$

$\therefore\ x = 2at_2^2 \qquad y = 3at_2$

$\therefore\ \left(\frac{y}{3a}\right)^2 \times 2a = x$

$\therefore\ 2y^2 = 9ax$ is locus of T

Sol 6: (C) Let P be $(at_1^2, 2at_1)$ and Q be $(at_2^2, 2at_2)$

$OP \perp OQ$

$\therefore\ \frac{2t_1}{t_1^2} \times \frac{2t_2}{t_2^2} = -1$

$\therefore\ t_2 = -\frac{4}{t_1}$

The equation of PQ is

$(y - at_1) = \frac{2}{t_1 - \frac{4}{t_1}}\ (x - at_1^2)$

$x_R = \left(t_1 - \frac{4}{t_1}\right) \times \frac{-2at}{2} + at_1^2 = 4a$

$\therefore$ R = (0, 4a) $\therefore$ OR = 4a

Sol 7: (C) $\sqrt{x^2 + y^2} - y = c$

$\therefore\ x^2 + y^2 = y^2 + 2cy + c^2$

$x^2 = 2c\left(y + \frac{c}{2}\right) \qquad \therefore$ Vertex is $\left(0, -\frac{c}{2}\right)$

It passes through $\left(c\sqrt{2}, \frac{c}{2}\right)$

Sol 8: (B) Equation of tangent is $y = mx + \frac{1}{m}$

(–1, 2) lies on it, then

$2 = -m + \frac{1}{m} \quad \Rightarrow m^2 + 2m - 1 = 0$

x = 2, so $y_1 - y_2 = 2(m_1 - m_2) + \frac{m_2 - m_1}{m_1m_2}$

$\therefore\ (y - y_2)^2 = (m_1 - m_2)^2\left(2 - \frac{1}{m_1m_2}\right)^2$

$= [(m_1 + m_2)^2 - 4m_1m_2]\left(2 - \frac{1}{m_1m_2}\right)^2$

$= [(m_1 + m_2)^2 - 4m_1m_2]\left(2 - \frac{1}{m_1m_2}\right)^2$

$(y_1 - y_2)^2 = (4 + 4)(2 + 1)^2 = 8 \times 9$

$\therefore |y_1 - y_2| = 6\sqrt{2}$

Sol 9: (B)

(A) is an ellipse

(B) $x^2 = 2 - \cos t$

$= 2 - \left(2\cos^2\frac{t}{2} - 1\right) = 2 - \left(\frac{y}{2} - 1\right)$

$\therefore\ x^2 = \frac{1}{2}(6 - y)$

$\therefore$ It is a parabola

(C) is a line in 1st quadrant

(D) $x = \sqrt{1 - \sin t}$, $y = \sin\frac{t}{2} + \cos\frac{t}{2}$

$\Rightarrow y^2 = \sin^2\frac{t}{2} + \cos^2\frac{t}{2} + 2\sin\frac{t}{2}\cos\frac{t}{2} = 1 + \sin t$

$= 1 + 1 - x^2$ (A circle)

Sol 10: (C) Let $A(at_1^2, 2at_2)$ & $B(at_2^2, 2at_2)$ be points on parabola. $P = (at_1 t_2, a(t_1 + t_2)) = (x, y)$

slope of AP is $\frac{1}{t_1}$ & slope of BP is $\frac{1}{t_2}$

$\tan(\theta_1 + \theta_2) = 1$

$\therefore \frac{\frac{1}{t_1} + \frac{1}{t_2}}{1 - \frac{1}{t_1 t_2}} = 1$

$\therefore \frac{t_1 + t_2}{t_1 t_2 - 1} = 1$

$\therefore \frac{y}{x - 1} = 1$

$\therefore\ x - y - 1 = 0$ is locus of P

Sol 11: (C) Parabola is $y^2 = 8x$. $a = 2$

The point P is $(2 \times (2 \times 1), 2(2 + 1))$

$\therefore$ The points of contact Q, R are

$(2 \times (2)^2, 2 \times 2 \times 2)$ & $(2 \times (1)^2, 2 \times 2 \times 1)$

$\therefore$ Q & R are (8, 8) & (2, 4)

$$\Delta = \frac{1}{2}\begin{vmatrix} 4 & 6 & 1 \\ 8 & 8 & 1 \\ 2 & 4 & 1 \end{vmatrix} = \frac{1}{2}|4| = 2$$

Sol 12: (A) Let $P = (at^2, 2at)$

$\because\ a = 2\quad P = (2t^2, 4t)$; $S = (2, 0)$

$SP = \sqrt{(2t^2 - 2)^2 + (4t)^2} = 6$

$\therefore 4[(t^2 - 1)^2 + 4t^2] = 36$

$\therefore\ (t^2 + 1)^2 = 9 \Rightarrow t^2 = 2$

The length of focal chord of a parabola is $a\left(t + \frac{1}{t}\right)^2$

$= 2\frac{(t^2 + 1)^2}{t^2} = 2.\frac{9}{2} = 9$

$\therefore\ \ell(SQ) = \ell(PQ) - \ell(SP) = 3$

Sol 13: (C) Let (h, k) be the point of intersection

$\therefore$ Chord of contact is $yk = 2(x + h)$

$\Rightarrow 2x - ky + 2h = 0 \Rightarrow 4x - 2ky + 4h = 0$

$\therefore\ k = \frac{7}{2}$ and $h = \frac{5}{2}$

$\therefore$ Point of contact is $\left(\frac{5}{2}, \frac{7}{2}\right)$

Sol 14: (D) The curve is $y^2 = 4x$ & $y > 0$

Normal to parabola is

$y = mx - 2m - m^3$

It passes through (21, 30)

$m^3 + 2m - 21m + 30 = 0$

$m^3 - 19m + 30 = 0$

$\Rightarrow m = 2, m = 3$ & $m = -5$

For $m = 2, 3$; y – coordinate < 0

$\therefore\ m = -5$ is only possible solution

Sol 15: (D) Let $P = (h, k)$

The chord of contact is given by

$ky = 2a(x + h)$

$\therefore x = \frac{k}{2a}y - h$

It is a tangent to $x^2 = 4by$

$\therefore -h = \frac{b}{\frac{k}{2a}} \Rightarrow -kh = 2ab$

$\therefore\ xy = -2ab$ is locus of P. It is a hyperbola.

Sol 16: (D) Let $P = (at^2, 2at)$

$\therefore M = (-a, 2at)$; $S = (a, 0)$

For SPM to be equilateral triangle

SM = PM (as PM = SP always)

$\therefore\ (2a)^2 + (2at)^2 = (at^2 + a)^2$

$(2a)^2 = (at^2 - a)^2 \Rightarrow at^2 = 3a \Rightarrow t = \sqrt{3}$

$SP = MP = at^2 + a = 4a$

Sol 17: (A) Let $P = (at^2, 2at)$

$\therefore (at^2 - a)^2 + (2at)^2 = (3)^2$ ... (i)

The length of focal chord $= a\left(t + \frac{1}{t}\right)^2 = 5$ ... (ii)

$(1) \Rightarrow (at^2 + a)^2 = 9$ ... (iii)

$a(t^2 + 1) = 3 \Rightarrow a(t^2 + 1) = 3$

$\therefore\ a \times \frac{9}{a^2 \times t^2} = 5 \ \Rightarrow t^2 = \frac{9}{5a}$

From (iii) $\left(\frac{9}{5} + a\right)^2 = 9$

$\Rightarrow a = 3 - \frac{9}{5} = \frac{6}{5}$ (Since a cannot be negative)

$\therefore$ Length of latus rectum $= 4a = \frac{24}{5}$

Sol 18: (D) Ordinate = abscissa

$\therefore\ at^2 = 2at \Rightarrow t = 2$

t_2 (other end of normal) $= -t - \frac{2}{t} = -3$

$P = (4a, 4a)$; $Q = (9a, -6a)$ and $S = (a, 0)$

$\therefore$ Slope of SP $= \frac{4}{3}$ and slope of SQ $= -\frac{3}{4}$

$\therefore$ Angle subtend at focus $= 90^o$

Sol 19: (D) Let P be $(at^2, 2at)$

$\therefore$ Equation of PA is $y = \frac{2x}{t}$

$M = (-a, 2at)$; $D = \left(-a, -\frac{2a}{t}\right)$

Slope of SM is $\frac{2at}{-2a} = -t$

and slope of SD is $\frac{-2a}{t \times (-2a)} = \frac{1}{t}$

$\therefore$ SD $\perp$ SM $\therefore$ angle between $= 90^o$

Sol 20: (D) Let equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

It passes through $(\alpha, 0)$ and $(\beta, 0)$

$\therefore\ a^2 + 2y\alpha + c = 0$ and $\beta^2 + 2g\beta + c = 0$

$\therefore C(\beta - \alpha) + a\beta(\alpha - \beta) = 0$

$\therefore C = \alpha\beta$

$\therefore$ Length of tangent $= \sqrt{800} = \sqrt{C} = \sqrt{\alpha\beta}$

$\alpha\beta = \frac{c}{a}\ \therefore$ LT $= \sqrt{\frac{c}{a}}$

Sol 21: (C) Let $P = (at_1^2, 2at_1)$ & $Q = (at_2^2, 2at_2)$

and PQ passes through $F(-a, b)$

$\therefore\ \frac{2at_1 - b}{2at_1^2 + a} = \frac{2at_2 - b}{at_2^2 + 9}$

$\therefore (2at_1 - b)(t_2 + 1) = (2at_2 - b)(t_1^2 + 1)$

$\therefore\ 2at_1t_2^2 ++ 2at_1 - bt_2^2 = 2at_2t_1^2 + 2at_2 - bt_1^2$

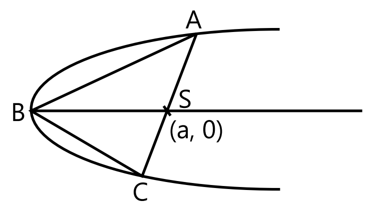
$\therefore\ 2at(t_2 - t_1) = 2a(t_2 - t_1) + b(t_2 - t_1)(t_1 + t_2)$

$\therefore\ 2at_1t_2 = 2a + b(t_1 + t_2)$

Let $T = (x, y) = (at_1t_2, a(t_1 + t_2))$

$\therefore\ 2x = 2a + \frac{by}{a} \Rightarrow 2a(x - a) = by$

Sol 22: (C)



Area of ΔABC = area of ΔABS + area ΔBCS

$= \frac{1}{2} \times BS \times h_1 + \frac{1}{2} BS \times h_2$

$= \frac{1}{2}\, ax\,(h_1 + h_2)$

$h_1 + h_2$ = diff. in y-coordinates of A & C

$\therefore\ \frac{1}{2} a|y_2 - y_1| = A$

$\therefore\ |y_2 - y_1| = \frac{2A}{a}$

Previous Years' Questions

Sol 1: (B) If $y = mx + c$ is normal to the parabola $y^2 = 4ax$,

Then $c = -2am - am^3$.

From given condition, $y^2 = 12x$

$\Rightarrow y^2 = 4 . 3 . x \Rightarrow a = 3$ and $x + y = k$

$\Rightarrow y = (-1)x + k \Rightarrow m = -1$

And $c = k$

$\therefore c = k = -2(3)(-1) - 3(-1)^3 = 9$

Sol 2: (C) Given, $y^2 = kx - 8$

$\Rightarrow y^2 = k\left(x - \frac{8}{k}\right)$

Shifting the origin

$Y^2 = kX$, where $Y = y$, $X = x - 8/k$.

Directrix of standard parabola is $X = -\frac{k}{4}$

Directrix of original parabola is $x = \frac{8}{k} - \frac{k}{4}$

Now, $x = 1$ also coincides with $x = \frac{8}{k} - \frac{k}{4}$

On solving, we get $k = 4$

Sol 3: (D) Given, $y^2 + 4y + 4x + 2 = 0$

$\Rightarrow (y + 2)^2 + 4x - 2 = 0$

$\Rightarrow (y + 2)^2 = -4\left(x - \frac{1}{2}\right)$

Replace $y + 2 = Y$, $x - \frac{1}{2} = X$

We have, $Y^2 = -4X$

This is a parabola with directrix at $X = 1$

$\Rightarrow x - \frac{1}{2} = 1 \Rightarrow x = \frac{3}{2}$

Sol 4: (C) Let P(h, k) be the midpoint of the line segment joining the focus (a, 0) and a general point Q(x, y) on the parabola. Then

$h = \frac{x + a}{2}$, $k = \frac{y}{2}$

$\Rightarrow x = 2h - a$, $y = 2k$

Substitute these values of x & y in $y^2 = 4ax$, we get

$4k^2 = 4a(2h - a)$

$\Rightarrow 4k^2 = 8ah - 4a^2$

$\Rightarrow k^2 = 2ah - a^2$

So, locus of P(h, k) is $y^2 = 2ax - a^2$

$\Rightarrow y^2 = 2a\left(x - \frac{a}{2}\right)$

Its directrix is $x - \frac{a}{2} = -\frac{a}{2}$

$\Rightarrow x = 0 \Rightarrow$ y-axis

Sol 5: (D) Tangent to the curve $y^2 = 8x$ is $y = mx + \frac{2}{m}$. Substituting this in $xy = -1$

$\Rightarrow x.\left(mx + \frac{2}{m}\right) = -1$

$\Rightarrow mx^2 + \frac{2}{m}x + 1 = 0$

Since, it has equal roots.

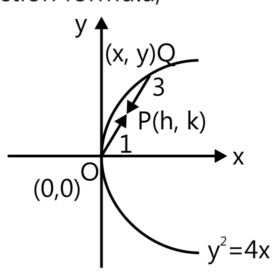
$\therefore D = 0$

$\Rightarrow \frac{4}{m^2} - 4m = 0 \Rightarrow m^3 = 1 \Rightarrow m = 1$

Hence, equation of common tangent is $y = x + 2$.

$\Rightarrow (x - y) = 8(x + y - 2)$

Sol 6: (C) By section formula,



$h = \frac{x + 0}{4}$, $k = \frac{y + 0}{4}$

$\therefore x = 4h$, $y = 4k$

Substituting in $y^2 = 4x$

$(4k)^2 = 4(4h) \Rightarrow k^2 = h$

Or $y^2 = x$ is required locus.

(C) Centroid of $\Delta PQR = \left(\frac{2}{3}, 0\right)$

Equation of circle passing through P,Q,R is

$(x-1)(x-1) + (y-2)(y+2) + \lambda(x-1) = 0$

$\Rightarrow 1 - 4 - \lambda = 0 \Rightarrow \lambda = -3$

$\therefore$ required equation of circle is

$x^2 + y^2 - 5x = 0$

$\therefore$ Centre $\left(\frac{5}{2}, 0\right)$ and radius $\frac{5}{2}$.

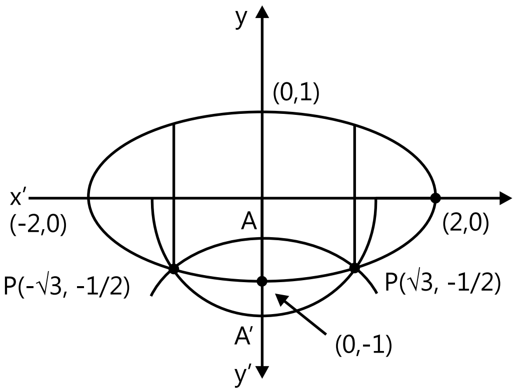
Sol 7: (B, C) The equation $x^2 + 4y^2 = 4$ represents an ellipse with 2 and 1 as semi-major and semi-minor axes and eccentricity $\frac{\sqrt{3}}{2}$.

Thus, the ends of latus rectum are $\left(\sqrt{3}, \frac{1}{2}\right)$

and $\left(\sqrt{3}, -\frac{1}{2}\right)$, $\left(-\sqrt{3}, \frac{1}{2}\right)$ and $\left(-\sqrt{3}, -\frac{1}{2}\right)$.

According to the question, we consider only

P $\left(-\sqrt{3}, -\frac{1}{2}\right)$ and Q$\left(\sqrt{3}, -\frac{1}{2}\right)$, y, $y_2 < 0$



Now,PQ = $2\sqrt{3}$

Thus, the coordinates of the vertex of the parabolas are $A\left(0, \frac{-1+\sqrt{3}}{2}\right)$ and $A'\left(0, \frac{-1-\sqrt{3}}{2}\right)$ and corresponding equations are

$(x-0)^2 = -4.\ \frac{\sqrt{3}}{2}\left(y + \frac{1-\sqrt{3}}{2}\right)$

and $(x-0)^2 = 4.\ \frac{\sqrt{3}}{2}\left(y - \frac{1-\sqrt{3}}{2}\right)$

i.e.,$x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$ and $x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$

Sol 8: (A,B,D) Normal to $y^2 = 4x$, is

$y = mx - 2m - m^3$ which passes through (9, 6)

$\Rightarrow 6 = 9m - 2m - m^3$

$\Rightarrow m^3 - 7m + 6 = 0$

$\Rightarrow m = 1, 2, -3$

$\therefore$ Equation of normals are,

$y - x + 3 = 0$, $y + 3x - 33 = 0$ and $y - 2x + 12 = 0$

Sol 9: The coordinates of the extremities of the latus rectum of y2 = 4ax are (1, 2) and (1, -2).

The equations of tangents at these points are gievn by

y.2 = 4(x+1)/2

This gives 2y = 2(x + 1) ... (i)

and y(-2) = 4(x+1)/2

which gievs -2y = 2(x + 1) ... (ii)

The points of intersection of these tangents can be obtained by solving these two equatiosn simultaneously.

Therefore, -2(x +1) = 2(x + 1)

which gives 0 = 4(x +1)

this yields x = -1 and y = 0.

Hence, the required point is (-1, 0).

Sol 10: Let the point be $Q(x, x^2)$ on $x^2 = y$ whose distance from (0, c) is minimum.

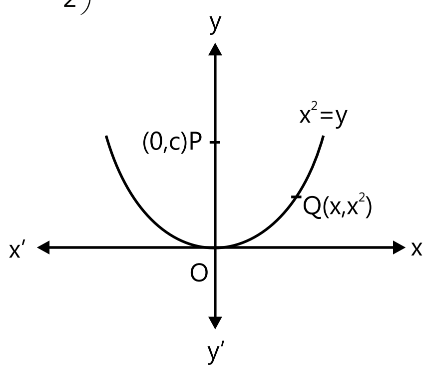
Now, $PQ^2 = x^2 + (x^2 - c)^2$

Let $f(x) = x^2 + (x^2 - c)^2$ ...(i)

$f'(x) = 2x + 2(x^2 - c).2x$

$= 2x(1 + 2x^2 - 2c)$

$= 4x\left(x^2 - c + \frac{1}{2}\right)$



$= 4x\left(x - \sqrt{c - \frac{1}{2}}\right)\left(x + \sqrt{c - \frac{1}{2}}\right)$

When $c > \frac{1}{2}$

For maxima, put $f'(x) = 0$

$\Rightarrow 4x\left(x^2 - c + \frac{1}{2}\right)$

$\Rightarrow x = 0,\ x = \pm\sqrt{c - \frac{1}{2}}$

Now, $f''(x) = 4\left[x^2 - c + \frac{1}{2}\right] + 4x[2x]$

At $x = \pm\sqrt{c - \frac{1}{2}}$

$f''(x) > 0$.

$\therefore f(x)$ is minimum

Hence, minimum value of f(x) = |PQ|

$$= \sqrt{\left(\sqrt{c-\frac{1}{2}}\right)^2 + \left(\left(\sqrt{c-\frac{1}{2}}\right)^2 - c\right)^2}$$

$$= \sqrt{c-\frac{1}{2}+\left(c-\frac{1}{2}-c\right)^2} = \sqrt{c-\frac{1}{4}},\ \frac{1}{2} \le c \le 5.$$

Sol 11: Equation of normal to $x^2 = 4y$ is $x = my - 2m - m^3$ and passing through (1, 2).

$\therefore 1 = 2m - 2m - m^3$

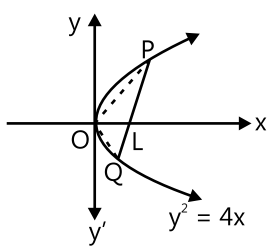
$\Rightarrow m^3 = -1$ or $m = -1$

Thus, the required equation of normal is, $x = -y + 2 + 1$ or $x + y = 3$.

Sol 12: Let the equation of chord OP be $y = mx$ and then, equation of chord will be $y = -\frac{1}{m}x$ and P is point of intersection of $y = mx$ and $y^2 = 4x$ is $\left(\frac{4}{m^2}, \frac{4}{m}\right)$ and Q is point intersection of $y = -\frac{1}{mn}$ and $y^2 = 4x$ is $(4m^2, -4m)$

Now, equation of PQ is

$$y + 4m = \frac{\frac{4}{m}+4m}{\frac{4}{m^2}-4m^2}(x - 4m^2)$$



$\Rightarrow y + 4m = \frac{m}{1-m^2}(x - 4m^2)$

$\Rightarrow (1- m^2)y + 4m - 4m^3 = mx - 4m^3$

$\Rightarrow mx - (1 - m^2)y - 4m = 0$

This line meets x-axis, where $y = 0$

i.e., $x = 4 \Rightarrow OL = 4$ which is constant as independent of m.

Again, let (h, k) be the midpoint of PQ, then

$$h = \frac{4m^2 + \frac{4}{m^2}}{2} \text{ and } k = \frac{\frac{4}{m} - 4m}{2}$$

$$\Rightarrow h = 2\left(m^2 + \frac{1}{m^2}\right) \text{ \& } k = 2\left(\frac{1}{m} - m\right)$$

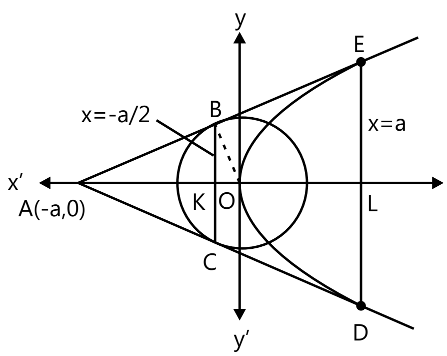
$$\Rightarrow h = 2\left(\left(m - \frac{1}{m}\right)^2 + 2\right) \text{ \& } k = 2\left(\frac{1}{m} - m\right)$$

Eliminating m, we get $2h = k^2 + 8$

Or $y^2 = 2(x - 4)$ is required equation of locus.

Sol 13: Equation of any tangent to the parabola, $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

This line will touch the circle $x^2 + y^2 = \frac{a^2}{2}$



If $\left(\frac{a}{m}\right)^2 = \frac{a^2}{2}(m^2 + 1)$ [Tangency condition]

$\Rightarrow \frac{1}{m^2} = \frac{1}{2}(m^2 + 1) \Rightarrow 2 = m^4 + m^2$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 2) = 0$

$\Rightarrow m^2 - 1 = 0, m^2 = -2$

$\Rightarrow m = \pm 1$ ($m^2 = -2$ is not possible)

Therefore, two common tangents are

$y = x + a$ and $y = -x - a$

These two intersect at A(–a, 0)

The chord of contact of A(–a, 0) for the circle

$x^2 + y^2 = a^2/2$ is

$(-a)x + 0.y = a^2/2 \Rightarrow x = -a/2$

Sol 14: Let $P(\alpha, \beta)$ be any point on the locus. Equation of pair of tangents from $P(\alpha, \beta)$ to the parabola $y^2 = 4ax$ is

$[by - 2a(x + \alpha)]^2 = (b^2 - 4a\alpha)(y^2 - 4ax)$

$[\because T^2 = S.S_1]$

$\Rightarrow b^2y^2+4a^2(x^2+a^2+2x.a)-4aby(x + \alpha)$

$= b^2y^2 - 4b^2ax - 4a\alpha y^2 + 16a^2\alpha x$

$\Rightarrow b^2y^2 + 4a^2x^2 + 4a^2\alpha^2$

$= b^2y^2 - 4b^2ax - 4a\alpha y^2 + 16a^2\alpha x$

$- 4abxy - 4ab\alpha y$ ...(i)

Now, coefficient of $x^2 = 4a^2$

Coefficient of $xy = -4a\beta$

Coefficient of $y^2 = 4a\alpha$

Again, angle between the two of Eq.(i) is given as 45º

$\therefore \tan 45° = \dfrac{2\sqrt{h^2 - ab}}{a+b}$

$\Rightarrow 1 = \dfrac{2\sqrt{h^2 - ab}}{a+b} \Rightarrow a + b = 2\sqrt{h^2 - ab}$

$\Rightarrow (a + b)^2 = 4(h^2 - ab)$

$\Rightarrow (4a^2+4a\alpha)^2 = 4[4a^2b^2-(4a^2)(4a\alpha)]$

$\Rightarrow 16a^2(a+\alpha)^2 = 4.4a^2[b^2-4a\alpha]$

$\Rightarrow \alpha^2 + 6a\alpha + a^2 - b^2 = 0$

$\Rightarrow (\alpha + 3a)^2 - b^2 = 8a^2$

Thus, the required equation of the locus $(x + 3a)^2 - y^2 = 8a^2$ which is hyperbola.

Sol 15: Given equation can be written as

$(y - 1)^2 = 4(x - 1)$

Whose parametric coordinate are

$x - 1 = t^2$ and $y - 1 = 2t$

i.e., $P(1 + t^2, 1 + 2t)$

$\therefore$ Equation of tangent at P is,

$t(y - 1) = x - 1 + t^2$, which meets the directrix $x = 0$ at Q.

$\Rightarrow y = 1 + t - \dfrac{1}{t}$

or $Q\left(0, 1 + t - \dfrac{1}{t}\right)$

Let R(h, k) which divides QP externally in the ratio $\dfrac{1}{2} : 1$ or Q is mid point of RP,

$\Rightarrow 0 = \dfrac{h + t^2 + 1}{2}$ or $t^2 = -(h+1)$ ...(i)

and $1 + t - \dfrac{1}{t} = \dfrac{k + 2t + 1}{2}$

or $t = \dfrac{2}{1-k}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii),

$\dfrac{4}{(1-k)^2} + (h + 1) = 0$

or $(k - 1)^2(h + 1) + 4 = 0$

$\therefore$ Locus of a point is $(x + 1)(y - 1)^2 + 4 = 0$

Sol 16: (A) Let any point P on the Parabola $y^2 = 4x$ be $(t^2, 2t)$

Eq. of tangent is $ty = x + t^2$

Now, this equation is tangent to $x^2 = -32y$, then

$x^2 = -32\left(\dfrac{x + t^2}{t}\right)$

$\Rightarrow x^2 + 32x + 32t^2 = 0$

$\Rightarrow D = 0$

$\Rightarrow (32)^2 - 4 \times 32t^2 \times t = 0 \Rightarrow t^3 = 8$

$\Rightarrow t = 2$

Slope of tangent $= \dfrac{1}{t} = \dfrac{1}{2}$

JEE Advanced/Boards

Exercise 1

Sol 1: Slope of tangent || to L =2

Slope of tangent $\perp$ to L = $-\dfrac{1}{2}$

$P : y^2 = 16x \Rightarrow a = 4$

$\therefore$ Equation of tangent || to L is

$y = 2x + \dfrac{4}{2}$ $\therefore$ $y = 2(x + 1)$

Equation of tangent $\perp$ to L is $y = -\dfrac{1}{2}x + \dfrac{4}{-\frac{1}{2}}$

$\therefore$ $2y + x + 16 = 0$

Point of contact $= \left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

For || line $P_1 = \left(\dfrac{4}{2^2}, \dfrac{2\times 4}{2}\right) = (1, 4)$

For $\perp$ line $P_1 = \left(\dfrac{4}{\left(\frac{1}{2}\right)^2}, \dfrac{2\times 4}{-\frac{1}{2}}\right) = (16, -16)$

Sol 2 Let equation be $y = mx + \frac{3}{m}$

It passes through (2, 5)

$\therefore 5 = 2m + \frac{3}{m}$

$\therefore 2m^2 - 5m + 3 = 0$

$\therefore 2m^2 - 2m - 3m + 3 = 0$

$\therefore m = 1$ or $m = \frac{3}{2}$

$\therefore$ Equation of tangents are

$y = x + 3$ and $y = \frac{3x}{2} + 2 \Rightarrow 3x - 2y + 4 = 0$

Sol 3: Let two points be $(at_1^2, 2at_1)$ & $(at_2^2, 2at_1)$ point of intersection is $(a(t_1t_2), a(t_1 + t_2))$ slope of $T_1 = \frac{1}{t_1}$ & that of $T_2 = \frac{1}{t_2}$

$$\therefore \tan\alpha = \frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1t_2}}$$

$$\therefore \tan\alpha = \left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$$

$$\therefore \tan^2\alpha = \frac{(t_1 + t_2)^2 - 4t_1t_2}{(1 + t_1t_2)^2}$$

$\frac{x}{a} = t_1t_2$ & $\frac{y}{a} = (t_1 + t_2)$

$$\therefore \tan^2\alpha = \frac{\left(\frac{y}{a}\right)^2 - \frac{4x}{a}}{\left(1 + \frac{x}{a}\right)^2}$$

$$\tan^2\alpha = \frac{y^2 - 4ax}{(x + a)^2}$$

$\therefore (x + a)^2 \sin^2\alpha = \cos^2\alpha\,(y^2 - 4ax)$

Sol 4: Let $y = mx + \frac{a}{m}$ be tangent to

$y^2 = 4ax$ the tangent to $x^2 = 4by$ is

$y = m_1x - bm_1^2$ where m_1 is slope of tangent $m_1m = -1$

$\Rightarrow m_1 = -\frac{1}{m}$

$\therefore y = -\frac{1}{m}x - \frac{a}{m^2}$

$\Rightarrow m^2y + mx + b = 0$ & $m^2x - my + a = 0$

both have a common root

$\therefore (c_1a_2 - a_2c_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

$\Rightarrow (ay - bx)^2 = (ax + by)(-y^2 - x^2)$

$\therefore (x^2 + y^2)(ax + by) + (ay - bx)^2 = 0$

is the locus of (h, k)

Sol 5: For a point $(at_1^2, 2at_1)$ the equation of normal is

$y = -tx + 2at + at^3$

$\therefore$ Interception axis $= (2a + at^2, 0)$

$\therefore M = (a + at, at) = (x, y)$

$\therefore x = \frac{a + y^2}{a}$

$\therefore y^2 = a(x - a)$

Vertex is (a, 0) and latus rectum $= \frac{a}{4} \times 4 = a$

Sol 6: The equation of normals is $y = mx - 2am - am^3$

P(h, k) satisfies it

$\therefore am^3 + m(2a - h) + k = 0$

$m_1 + m_2 + m_3 = 0$

$m_1m_2 = -1$

$m_1m_2m_3 = -\frac{k}{a} \Rightarrow m_3 = \frac{k}{a}$

$m_1m_2 + m_3(m_1 + m_2) = \frac{2a - h}{a}$

$\Rightarrow m_1m_2 - m_3^2 = \frac{2a - h}{a} \Rightarrow -(1 + m_3^2) = \frac{2a - h}{a}$

$\Rightarrow 1 + \frac{k^2}{a^2} = \frac{h - 2a}{a} \Rightarrow k^2 = a(h - 3a)$ or $y^2 = a(x - 3a)$

Sol 7: Let $P = (at_1^2, 2at_1)$

$R = (at_0^2, 2at_0)$

$Q = (at_0^2, 2at_0)$

$$\therefore \frac{2a(t_1 - t_0)}{a(t_1^2 - t_0^2)} \times \frac{2a(t_2 - t_0)}{a(t_2^2 - t_0^2)} = -1$$

$$\frac{4}{(t_1 + t_0)}(t_2 + t_0) = -1$$

The equation of chord PQ is $\frac{y - 2at_1}{x - at_1^2} = \frac{2}{t_1 + t_2}$

$t_2 = -t_0 - \frac{4}{t_1 - t_0}$

$\therefore\ y - 2at_1 = \frac{2(x - at_1^2)}{t_1 - t_0 - \frac{4}{t_1 + t_0}}$

$\therefore\ (y - 2at_1)\ (t_1^2 - t_0^2 - 4) = 2(t_1 + t_0)\ (x - at_1^2)$

$\therefore\ t_1^2 y - t_0^2 y - 4y - 2at_1^3 + 2at_1 t_0^2 + 8at_1$

$= 2(t_1x - at_1^3 + t_0x - at_1^2 t_0)$

$\therefore\ t_1^2 y - t_0^2 y - 4y + 2at_1t_0^2 + 8at_1$

$= 2t_1x + 2t_0x - 2at_1^2 t_0$

$t_1^2 y + 2at_1t_0(t_1 + t_0) + 8at_1 - 2t_1x$

$= t_0^2 y + 4y + 2t_0x$

$\therefore\ t_1(t_1y + 2at_0t_1 + 2at_0^2 + 8a - 2a)$

$= t_0^2 y + 4y + 2t_0x$

$\therefore$ It passes through intersection if $2t_0x + 4y + t_0^2 y = 0$

$t_1y - 2x + 2at_0t_1 + 2at_0^2 + 8a = 0$

$\therefore$ The point of intersection which is the required fixed point $(a(t_0^2 + 4), -2at_0)$

Sol 8: Equation of tangent is $y = mx + \frac{a}{m}$

Equation of $\perp$ line through origin is

$y = -\frac{1}{m}x$

$\therefore y^2 = 4a(-my)$

$\therefore y = -4am$ is the y-coordinate of Q. & $x = 4am^2$

$\therefore\ OQ = \sqrt{(4am)^2 + \left(4am^2\right)^2} = 4am\ \sqrt{1 + m^2}$

$OP = \frac{a}{m\sqrt{1 + m^2}}$ ($\perp$ distance of (0, 0) from line

$y = mx + \frac{a}{m}$)

$\therefore$ OP × OQ = $4a^2$ = constant

Sol 9: $P = (2(3)^2, 4(3))$

$\therefore$ Parameter $(t_1) = 3$

$t_2 = -t_1 - \frac{2}{t_1} = -3 - \frac{2}{3} = -\frac{11}{3}$

$\therefore\ Q = \left(\frac{2 \times 121}{9}, \frac{-44}{3}\right)$

$\therefore\ PQ = \sqrt{\left(18 - \frac{2 \times |2|}{9}\right)^2 + \left(12 + \frac{44}{3}\right)^2}$

$= \frac{1}{9}\sqrt{(80)^2 + 9(80)^2} = \frac{80}{9}\sqrt{10}$

$\therefore 9PQ = 80\sqrt{10}$

Sol 10: O = (0, 0), L = (2a, a)

Let H = (h, 0)

$\therefore\ \frac{a}{2a - h} \times \frac{1}{2} = -1 \Rightarrow a = -4a + h$

$\therefore\ h = 5a$

$\therefore$ H = (5a, 0)

$\therefore$ Length of double ordinate $= 2\sqrt{4a \times 5a} = 4a\sqrt{5}$

Sol 11: $y^2 = 4ax$

Equation of normal at $(at^2, 2at)$ is

$y = -tx + 2at + at^3$

It meets y = 0 at G

$\therefore G = (2a + at^2, 0)$

$QG = \sqrt{4ax} = \sqrt{4a(2a + at^2)}$

$PG = \sqrt{(at^2 - (2a + at^2))^2 + (2at)^2}$

$\therefore\ QG^2 - PG^2 = 4a(2a + at)^2 - (4a^2 + 4a^2t^2)$

$= 8a^2 - 4a^2 = 4a^2$

Sol 12: The equation of tangent to $y^2 = 4ax$ of slope m is $y = mx + \frac{a}{m}$

$\therefore\ xm^2 - my + a = 0$ ...(i)

The equation of normal to $x^2 = 4by$ of slope m is $y = mx + 2b + \frac{b}{m^2}$

$\therefore\ m^3x + (2b - y)m^2 + b = 0$ ...(ii)

Let the point (x, y) satisfy both the equation.

$\therefore$ From (i) $m_1m_2 = \frac{a}{x}$ & $m_1 + m_2 = \frac{y}{x}$

These two tangents are normal to $x^2 = 4by$

$\therefore\ m_1, m_2$ satisfy (ii)

$\therefore m_1m_2m_3 = -\frac{b}{x} \Rightarrow m_3 = -\frac{b}{a}$

$m_1 + m_2 + m_3 = \frac{y - 2b}{x} \Rightarrow \frac{y}{x} + m_3 = \frac{y}{x} - \frac{2b}{x}$

$\Rightarrow m_3 = -\frac{2b}{x}$ & $m_1m_2 + m_3(m_1 + m_2) = 0$

$\therefore\ m_3 = -\frac{a}{y} \Rightarrow -\frac{b}{a} = -\frac{2b}{x} = -\frac{a}{y}$

$\Rightarrow x = 2ay = \frac{a^2}{b}$

Now m_1, m_2 are distinct & real

$\therefore$ D of equation (i) > 0

$\therefore\ y^2 - 4ax > 0 \Rightarrow \frac{a^4}{b^2} > 8a^2 \Rightarrow a^2 > 8b+2$

Sol 13: Let point be $P(at^2, 2at)$

The directrix is $x = -a$

The equation of tangent is $y = \frac{1}{x}x + at$

$\therefore$ The point where is meets the directrix is

$Q = \left(-a, a\left(t - \frac{1}{t}\right)\right)$

$M = (x, y) = \frac{(a(t^2-1))}{2}, \frac{a}{2}(3t - \frac{1}{t})]$ midpoint

$t^2 = \frac{2x+1}{a}$ $2yt = a\,(3t^2 - 1)$

$\therefore\ 4y^2 \times \left(\frac{2x+a}{a}\right) = a^2\left(3\times\left(\frac{2x}{a}+1\right)-1\right)^2$

$\therefore\ 4y^2\ \frac{(2x+a)}{a} = 2^2(3x + a)^2$

$\therefore\ y^2(2x + a) = a(3x + a)^2$

Sol 14: Parabola $y^2 = 12x$

$a = 3$

$\therefore\ P = (3, 6)\ \therefore\ t = 1$

$\therefore\ t_2$ to the other point $= -t - \frac{2}{t} = -3$

$\therefore\ Q = (a(-3)^2, 2a(-3)) = (27, -18)$

$\therefore$ Equation of circle with PQ as diameter is

$(x - 3)(x - 27) + (y - 6)(y + 18) = 0$

$\therefore\ x^2 + y^2 - 30x + 12y - 27 = 0$

Sol 15: Let upper end of latus rectum be P &

$Q = (4a, 4a)$

$\therefore\ P = (a, 2a)\ \therefore\ t = 1$

& for Q $(4a, 4a)\ \therefore\ t_2 = 2$

$\therefore$ let t_3 be the other end of normal for P

$\therefore\ t_3 = -1 - \frac{2}{1} = -3$

And let t_4 be the other end of normal for Q

$\therefore\ t_4 = -2 - \frac{2}{2} = -3$

$\therefore\ t_3 = t_4$

$\therefore$ The both the normals intersect on parabola it self

Sol 16: Let O be vertex & P be $(at^2, 2at)$

$\therefore$ Slope of OP is $\frac{2}{t}$

Let Q be $(at_2^2, 2at_2)$

$OQ \perp OP \Rightarrow \frac{2}{t}\times\frac{2}{t_2} = -1 \Rightarrow t_2 = -\frac{4}{t}$

$Q = \left(\frac{16a}{t^2}, -\frac{8a}{t}\right)$

Now let R be to other end of rectangle (x, y) since it is a rectangle & $OP \perp OQ$

$\therefore$ Midpoint of R & O = midpoint of P & Q

$\therefore\ \left(\frac{x}{2}, \frac{y}{2}\right) = \left(\frac{at^2 + \frac{16a}{t^2}}{2}, \frac{2a\left(t - \frac{4}{t}\right)}{2}\right)$

$\therefore\ x = a\left(\left(t - \frac{4}{t}\right)^2 + 8\right);\quad y = 2a\left(t - \frac{4}{t}\right)$

$\therefore x = a\left(\frac{y^2}{4a^2} + 8\right);\quad \therefore\ y^2 = \frac{(x-8a)\times 4a^2}{a}$

$\therefore\ y^2 = 4a(x - 8a)$ is the locus of other end

Sol 17: The equation of normals is $y = mx - 2am - am^3$

$\Rightarrow a = 1$ & (15, 12) lies on it

$\therefore\ 12 = 15m - 2m - m^3$

$\therefore\ m^3 - 13m + 12 = 0$

$(m - 1)(m^2 + m - 12) = 0$

$(m - 1)(m + 4)(m - 3) = 0$

$\therefore$ 1, – 4, 3 are three possible normals

$\therefore$ Equation is $(y - 12) = (x - 15) \Rightarrow y = x - 3$

and $(y - 12) = 3(x - 15) \Rightarrow y = 3x - 33$

and $(y - 12) = -4(x - 15) \Rightarrow 4x + y - 72 = 0$

Sol 18:

Centre of circle = (0, 0) diameter $= 2r = \frac{3}{4}\times 4a$

$\therefore\ r = \frac{3a}{2}$

$\therefore$ Equation of circle is $x^2 + y^2 = \left(\frac{3a}{2}\right)^2$

$\Rightarrow 4x^2 + 16ax - 9a^2 = 0$

$\Rightarrow 4x^2 + 18ax - 2ax - 9a^2 = 0$

$\Rightarrow (2x - a)(2x + 9a) = 0$

But $x = -\frac{9a}{2}$ is not possible as y becomes imaginary

$\therefore\ x = \frac{a}{2}$ is the abscissa of the two points of intersection

$\therefore$ The common chord bisects the line joining Vertex.

Sol 19: Let $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

Let R be $(aT^2, 2aT)$ on the parabola $y^2 = 4ax$

$T = -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2} \Rightarrow t_1t_2 = 2$

Tangents at P and Q intersect at T $(at_1t_2, a(t_1 + t_2))$

$T(2a, a(t_1 + t_2))$

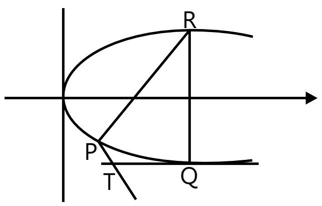
Coordinates of R the point of intersection, are

$\left(2a + a\left(t_1^2 + t_2^2 + t_1t_2\right), -at_1t_2\left(t_1 + t_2\right)\right)$

$\equiv\left(4a + a\left(t_1^2 + t_2^2\right), 2a\left(t_1 + t_2\right)\right)$

$\Rightarrow \angle TPR = \Rightarrow \angle TQR = 90^\circ \Rightarrow \angle TPR + \angle TQR = 180^\circ$

$\Rightarrow$ Quadrilateral TPQR is a cyclic quadrilateral & the centre of circle lies on the midpoint of TR.



Let midpoint be M (h, k)

$\therefore\ 2h = 2a + 4a + a(t_1^2 + t_2^2)$

$\frac{2h - 6a}{a} = (t_1 + t_2^2) - 2t_1t_2$

$\frac{2h - 2a}{a} = (t_1 + t_2)^2 (\because t_1t_2 = 2)$

$2k = a(t_1 + t_2) - 2a(t_1 + t_2)$

$\frac{2k}{a} = (t_1 - t_2)$

$\therefore \left(\frac{2h - 2a}{9}\right) = \left(\frac{-2k}{a}\right)^2$

$\therefore\ 2R^2 = a(h - a)$

$\therefore$ Locus of M(h, k) is $2y^2 = a(x - a)$

Sol 20: The focus of $x^2 = 4y$ is (0, 1)

$\therefore$ Tangent to parabola at (6, 9) is $6x = 2(y + 9)$

$\therefore\ 3x - y - 9 = 0$

$\therefore$ Equation of normal is

$(y - 9) = -\frac{1}{3}(x - 6) \Rightarrow x + 3y - 33 = 0$

Centre lies on it $\therefore\ g + 3f + 33 = 0$

$(-g - 6)^2 + (-t - 9)^2 = (-g - 0)^2 + (-f - 1)^2$

$\therefore\ 12g + 36 + 18 + 81 = 2f + 1$

$\therefore\ 12g + 16f + 116 = 0$

$\Rightarrow 3g + 4f + 29 = 0$

Solving we get g = 9 & f = – 14

$\therefore\ r = \sqrt{(-g)^2 + (-f - 1)^2} = \sqrt{g^2 + 13^2} = \sqrt{250}$

$\therefore C = g^2 + f^2 - r^2 = 27$

$\therefore$ Equation of circle is $x^2 + y^2 + 18x - 28y + 27 = 0$

Sol 21: Let $P(at_1^2, 2at_1)$

since Q is the other end of the normal from P

$t_2 = -t_1 - \frac{2}{t_1}$

$Q = \left[a\left(t_1 + \frac{2}{t_1}\right)^2, -2a\left(t_1 + \frac{2}{t_1}\right)\right]$

$T = \left(at_1 \times \left(-t_1 - \frac{2}{t_1}\right), a\left(\frac{-2}{t_1}\right)\right)$

$\therefore$ The x-coordinate of midpoint of T & P is

$x = -\frac{at_1^2 - 2a + at_1^2}{2} = -a$

$\therefore$ TP is bisected by directrix

Sol 22: Let $P = (at_1^2, 2at_1)$

The other end of normal chord $Q = (at_2^2, 2at_2)$

$\therefore\ t_2 = -t_1 - \frac{2}{t_1}$

$\therefore\ Q = \left[a\left(t_1 + \frac{2}{t_1}\right)^2, -2a\left(t_1 + \frac{2}{t_1}\right)\right]$

Let M(x, y) be midpoint of P & Q

$\therefore\quad y = \frac{-2a}{t_1} \qquad \therefore\ t_1 = -\frac{2a}{y}$

$\Rightarrow 2x = a\times\left(\frac{-2a}{y}\right)^2 + a\left(\frac{-2a}{y}-\frac{y}{a}\right)^2$

$\therefore\ 2x = \frac{4a^3}{y^2} + \frac{4a^3}{y^2} + \frac{y^2}{a} + 4a$

$\therefore\ x - 2a = \frac{4a^3}{y^2} + \frac{y^2}{2a}$

Sol 23: Let A = $(at_1^2, 2at_1)$ a = 2

B = $(at_2^2, 2at_2)$

Tangent at A is $y = \frac{1}{t_1}x + at_1$

Tangent at B is $y = \frac{1}{t_2}x + at_2$

The y-coordinate of point of intersection of A & tangent at vertex (x = 0) is

$y_1 = (2t_1)$; $y_2 = 2t_2$

$\therefore PQ = 2(t_1 - t_2) = 4$

$\therefore\ |t_1 - t_2| = 2$

& point of intersect is $(x, y) = (2t_1t_2, 2(t_1 + t_2)$

$(t_1 + t_2)^2 - 4t_1t_2 = (t_1 - t_2)^2$

$\therefore\ \frac{y^2}{4} - 2x = 4$

$\therefore\ y^2 - 8x = 16$

$\therefore\ y^2 = 8(x+2)$ is the locus of point of intersection

Sol 24: Let m be the slope of 1st line

$\therefore -\frac{1}{m}$ is the slope of the other line equation of tangents is $y = mx + \frac{a}{m}$

and $y = -\frac{1}{m}x - am$

$\Rightarrow T(-a, 2am)$ & $T'\left(-a, \frac{-a}{m}\right)$

$\therefore$ Point of intersection of tangents

$M = \left(-a, \frac{a}{m} - am\right)$

One line passing through (a, 0) with slope m is

$(y) = m(x - a)$

And $\perp$ line is $(y) = -\frac{1}{m}(x - a)$

$\therefore\ T = (-a, -2am)$

$T' = \left(-a, \frac{2a}{m}\right)$

$\therefore$ Midpoint of T and T′ is

$A = \left(-a, \frac{a}{m} - am\right)$

$\therefore$ Point of intersection of tangents is the midpoint of T and T′

Sol 25: Let $P = (at_1^2, 2at_1)$

$Q = (at_2^2, 2at_2)$

Slope of PQ $= \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2} = 1$

$\therefore\ t_1 + t_2 = 2$

Point of intersection of points with parameter t_1 & t_2 is

$(2a + a(t_1^2 + t_2^2 + t_{12}), -at_1t_2(t_1 + t_2))$

$\therefore\ x = 2a + a((t_1 + t_2)^2 - t_1t_2)$

$y = -a(t_1 + t_2) \times t_1t_2$

$\therefore\ x \times (t_1 + t_2) - y = 2a(t_1 + t_2) + a(t_1 + t_2)^2 = 0$

$2x - y = 2a \times 2 + a \times 8$

Putting a = 1

$\therefore$ 2x − y =12 is the locus of point of intersection of normals.

Sol 26: Let $A(am_1^2, 2am_1)$, $B(am_2^2, -2am_2)$ and $C(am_3^2, -2am_3)$ be points on parabola $y^2 = 4ax$

Let point of intersection of normals be (h, k) then

$am^3 + (2a - h)m + k = 0$

$m_1 + m_2 + m_3 = 0$ ... (i)

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - h)}{a}$ ... (ii)

$m_1m_2m_3 = -\frac{k}{a}$ ... (iii)

Let equation of circle through ABC be

$x^2 + y^2 + 2gx + 2fy + c = 0$

The point $(am^2, -2am)$ lies on it

$a^2m^4 + (4a^2 + 2ag)m^2 - 4afm + c = 0$ ... (iv)

$\therefore\ m_1 + m_2 + m_3 + m_4 = 0$

$\therefore$ From (i)

$m_4 = 0 - 0 = 0$

$\therefore$ (0, 0) two lies on circle

$\therefore\ c = 0$

From (4) $a^2m^4 + (4a^2 + 2ag)m^2 - 4afm = 0$

$\Rightarrow am^3 + (4a + 2g)m - 4f = 0$ ... (v)

Now, equation A & 5 are identicals

$\therefore\ 1 = \dfrac{4a+2g}{2a-b} = -\dfrac{4t}{k}$

$\therefore\ 2g = -(2a + h)$

$2t = -\dfrac{k}{2}$

$\therefore$ The equation of circle is $x^2 + y^2 - (2a + h)x - \dfrac{k}{2}y = 0$

or $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$

Sol 27: Let $P(at_1^2, 2at_1)$ and Q be $(at_2^2, 2at_2)$

Now chord PQ passes through A(a, 3a)

$\therefore\ \dfrac{2at_2 - 3a}{at_2^2 - a} = \dfrac{2at_1 - 3a}{at_1^2 - a}$

$\therefore\ \dfrac{2t_2 - 3}{t_2^2 - 1} = \dfrac{2t_1 - 3}{t_1^2 - 1}$

$2t_1^2t_2 + 3t_1^2 - 2t_1 + 3 = 2t_1t_2^2 - 3t_2^2 - 2t + 3$

$\therefore\ 2t_1t_2(t_2 - t_1) - 3(t_2 - t_1)(t_2 + t_1) + 2(t_2 - t_1) = 0$

$\therefore\ 2t_1t_2 - 3(t_1 + t_2) + 2 = 0$

Point of intersection of tangent at t_1t_2 is

$(at_1t_2, a(t_1 + t_2))$

$\therefore\ x = at_1t_2$ and $y = a(t_1 + t_2)$

$\therefore\ 2\ \dfrac{x}{a} - \dfrac{3y}{a} + 2 = 0$

$\therefore\ 2x - 3y + 2a = 0$ is the locus of point of intersection of tangent

Sol 28: Equation of tangent is

$y = mx + \dfrac{a}{m} \quad \Rightarrow\ mx - y + \dfrac{a}{m}$

$\therefore$ Midpoint of P & Q is foot of $\perp$ of (0, 0) on the tangent let M be (x, y)

$\therefore\ \dfrac{x-0}{m} = \dfrac{y-0}{-1} = -\dfrac{\left(\frac{a}{m}\right)}{m^2+1}$

$\therefore\ x = -\dfrac{a}{m^2+1}\ y = \dfrac{a}{m(m^2+1)}$

$\therefore\ m^2 = -1 - \dfrac{a}{x}$

$\therefore\ y^2 = \dfrac{a^2}{-\left(1+\frac{a}{x}\right)\left(1-1-\frac{a}{x}\right)^2} \quad y^2 = \dfrac{a^2}{-\frac{(x+a)}{x}\frac{a^2}{x^2}}$

$\therefore\ y^2(x + a) + x^3 = 0$

$\Rightarrow x(x^2 + y^2) + ay^2 = 0$

Sol 29: Let P be $(at_1^2, 2at_1)$ & Q be $(at_2^2, 2at_2)$

$\therefore$ OP $\perp$ OQ

$\Rightarrow \dfrac{2}{t_1}\times\dfrac{2}{t_2} = -1 \Rightarrow t_2 = -\dfrac{4}{t_1}$

$\therefore$ Equation of PQ is

$\dfrac{y-2at_1}{x-at_1^2} = \dfrac{2(t_2-t_1)}{(t_2^2-t_1^2)}$

$\therefore\ \dfrac{y-2at_1}{x-at_1^2} = \dfrac{2}{\left(t_1 - \frac{4}{t_1}\right)}$

$\therefore\ (y - 2at_1)(t_1^2 - 4) = 2t_1(x - at_1^2)$

$\therefore\ t_1^2y - 4y - 2at_1^3 + 8at_1 = 2t_1x - 2at_1^3$

$\therefore\ t_1(t_1y + 8a - 2x) - 4y = 0$

$\therefore$ The line passes through point of intersection of

$t_1y + 8a - 2x = 0$ & $y = 0$

The point is (4a, 0)

Let M = (x, y) be midpoint

$\therefore\ (x, y) = \dfrac{a(t_1^2+t_2^2)}{2}, \dfrac{2a(t_1+t_2)}{2}$

$2x = a(t_1^2 + t_2^2) = a(t_1 + t_2)^2 - 2t_1t_2)$

$t_1 t_2 = -4$

$\therefore\ 2x = a(t_1 + t_2)^2 + 8$

$y = a(t_1 + t_2)$

$\therefore\ 2x = a\left(\dfrac{y^2}{a^2} + 8\right)$

$y^2 = 2ax - 8a^2 = 2a(x - 4a)$ is the locus of M.

Exercise 2

Single Correct Choice Type

Sol 1: (C) Let equation of tangent be

$y = mx + \dfrac{a}{m}$ directrix is $x = -a$ & latus rectum is

$x = a$; S = (a, 0)

$U = \left(-a, a\left(\dfrac{1}{m} - m\right)\right)$; $V = \left(a, a\left(m + \dfrac{1}{m}\right)\right)$

$SU = \sqrt{4a^2 + a^2\left(m^2 - 2 + \dfrac{1}{m^2}\right)} = m + \dfrac{1}{m}$

$SV = m + \frac{1}{m}$

$\therefore$ It is always an is isosceles triangle

Angle between SU & SV is not always 90° as slope of SV = ∞ and slope of SU depends on m

$\therefore$ It is just an isosceles triangle

Sol 2: (B) Let points be $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$

$\therefore\ (a(t_1^2 - 1))^2 + 4a^2t_1^2 = 16$

& $(a(t_2^2 - 1))^2 + 4at_2^2 = 81$

$\therefore (at_1^2 + a)^2 = 16$

$\therefore\ at_1^2 + a = \pm 4$ ... (i)

And $at_2^2 + a = \pm 9$ ... (ii)

The point of intersection is $(at_1t_2, a(t_1 + t_2))$

$\therefore\ PS^2 = a^2(t_1t_2 - 1)^2 + a^2(t_1 + t_2)^2$

$= a^2(t_1^2t_2^2 + t_1^2t_2^2 + 1) = 1 \times 2 = 36$

Sol 3: (A) Let (h, k) be a point on line (2x + y = 4)

$\therefore$ Chord of contact is ky = – 2(x + h)

$\therefore$ 2x + ky = – 2h & (h,k) also satisfy

2h + k = 4 & 2h + ky = – 2x

$\therefore$ It passes through y = 1 & x = – 2

$\therefore$ (–2, 1)

Sol 4: (D) Parabola is $y^2 = ax$

$\therefore\ a' = \frac{a}{4}$

Tangent is $y = mx + \frac{a}{4m}$

or $mx - y + \frac{a}{4m} = 0$

$\therefore$ Foot of $\perp$ from (0, 0) is (x, y)

$\therefore\ \frac{x-0}{m} = \frac{y-0}{-1} = -\left(\frac{a}{4m(m^2+1)}\right)$

$\therefore\ x = \frac{-a}{4(m^2+1)}$ and $y = \frac{a}{4m(m^2+1)}$

$\Rightarrow m = -\frac{x}{y} \Rightarrow x = \frac{-ay^2}{4(x^2+y^2)}$

$\therefore 4x(x^2 + y^2) + ay^2 = 0$

Sol 5: (D) $(y + 2)^2 = 6(x + 1)$

let y = Y – 2 & x = X – 1

$\therefore\ Y^2 = 6X$

The locus of $\perp$ tangents is $X = -a = -\frac{3}{2}$

$\therefore\ x + 1 = -\frac{3}{2} \Rightarrow 2x + 5 = 0$

Sol 6: (C) The tangent at (x_1, y_1) to $y^2 = 4ax$

$\Rightarrow yy_1 = 2a(x + x_1)$

$\Rightarrow 2ax - yy_1 + 2ax_1 = 0$ ...(i)

Let (h, k) be midpoint

$\therefore$ Locus of chord to $y^2 = 4a(x + b)$ with M(h, k) is

$yk - 2a(x + h) - 4ab = k^2 - 4ah - 4ab$

$\Rightarrow ky - 2ax = k^2 - 2ah$

or $2ax - ky + k^2 - 2ah = 0$

$\therefore\ 2ax - yy_1 + 2ax_1 = 0$

And $2ax - ky + k^2 - 2ah = 0$ represent the same line

$\therefore\ k = y_1$ & $k^2 - 2ah = 2ax_1$

$2ah = y_1^2 - 2ax_1$

$y_1^2 - 4ax_1 = 0 \Rightarrow y_1^2 - 2ax_1 = 2ax_1$

$\Rightarrow 2ah = 2ax_1 \Rightarrow h = x_1$

Sol 7: (C) For closest points normal to parabola should be normal to circle equation of normal at $(at^2, 2at)$ is

$y = -tx + 2at + at^3$

a = 1

$\therefore\ y = -tx + 2t + t^3$ is equation of normal it should pass through (0, 2)

$\therefore 12 = 2t + t^3 \quad \Rightarrow t^3 + 2t - 12 = 0$

$(t - 2)(t^2 + 2t + 6) = 0$

t = 2 is only possible solution

$\therefore$ The point on parabola closest to the circle is (4, 4)

Sol 8: (C) $P = (at^2, 2at)$

$\therefore$ Equation of focal chord is $y = \frac{2at}{a(t^2-1)}(x - a)$

$2tx - (t^2 - 1)y - 2at = 0$

The distance from (0, 0) is

$\frac{2t}{\sqrt{(t^2+1)^2}} = P$

$\therefore\ \frac{|2at|}{t^2+1} = P$

Now length of focal chord is $a\left(t+\frac{1}{t}\right)^2$

$\therefore \left(t+\frac{1}{t}\right) = \left|\frac{2a}{P}\right|$

$\therefore L_f = a \times \frac{4a^2}{P^2} = \frac{4a^3}{P^2}$

Sol 9: (B) Let the points be $A(am_1^2, -2am_1)$

& B = $(am_2^2 - 2am_2)$

$m_1m_2 = -1 \Rightarrow m_2 = -\frac{1}{m}$

The line joining A & B is

$(y + 2am_1) = \frac{2a(m_2 - m_1)}{a(m_1 - m_2)(m_1 + m_2)}(x - am_1^2)$

$y + 2am_1 = \frac{-2am_1}{m_1^2 - 1}(x - am_1^2)$

$\therefore ym_1^2 - y + 2am_1^3 - 2am_1 = -2m_1x + 2am_1^3$

$\therefore y + m_1(-m_1y + 2a - 2x) = 0$

$\therefore$ It passes through intersection of y = 0 &

$-m_1y + 2a - 2x = 0$

$\therefore$ It always passes through (a, 0)

Sol 10: (D) Let (h, k) be the point

The equation of normals through (h, k) is

$\therefore k = mh - 2am - am^3$

$\therefore am^3 + (2a - h)m + k = 0$

θ_1 and θ_2 are complimentary

$\therefore \tan(\theta_1 + \theta_2) = \tan(90°)$

$\therefore m_1m_2 = 1$

$m_1 + m_2 + m_3 = 0$

$m_1m_2 + m_3(m_1 + m_2) = \frac{2a - h}{a}$

$m_1m_2m_3 = -\frac{k}{a} \Rightarrow m_3 = -\frac{k}{a}$

$\therefore 1 - m_3^2 = \frac{2a - h}{a}$

$\therefore y^2 + 2a^2 - ax = a^2$

$\therefore y^2 = ax - a^2 \Rightarrow y^2 = a(x - a)$ is locus of P

Sol 11: (A) Let the tangent to parabola be

$x = my - \frac{a}{m}$ or $mx - m^2y + a = 0$ ... (i)

Let midpoint of A, B be (h, k)

$\therefore$ Equation of chord through (h, k) is

$xh - 2b(y + k) = h^2 - 4bk$

$hx - 2by + 2bk - h^2 = 0$ ... (ii)

Equation (i) and (ii) are the same lines

$\therefore \frac{m}{h} = \frac{m^2}{2b} = \frac{a}{2bk - h^2}$

$\therefore \frac{2ab}{2bk - h^2} = \frac{(ab)^2}{(2bk - h^2)^2}$

$\therefore 2b(2by - x^2) = ax^2$

$\therefore x^2(a + 2b) = 4b^2y$ is locus of M

Sol 12: (A) $4y^2 - 4y + 1 = 16x - 24$

$(2y - 1)^2 = 16\left(x - \frac{3}{2}\right) \Rightarrow \left(y - \frac{1}{2}\right)^2 = 4\left(x - \frac{3}{2}\right)$

let $x = X + \frac{3}{2}$ and $y = Y + \frac{1}{2}$

$\therefore Y^2 = 4aX \quad \Rightarrow a = 1$

The circle cuts the axis at (–a, 0) and (3a, 0)

$\therefore$ The points in original system are $\left(\frac{1}{2}, \frac{1}{2}\right)$ & $\left(\frac{9}{2}, \frac{1}{2}\right)$

Sol 13: (B) m = 1

$\therefore$ Equation of tangent is y = x + A and equation of normal is

$y = mx - 2Am - Am^3$

$m = 1 \quad \therefore y = x - 3A$

$\therefore \perp$ distance $= \frac{A - (-3A)}{\sqrt{2}} = 2\sqrt{2}A$

Sol 14: (B) $4a = 4 \therefore a = 1$

$\therefore$ Equation of latus rectum is y = 4

$\therefore$ Equation of directrix is y = 4 ± 2

Now focus = (4, 4)

When directrix is y = 6

$\therefore$ Tangent at vertex is y = 5

And parabola lies below y = 5

When directrix is y = 2

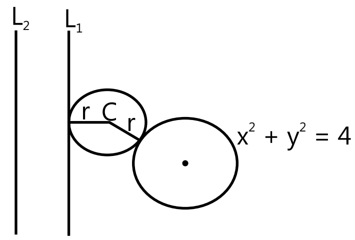
$\therefore$ Tangent at vertex x is y = 3

$\therefore$ Parabola lies above y = 3

But y-coordinate of point is (1)

$\therefore$ y = 6 is the directrix

Sol 15: (D) Consider a line L_2, 2 units to left of L_1 & parallel to L_1



$\therefore$ Distance of centre of circle c from L_2 = 2 + r

& distance of C from origin = 2 + r

$\therefore$ Locus of C is a parabola

Sol 16: (C) P = $(at^2, 2at)$

Equation of tangent is $y = \frac{x}{t} + at$

Equation of normal is $y = -tx + 2at + at^3$

$T = (-at^2, 0)$ & $G = (2a + at^2, 0)$

Since PT $\perp$ PG

$\therefore$ The circle passing through PTG will have its centre at midpoint of T and G

$\therefore$ C = (a, 0)

Slope PC = $\frac{2at}{a(t^2-1)} = \frac{2t}{t^2-1}$

$\therefore$ Slope of tangent at P = $\frac{1-t^2}{2t}$

Slope of tangent to parabola = $\frac{1}{t}$

$$\tan(\theta_1 - \theta_2) = \left|\frac{\frac{1}{t} - \frac{(1-t^2)}{2t}}{1 + \frac{1}{t}\times\frac{(1-t^2)}{2t}}\right| = \left|\frac{(1+t^2)t}{(1+t^2)}\right| = |t|$$

$\therefore$ $|\theta_1 - \theta_2| = \tan^{-1}t$

Sol 17: (A) Let P be (h, h)

$\therefore$ $h^2 - 4h = 0 \Rightarrow h = (4, 4)$ or $h = (0, 0)$

Let centre be (h, k)

Centre lies on normal at (4, 4)

$2t = 4 \Rightarrow t = 2$

$\therefore$ Equation of normal is $y = -2x + 4 + 8$

$\Rightarrow y + 2x = 12$

Centre passes through this

$\therefore$ k = 12 – 2h

$\therefore$ Centre is (h, 12 – 2h)

and distance from focus = distance from (4, 4)

$\therefore$ $(h-4)^2 + (8-2h)^2 = (h-1)^2 + (12-2h)^2$

$\therefore$ – 8h – 32h + 64 + 16

= – 2h – 48h + 144 + 10h = 145 – 80

$\therefore$ $h = \frac{13}{2}$

$\therefore$ Centre = $\left(\frac{13}{2}, -1\right)$ and radius = $\sqrt{\left(\frac{11}{2}\right)^2} = \frac{5\sqrt{5}}{2}$

$\therefore$ Equation of circle is $x^2+y^2-13x + 2y + 12 = 0$

Sol 18: (D) Three quarters of the latus rectum = 3a

$\therefore$ Equation of circle is $x^2 + y^2 = \left(\frac{3a}{2}\right)^2$

and equation of parabola is $y^2 = 4x$

$\therefore$ Point of intersection of parabola & circle is

$x^2 + 4ax - \left(\frac{3a}{2}\right)^2 = 0$

$\therefore$ $4x^2 + 16ax - 9a^2 = 0$

$4x^2 + 16ax - 2ax - 9a^2 = 0$

But $y^2 > 0$ & $x > 0$.

$\therefore$ $x = \frac{a}{2}$ is only possible solution

$\Rightarrow y = \pm\sqrt{2}a$

$\therefore$ $P = \left(\frac{a}{2}, \sqrt{2}a\right)$; $Q = \left(\frac{a}{2}, -\sqrt{2}a\right)$

$L_1 L_2 = 4a$

Area of trapezium = $\frac{1}{2}h\,(PQ + L_1L_2)$

$= \frac{1}{2}\times\frac{a}{2}\,(2\sqrt{2}a + 4a) = \left(\frac{2+\sqrt{2}}{2}\right)a^2$ Ans (D)

Multiple Correct Choice Type

Sol 19: (A, B, C, D)

Equation of normal is $y = mx - 2am - am^3$

or $am^2 + (2a - x)m + y = 0$

The points are $(am_i^2, -2am_i)$

$m_1 + m_2 + m_3 = 0$

Algebraic sum of ordinates is $-2a(m_1 + m_2 + m_3) = 0$

The y-coordinate of centroid of triangle is

$\frac{-2a}{3}(m_1 + m_2 + m_3) = 0$

$\therefore$ It lies on x-axis

Sol 20: (A, B) $y^2 - 2y - 4x - 7 = 0$

$\therefore (y-1)^2 = 4(x+2)$ a = 1

Its axis is x-axis

a′ of the 2nd parabola = 2 × a = 2

$\therefore$ Equation of 2nd parabola is $(x+2)^2 = \pm 8(y-1)$

$\therefore\ x^2 + 4x - 8y + 12 = 0$ & $x^2 + 4x + 8y - 4 = 0$

can be the equation of 2nd parabola

Sol 21: (A, B) Equation of parabola 1 is

$(x-a)^2 + (y-b)^2 = y^2$ and of parabola 2 is

$(x-a)^2 + (y-b)^2 = x^2$

Their common chord is such that $x^2 = y^2$

$\therefore \left(\frac{y}{x}\right)^2 = 1 \to y = \pm x$

$\therefore$ Slope = ±1

Sol 22: (B, C) For parabola $y^2 = 40x$, a = 10

$\therefore$ Equation of tangent to parabola is

$\therefore\ y = mx + \frac{10}{m}$

$\therefore\ \perp$ distance from origin is $\frac{10}{m\sqrt{1+m^2}}$

Since it is tangent to circle

$\therefore \perp$ from centre = radius

$\Rightarrow \frac{10^2}{m\sqrt{1+m^2}} = 5\sqrt{2}$

$\therefore 2 = m^2(1+m^2)$

$\therefore\ m^4 + m^2 - 2 = 0$

$m^4 + 2m^2 - m^2 - 2 = 0$

$\therefore\ m^2 = 1 \therefore m = \pm 1$

$\therefore$ Possible equation of tangents are y = x + 10

and x + y + 10 = 0

Sol 23: (A, C) $y^2 - 2y = 4x - 3$

$\therefore (y-1)^2 = 4\left(x - \frac{1}{2}\right)$

$\therefore$ Vertex is at $\left(\frac{1}{2}, 1\right)$

And axis is parallel to x-axis a = 1

$\therefore$ Focus is at $\left(\frac{3}{2}, 1\right)$

Sol 24: (A, D) The Parabola & circle both pass through origin

$\therefore$ The circle touches parabola at (0, 0)

$\therefore$ Centre of circle = (–b, 0)

If $a > 0 \therefore -b < 0 \therefore b > 0$ & if $a < 0$ $b < 0$

Sol 25: (A, D) Let P = $(at^2, 2at)$

$\therefore$ PA is $y = \frac{2x}{t}$

M = (–a, 2at) and D = $\left(-a, -\frac{2a}{t}\right)$, end points of diameter

$\therefore$ Equation of circle is

$(x+a)(x+a) + (y-2at)\left(y + \frac{2a}{t}\right) = 0,$

It intersects x-axis therefore satisfying y = 0

$(x+a)^2 = 4a^2$

$\therefore\ x = 2a - a$ or $x = -2a - a$

$\therefore$ It intersects x-axis at (a, 0) & (–3a, 0)

Previous Years' Questions

Sol 1: (C) Given curves are $x = t^2 + t + 1$ ... (i)

and $y^2 = t^2 - t + 1$ ... (ii)

On subtracting Eq. (ii) from Eq. (i),

$x - y = 2t$

Now, substituting the value of 't' in (i)

$\Rightarrow x = \left(\frac{x-y}{2}\right)^2 + \left(\frac{x-y}{2}\right) + 1$

$\Rightarrow 4x = (x-y)^2 + 2x - 2y + 4$

$\Rightarrow (x-y)^2 = 2(x+y-2)$

$\Rightarrow x^2 + y^2 - 2xy - 2x - 2y + 4 = 0$

Now, $\Delta = 1 \cdot 1 \cdot 4 + 2 \cdot (-1)(-1)(-1)$

$-1 \times (-1)^2 - 1 \times (-1)^2 - 4(-1)^2$

$= 4 - 2 - 1 - 1 - 4 = -4$

$\therefore \Delta \neq 0$

and $ab - h^2 = 1 \cdot 1 - (-1)^2 = 1 - 1 = 0$

Hence, it represents a equation of a parabola.

Sol 2: (C) Any tangent to $y^2 = 4x$ is of the form $y = mx + \frac{1}{m}$, $(\because a = 1)$ and this touches the circle

$(x-3)^2 + y^2 = 9$,

If $\left| \frac{m(3) + \frac{1}{m} - 0}{\sqrt{m^2+1}} \right| = 3$

[$\therefore$ Centre of the circle is (3,0) and radius is 3]

$\Rightarrow \frac{3m^2+1}{m} = \pm 3\sqrt{m^2+1}$

$\Rightarrow 3m^2 + 1 = \pm 3m\sqrt{m^2+1}$

$\Rightarrow 9m^4 + 1 + 6m^2 = 9m^2(m^2+1)$

$\Rightarrow 9m^4 + 1 + 6m^2 = 9m^4 + 9m^2$

$\Rightarrow 3m^2 = 1$

$\Rightarrow m = \pm \frac{1}{\sqrt{3}}$

If the tangent touches the parabola and circle above the x-axis, then slope m should be positive.

$\therefore m = \frac{1}{\sqrt{3}}$ and the equation is $y = \frac{1}{\sqrt{3}}x + \sqrt{3}$

or $\sqrt{3}y = x + 3$.

Sol 3: (A) Here, the focal chord of $y^2 = 16x$ is tangent to circle $(x-6)^2 + y^2 = 2$

$\Rightarrow$ Focus of parabola as (a,0) ie. (4,0)

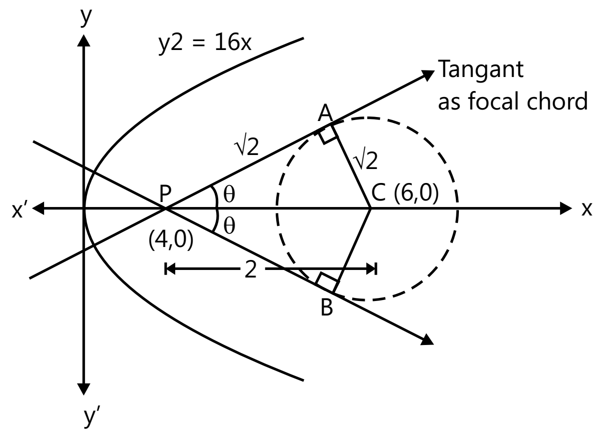
Now, tangents are drawn from (4,0) to $(x-6)^2 + y^2 = 2$

Since, PA is tangent to circle.

$\therefore$ $\tan\theta$ = slope of tangent

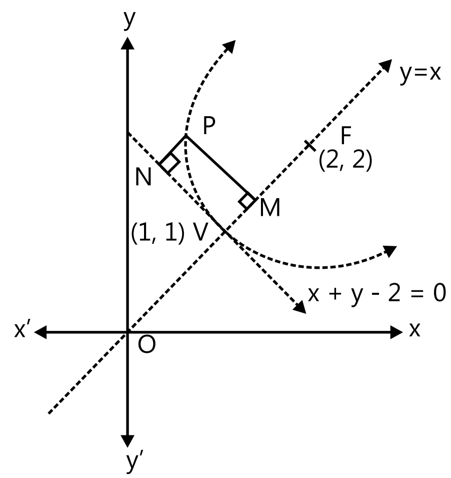
$\tan\theta = \frac{AC}{AP} = \frac{\sqrt{2}}{\sqrt{2}} = 1$,

Slope of other tangent $= -\tan\theta = -1$



$\therefore$ Slope of focal chords as tangent to circle= ± 1

Sol 4: (A)



Since, distance of vertex from origin is $\sqrt{2}$ and focus is $2\sqrt{2}$

$\therefore$ V(1,1) and F(2,2) (ie, lying on y = x)

Where, length of latusrectum

$= 4a = 4\sqrt{2}$ $\qquad (\because a = \sqrt{2})$

$\therefore$ By definition of parabola

$PM^2 = (4a)(PN)$

Where, PN is length of perpendicular upon x+y-2=0 (ie, tangent at vertex).

$\Rightarrow \frac{(x-y)^2}{2} = 4\sqrt{2}\left(\frac{x+y-2}{\sqrt{2}}\right)$

$\Rightarrow (x-y)^2 = 8(x+y-2)$

Sol 5: Since, equation of normal to the parabola $y^2 = 4ax$ is $y + xt = 2at + at^3$ passes through (3, 0)

$\Rightarrow 3t = 2t + t^3$ $(\because a = 1)$

$\Rightarrow t = 0, 1, -1$

$\therefore$Coordinates of the normals are

P(1, 2), Q(0, 0), R(1, –2). Thus,

(A) Area of $\Delta PQR = \frac{1}{2}\times 1\times 4=2$

(C) Centroid of $\Delta PQR = \left(\frac{2}{3},0\right)$

Equation of circle passing through P, Q, R is

$(x-1)(x-1) + (y-2)(y+2) + \lambda(x-1) = 0$

$\Rightarrow 1-4-\lambda = 0 \quad \Rightarrow \lambda = -3$

$\therefore$ Required equation of circle is $x^2 + y^2 - 5x = 0$

$\therefore$ Centre $\left(\frac{5}{2},0\right)$ and radius $\frac{5}{2}$

Sol 6: (A, B) The equation of tangent to $y = x^2$, be

$y = mx - \frac{m^2}{4}$.

Putting in $y = -x^2 + 4x - 4$, we should only get one value of x i.e.,

Discriminant must be zero.

$\therefore mx - \frac{m^2}{4} = -x^2 + 4x - 4$

$\Rightarrow x^2 + x(m-4) + 4 - \frac{m^2}{4} = 0$

D = 0

Now,$(m-4)^2 - (16 - m^2) = 0$

$\Rightarrow 2m(m-4) = 0$

$\Rightarrow m = 0, 4$

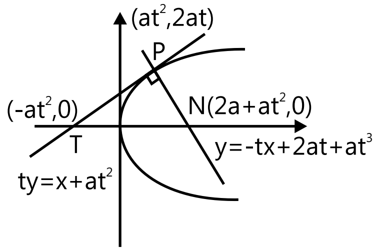
$\therefore y = 0$ and $y = 4(x-1)$ are the required tangents.

Hence, (A) and (B) are correct answers.

Sol 7: (A, D) Equation of tangent and normal at point $P(at^2, 2at)$ is $ty = x = at^2$ and $y = -tx + 2at + at^3$

Let centroid of ΔPTN is R(h, k)

$\therefore h = \frac{at^2 + (-at^2) + 2a + at^2}{3} = \frac{2a + at^2}{3}$



$k = \frac{2at}{3}$

$\Rightarrow 3h = 2a + a.\left(\frac{3k}{2a}\right)^2 \Rightarrow 3h = 2a + \frac{9k^2}{4a}$

$\Rightarrow 9k^2 = 4a(3h - 2a)$

$\therefore$Locus of centroid is $y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

$\therefore$Vertex $\left(\frac{2a}{3},0\right)$;

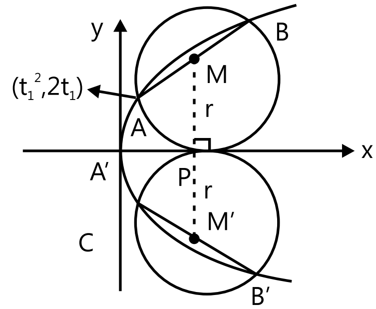
Directrix $x - \frac{2a}{3} = -\frac{a}{3} \Rightarrow x = \frac{a}{3}$

Latus rectum $= \frac{4a}{3}$

$\therefore$ Focus $\left(\frac{a}{3}+\frac{2a}{3},0\right)$ i.e., (a, 0)

Sol 8: (C, D) Here, coordinates of

$M = \left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$ i.e., mid point of chord AB.



$MP = t_1 + t_2 = r$ ...(i)

$m_{AB} = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \frac{2}{t_2 + t_1}$

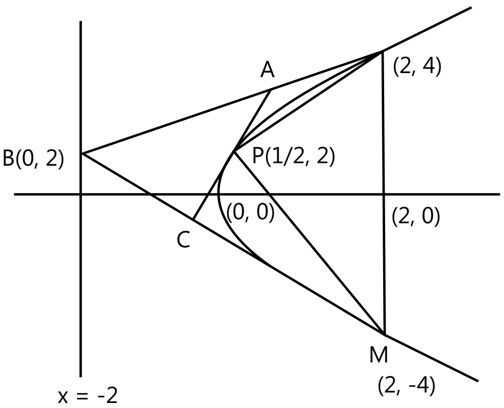
(When AB is chord)

$\Rightarrow m_{AB} = \frac{2}{r}$ [from Eq.(i)]

Also, $m_{A'B'} = -\frac{2}{r}$

(When A′B′ is chord)

Sol 9: (2) y= 8x = 4.2.x



$$\frac{\Delta LPM}{\Delta ABC} = 2$$

$$\frac{\Delta_1}{\Delta_2} = 2$$

Sol 10: If three different normals are drawn from (h, 0) to $y^2 = 4x$. Then, equation of normals are

$y = mx - 2m - m^3$ which passes through (h, 0)

$\Rightarrow mh - 2m - m^3 = 0$

$\Rightarrow h = 2 + m^2$

$2 + m^2 \geq 2$

$\therefore$ h > 2 (Neglect equality as if $2 + m^2 = 2 \Rightarrow m = 0$)

Therefore, three normals are coincident.

$\therefore$ h > 2

Sol 11: We know that, normal for $y^2 = 4ax$ is given by, $y = mx - 2am - am^3$.

$\therefore$Equation of normal for $y^2 = x$ is

$$y = mx - \frac{m}{2} - \frac{m^3}{4} \quad \left(\because a = \frac{1}{4}\right)$$

Since, normal passes through (c, 0)

$$\therefore mc - \frac{m}{2} - \frac{m^3}{4} = 0$$

$$\Rightarrow m\left(c - \frac{1}{2} - \frac{m^2}{4}\right) = 0$$

$$\Rightarrow m = 0 \text{ or } m^2 = 4\left(c - \frac{1}{2}\right)$$

$\Rightarrow$ m = 0, the equation of normal is y = 0

Also, $m^2 \geq 0$

$$\Rightarrow c - \frac{1}{2} \geq 0 \Rightarrow c \geq \frac{1}{2}$$

$$\text{At}, c = \frac{1}{2} \Rightarrow m = 0$$

Now, for other normals to be perpendicular to each other, we must have $m_1 . m_2 = -1$

or $\frac{m^2}{4} + \left(\frac{1}{2} - c\right) = 0$, has $m_1m_2 = -1$

$$\Rightarrow \frac{\left(\frac{1}{2} - c\right)}{1/4} = -1$$

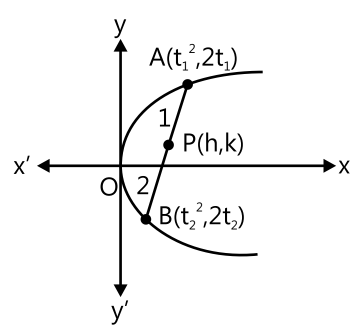
$$\Rightarrow \frac{1}{2} - c = -\frac{1}{4} \Rightarrow c = \frac{3}{4}$$

Sol 12: Let $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$ be coordinates of the end points of a chord of the parabola $y^2 = 4x$ having slope 2.

Now, slope of AB is

$$m = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)}$$

$$= \frac{2}{t_2 + t_1}$$



But m = 2(given)

$$\Rightarrow 2 = \frac{2}{t_2 + t_1}$$

$\Rightarrow t_1 + t_2 = 1$ ...(i)

Let P(h, k) be a point on AB such that, it divides AB internally in the ratio 1 : 2.

Then, $h = \frac{2t_1^2 + t_2^2}{2+1}$ and $k = \frac{2(2t_1) + 2t_2}{2+1}$

$\Rightarrow 3h = 2t_1^2 + t_2^2$ ...(ii)

and $3k = 4t_1 + 2t_2$ ...(iii)

On substituting value of t_1 from Eq. (i) in Eq. (iii)

$3k = 4(1 - t_2) + 2t_2$

$\Rightarrow 3k = 4 - 2t_2$

$\Rightarrow t_2 = 2 - \frac{3k}{2}$ ...(iv)

On substituting $t_1 = 1 - t_2$ in Eq.(ii), we get

$3h = 2(1-t_2)^2 + t_2^2$

$= 2(1 - 2t_2 + t_2^2) + t_2^2 = 3t_2^2 - 4t_2 + 2$

$$= 3\left(t_2^2 - \frac{4}{3}t_2 + \frac{2}{3}\right) = 3\left[\left(t_2 - \frac{2}{3}\right)^2 + \frac{2}{3} - \frac{4}{9}\right] = 3\left(t_2 - \frac{2}{3}\right)^2 + \frac{2}{3}$$

$$\Rightarrow 3h - \frac{2}{3} = 3\left(t_2 - \frac{2}{3}\right)^2 \quad \Rightarrow 3\left(h - \frac{2}{9}\right) = 3\left(2 - \frac{3k}{2} - \frac{2}{3}\right)^2$$

[From Eq. (iv)]

$$\Rightarrow 3\left(h - \frac{2}{9}\right) = 3\left(\frac{4}{3} - \frac{3k}{2}\right)^2 \Rightarrow \left(h - \frac{2}{9}\right) = \frac{9}{4}\left(k - \frac{8}{9}\right)^2$$

$$\Rightarrow \left(k - \frac{8}{9}\right)^2 = \frac{4}{9}\left(h - \frac{2}{9}\right)$$

On generating, we get the required locus

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$$

This represents a parabola with vertex at $\left(\frac{2}{9}, \frac{8}{9}\right)$

Sol 13: Let the three points on the parabola be $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ and $C(at_3^2, 2at_3)$.

Equation of the tangent to the parabola at $(at^2, 2at)$ is

$ty = x + at^2$

Therefore, equations of tangents at A and B are

$t_1y = x + at_1^2$ ...(i)

And $t_2y = x + at_2^2$ ...(ii)

From Eqs. (i) and (ii)

$t_1y = t_2y - at_2^2 + at_1^2$

$\Rightarrow t_1y - t_2y = at_1^2 - at_2^2$

$\Rightarrow y = a(t_1 + t_2) (\because t_1 \neq t_2)$

And $t_1a(t_1 + t_2) = x + at_1^2$

[from Eq.(i)] $\Rightarrow x = at_1t_2$

Therefore, coordinates of P are $(at_1t_2, a(t_1 + t_2))$

Similarly, the coordinates of Q and R are respectively

$[at_2t_3, a(t_2 + t_3)]$ and $[at_1t_3, a(t_1 + t_3)]$

Let Δ_1 = Area of the triangle ABC

$$= \frac{1}{2}\left\|\begin{matrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{matrix}\right\|$$

Applying $R_3 \to R_3 - R_2$ and

$R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \frac{1}{2}\left\|\begin{matrix} at^2 & 2at_1 & 1 \\ a(t_2^2 - t_1^2) & 2a(t_2 - t_1) & 0 \\ a(t_3^2 - t_2^2) & 2a(t_3 - t_2) & 0 \end{matrix}\right\|$$

$$= \frac{1}{2}\left\|\begin{matrix} a(t_2^2 - t_1^2) & 2a(t_2 - t_1) \\ a(t_3^2 - t_2^2) & 2a(t_3 - t_2) \end{matrix}\right\|$$

$$= \frac{1}{2}a.2a\left\|\begin{matrix} (t_2 - t_1)(t_2 + t_1) & (t_2 - t_1) \\ (t_3 - t_2)(t_3 + t_2) & (t_3 - t_2) \end{matrix}\right\|$$

$$= a^2(t_2 - t_1)(t_3 - t_2)\left\|\begin{matrix} t_2 + t_1 & 1 \\ t_3 + t_2 & 1 \end{matrix}\right\|$$

$= a^2|(t_2 - t_1)(t_3 - t_2)(t_1 - t_3)|$

Again, let Δ_2 = area of the triangle PQR

$$= \frac{1}{2}\left\|\begin{matrix} at_1t_2 & a(t_1 + t_2) & 1 \\ at_2t_3 & a(t_2 + t_3) & 1 \\ at_3t_1 & a(t_3 + t_1) & 1 \end{matrix}\right\| = \frac{1}{2}a.a\left\|\begin{matrix} t_1t_2 & (t_1 + t_2) & 1 \\ t_2t_3 & (t_2 + t_3) & 1 \\ t_3t_1 & (t_3 + t_1) & 1 \end{matrix}\right\|$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$= \frac{a^2}{2}\left\|\begin{matrix} t_1t_2 & t_1 + t_2 & 1 \\ t_2(t_3 - t_1) & t_3 - t_1 & 0 \\ t_3(t_1 - t_2) & t_1 - t_2 & 0 \end{matrix}\right\|$$

$$= \frac{a^2}{2}(t_3 - t_1)(t_1 - t_2) \times \left\|\begin{matrix} t_1t_2 & t_1 + t_2 & 1 \\ t_2 & 1 & 0 \\ t_3 & 1 & 0 \end{matrix}\right\|$$

$$= \frac{a^2}{2}(t_3 - t_1)(t_1 - t_2)\left\|\begin{matrix} t_2 & 1 \\ t_3 & 1 \end{matrix}\right\|$$

$$= \frac{a^2}{2}|(t_3 - t_1)(t_1 - t_2)(t_2 - t_3)|$$

Therefore,

$$\frac{\Delta_1}{\Delta_2} = \frac{a^2|(t_2 - t_1)(t_3 - t_2)(t_1 - t_3)|}{\frac{1}{2}a^2|(t_3 - t_1)(t_1 - t_2)(t_2 - t_3)|} = 2$$

Sol 14: Let coordinates of P be $(t, t^2 + 1)$

Reflection of P in y = x is $P_1(t^2+1, t)$

Which clearly lies on $y^2 = x - 1$

Similarly, let coordinates of Q be $(s^2 + 1, s)$

Its reflection in y = x is

$Q_1(s, s^2 + 1)$ which lies on $x^2 = y - 1$

We have,

$PQ_1^2 = (t - s)^2 + (t^2 - s^2)^2 = P_1Q^2$

$\Rightarrow PQ_1 = P_1Q$

Also, $PP_1 \parallel QQ_1$

($\because$ both perpendicular to y = x)

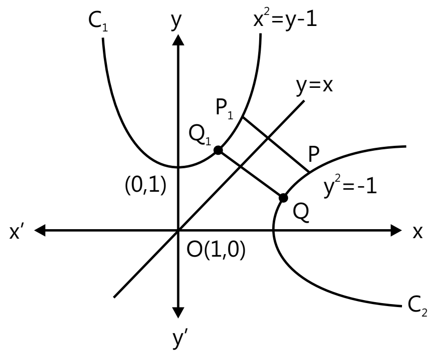
Thus, PP_1QQ_1 is an isosceles trapezium.

Also, P lies on PQ_1 and Q lies on P_1Q, we have $PQ \geq \min\{PP_1, QQ_1\}$

Let us take $\min\{PP_1, QQ_1\} = PP_1$

$\therefore PQ^2 \Rightarrow PP_1^2 = (t^2+1-t)^2 + (t^2+1-t)^2$

$= 2(t^2+1-t)^2 = f(t)$ (say)



We have,

$f'(t) = 4(t^2+1-t)(2t-1) = 4[(t-1/2)^2+3/4][2t-1]$

Now, $f'(t) = 0 \Rightarrow t = 1/2$

Also, $f'(t) < 0$ for $t < 1/2$ and $f'(t) > 0$ for $t > 1/2$

Corresponding to t = 1/2, point P_0 on C_1 is (1/2, 5/4) and P_1 (which we take as Q_0) on C_2 are (5/4, 1/2). Note that $P_0Q_0 \leq PQ$ for all pairs of (P, Q) with P on C_1 and Q on C_2.

Sol 15: We know, equation of normal to $y^2=4ax$ is

$y = mx - 2am - am^3$

Thus, equation of normal to $y^2 = 4x$ is,

$y = mx - 2m - m^3$, let it passes through (h, k).

$\Rightarrow k = mh - 2m - m^3$

Or $m^3 + m(2 - h) + k = 0$ ...(i)

Here, $m_1 + m_2 + m_3 = 0$,

$m_1m_2+m_2m_3+m_3m_1 = 2-h$

$m_1m_2m_3 = -k$, where $m_1m_2 = \alpha$

$\Rightarrow m_3 = -\frac{k}{\alpha}$ it must satisfy Eq.(i)

$\Rightarrow -\frac{k^3}{\alpha^3} - \frac{k}{\alpha}(2 - h) + k = 0$

$\Rightarrow k^2 = \alpha^2 h - 2\alpha^2 + \alpha^3$

$\Rightarrow y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$

On comparing with $y^2 = 4x$

$\Rightarrow \alpha^2 = 4$ and $-2\alpha^2 + \alpha^3 = 0 \quad \Rightarrow \alpha = 2$

Sol 16: (A,B,D) Equation of normal to parabola $y^2 = 4x$ is given by

$y = mx - 2m - m^3$

It passes through Point (9, 6)

$6 = 9m - 2m - m^3$

$\Rightarrow m^3 - 7m + 6 = 0$

$\Rightarrow m^3 - 1 - 7m + 7 = 0$

$\Rightarrow (m-1)(m^2+1+m) - 7(m-1) = 0$

$\Rightarrow (m-1)(m^2+m-6) = 0$

$\Rightarrow (m-1)(m+3)(m-2) = 0$

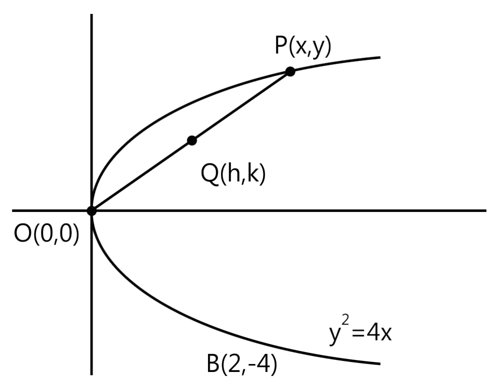
$\Rightarrow m = -3, 1, 2$

$\therefore$ The equations of normal are

$y - x + 3 = 0$, $y + 3x - 33 = 0$ and $y - 2x + 12 = 0$

Sol 17: (C) let Co-ordinates of point Q are (h, k)

According to the given condition



$h = \frac{0\times3+x\times1}{1+3} = \frac{x}{4} \Rightarrow x = 4h$

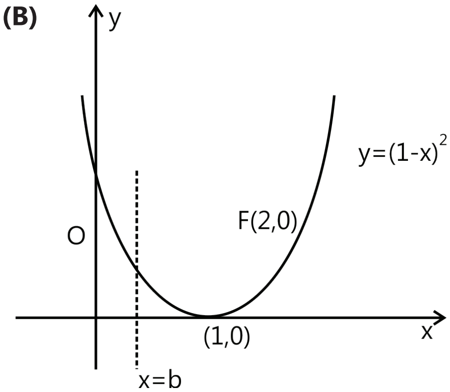
$k = \frac{0\times3+y\times1}{1+3} = \frac{y}{4} \Rightarrow y = 4k$

P (x, y) lies on the Parabola $y^2 = 4x$

$(4k)^2 = 4(4h) \Rightarrow k^2 = h$

$\Rightarrow$ Locus is $\boxed{y^2 = x}$

Sol 18: (B)



$$R_1 = \int_o^b (1-x)^2\,dx$$

$$R_2 = \int_b^1 (1-x)^2\,dx$$

Given, $R_1 - R_2 = \frac{1}{4}$

$$\int_o^b (1-x)^2\,dx - \int_b^1 (1-x)^2\,dx = \frac{1}{4}$$

$$\Rightarrow \left[-\frac{(1-x)^3}{3}\right]_o^b + \left[\frac{(1-x)^3}{3}\right]_b^1 = \frac{1}{4}$$

$$\Rightarrow \frac{-1}{3}\left[(1-b)^3 - 1\right] + \left[0 - \frac{(1-b)^3}{3}\right] = \frac{1}{4}$$

$$\Rightarrow \frac{-2}{3}(1-b)^3 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

$$\Rightarrow 2(1-b)^3 = \frac{1}{4} \qquad \Rightarrow (1-b)^3 = \frac{1}{3}$$

$$\Rightarrow (1-b) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$$

Sol 19: (2) The end points of latus rectum are A (2, 4), B (2, -4)

Area of Δ formed by A (2, 4), B (2, -4) and $P\left(\frac{1}{2}, 2\right)$

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} 2 & 4 & 1 \\ 2 & -4 & 1 \\ \frac{1}{2} & 2 & 1 \end{vmatrix}$$

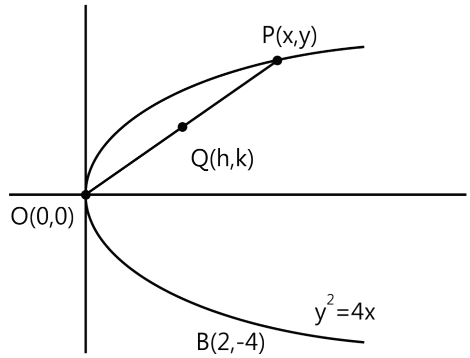
$$= \frac{1}{2}\left[2(-4-2) - 4\left(2 - \frac{1}{2}\right) + 1\left(4 + 4 \times \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[12 - 6 + 6\right] = 6 \text{ sq. units}$$

Tangents at end points of latus rectum are y = x + 2 and –y = x + 2, intersection point (-2, 0)

Equation of tangent at $P\left(\frac{1}{2}, 2\right)$ is given by y = 2x + 1. Points of intersection of tangent at $P\left(\frac{1}{2}, 2\right)$ and tangents at latus rectum are (-1, -1) and (1, 3)

Area of Δ formed by Points (-2, 0), (-1, -1) and (1, 3)



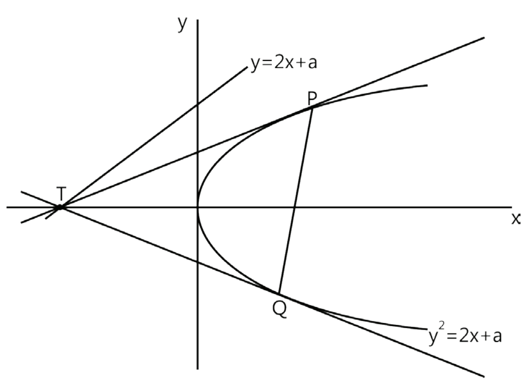
$$\Delta_2 = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(-1-3) + 0 + 1(-3+1)\right] = \frac{1}{2}\left[8-2\right] = 3$$

$$\Rightarrow \frac{\Delta_1}{\Delta_2} = \frac{6}{3} = 2$$

Sol 20: (B) Let Point P be $(at^2, 2at)$ then other end of focal Chord Q is $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

We know that point of intersection of tangent at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is given by $\left[at_1t_2, a(t_1+t_2)\right]$



$$\Rightarrow T\left[-a, a\left(t - \frac{1}{t}\right)\right]$$, which lies on y = 2x + a

$$\Rightarrow a\left(t-\frac{1}{t}\right)=-2a+a=-a$$

$$\Rightarrow t-\frac{1}{t}=-1 \quad = t^2+\frac{1}{t^2}=1+2$$

$$\Rightarrow \left(t+\frac{1}{t}\right)^2=5$$

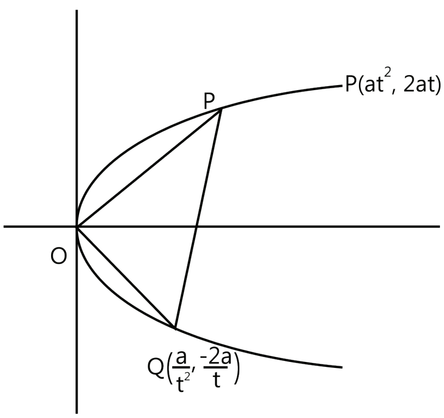
Length of Chord $=a\left(t+\frac{1}{t}\right)^2 = 5a$

Sol 21: (D) Slope $OP=\frac{2}{t}$

Slope $OQ=-2t$

$$\tan\theta=\frac{\frac{2}{t}-(-2t)}{1+\frac{2}{t}(-2t)} = \frac{\frac{2}{t}+2t}{1-4}=\frac{2\left(\frac{1}{t}+t\right)}{-3}$$

$$=\frac{2\times\sqrt{5}}{-3}=-\frac{2\sqrt{5}}{3}$$



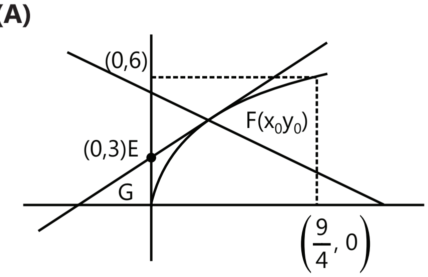
$$\left(t+\frac{1}{t}\right)^2=5$$

From previous question

Since, $t-\frac{1}{t}=-1$

$$\Rightarrow t+\frac{1}{t}=\sqrt{5}$$

Sol 22: (A)



Tangent at F $\quad yt = x + 4t^2$

$a : x = 0 \quad\quad y = 4t\ (0, 4t)$

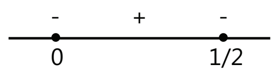
$(4t^2, 8t)$ satisfies the line $8t = 4mt^2 + 3$

$4mt^2 - 8t + 3 = 0$

$$\text{Area}=\frac{1}{2}\begin{vmatrix}0 & 3 & 1\\ 0 & 4t & 1\\ 4t^2 & 8t & 1\end{vmatrix}=\frac{1}{2}\left(4t^2(3-4t)\right)=2t^2(3-4t)$$

$$A=2\left[3t^2-4t^3\right]$$

$$\frac{dA}{dt}=2\left[6t-12t^2\right]=24t(1-2t)$$



$t = 1/2$ maxima

$G(0, 4t) \quad \Rightarrow \quad G(0, 2)$

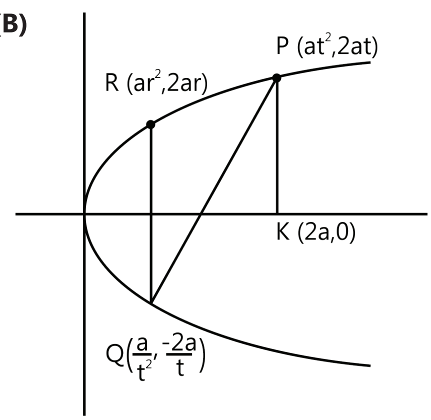
$y_1 = 2$

$(x_0, y_0) = (4t^2, 8t) = (t, 4)$

$y_0 = 4$

$$\text{Area}=2\left(\frac{3}{4}-\frac{1}{2}\right)=2\left(\frac{3-2}{4}\right)=\frac{1}{2}$$

Sol 23: (B)



$PK \parallel QR$

$$\frac{2at-0}{at^2-2a}=\frac{2ar+\frac{2a}{t}}{ar^2-\frac{a}{t^2}}$$

$$\Rightarrow \frac{2t}{t^2-2}=\frac{2\left(r+\frac{1}{t}\right)}{\left(r+\frac{1}{t}\right)\left(r-\frac{1}{t}\right)} \quad \Rightarrow \frac{t}{t^2-2}=\frac{1}{r-\frac{1}{t}}$$

$$\Rightarrow tr-1=t^2-2 \quad \Rightarrow tr=t^2-1 \Rightarrow r=\frac{t^2-1}{t}$$

Sol 24: (B)

Equation of tangent at O

$ty = x + at^2$ ... (i)

Equation of normal at S

$y = -sx + 2as + as^3$ ... (ii)

From (i) and (ii)

$$y = -s\left[ty - at^2\right] + 2as + as^2$$

$$\Rightarrow y = -sty + ast^2 + 2as + as^3$$

$$\Rightarrow y = -y + at + \frac{2a}{t} + \frac{a}{t^3} \quad [st = 1]$$

$$\Rightarrow 2y = \frac{at^4 + 2at^2 + a}{t^3} \qquad \Rightarrow y = \frac{a\left(1 + t^2\right)^2}{2t^3}$$

Sol 25: The given Parabola $y^2 = 4x$ has vertex (0, 0) and focus (1, 0)

Image of Vertex (0, 0) about the given line

x + y + 4 = 0 is given by

$$\frac{x-0}{1} = \frac{y-0}{1} = -2\frac{[0+0+4]}{1^2+1^2}$$

$$\Rightarrow \frac{x-0}{1} = \frac{y-0}{1} = -4$$

$$\Rightarrow V'(x, y) \equiv (-4, -4)$$

Image of focus (1, 0) about the line

x + y + 4 = 0

$$\frac{x-1}{1} = \frac{y-0}{1} = -2\frac{[1+0+4]}{1^2+1^2} = -5$$

$$\Rightarrow F'(x, y) \equiv (-4, -5)$$

The line y = 5 Passes through the focus of the parabola C, so y = -5 is latus rectum of Parabola (C), and. A and B are end points of latus rectum.

The length of latus of C is same as of $y^2 = 4x$

$$\Rightarrow AB = 4a = 4 \times 1 = 4$$