Chapter

5.2

Ellipse

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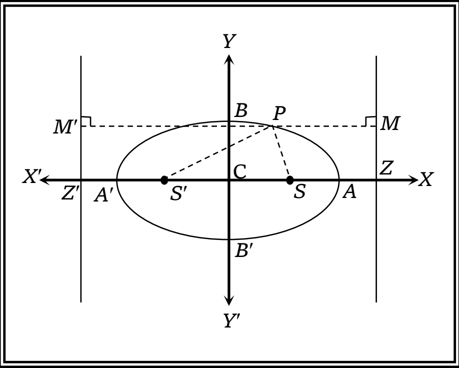


The Greeks particularly Archimedes (287-212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles.

Kepler was first to declare that the planets of our solar system travel around the sun in elliptic path. The ellipse is also used in many art forms and in the construction of bridges. Due to our knowledge of the ellipse, it is now possible to predict accurately the time and place of solar and lunar eclipses.

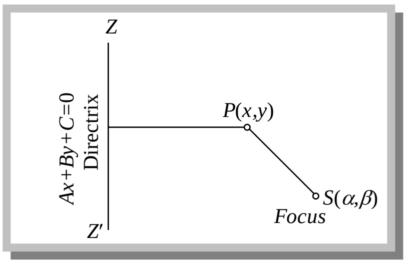
5.2 Ellipse

5.2.1 Definition

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point is in constant ratio (<1) to its distance from a fixed line. The fixed point is called the **focus** and fixed line is called the **directrix** and the constant ratio is called the **eccentricity** of the ellipse, denoted by (e).

In other words, we can say an ellipse is the locus of a point which moves in a plane so that the sum of its distances from two fixed points is constant and is more than the distance between the two fixed points.

Let $S(\alpha, \beta)$ is the focus, ZZ' is the directrix and P is any point on the ellipse. Then by definition,



$$\frac{SP}{PM} = e \Rightarrow SP = e.PM$$

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e\,\frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

Squaring both sides, $(A^2 + B^2)[(x-\alpha)^2 + (y-\beta)^2] = e^2(Ax + By + C)^2$

Note : ❑ The condition for second degree equation in x and y to represent an ellipse is that $h^2 - ab < 0$ and

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$$

Example: 1 The equation of an ellipse whose focus is (–1, 1), whose directrix is $x - y + 3 = 0$ and whose eccentricity is $\frac{1}{2}$, is given by **[MP PET 1993]**

(a) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

(b) $7x^2 - 2xy + 7y^2 - 10x + 10y + 7 = 0$

(c) $7x^2 - 2xy + 7y^2 - 10x - 10y - 7 = 0$

(d) $7x^2 - 2xy + 7y^2 + 10x + 10y - 7 = 0$

Solution: (a) Let any point on it be (x, y) then by definition,

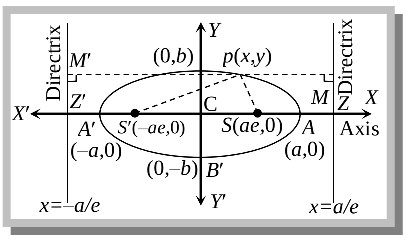
$$\sqrt{(x+1)^2 + (y-1)^2} = \frac{1}{2}\left|\frac{x-y+3}{\sqrt{1^2+1^2}}\right|$$

Squaring and simplifying, we get

$7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$, which is the required ellipse.

5.2.2 Standard equation of the Ellipse

Let S be the focus, ZM be the directrix of the ellipse and $P(x, y)$ is any point on the ellipse, then by definition



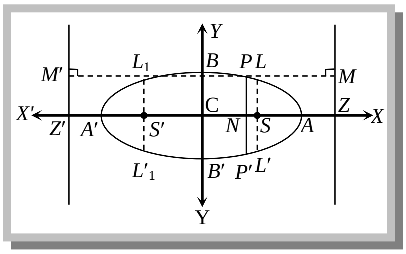
$$\frac{SP}{PM} = e \Rightarrow (SP)^2 = e^2(PM)^2$$

$$(x - ae)^2 + (y-0)^2 = e^2\left(\frac{a}{e} - x\right)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1-e^2)$

Since $e<1$, therefore $a^2(1-e^2)<a^2 \Rightarrow b^2<a^2$. Some terms related to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$:

(1) **Centre:** The point which bisects each chord of the ellipse passing through it, is called centre $(0,0)$ denoted by C.



(2) **Major and minor axes:** The diameter through the foci, is called the major axis and the diameter bisecting it at right angles is called the minor axis. The major and minor axes are together called principal axes.

Length of the major axis $AA'=2a$, Length of the minor axis $BB'=2b$

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is symmetrical about both the axes.

(3) **Vertices:** The extremities of the major axis of an ellipse are called vertices.
The coordinates of vertices A and A' are $(a, 0)$ and $(-a, 0)$ respectively.

(4) **Foci:** S and S' are two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively. Distance between foci $SS'=2ae$.

(5) **Directrices:** ZM and $Z'M'$ are two directrices of the ellipse and their equations are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively. Distance between directrices $ZZ'=\frac{2a}{e}$.

(6) **Eccentricity of the ellipse:** For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

we have $b^2=a^2(1-e)^2 \Rightarrow e^2=1-\frac{b^2}{a^2}=1-\frac{4b^2}{4a^2}=1-\left(\frac{2b}{2a}\right)^2$; $e=\sqrt{1-\left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}$

This formula gives the eccentricity of the ellipse.

(7) **Ordinate and double ordinate:** Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that PN produced meets the ellipse at P'. Then PN is called the ordinate of P and PNP' the double ordinate of P.

If abscissa of P is h, then ordinate of P, $\frac{y^2}{b^2}=1-\frac{h^2}{a^2} \Rightarrow y=\frac{b}{a}\sqrt{(a^2-h^2)}$ (For first quadrant)

And ordinate of P' is $y=\frac{-b}{a}\sqrt{(a^2-h^2)}$ (For fourth quadrant)

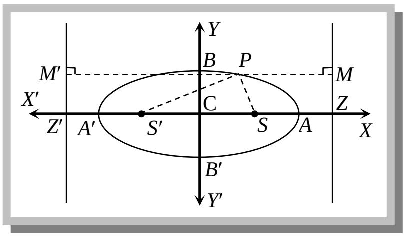
Hence coordinates of P and P' are $\left(h, \frac{b}{a}\sqrt{(a^2-h^2)}\right)$ and $\left(h, \frac{-b}{a}\sqrt{(a^2-h^2)}\right)$ respectively.

(8) **Latus-rectum:** Chord through the focus and perpendicular to the major axis is called its latus rectum.
The double ordinates LL' and L_1L_1' are latus rectum of the ellipse.

Length of latus rectum $LL'=L_1L_1'=\frac{2b^2}{a}$ and end points of latus-rectum are $L=\left(ae, \frac{b^2}{a}\right), L'=\left(ae, \frac{-b^2}{a}\right)$ and $L_1=\left(-ae, \frac{b^2}{a}\right)$; $L_1'=\left(-ae, \frac{-b^2}{a}\right)$

(9) **Focal chord:** A chord of the ellipse passing through its focus is called a focal chord.

(10) **Focal distances of a point:** The distance of a point from the focus is its focal distance. The sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.



Let $P(x_1, y_1)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$SP = ePM = e\left(\frac{a}{e}-x_1\right) = a - ex_1 \text{ and } S'P = ePM' = e\left(\frac{a}{e}+x_1\right) = a + ex_1$$

$\therefore \quad SP + S'P = (a-ex_1)+(a+ex_1) = 2a = AA' =$ major axis.

Example: 2 The length of the latus-rectum of the ellipse $5x^2+9y^2=45$ is **[MNR 1978, 80, 81; Kurukshetra CEE 1999]**

(a) $\sqrt{5}/4$ (b) $\sqrt{5}/2$ (c) $5/3$ (d) $10/3$

Solution: (d) Here the ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

Here $a^2=9$ and $b^2=5$. So, latus-rectum $=\frac{2b^2}{a}=\frac{2(5)}{3}=\frac{10}{3}$.

Example: 3 In an ellipse the distance between its foci is 6 and its minor axis is 8. Then its eccentricity is **[EAMCET 1994]**

(a) $\frac{4}{5}$ (b) $\frac{1}{\sqrt{52}}$ (c) $\frac{3}{5}$ (d) $\frac{1}{2}$

Solution: (c) Distance between foci $=6 \Rightarrow 2ae=6 \Rightarrow ae=3$, Minor axis $=8 \Rightarrow 2b=8 \Rightarrow b=4 \Rightarrow b^2=16$

From $b^2=a^2(1-e^2), \Rightarrow 16=a^2-a^2e^2 \Rightarrow a^2-9=16 \Rightarrow a=5$

Hence, $ae=3 \Rightarrow e=\frac{3}{5}$

Example: 4 What is the equation of the ellipse with foci $(\pm 2, 0)$ and eccentricity $\frac{1}{2}$ **[DCE 1999]**

(a) $3x^2+4y^2=48$ (b) $4x^2+3y^2=48$ (c) $3x^2+4y^2=0$ (d) $4x^2+3y^2=0$

Solution: (a) Here $ae=\pm 2, \because e=\frac{1}{2}, \quad \therefore a=\pm 4$

Form $b^2=a^2(1-e^2) \Rightarrow b^2=16\left(1-\frac{1}{4}\right) \Rightarrow b^2=12$

Hence, the equation of ellipse is $\frac{x^2}{16}+\frac{y^2}{12}=1$ or $3x^2+4y^2=48$

Example: 5 If $P(x,y), F_1=(3,0)$, $F_2=(-3,0)$ and $16x^2+25y^2=400$, then PF_1+PF_2 equals **[IIT 1998]**

(a) 8 (b) 6 (c) 10 (d) 12

Solution: (c) We have $16x^2+25y^2=400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=25$ and $b^2=16$

This equation represents an ellipse with eccentricity given by $e^2=1-\frac{b^2}{a^2}=1-\frac{16}{25}=\frac{9}{25} \Rightarrow e=3/5$

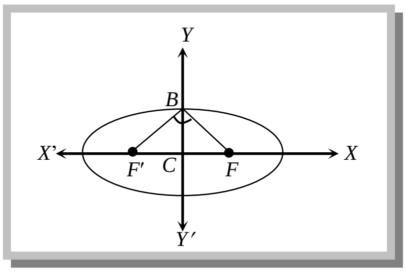
So, the coordinates of the foci are $(\pm ae, 0)$ i.e. $(3,0)$ and $(-3,0)$, Thus, F_1 and F_2 are the foci of the ellipse.

Since, the sum of the focal distance of a point on an ellipse is equal to its major axis, $\therefore PF_1+PF_2=2a=10$

Example: 6 An ellipse has OB as semi minor axis. F and F' are its foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is **[Pb. CET 2001, IIT 1997, DCE 2002]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Solution: (b) Since $\angle FBF' = \frac{\pi}{2}$



$\therefore \quad \angle FBC = \angle F'BC = \frac{\pi}{4}$

$\therefore \quad CB = CF \Rightarrow b = ae$

$\Rightarrow b^2 = a^2e^2 \Rightarrow a^2(1-e^2) = a^2e^2$

$\Rightarrow 1 - e^2 = e^2 \Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$.

Example: 7 Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then the maximum value of A is **[IIT 1994]**

(a) $2abe$ (b) abe (c) $\frac{1}{2}abe$ (d) None of these

Solution: (b) Let $P(a\cos\theta, b\sin\theta)$ and $F_1(-ae, 0)$, $F_2(ae, 0)$

$$A = \text{Area of } \Delta PF_1F_2 = \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix} = \frac{1}{2}|\ 2aeb\sin\theta| = aeb\,|\sin\theta|$$

$\therefore$ A is maximum, when $|\sin\theta| = 1$.

Hence, maximum value of $A = abe$

Example: 8 The eccentricity of an ellipse, with its centre at the origin is $\frac{1}{2}$. If one of the directrices is $x = 4$, then the equation of the ellipse is **[AIEEE 2004]**

(a) $4x^2 + 3y^2 = 1$ (b) $3x^2 + 4y^2 = 12$ (c) $4x^2 + 3y^2 = 12$ (d) $3x^2 + 4y^2 = 1$

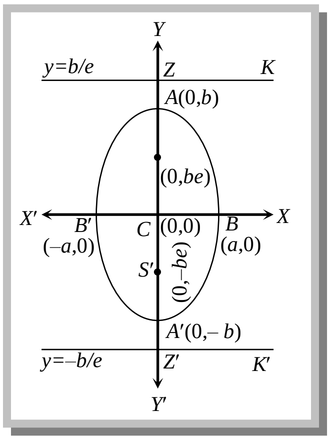
Solution: (b) Given $e = \frac{1}{2}, \frac{a}{e} = 4$. So, $a = 2 \Rightarrow a^2 = 4$

From $b^2 = a^2(1-e^2) \Rightarrow b^2 = 4\left(1 - \frac{1}{4}\right) = 4 \times \frac{3}{4} = 3$

Hence the equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$, *i.e.* $3x^2 + 4y^2 = 12$

5.2.3 Equation of Ellipse in other form

In the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$ or $a^2 > b^2$ (denominator of x^2 is greater than that of y^2), then the major and minor axis lie along x-axis and y-axis respectively. But if $a < b$ or $a^2 < b^2$ (denominator of x^2 is less than that of y^2), then the major axis of the ellipse lies along the y-axis and is of length $2b$ and the minor axis along the x-axis and is of length $2a$.



The coordinates of foci S and S' are (0, be) and (0, – be) respectively.

The equation of the directrices ZK and $Z'K'$ are $y = \pm b/e$ and eccentricity e is given by the formula $a^2 = b^2(1-e^2)$ or $e = \sqrt{1 - \frac{a^2}{b^2}}$

Difference between both ellipse will be clear from the following table.

Ellipse / **Basic fundamentals**	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For $a > b$	**For $b > a$**
Centre	$(0, 0)$	$(0, 0)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus-rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	$(a\cos\phi, b\sin\phi)$	$(a\cos\phi, b\sin\phi)$ $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = b - ey_1$ and $S'P = b + ey_1$
Sum of focal radii $SP + S'P =$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$2a/e$	$2b/e$
Tangents at the vertices	$x = -a$, $x = a$	$y = b$, $y = -b$

Example: 9 The equation of a directrix of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is

(a) $y = \frac{25}{3}$ (b) $x = 3$ (c) $x = -3$ (d) $x = \frac{3}{25}$

Solution: (a) From the given equation of ellipse $a^2 = 16, b^2 = 25$ (since $b > a$)

So, $a^2 = b^2(1-e^2)$, $\therefore$ $16 = 25(1-e^2) \Rightarrow 1-e^2 = \frac{16}{25} \Rightarrow e^2 = \frac{9}{25} \Rightarrow e = \frac{3}{5}$

$\therefore$ One directrix is $y = \frac{b}{e} = \frac{5}{3/5} = \frac{25}{3}$

Example: 10 The distances from the foci of $P(x_1, y_1)$ on the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$ are

(a) $4 \pm \frac{5}{4}y_1$ (b) $5 \pm \frac{4}{5}x_1$ (c) $5 \pm \frac{4}{5}y_1$ (d) None of these

Solution: (c) For the given ellipse $b > a$, so the two foci lie on y-axis and their coordinates are $(0, \pm be)$,

Where $b = 5, a = 3$. So $e = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

The focal distances of a point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Where $b^2 > a^2$ are given by $b \pm ey_1$. So, Required distances are $b \pm ey_1 = 5 \pm \frac{4}{5}y_1$.

5.2.4 Parametric form of the Ellipse

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then the equation of ellipse in the parametric form will be given by $x=a\cos\phi, y=b\sin\phi$, where ϕ is the eccentric angle whose value vary from $0\le\phi<2\pi$. Therefore coordinate of any point P on the ellipse will be given by $(a\cos\phi, b\sin\phi)$

Example: 11 The curve represented by $x=3(\cos t+\sin t)$, $y=4(\cos t-\sin t)$ is **[EAMCET 1988; DCE 2000]**

(a) Ellipse (b) Parabola (c) Hyperbola (d) Circle

Solution: (a) Given, $x=3(\cos t+\sin t), y=4(\cos t-\sin t) \Rightarrow \frac{x}{3}=(\cos t+\sin t), \frac{y}{4}=(\cos t-\sin t)$

Squaring and adding, we get $\frac{x^2}{9}+\frac{y^2}{16}=(1+\sin 2t)+(1-\sin 2t) \Rightarrow \frac{x^2}{9}+\frac{y^2}{16}=2$, which represents ellipse.

Example: 12 The distance of the point 'θ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from a focus is

(a) $a(e+\cos\theta)$ (b) $a(e-\cos\theta)$ (c) $a(1+e\cos\theta)$ (d) $a(1+2e\cos\theta)$

Solution: (c) Focal distance of any point $P(x,y)$ on the ellipse is equal to $SP=a+ex$. Here $x=\cos\theta$.

Hence, $SP=a+ae\cos\theta=a(1+e\cos\theta)$

5.2.5 Special forms of an Ellipse

(1) If the centre of the ellipse is at point (h,k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

If we shift the origin at (h, k) without rotating the coordinate axes, then $x=X+h$ and $y=Y+k$

(2) If the equation of the curve is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$ where $lx+my+n=0$ and $mx-ly+P=0$ are perpendicular lines, then we substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X, \frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

Example: 13 The foci of the ellipse $25(x+1)^2+9(y+2)^2=225$ are **[MP PET 1998, UPSEAT 1991, 2000]**

(a) $(-1,2),(6,1)$ (b) $(-1,-2),(1,6)$ (c) $(1,-2),(1,-6)$ (d) $(-1,2),(-1,-6)$

Solution: (d) Given ellipse is $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{25}=1$ *i.e.* $\frac{X^2}{9}+\frac{Y^2}{25}=1$, where $X=x+1$ and $Y=y+2$

Here $a^2=25, b^2=9$ [Type : $\frac{X^2}{b^2}+\frac{Y^2}{a^2}=1$]

Eccentricity is given by $e^2=\frac{a^2-b^2}{a^2}=\frac{25-9}{25}=\frac{16}{25}$, $\therefore\ e=\frac{4}{5}$

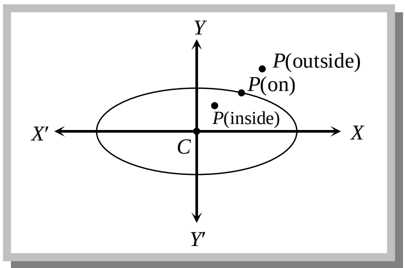
Foci are given by $Y=\pm ae=\pm 5\left(\frac{4}{5}\right)=\pm 4$

$X=0 \Rightarrow y+2=\pm 4 \Rightarrow y=-2\pm 4=-6$ or 2

$x+1=0 \Rightarrow x=-1$. Hence foci are (–1, –6) or (–1, 2).

5.2.6 Position of a point with respect to an Ellipse

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse. The point lies outside, on or inside the ellipse as if $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0, =0, <0$



Example: 14 Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then **[IIT 1994]**

(a) Q lies inside C but outside E (b) Q lies outside both C and E
(c) P lies inside both C and E (d) P lies inside C but outside E

Solution: (d) The given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$. The value of the expression $\frac{x^2}{9}+\frac{y^2}{4}-1$ is positive for $x=1, y=2$ and negative for $x=2, y=1$. Therefore P lies outside E and Q lies inside E. The value of the expression x^2+y^2-9 is negative for both the points P and Q. Therefore P and Q both lie inside C. Hence P lies inside C but outside E.

5.2.7 Intersection of a Line and an Ellipse

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ......(i) and the given line be $y=mx+c$ ......(ii)

Eliminating y from equation (i) and (ii), then $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

i.e., $(a^2m^2+b^2)x^2+2mca^2x+a^2(c^2-b^2)=0$

The above equation being a quadratic in x, its discriminant $=4m^2c^2a^4-4a^2(a^2m^2+b^2)(c^2-b^2)$

$$=b^2\{(a^2m^2+b^2)-c^2\}$$

Hence the line intersects the ellipse in two distinct points if $a^2m^2+b^2>c^2$ in one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$.

5.2.8 Equations of Tangent in Different forms

(1) **Point form:** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

(2) **Slope form:** If the line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2=a^2m^2+b^2$. Hence, the straight line $y=mx\pm\sqrt{a^2m^2+b^2}$ always represents the tangents to the ellipse.

Points of contact: Line $y=mx\pm\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$$

(3) **Parametric form:** The equation of tangent at any point $\phi(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$

Note : ❑ The straight line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2l^2 + b^2m^2 = n^2$.

❑ The line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ and that point of contact is $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$.

❑ Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

❑ The tangents at the extremities of latus-rectum of an ellipse intersect on the corresponding directrix.

Important Tips

☞ *A circle of radius r is concentric with the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *then the common tangent is inclined to the major axis at an angle* $\tan^{-1}\sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$.

☞ *The locus of the foot of the perpendicular drawn from centre upon any tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ *or* $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ *(in polar coordinates)*

☞ *The locus of the mid points of the portion of the tangents to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *intercepted between the axes is* $a^2y^2 + b^2x^2 = 4x^2y^2$.

☞ *The product of the perpendiculars from the foci to any tangent of an ellipse is equal to the square of the semi minor axis, and the feet of these perpendiculars lie on the auxiliary circle.*

Example: 15 The number of values of 'c' such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is **[IIT 1998]**

(a) 0 (b) 1 (c) 2 (d) Infinite

Solution: (c) We know that the line $y = mx + c$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *iff* $c^2 = a^2m^2 + b^2$

Here, $a^2 = 4, b^2 = 1, m = 4 \quad \therefore c^2 = 64 + 1 \Rightarrow c = \pm\sqrt{65}$

Example: 16 On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are **[IIT 1999]**

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(\frac{-2}{5}, \frac{1}{5}\right)$ (c) $\left(\frac{-2}{5}, \frac{-1}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{-1}{5}\right)$

Solution: (b,d) Ellipse is $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1 \Rightarrow a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$. The equation of its tangent is $4xx' + 9yy' = 1$

$\therefore \ m = -\frac{4x'}{9y'} = \frac{8}{9} \Rightarrow x' = -2y'$ and $4x'^2 + 9y'^2 = 1 \Rightarrow 4x'^2 + 9\frac{x'^2}{4} = 1 \Rightarrow x' = \pm\frac{2}{5}$

When $x' = \frac{2}{5}$, then $y' = \frac{-1}{5}$ and when $x' = \frac{-2}{5}$, then $y' = \frac{1}{5}$.

Hence points are $\left(\frac{2}{5}, \frac{-1}{5}\right), \left(\frac{-2}{5}, \frac{1}{5}\right)$

Example: 17 If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts equal lengths l on the axes, then l=

(a) a^2+b^2 (b) $\sqrt{a^2+b^2}$ (c) $(a^2+b^2)^2$ (d) None of these

Solution: (b) The equation of any tangent to the given ellipse is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

This line meets the coordinate axes at $P\left(\frac{a}{\cos\theta},0\right)$ and $Q\left(0,\frac{b}{\sin\theta}\right)$

$\therefore \frac{a}{\cos\theta}=l=\frac{b}{\sin\theta}\Rightarrow \cos\theta=\frac{a}{l}$ and $\sin\theta=\frac{b}{l}\Rightarrow \cos^2\theta+\sin^2\theta=\frac{a^2}{l^2}+\frac{b^2}{l^2}\Rightarrow l^2=a^2+b^2\Rightarrow l=\sqrt{a^2+b^2}$.

Example: 18 The area of the quadrilateral formed by the tangents at the end points of latus- rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is

IIT Screening 2003]

(a) 27/4 *sq.* units (b) 9 *sq.* units (c) 27/2 *sq.* units (d) 27*sq.* units

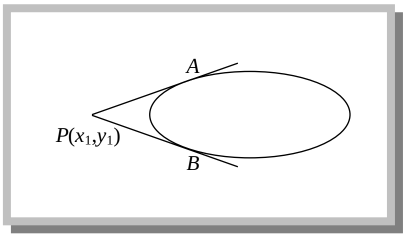
Solution: (d) By symmetry the quadrilateral is a rhombus. So area is four times the area of the right angled triangle formed by the tangents and axes in the 1st quadrant.

Now $ae=\sqrt{a^2-b^2}\Rightarrow ae=2\Rightarrow$ Tangent (in the first quadrant) at one end of latus rectum $\left(2,\frac{5}{3}\right)$ is $\frac{2}{9}x+\frac{5}{3}\cdot\frac{y}{5}=1$

i.e. $\frac{x}{9/2}+\frac{y}{3}=1$. Therefore area $=4.\frac{1}{2}.\frac{9}{2}.3=27\,sq.$ units.

5.2.9 Equation of Pair of Tangents $SS_1=T^2$

Pair of tangents: Let $P(x_1,y_1)$ be any point lying outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let a pair of tangents *PA*, *PB* can be drawn to it from *P*.



Then the equation of pair of tangents *PA* and *PB* is $SS_1=T^2$

where $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

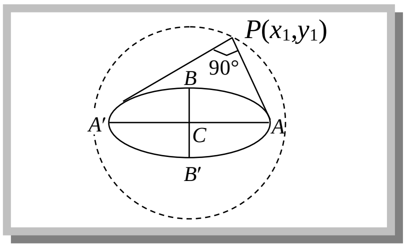
$$S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0$$

$$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

Director circle: The director circle is the locus of points from which perpendicular tangents are drawn to the ellipse.

Let $P(x_1,y_1)$ be any point on the locus. Equation of tangents through $P(x_1,y_1)$ is given by $SS_1=T^2$

i.e., $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left[\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right]^2$



They are perpendicular, So coeff. of x^2 + coeff. of $y^2=0$

$\therefore \left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)=0$ or $x_1^2+y_1^2=a^2+b^2$

Hence locus of $P(x_1,y_1)$ *i.e.*, equation of director circle is $x^2+y^2=a^2+b^2$

Example: 19 The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2+2y^2=5$ is **[UPSEAT 2001]**

(a) $\tan^{-1}(12/5)$ (b) $\tan^{-1}(6/\sqrt{5})$ (c) $\tan^{-1}(12/\sqrt{5})$ (d) $\tan^{-1}(6/5)$

Solution: (c) The combined equation of the pair of tangents drawn from (1,2) to the ellipse $3x^2+2y^2=5$ is $(3x^2+2y^2-5)(3+8-5)=(3x+4y-5)^2$ **[**using $SS_1=T^2$ **]**

$\Rightarrow\ 9x^2-24xy-4y^2+.......=0$

The angle between the lines given by this equation is $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$

Where $a=9$, $h=-12$, $b=-4$ $\Rightarrow$ $\tan\theta=12/\sqrt{5}\Rightarrow\theta=\tan^{-1}(12/\sqrt{5})$

Example: 20 The locus of the point of intersection of the perpendicular tangents to the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ is **[Karnataka CET 2003]**

(a) $x^2+y^2=9$ (b) $x^2+y^2=4$ (c) $x^2+y^2=13$ (d) $x^2+y^2=5$

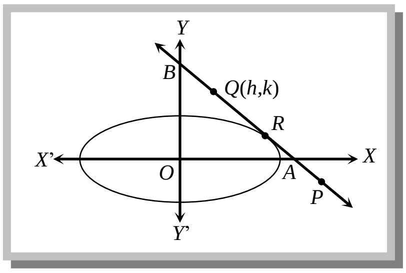
Solution: (c) The locus of point of intersection of two perpendicular tangents drawn on the ellipse is $x^2+y^2=a^2+b^2$, which is called "director circle".

Given ellipse is $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$. $\therefore$ Locus is $x^2+y^2=9+4$, *i.e.* $x^2+y^2=13$.

Example: 21 The locus of the middle point of the intercept of the tangents drawn from an external point to the ellipse $x^2+2y^2=2$ between the coordinate axes, is **[IIT Screening 2004]**

(a) $\dfrac{1}{x^2}+\dfrac{1}{2y^2}=1$ (b) $\dfrac{1}{4x^2}+\dfrac{1}{2y^2}=1$ (c) $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$ (d) $\dfrac{1}{2x^2}+\dfrac{1}{y^2}=1$

Solution: (c) Let the point of contact be $R\equiv(\sqrt{2}\cos\theta,\sin\theta)$



Equation of tangent AB is $\dfrac{x}{\sqrt{2}}\cos\theta+y\sin\theta=1$

$\Rightarrow$ $A\equiv(\sqrt{2}\sec\theta,0)$; $B\equiv(0,\text{coses}\,\theta)$

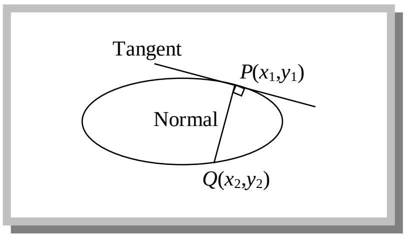
Let the middle point Q of AB be (h, k).

$\Rightarrow$ $h=\dfrac{\sec\theta}{\sqrt{2}},k=\dfrac{\text{cosec}\,\theta}{2}\Rightarrow\cos\theta=\dfrac{1}{h\sqrt{2}},\sin\theta=\dfrac{1}{2k}\Rightarrow\dfrac{1}{2h^2}+\dfrac{1}{4k^2}=1$

Thus required locus is $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$

5.2.10 Equations of Normal in Different forms

(1) **Point form:** The equation of the normal at (x_1,y_1) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=a^2-b^2$.



(2) **Parametric form:** The equation of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at $(a\cos\phi,b\sin\phi)$ is $ax\sec\phi-by\,\text{cosec}\phi=a^2-b^2$.

(3) **Slope form:** If m is the slope of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then the equation of normal is

$$y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$$

The coordinates of the point of contact are $\left(\frac{\pm a^2}{\sqrt{a^2 + b^2m^2}}, \frac{\pm mb^2}{\sqrt{a^2 + b^2m^2}}\right)$

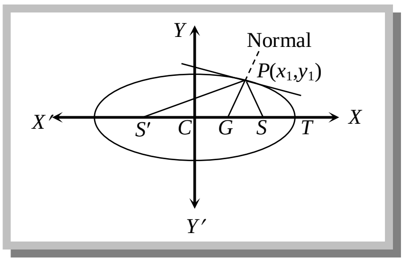
Note : ❑ If $y = mx + c$ is the normal of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then condition of normality is $c^2 = \frac{m^2(a^2 - b^2)^2}{(a^2 + b^2m^2)}$.

❑ The straight line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \left(\frac{a^2 - b^2}{n^2}\right)^2$.

❑ Four normals can be drawn from a point to an ellipse.

Important Tips

☞ *If S be the focus and G be the point where the normal at P meets the axis of an ellipse, then* $SG = e.SP$ *, and the tangent and normal at P bisect the external and internal angles between the focal distances of* P.



☞ *Any point P of an ellipse is joined to the extremities of the major axis then the portion of a directrix intercepted by them subtends a right angle at the corresponding focus.*

☞ *With a given point and line as focus and directrix, a series of ellipse can be described. The locus of the extermities of their minor axis is a parabola.*

☞ *The equations to the normals at the end of the latera recta and that each passes through an end of the minor axis, if* $e^4 + e^2 + 1 = 0$

☞ *If two concentric ellipse be such that the foci of one be on the other and if e and e' be their eccentricities. Then the angle between their axes is* $\cos^{-1}\sqrt{\frac{e^2 + e'^2 - 1}{ee'}}$.

Example: 22 The equation of normal at the point (0, 3) of the ellipse $9x^2 + 5y^2 = 45$ is **[MP PET 1998]**

(a) $y - 3 = 0$ (b) $y + 3 = 0$ (c) x-axis (d) y-axis

Solution: (d) For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, equation of normal at point (x_1, y_1), is $\frac{(x - x_1)a^2}{x_1} = \frac{(y - y_1)b^2}{y_1}$

Here, $(x_1, y_1) = (0, 3)$ and $a^2 = 5$, $b^2 = 9$, Therefore $\frac{(x - 0)}{0}.5 = \frac{(y - 3)}{3}.9$ or $x = 0$ *i.e.*, y-axis.

Example: 23 If the normal at any point P on the ellipse cuts the major and minor axes in G and g respectively and C be the centre of the ellipse, then **[Kurukshetra CEE 1998]**

(a) $a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2$ (b) $a^2(CG)^2 - b^2(Cg)^2 = (a^2 - b^2)^2$

(c) $a^2(CG)^2 - b^2(Cg)^2 = (a^2 + b^2)^2$ (d) None of these

Solution: (a) Let at point (x_1, y_1) normal will be $\frac{(x - x_1)}{x_1}a^2 = \frac{(y - y_1)b^2}{y_1}$

At G, $y = 0 \Rightarrow x = CG = \frac{x_1(a^2 - b^2)}{a^2}$ and at g, $x = 0 \Rightarrow y = Cg = \frac{y_1(b^2 - a^2)}{b^2}$

$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \Rightarrow a^2(CG)^2 + b(Cg)^2 = (a^2 - b^2)^2$.

Example: 24 The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the positive end of the latus-rectum is

(a) $x + ey + e^3 a = 0$ (b) $x - ey - e^3 a = 0$ (c) $x - ey - e^2 a = 0$ (d) None of these

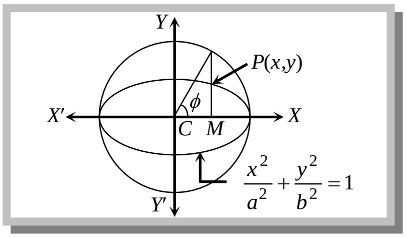
Solution: (b) The equation of the normal at (x_1, y_1) to the given ellipse is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$. Here, $x_1 = ae$ and $y_1 = \frac{b^2}{a}$

So, the equation of the normal at positive end of the latus- rectum is

$$\frac{a^2 x}{ae} - \frac{b^2 y}{b^2 / a} = a^2 e^2 \quad [\because\ b^2 = a^2(1 - e^2)] \Rightarrow \frac{ax}{e} - ay = a^2 e^2 \Rightarrow x - ey - e^3 a = 0$$

5.2.11 Auxiliary Circle

The circle described on the major axis of an ellipse as diameter is called an auxiliary circle of the ellipse.



If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse, then its auxiliary circle is $x^2 + y^2 = a^2$

Eccentric angle of a point: Let P be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to meet the auxiliary circle in Q. Join CQ. The angle $\angle XCQ = \phi$ is called the eccentric angle of the point P on the ellipse.

Note that the angle $\angle XCP$ is not the eccentric angle of point P.

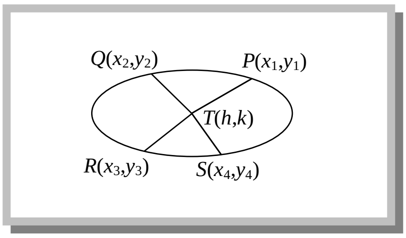
5.2.12 Properties of Eccentric angles of the Co-normal points

(1) The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to odd multiple of π.

(2) If α, β, γ are the eccentric angles of three points on the ellipse, the normals at which are concurrent, then $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$.

(3)**Co-normal points lie on a fixed curve:** Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ be co-normal points, then $PQRS$ lie on the curve $(a^2 - b^2)xy + b^2 kx - a^2 hy = 0$

This curve is called Apollonian rectangular hyperbola.



Note : ❑ The feet of the normals from any fixed point to the ellipse lie at the intersections of the apollonian rectangular hyperbola with the ellipse.

Important Tips

☞ *The area of the triangle formed by the three points, on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *whose eccentric angles are* θ, ϕ *and* ψ *is* $2ab\sin\left(\frac{\phi-\psi}{2}\right)\sin\left(\frac{\psi-\theta}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$.

☞ *The eccentricity of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is given by* $2\cot w=\frac{e^2\sin 2\theta}{\sqrt{(1-e^2)}}$, *where w is one of the angles between the normals at the points whose eccentric angles are* θ *and* $\frac{\pi}{2}+\theta$.

Example: 25 The eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$, whose distance from the centre of the ellipse is 2, is

[WB JEE 1990]

(a) $\pi/4$ (b) $3\pi/2$ (c) $5\pi/3$ (d) $7\pi/6$

Solution: (a) Let θ be the eccentric angle of the point P. Then the coordinates of P are $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

The centre of the ellipse is at the origin, It is given that $OP = 2$

$\Rightarrow \sqrt{6\cos^2\theta+2\sin^2\theta}=2 \Rightarrow 6\cos^2\theta+2\sin^2\theta=4 \Rightarrow 3\cos^2\theta+\sin^2\theta=2 \Rightarrow 2\sin^2\theta=1$

$\Rightarrow \sin^2\theta=\frac{1}{2} \Rightarrow \sin\theta=\pm\frac{1}{\sqrt{2}} \Rightarrow \theta=\pm\pi/4$

Example: 26 The area of the rectangle formed by the perpendiculars from the centre of the ellipse to the tangent and normal at the point-whose eccentric angle is $\pi/4$, is

(a) $\left(\frac{a^2-b^2}{a^2+b^2}\right)ab$ (b) $\left(\frac{a^2+b^2}{a^2-b^2}\right)ab$ (c) $\frac{1}{ab}\left(\frac{a^2-b^2}{a^2+b^2}\right)ab$ (d) $\frac{1}{ab}\left(\frac{a^2+b^2}{a^2-b^2}\right)ab$

Solution: (a) The given point is $(a\cos\pi/4, b\sin\pi/4)$ *i.e.* $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

So, the equation of the tangent at this point is $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ ......(i)

$\therefore\ p_1 =$ length of the perpendicular form (0, 0) on (i) $=\left|\frac{\frac{0}{a}+\frac{0}{b}-\sqrt{2}}{\sqrt{1/a^2+1/b^2}}\right|=\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$

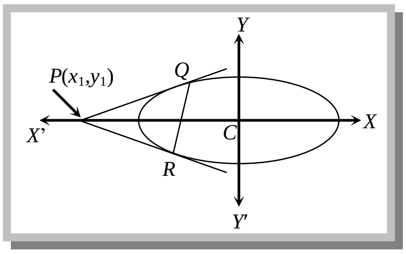
Equation of the normal at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ is $\frac{a^2x}{a/\sqrt{2}}-\frac{b^2y}{b/\sqrt{2}}=a^2-b^2 \Rightarrow \sqrt{2}ax-\sqrt{2}by=a^2-b^2$ .....(ii)

Therefore, $p_2 =$ length of the perpendicular form (0, 0) on (ii) $=\frac{a^2b^2}{\sqrt{(\sqrt{2}a)^2+(-\sqrt{2}b)^2}}=\frac{a^2-b^2}{\sqrt{2(a^2+b^2)}}$

So, area of the rectangle $=p_1p_2=\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}\times\frac{a^2-b^2}{\sqrt{2(a^2+b^2)}}=\left(\frac{a^2-b^2}{a^2+b^2}\right)ab$

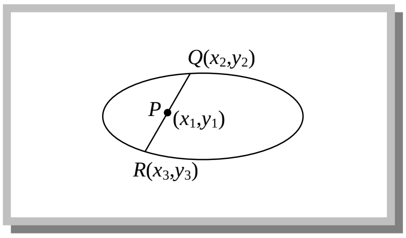
5.2.13 Chord of Contact

If PQ and PR be the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact QR is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T=0$ at (x_1, y_1)



5.2.14 Equation of Chord with Mid point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid point be (x_1, y_1) is $T = S_1$, where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0$



5.2.15 Equation of the Chord joining two points on an Ellipse

Let $P(a\cos\theta, b\sin\theta)$; $Q(a\cos\phi, b\sin\phi)$ be any two points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then ,the equation of the chord joining these two points is $y-b\sin\theta=\frac{b\sin\phi-b\sin\theta}{a\cos\phi-a\cos\theta}(x-a\cos\theta)$

Thus, the equation of the chord joining two points having eccentric angles θ and ϕ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$

Note : ❑ If the chord joining two points whose eccentric angles are α and β cut the major axis of an ellipse at a distance 'c' from the centre, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}$.

❑ If α and β be the eccentric angles of the extremities of a focal chord of an ellipse of eccentricity e, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\frac{1\mp e}{1\pm e}=0$.

Example: 27 What will be the equation of that chord of ellipse $\frac{x^2}{36}+\frac{y^2}{9}=1$ which passes from the point $(2,1)$ and bisected on the point

[UPSEAT 1999]

(a) $x+y=2$ (b) $x+y=3$ (c) $x+2y=1$ (d) $x+2y=4$

Solution: (d) Let required chord meets to ellipse on the points P and Q whose coordinates are (x_1, y_1) and (x_2, y_2) respectively

$\because$ Point (2,1) is mid point of chord PQ

$\therefore\ 2=\frac{1}{2}(x_1+x_2)$ or $x_1+x_2=4$ and $1=\frac{1}{2}(y_1+y_2)$ or $y_1+y_2=2$

Again points (x_1, y_1) and (x_2, y_2) are situated on ellipse; $\therefore\ \frac{x_1^2}{36}+\frac{y_1^2}{9}=1$ and $\frac{x_2^2}{36}+\frac{y_2^2}{9}=1$

On subtracting $\frac{x_2^2-x_1^2}{36}+\frac{y_2^2-y_1^2}{9}=0$ or $\frac{y_2-y_1}{x_2-x_1}=-\frac{(x_2+x_1)}{4(y_2+y_1)}=\frac{-4}{4\times 2}=\frac{-1}{2}$

$\therefore$ Gradient of chord $PQ=\frac{y_2-y_1}{x_2-x_1}=\frac{-1}{2}$

Therefore, required equation of chord PQ is as follows, $y-1=-\frac{1}{2}(x-2)$ or $x+2y=4$

Alternative: $S_1=T$ (If mid point of chord is known)

$\therefore\ \frac{2^2}{36}+\frac{1^2}{9}-1=\frac{2x}{36}+\frac{1y}{9}-1 \Rightarrow x+2y=4$

Example: 28 What will be the equation of the chord of contact of tangents drawn from (3, 2) to the ellipse $x^2 + 4y^2 = 9$

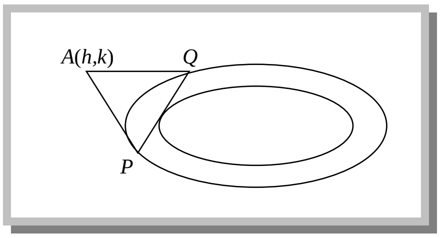
(a) $3x + 8y = 9$ (b) $3x + 8y = 25$ (c) $3x + 4y = 9$ (d) $3x + 8y + 9 = 0$

Solution: (a) The required equation is $T = 0$ *i.e.*, $3x + 4(2y) - 9 = 0$ or $3x + 8y = 9$.

Example: 29 A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. The angle between the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ is **[IIT 1997]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution: (a) The given ellipse $x^2 + 4y^2 = 4$ can be written as $\frac{x^2}{4} + \frac{y^2}{1} = 1$ .....(i)



Any tangent to ellipse (i) is $\frac{x}{2}\cos\theta + y\sin\theta = 1$ .....(ii)

Second ellipse is $x^2 + 2y^2 = 6$, *i.e.* $\frac{x^2}{6} + \frac{y^2}{3} = 1$ .....(iii)

Let the tangents at P, Q meet at (h, k).

$\therefore$ Equation of *PQ*, *i.e.* chord of contact is $\frac{hx}{6} + \frac{ky}{3} = 1$ .....(iv)

Since (ii) and (iv) represent the same line, $\therefore \frac{h/6}{(\cos\theta)/2} = \frac{k/3}{\sin\theta} = \frac{1}{1} \Rightarrow h = 3\cos\theta$ and $k = 3\sin\theta$

So, $h^2 + k^2 = 9$ or $x^2 + y^2 = 9$ is the locus of (h, k) which is the director circle of the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$

$\therefore$ The angle between the tangents at P and Q will be $\pi/2$.

Example: 30 The locus of mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is **[EAMCET 1995]**

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$ (c) $x^2 + y^2 = a^2 + b^2$ (d) None of these

Solution: (a) Let (h, k) be the mid point of a focal chord. Then its equation is $S_1 = T$ or $\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. This passes through $(ae, 0)$,

$\therefore \frac{hae}{a^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. So, locus of (h, k) is $\frac{xe}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Example: 31 If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then the eccentricity of the ellipse is

(a) $\frac{\cos\alpha + \cos\beta}{\cos(\alpha - \beta)}$ (b) $\frac{\sin\alpha - \sin\beta}{\sin(\alpha - \beta)}$ (c) $\frac{\cos\alpha - \cos\beta}{\cos(\alpha - \beta)}$ (d) $\frac{\sin\alpha + \sin\beta}{\sin(\alpha + \beta)}$

Solution: (d) The equation of a chord joining points having eccentric angles α and β is given by

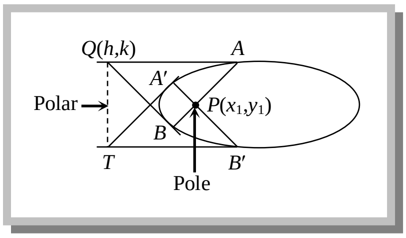
$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

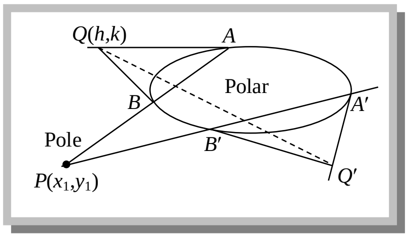
If it passes through $(ae, 0)$ then $e\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

$$\Rightarrow e = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow e = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow e = \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$$

5.2.16 Pole and Polar

Let $P(x_1, y_1)$ be any point inside or outside the ellipse. A chord through P intersects the ellipse at A and B respectively. If tangents to the ellipse at A and B meet at $Q(h,k)$ then locus of Q is called polar of P with respect to ellipse and point P is called pole.



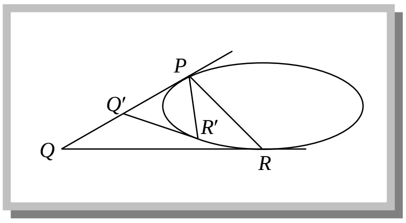


Equation of polar: Equation of polar of the point (x_1, y_1) with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad (i.e.\ T=0)$$

Coordinates of pole: The pole of the line $lx+my+n=0$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$$



Note : ❑ The polar of any point on the directrix, passes through the focus.

❑ Any tangent is the polar of its own point of contact.

Properties of pole and polar

(1) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

(2) If the pole of a line $l_1x+m_1y+n_1=0$ lies on the another line $l_2x+m_2y+n_2=0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(3) Pole of a given line is same as point of intersection of tangents at its extremities.

Example: 32 The pole of the straight line $x+4y=4$ with respect to ellipse $x^2+4y^2=4$ is **[EAMCET 2002]**

(a) (1, 4) (b) (1, 1) (c) (4, 1) (d) (4, 4)

Solution: (b) Equation of polar of (x_1, y_1) *w.r.t* the ellipse is $xx_1+4yy_1=4$ .....(i)

Comparing with $x+4y=4$ .....(ii)

$\frac{x_1}{1}=\frac{4y_1}{4}=1 \Rightarrow x_1=1, y_1=1$. $\therefore$ Coordinates of pole $(x_1, y_1)=(1, 1)$

Example: 33 If the polar with respect to $y^2=4ax$ touches the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, the locus of its pole is **[EAMCET 1995]**

(a) $\frac{x^2}{\alpha^2}-\frac{y^2}{(4a^2\alpha^2/\beta^2)}=1$ (b) $\frac{x^2}{\alpha^2}+\frac{\beta^2y^2}{4a^2}=1$

(c) $\alpha^2x^2+\beta^2y^2=1$ (d) None of these

Solution: (a) Let $P(h,k)$ be the pole. Then the equation of the polar is $ky = 2a(x+h)$ or $y = \frac{2a}{k}x + \frac{2ah}{k}$.

This touches $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, So $\left(\frac{2ah}{k}\right)^2 = \alpha^2\left(\frac{2a}{k}\right)^2+\beta^2$, (using $c^2 = a^2m^2+b^2$)

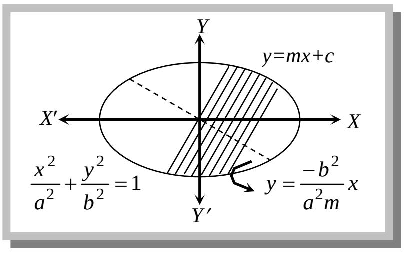
$\Rightarrow 4a^2h^2 = 4a^2\alpha^2 + k^2\beta^2$. So, locus of (h,k) is $4a^2x^2 = 4a^2\alpha^2+\beta^2y^2$ or $\frac{x^2}{\alpha^2} - \frac{y^2}{\left(\frac{4a^2\alpha^2}{\beta^2}\right)} = 1$

5.2.17 Diameter of the Ellipse

Definition : The locus of the mid- point of a system of parallel chords of an ellipse is called a diameter and the chords are called its double ordinates *i.e.* A line through the centre of an ellipse is called a diameter of the ellipse.

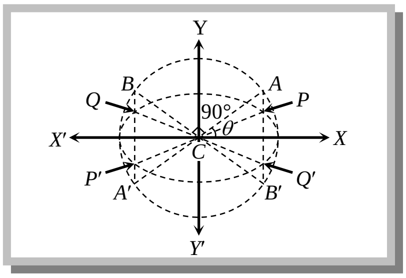
The point where the diameter intersects the ellipse is called the vertex of the diameter.

Equation of a diameter to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$: Let $y = mx + c$ be a system of parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where m is a constant and c is a variable.



The equation of the diameter bisecting the chords of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = -\frac{b^2}{a^2m}x$, which is passing through (0, 0).

Conjugate diameter: Two diameters of an ellipse are said to be conjugate diameter if each bisects all chords parallel to the other.



Conjugate diameter of circle *i.e.* AA' and BB' are perpendicular to each other. Hence, conjugate diameter of ellipse are PP' and QQ'. Hence, angle between conjugate diameters of ellipse $> 90^o$.

Now the coordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi); P'(-a\cos\phi, -b\sin\phi); Q(-a\sin\phi, b\cos\phi); Q'(a\sin\phi, -b\cos\phi)$

If $y = m_1x$ and $y = m_2x$ be two conjugate diameters of an ellipse, then $m_1m_2 = \frac{-b^2}{a^2}$

(1) **Properties of diameters**

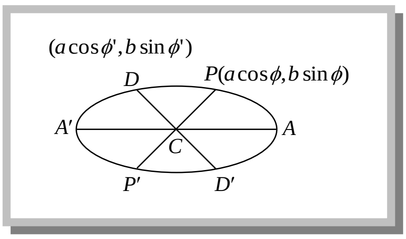
(i) The tangent at the extremity of any diameter is parallel to the chords it bisects or parallel to the conjugate diameter.

(ii) The tangent at the ends of any chord meet on the diameter which bisects the chord.

(2) **Properties of conjugate diameters**

(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle,

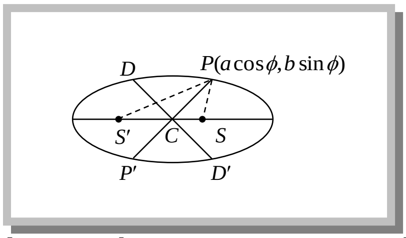
i.e. $\phi - \phi' = \frac{\pi}{2}$



(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi axes of the ellipse, *i.e.* $CP^2 + CD^2 = a^2 + b^2$

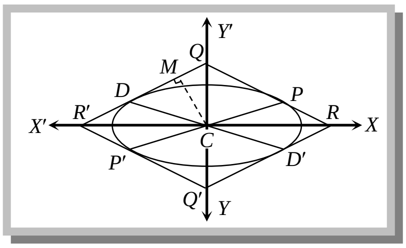
(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point,

i.e., $SP.S'P = CD^2$



(iv) The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes, *i.e.*

Area of parallelogram $=(2a)(2b)=$ Area of rectangle contained under major and minor axes.



(v) The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid point is $(x_{1,}y_1)$, *i.e.* chord is $T = S_1$.

(3) **Equi-conjugate diameters:** Two conjugate diameters are called equi-conjugate, if their lengths are equal *i.e.* $(CP)^2 = (CD)^2$

$$\therefore \ a^2\cos^2\phi + b^2\sin^2\phi = a^2\sin^2\phi + b^2\cos^2\phi$$

$$\Rightarrow \ a^2(\cos^2\phi - \sin^2\phi) - b^2(\cos^2\phi - \sin^2\phi) = 0 \Rightarrow (a^2 - b^2)(\cos^2\phi - \sin^2\phi) = 0$$

$$\because \ (a^2 - b^2) \neq 0, \ \therefore \cos 2\phi = 0. \ \text{So, } \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\therefore \quad (CP) = (CD) = \sqrt{\frac{(a^2+b^2)}{2}} \text{ for equi-conjugate diameters.}$$

Important Tips

☞ *If the point of intersection of the ellipses* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and* $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ *be at the extremities of the conjugate diameters of the former, then* $\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=2$

☞ *The sum of the squares of the reciprocal of two perpendicular diameters of an ellipse is constant.*

☞ *In an ellipse, the major axis bisects all chords parallel to the minor axis and vice-versa, therefore major and minor axes of an ellipse are conjugate diameters of the ellipse but they do not satisfy the condition* $m_1.m_2 = -b^2/a^2$ *and are the only perpendicular conjugate diameters.*

Example: 34 If one end of a diameter of the ellipse $4x^2 + y^2 = 16$ is $(\sqrt{3}, 2)$, then the other end is

(a) $(-\sqrt{3}, 2)$ (b) $(\sqrt{3}, -2)$ (c) $(-\sqrt{3}, -2)$ (d) $(0, 0)$

Solution: (c) Since every diameter of an ellipse passes through the centre and is bisected by it, therefore the coordinates of the other end are $(-\sqrt{3}, -2)$.

Example: 35 If θ and ϕ are eccentric angles of the ends of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\theta - \phi$ is equal to

(a) $\pm\frac{\pi}{2}$ (b) $\pm\pi$ (c) 0 (d) None of these

Solution: (a) Let $y = m_1 x$ and $y = m_2 x$ be a pair of conjugate diameter of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P(a\cos\theta,\, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be ends of these two diameters. Then $m_1 m_2 = \frac{-b^2}{a^2}$

$\Rightarrow \frac{b\sin\theta - 0}{a\cos\theta - 0}\times\frac{b\sin\phi - 0}{a\cos\phi - 0} = \frac{-b^2}{a^2} \Rightarrow \sin\theta\sin\phi = -\cos\theta\cos\phi \Rightarrow \cos(\theta-\phi)=0 \Rightarrow \theta-\phi = \pm\pi/2$.

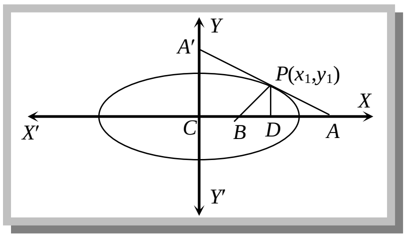
5.2.18 Subtangent and Subnormal

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively.

Length of subtangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $DA = CA - CD = \frac{a^2}{x_1} - x_1$

Length of sub-normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$BD = CD - CB = x_1 - \left(x_1 - \frac{b^2}{a^2}x_1\right) = \frac{b^2}{a^2}x_1 = (1-e^2)x_1$.



Note : ❑ The tangent and normal to any point of an ellipse bisects respectively the internal and external angles between the focal radii of that point.

Example: 36 Length of subtangent and subnormal at the point $\left(\frac{-5\sqrt{3}}{2}, 2\right)$ of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ are

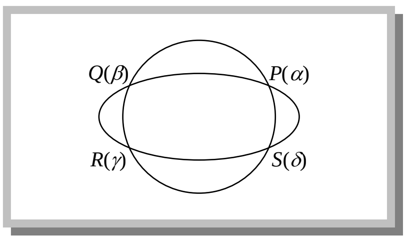
(a) $\left(\frac{5\sqrt{3}}{2}-\frac{10}{\sqrt{3}}\right), \frac{8\sqrt{3}}{5}$ (b) $\left(\frac{5\sqrt{3}}{2}+\frac{10}{\sqrt{3}}\right), \frac{8\sqrt{3}}{10}$ (c) $\left(\frac{5\sqrt{3}}{2}+\frac{12}{\sqrt{3}}\right), \frac{16\sqrt{3}}{5}$ (d) None of thee

Solution: (a) Here $a^2 = 25, b^2 = 16, x_1 = \frac{-5\sqrt{3}}{2}$. Length of subtangent $= \left|\frac{a^2}{x_1} - x_1\right| = \left|\frac{25}{-5\sqrt{3}/2} + \frac{5\sqrt{3}}{2}\right| = \left|\frac{5\sqrt{3}}{2} - \frac{10}{\sqrt{3}}\right|$.

Length of subnormal $= \left|\frac{b^2}{a^2}x_1\right| = \left|\frac{16}{25}\left(\frac{-5\sqrt{3}}{2}\right)\right| = \left|\frac{8\sqrt{3}}{5}\right|$

5.2.19 Concyclic points

Any circle intersects an ellipse in two or four points. They are called concyclic points and the sum of their eccentric angles is an even multiple of π .



If $\alpha, \beta.\gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse, then $\alpha+\beta+\gamma+\delta = 2n\pi$, where n is any integer.

Note : ❑ The common chords of a circle and an ellipse are equally inclined to the axes of the ellipse.

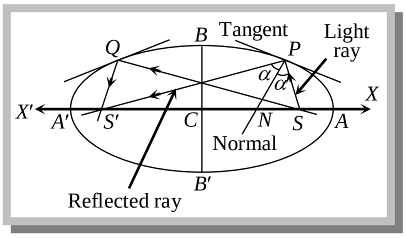
Important Tips

☞ *The centre of a circle* $x^2+y^2+2gx+2fy+c=0$ *passing through the three points, on an ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *(whose eccentric angles are* a,β,γ *) is* $-g=\left(\frac{a^2-b^2}{4a}\right)\{\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)\}$ *and* $-f=\left(\frac{b^2-a^2}{4a}\right)\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}$

☞ $P'CP$ *and* $D'CD$ *are conjugate diameters of an ellipse and* α *is the eccentric angles of P. Then the eccentric angles of the point where the circle through* P, P', D *again cuts the ellipse is* $\pi/2-3\alpha$.

5.2.20 Reflection property of an Ellipse

Let S and S' be the foci and PN the normal at the point P of the ellipse, then $\angle SPS'=\angle SQS'$. Hence if an incoming light ray aimed towards one focus strike the concave side of the mirror in the shape of an ellipse then it will be reflected towards the other focus.



Example: 37 A ray emanating from the point $(-3,0)$ is incident on the ellipse $16x^2+25y^2=400$ at the point P with ordinate 4. Then the equation of the reflected ray after first reflection is

(a) $4x+3y=12$ (b) $3x+4y=12$ (c) $4x-3y=12$ (d) $3x-4y=12$

Solution: (a) For point P y-coordinate =4

Given ellipse is $16x^2+25y^2=400$

$16x^2+25(4)^2=400$, $\therefore x=0$

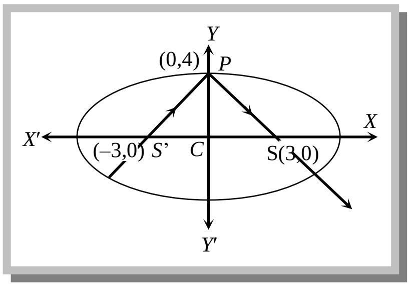
$\therefore$ co-ordinate of P is (0, 4)

$e^2=1-\frac{16}{25}=\frac{9}{25}$

$\therefore\ e=\frac{3}{5}$

$\therefore$ Foci $(\pm ae,0)$, *i.e.* $(\pm 3,0)$

$\therefore$ Equation of reflected ray $(i.e. PS)$ is $\frac{x}{3}+\frac{y}{4}=1$ or $4x+3y=12$.



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