CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

Syllabus

Definition, range, domain, principal value branch. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

Chapter Analysis

Topics	2016		2017		2018
	Delhi	OD	Delhi	OD	Delhi/OD
Solution of equations of Inverse Trigonometric functions	**1 Q. (1 Mark) 1 Q. (4 Marks)**	**1 Q. (1 Mark) 1 Q. (4 Marks)**	**1 Q. (1 Mark)**	**1 Q. (2 Marks) 1 Q. (4 Marks)**	–
Properties of Trigonometric Functions	–	–	**1 Q. (4 Marks)**	–	–

Revision Notes

As we have learnt in class XI, the domain and range of trigonometric functions are given below :

S. No.	Function	Domain	Range
(i)	sine	R	$[-1, 1]$
(ii)	cosine	R	$[-1, 1]$
(iii)	tangent	$R - \left\{x : x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$	R
(iv)	cosecant	$R - \{x : x = n\pi, n \in Z\}$	$R - (-1, 1)$
(v)	secant	$R - \left\{x : x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$	$R - (-1, 1)$
(vi)	cotangent	$R - \{x : x = n\pi, n \in Z\}$	R

(1) Inverse function : We know that if $f : X \to Y$ such that $y = f(x)$ is one-one and onto, then we define another function $g : Y \to X$ such that $x = g(y)$, where $x \in X$ and $y \in Y$, which is also one-one and onto.

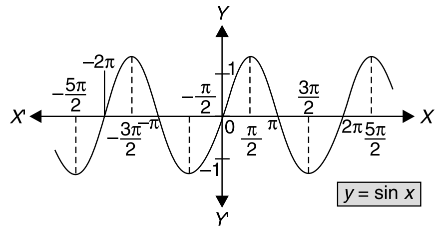
In such a case, Domain of g = Range of f

and Range of g = Domain of f

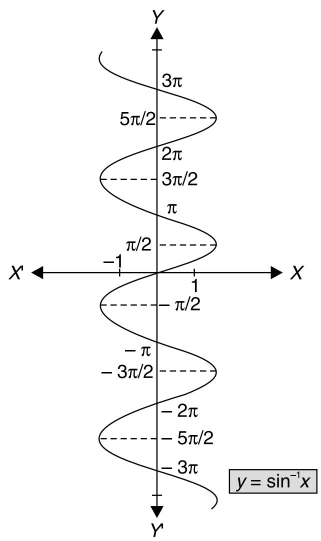
g is called the inverse of f $\quad g = f^{-1}$

or Inverse of $g = g^{-1} = (f^{-1})^{-1} = f$

The graph of sine function is shown here :

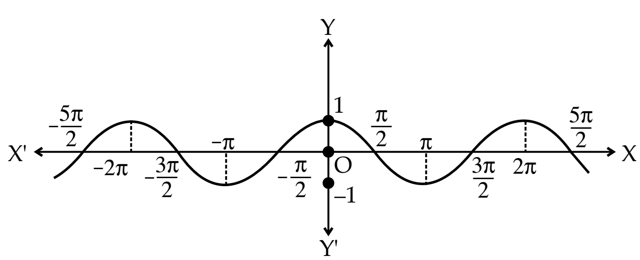


Principal value of branch function $\sin^{-1}$: It is a function with domain [– 1, 1] and range $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and so on corresponding to each interval, we get a branch of the function $\sin^{-1} x$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the principal value branch. Thus, $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

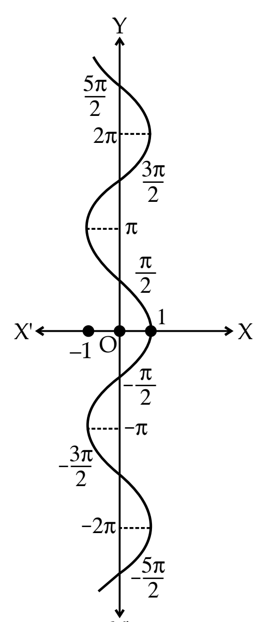


Principal value branch of function $\cos^{-1}$: The graph of the function $\cos^{-1}$ is as shown in figure. Domain of the function $\cos^{-1}$ is [–1, 1]. Its range in one of the intervals (– π, 0), (0, π), (π, 2π), etc. is one-one and onto with the range [– 1, 1]. The branch with range (0, π) is called the principal value branch of the function $\cos^{-1}$.

Thus, $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$



$y = \cos x$

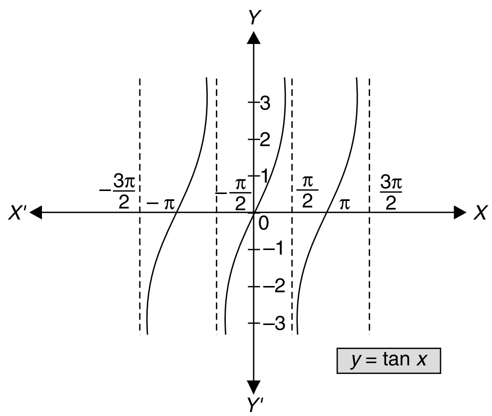


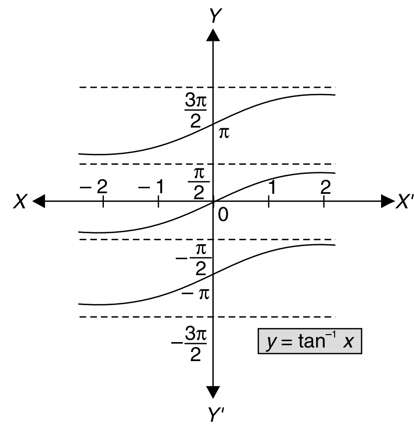
$y = \cos^{-1} x$

Principal value branch function $\tan^{-1}$: The function $\tan^{-1}$ is defined whose domain is set of real numbers and range is one of the intervals,

$$\left(\frac{-3\pi}{2},\frac{\pi}{2}\right),\left(\frac{-\pi}{2},\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right),\ldots..$$

Graph of the function is as shown in the figure :

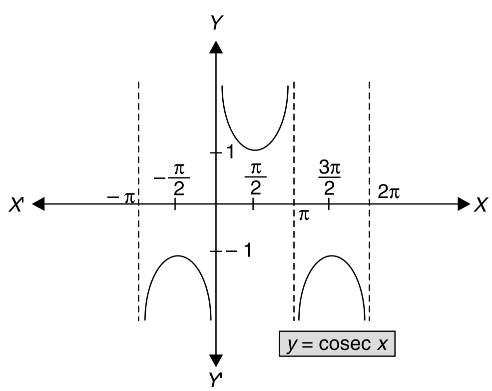


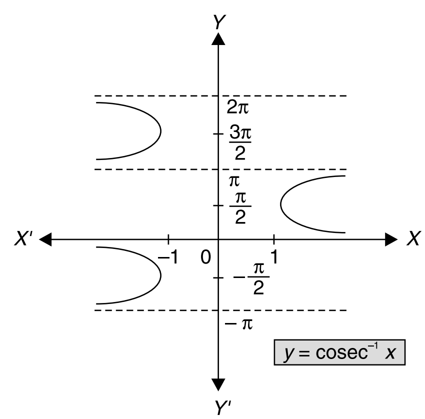


The branch with range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is called the principal value branch of function $\tan^{-1}$. Thus, $\tan^{-1}: R \rightarrow \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

Principal value branch of function cosec^{-1} : The graph of function cosec^{-1} is shown in the figure. The cosec^{-1} is defined on a function whose domain is $R - (-1, 1)$ and the range is any one of the interval,

$$\left[\frac{-3\pi}{2},\frac{-\pi}{2}\right]-\{\pi\},\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\},\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-\{\pi\},\ldots$$



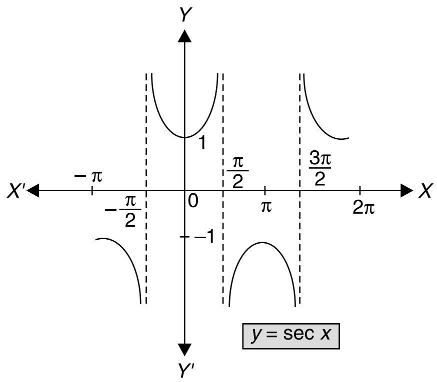


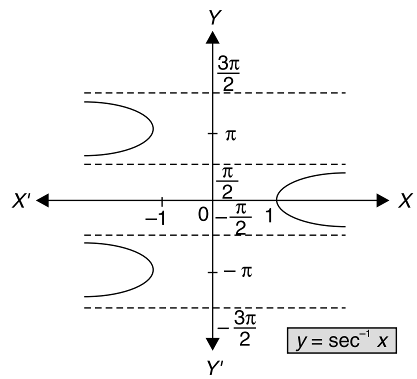
The function corresponding to the range $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ is called the principal value branch of cosec^{-1}.

Thus, $\text{cosec}^{-1}: R-(-1,1) \rightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$.

Principal value branch of function $\sec^{-1}$: The graph of function $\sec^{-1}$ is shown in figure. The $\sec^{-1}$ is defined as a function whose domain $R - (-1, 1)$ and range is $[-\pi, 0] - \left[\frac{-\pi}{2}\right]$, $[0, \pi]-\left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi]-\left\{\frac{3\pi}{2}\right\}$, etc. Function corresponding to range $[0, \pi]-\left\{\frac{\pi}{2}\right\}$ is known as the principal value branch of $\sec^{-1}$.

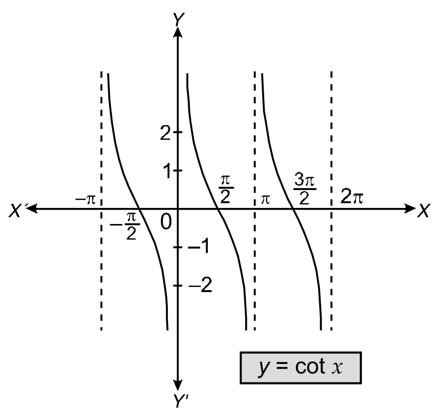
Thus, $\sec^{-1} : R-(-1,1) \rightarrow [0,\pi]-\left\{\frac{\pi}{2}\right\}$

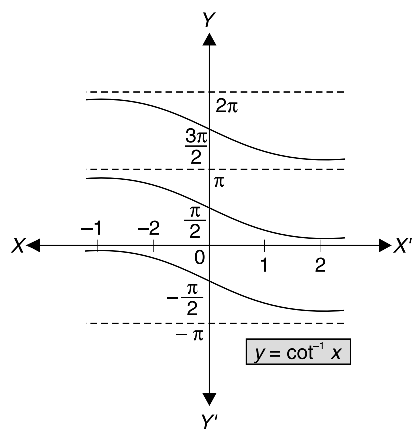




The principal value branch of function cot^{-1} :

The graph of function cot^{-1} is shown below :





The cot^{-1} function is defined on function whose domain is R and the range is any of the intervals, $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, ...

The function corresponding to $(0, \pi)$ is called the principal value branch of the function cot^{-1}.

Then, $\cot^{-1} : R \longrightarrow (0, \pi)$

The principal value branch of trigonometric inverse functions is as follows :

Inverse Function	Domain	Principal Value Branch
$\sin^{-1}$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
cosec^{-1}	$R-(-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
$\sec^{-1}$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\tan^{-1}$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$

2. Elementary Properties of Inverse Trigonometric Functions :

Basic Introduction : A function $f : A \longrightarrow B$ is said to be invertible if f is bijective (*i.e.*, one-one and onto). The inverse of the function f is denoted by $f : B \longrightarrow A$ such that $f^{-1}(y) = x$ if $f(x) = y$, $\forall\, x \in A$, $y \in B$. As trigonometric functions are many-one so, their inverse doesn't exist. But they become one-one onto by restricting their domains. Therefore, inverse of trigonometric functions are defined with restricted domains. In fact, in the discussion below we have used all the restrictions required so that the inverse of the concerned trigonometric functions do exist. If these restrictions are removed, the terms will represent **Inverse Trigonometric Relations** and not the functions. Note that the inverse trigonometric functions are also called as **Inverse Circular Functions**.

List of Formulae and their proofs for Inverse Trigonometric Functions :

A. **(a)** $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right), x \in [-1, 1]$

(b) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(c) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right), x \in [-1, 1]$

(d) $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(e) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right), & x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

(f) $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

PROOF : (a) Let $\sin^{-1}(x) = \theta$, then $\sin\theta = x$

or $\text{cosec}\,\theta = \frac{1}{x}$

or $\theta = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

$\therefore$ $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$ **Hence Proved.**

Other results can be proved in the same way.

B. **(a)** $\sin^{-1}(-x) = -\sin^{-1}x, x \in [-1, 1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, x \in R$

(d) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, |x| \geq 1$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \geq 1$

(f) $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

PROOF : (b) Let $\cos^{-1}(-x) = \theta$, then $\cos\theta = -x$

or $-\cos\theta = x$ or $\cos(\pi - \theta) = x$ or $\cos^{-1}x = \pi - \theta$

or $\theta = \pi - \cos^{-1}x$ or $\cos^{-1}(-x) = \pi - \cos^{-1}x$ **Hence Proved.**

Other results can be proved in the same way.

C. **(a)** $\sin^{-1}(\sin x) = x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(b) $\cos^{-1}(\cos x) = x, 0 \leq x \leq \pi$

(c) $\tan^{-1}(\tan x) = x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

(d) $\text{cosec}^{-1}(\text{cosec}\,x) = x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$

(e) $\sec^{-1}(\sec x) = x, 0 \leq x \leq \pi, x \neq \frac{\pi}{2}$

(f) $\cot^{-1}(\cot x) = x, 0 < x < \pi$

PROOF : (a) Let $\sin^{-1}(x) = \theta$ ...(i)

then, $\sin\theta = x$ ...(ii)

Substituting the value of x in (i) from (ii) we get,

$\sin^{-1}\sin\theta = \theta$ (Replacing θ by x)

$\therefore$ $\sin^{-1}\sin x = x$ **Hence Proved.**

For other results we can proceed similarly

D. **(a)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$

(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in R$

(c) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, |x| \geq 1$ *i.e.*, $x \leq -1$ or $x \geq 1$

PROOF : (a) Let $\sin^{-1}(x) = \theta$, then $\sin\theta = x$

or $\cos\left(\frac{\pi}{2} - \theta\right) = x$

or $\left(\frac{\pi}{2} - \theta\right) = \cos^{-1}x$

or $\frac{\pi}{2} = \cos^{-1}x + \theta$

$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, -1 \leq x \leq 1$ **Hence Proved.**

Similarly, proceed for other results.

E. **(a)** $\sin(\sin^{-1}x) = x;\ \forall\ x \in [-1, 1]$

(b) $\cos(\cos^{-1}x) = x;\ \forall\ x \in [-1, 1]$

(c) $\tan(\tan^{-1}x) = x;\ \forall\ x \in R$

(d) $\text{cosec}(\text{cosec}^{-1}x) = x;\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

(e) $\sec(\sec^{-1}x) = x;\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

(f) $\cot(\cot^{-1}x) = x;\ \forall\ x \in R$

F. **(a)** $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(b) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$

(c) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & x > 0,\ y > 0,\ xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & x < 0,\ y < 0,\ xy > 1 \end{cases}$$

(d) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & x > 0,\ y < 0,\ xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & x < 0,\ y > 0,\ xy < -1 \end{cases}$$

(e) $$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

G. **(a)** $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right),\ |x| \le 1$

(b) $2\tan^{-1} x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right),\ |x| \ge 0$

(c) $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right),\ -1 < x < 1$

PROOF (a) : Let $\tan^{-1} x = \theta$, then $\tan\theta = x$

As $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ or $\sin 2\theta = \dfrac{2x}{1+x^2}$

i.e., $2\theta = \sin^{-1}\dfrac{2x}{1+x^2}$ or $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ **Hence Proved.**

Other results can also be proved in the same way.

(3) Principal Value :

Numerically smallest angle is known as the principal value.

Finding the principal value : For finding the principal value, following algorithm can followed :

STEP 1 : First draw a trigonometric circle and mark the quadrant in which the angle may lie.

STEP 2 : Select anti-clockwise direction for 1st and 2nd quadrants and clockwise direction for 3rd and 4th quadrants.

STEP 3 : Find the angles in the first rotation.

STEP 4 : Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.

STEP 5 : In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

The principal value is never numerically greater than π.

(4) To simplify inverse trigonometrical expressions, following substitutions can be considered :

Expression	Substitution
$a^2 + x^2$ or $\sqrt{a^2+x^2}$	$x = a \tan\theta$ or $x = a \cot\theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a \sin\theta$ or $x = a \cos\theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a \sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2 \cos 2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a \sin^2\theta$ or $x = a \cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a \tan^2\theta$ or $x = a \cot^2\theta$

Note the following and keep them in mind :

- The symbol $\sin^{-1} x$ is used to denote the **smallest angle** whether positive or negative, such that the sine of this angle will give us x. Similarly $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are defined.
- You should note that $\sin^{-1} x$ can be written as **arcsin** x. Similarly, other Inverse Trigonometric Functions can also be written as arccos x, arctan x, arcsec x etc.
- Also note that $\sin^{-1} x$ (and similarly other Inverse Trigonometric Functions) **is entirely different from (sin x)$^{-1}$**. In fact, $\sin^{-1} x$ is the measure of an angle in Radians whose sine is x whereas $(\sin x)^{-1}$ is $\frac{1}{\sin x}$ (which is obvious as per the laws of exponents).
- Keep in mind that these inverse trigonometric relations are **true only in their domains** *i.e.*, they are valid only for some values of 'x' for which inverse trigonometric functions are well defined.

Know the Formula

TRIGONOMETRIC FORMULAE (ONLY FOR REFERENCE) :

- **Relation between trigonometric ratios**

  (a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ **(b)** $\tan\theta = \frac{1}{\cot\theta}$ **(c)** $\cot\theta = \frac{\cos\theta}{\sin\theta}$

  (d) $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ **(e)** $\sec\theta = \frac{1}{\cos\theta}$

- **Trigonometric Identities**

  (a) $\sin^2\theta + \cos^2\theta = 1$ **(b)** $\sec^2\theta = 1 + \tan^2\theta$ **(c)** $\text{cosec}^2\theta = 1 + \cot^2\theta$

- **Addition/subtraction/ formulae & some related results**

  (a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ **(b)** $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

  (c) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

  (d) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

  (e) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ **(f)** $\cot(A \pm B) = \frac{\cot B \cot A \mp 1}{\cot B \pm \cot A}$

- **Multiple angle formulae involving 2 A & 3 A**

(a) $\sin 2A = 2\sin A\cos A$

(b) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(c) $\cos 2A = \cos^2 A - \sin^2 A$

(d) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

(e) $\cos 2A = 2\cos^2 A - 1$

(f) $2\cos^2 A = 1 + \cos 2A$

(g) $\cos 2A = 1 - 2\sin^2 A$

(h) $2\sin^2 A = 1 - \cos 2A$

(i) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

(j) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(k) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(l) $\sin 3A = 3\sin A - 4\sin^3 A$

(m) $\cos 3A = 4\cos^3 A - 3\cos A$

(n) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

- **Transformation of sums/differences into products & vice-versa :**

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(e) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(f) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(g) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(h) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

- **Relations in different measures of Angle**

(a) Angle in Radian Measure = (Angle in degree measure) $\times \frac{\pi}{180}^{\circ}$ rad

(b) Angle in Degree Measure = (Angle in radian measure) $\times \frac{180}{\pi}^{\circ}$

(c) θ (in radian measure) $= \dfrac{l}{r} = \dfrac{\text{arc}}{\text{radius}}$

Also following are of importance as well :

(a) 1 right angle = 90°

(b) 1° = 60′, 1′ = 60″

(c) $1° = \frac{\pi}{180}^{\circ} = 0{\cdot}01745$ radians (Approx.)

(d) 1 radian = 57°17′45″ or 206265 seconds.

- **General Solutions :**

(a) $\sin x = \sin y$ Or $x = n\pi + (-1)^n y$, where $n \in Z$.

(b) $\cos x = \cos y$ Or $x = 2n\pi \pm y$, where $n \in Z$.

(c) $\tan x = \tan y$ Or $x = n\pi + y$, where $n \in Z$.

- **Relation in Degree & Radian Measures :**

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0°	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)

- **Trigonometric Ratio of Standard Angles :**

Degree	**0°**	**30°**	**45°**	**60°**	**90°**
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot x	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosec x	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

➢ **Trigonometric Ratios of Allied Angles :**

Angles (→) **T – Ratios (↓)**	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$ **or** $-\theta$	$2\pi+\theta$
sin	cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ
cos	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ	cos θ	cos θ
tan	cot θ	– cot θ	– tan θ	tan θ	cot θ	– cot θ	– tan θ	tan θ
cot	tan θ	– tan θ	– cot θ	cot θ	tan θ	– tan θ	– cot θ	cot θ
sec	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ	sec θ	sec θ
cosec	sec θ	sec θ	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ

Objective Type Questions

(1 mark each)

Q. 1. Which of the following is the principal value branch of $\cos^{-1}x$?

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **(b)** $[0, \pi]$

(c) $[0, \pi]$ **(d)** $(0, \pi)-\left\{\frac{\pi}{2}\right\}$

[NCERT Exemp. Ex. 2.3, Q. 20, Page 37]

Ans. Correct option : (c)

Explanation : As we know that the principal value of $\cos^{-1}x$ is $[0, \pi]$.

$y = \cos^{-1}x$

Q. 2. Which of the following is the principal value branch of $\text{cosec}^{-1}x$?

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **(b)** $[0, \pi]-\left\{\frac{\pi}{2}\right\}$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **(d)** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

[NCERT Exemp. Ex. 2.3, Q. 21, Page 37]

Ans. Correct option : (d)

Explanation : As we know that the principal value of $\text{cosec}^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$.

$y = \text{cosec}^{-1}x$

Q. 3. If $3\tan^{-1}x + \cot^{-1}x = \pi$, then x equals

(a) 0 **(b)** 1

(c) –1 **(d)** $\frac{1}{2}$

[NCERT Exemp. Ex. 2.3, Q. 22, Page 37]

Ans. Correct option : (b)

Explanation : Given that,

$$3\tan^{-1}x + \cot^{-1}x = \pi$$

Now, we have,

$$3\tan^{-1}x + \cot^{-1}x = \pi$$

$$\Rightarrow 2\tan^{-1}x + (\tan^{-1}x + \cot^{-1}x) = \pi$$

Using this property, $\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$

we get

$\Rightarrow 2\tan^{-1}x+\left(\frac{\pi}{2}\right)=\pi$

$\Rightarrow \quad 2\tan^{-1}x=\pi-\frac{\pi}{2}$

$\Rightarrow \quad 2\tan^{-1}x=\frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x=\frac{\pi}{4}$

$\Rightarrow \quad x=\tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad x=1$

Q. 4. The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

(a) $\frac{3\pi}{5}$ **(b)** $\frac{-7\pi}{5}$

(c) $\frac{\pi}{10}$ **(d)** $\frac{-\pi}{10}$

[NCERT Exemp. Ex. 2.3, Q. 23, Page 37]

Ans. Correct option : (d)

Explanation : Let,

$=\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]=\sin^{-1}\left[\cos\left(6\pi+\frac{3\pi}{5}\right)\right]$

$=\sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right]$

$[\because \cos(2n\pi+\theta)=\cos\theta$

$=\sin^{-1}\left[\cos\left(\frac{\pi}{2}+\frac{\pi}{10}\right)\right]$

$=\sin^{-1}\left(-\sin\frac{\pi}{10}\right)$

$\left[\because \cos\left(\frac{\pi}{2}+x\right)=-\sin x\right]$

$=-\sin^{-1}\left(\sin\frac{\pi}{10}\right)$

$[\because \sin^{-1}(-x)=-\sin^{-1}x]$

$=-\frac{\pi}{10}$

$\left[\because \sin^{-1}(\sin x)=x, x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$

Q. 5. The domain of function $\cos^{-1}(2x-1)$ is

(a) [0,1] **(b) [–1, 1]**

(c) (–1,1) **(d) [0, π]**

[NCERT Exemp. Ex. 2.3, Q. 24, Page 38]

Ans. Correct option : (a)

Explanation :

We have $\cos^{-1}(2x-1)$

$\Rightarrow -1\le 2x-1\le 1$ $\quad [\because x\in[-1,1]]$

$\Rightarrow 0\le 2x\le 2$

$\Rightarrow 0\le x\le 1$

$\Rightarrow x\in[0,1]$

Q. 6. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is

(a) $\frac{\pi}{2}$ **(b)** $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$ **(d)** $\frac{7\pi}{2}$

[NCERT Exemp. Ex. 2.3, Q. 28, Page 38]

Ans. Correct option : (a)

Explanation : We have, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$

$=\cos^{-1}\left[\cos\left(2\pi-\frac{\pi}{2}\right)\right]$

$\left[\because \cos\left(2\pi-\frac{\pi}{2}\right)=\cos\frac{\pi}{2}\right]$

$=\cos^{-1}\cos\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$

$[(\therefore \cos^{-1})(\cos x)=x, x\in[0,\pi]]$

$=\cos^{-1}\left(\cos\frac{\pi}{2}\right)$

$=\frac{\pi}{2}$ $[\because \cos^{-1}(\cos x)=x \text{ if } x\in[0,\pi]]$

$\Rightarrow \cos^{-1}\left(\cos\frac{3\pi}{2}\right)=\frac{\pi}{2}$

Q. 7. The value of expression $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{6}$ **(b)** $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$ **(d) 1**

[NCERT Exemp. Ex. 2.3, Q. 29, Page 38]

Ans. Correct option : (b)

Explanation : We have, $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)$

$=2\sec^{-1}\sec\frac{\pi}{3}+\sin^{-1}\sin\frac{\pi}{6}$

$=2\times\frac{\pi}{3}+\frac{\pi}{6}$ $[\because \sec^{-1}(\sec x)=x \text{ and } \sin^{-1}(\sin x)=x]$

$=\frac{4\pi+\pi}{6}$

$=\frac{5\pi}{6}$

Q. 8. If $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$, then $\cot^{-1}x+\cot^{-1}y$ equals to

(a) $\frac{\pi}{5}$ **(b)** $\frac{2\pi}{5}$

(c) $\frac{3\pi}{5}$ **(d)** π

[NCERT Exemp. Ex. 2.3, Q. 30, Page 38]

Ans. Correct option : (a)

Explanation : We have,

$$\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5}$$

$$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow -\left(\cot^{-1}x + \cot^{-1}y\right) = \frac{4\pi}{5} - \pi$$

$$\Rightarrow \cot^{-1}x + \cot^{-1}y = -\left(-\frac{\pi}{5}\right)$$

$$\Rightarrow \cot^{-1}x + \cot^{-1}y = \frac{\pi}{5}$$

Very Short Answer Type Questions (1 mark each)

Q. 1. Write the principal value of $\cos^{-1}$ [cos (680°)].
R&U [Delhi Set I Comptt. 2014]
[NCERT Exemplar]

Sol. $\cos^{-1}[\cos(680°)] = 40°$ **1**
[CBSE Marking Scheme 2014]

Detailed Answer :

$\cos^{-1}[\cos(680°)] = \cos^{-1}[\cos(720° - 40°)]$

$= \cos^{-1}(\cos 40°)$ ½

$[\because (720° - 40°)$ lies in IV Quadrant]

$= 40°$ [since $\cos^{-1}(\cos\theta) = \theta$, where $\theta \in [0, \pi]$ ½

Q. 2. Write the principal value of $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.
R&U [O.D. Set I Comptt. 2014]
[NCERT Exemplar]

Sol. $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = -\frac{\pi}{4}$ **1**
[CBSE Marking Scheme 2014]

Detailed Answer :

$$\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = \tan^{-1}[-1]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$= -\frac{\pi}{4}$$ ½

$\because \tan^{-1}(\tan\theta) = \theta \ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. ½

Q. 3. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of *x*. **R&U [NCERT][Delhi Set I, II, III 2014]**

Sol. $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$ or $x = \frac{1}{5}$ **1**
[CBSE Marking Scheme 2014]

Detailed Answer :

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

or $\sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}(1)$

or $\sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$

or $x = \frac{1}{5}$ ½

$\left[\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right]$ ½

Q. 4. Find the value of the following :

$$\cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right).$$

R&U [O.D. Set II Comptt. 2014]

Sol. $\cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right) = \sqrt{3}$ **1**
[CBSE Marking Scheme 2014]

Detailed Answer :

$$\cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right) = \cot\left[\frac{\pi}{2} - 2\cot^{-1}\left(\cot\frac{\pi}{6}\right)\right]$$

$$= \cot\left[\frac{\pi}{2} - 2\left(\frac{\pi}{6}\right)\right]$$ ½

$[\because \cot^{-1}(\cot\theta) = \theta \ \forall\ \theta \in (0, \pi)]$

$$= \cot\left[\frac{\pi}{2} - \frac{\pi}{3}\right]$$

$$= \cot\left(\frac{\pi}{6}\right)$$

$$= \sqrt{3}$$ ½

Q. 5. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.
R&U [Foreign Set I, II, III, 2014]

Sol. $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \pi$ **1**
[CBSE Marking Scheme 2014]

Commonly Made Error

- Mostly candidates attempt to solve by using formula of $2\sin^{-1}x$ but could not succeed further.

Answering Tips

- Clarify the conversion of inverse circular functions. Also provide sufficient practice for conversion by using formulae.

Detailed Answer :

$\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

$= \cos^{-1}\left[-\cos\left(\frac{\pi}{3}\right)\right]+2\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

$= \cos^{-1}\left[\cos\left(\pi-\frac{\pi}{3}\right)\right]+2\left(\frac{\pi}{6}\right)$ ½

$\left[\because \sin^{-1}(\sin\theta)=\theta\,\forall\,\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$= \pi-\frac{\pi}{3}+\frac{\pi}{3}$ $[\because \cos^{-1}(\cos\theta)=\theta\ \forall\ \theta\in(0,\pi)]$

$= \pi$ ½

Q. 6. Write the principal value of

$$\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\right].$$

R&U **[O.D. Set I Comptt. 2014]**

Sol. $\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\right]=\frac{5\pi}{6}$ 1

[CBSE Marking Scheme 2014]

Detailed Answer :

$\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]+\cos^{-1}\left[-\cos\left(\frac{\pi}{3}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(\pi-\frac{\pi}{3}\right)\right]$ ½

$= \frac{\pi}{6}+\frac{2\pi}{3} = \frac{\pi}{6}+\frac{4\pi}{6}$ $[\because \cos^{-1}(\cos\theta)=\theta\ \forall\ \theta\in(0,\pi)]$

$= \frac{5\pi}{6}$ ½

Q. 7. Write the principal value of the following :

$\left[\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(1)\right]$.

R&U **[S.Q.P. Set I Comptt. 2013]**

Sol. $\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(1)=-\frac{\pi}{12}$ 1

[CBSE Marking Scheme 2013]

Detailed Answer :

$\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(1)$

$= \tan^{-1}\left[-\tan\left(\frac{\pi}{3}\right)\right]+\tan^{-1}\left(\tan\frac{\pi}{4}\right)$ ½

$=-\frac{\pi}{3}+\frac{\pi}{4}=-\frac{\pi}{12}$ $\left[\because \tan^{-1}(\tan\theta)=\theta\,\forall\,\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$ ½

Q. 8. Evaluate : $\sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right]$.

R&U **[NCERT][O.D. Set I Comptt. 2013]**
[NCERT Exemplar]

Sol. $\sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right] = \frac{2\pi}{5}$ 1

[CBSE Marking Scheme 2013]

Detailed Answer :

$\sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right] = \sin^{-1}\left[\sin\left(\pi-\frac{3\pi}{5}\right)\right]$ ½

$\left(\because \frac{3\pi}{5}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

$\therefore$ $\sin^{-1}\left(\sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$ ½

Q. 9. Write the principal value of $\tan^{-1}(1)$ + $\cos^{-1}\left(-\frac{1}{2}\right)$. R&U **[NCERT][Delhi Set I, 2013]**

Sol. $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right) = \frac{11\pi}{12}$ 1

[CBSE Marking Scheme 2013]

Detailed Answer :

$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(-\cos\frac{\pi}{3}\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left[\cos\left(\pi-\frac{\pi}{3}\right)\right]$ ½

$\left[\because \tan^{-1}(\tan\theta)=\theta\,\forall\,\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right.$

$\left.\&\ \cos^{-1}(\cos\theta)=\theta\,\forall\,\theta\in(0,\pi)\right]$

$= \frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}$ ½

Q. 10. Write the value of $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right]$.

R&U **[Delhi Set I, 2013]**

Sol. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right] = \frac{5}{12}$ 1

[CBSE Marking Scheme 2013]

Alternative Method :

$$\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right] = \tan\left[\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\frac{1}{25}}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)\right] \quad \frac{1}{2}$$

$$\left[\because \tan^{-1}(\tan\theta) = \theta \, \forall \, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$= \left(\frac{5}{12}\right) \quad \frac{1}{2}$$

Q. 11. Write the principal value of $\tan^{-1}(\sqrt{3}) + \cot^{-1}(-\sqrt{3})$. R&U **[Delhi & O.D., 2018] [NCERT][O.D. Set I, 2013]**

Sol. $$\frac{\pi}{3}+\left(\pi-\frac{\pi}{6}\right) = \frac{5\pi}{6} \quad 1$$

Note : $\frac{1}{2}$ m. for any one of the two correct values and $\frac{1}{2}$ m. for final answer

[CBSE Marking Scheme 2018]

Alternative Method :

$$\tan^{-1}(\sqrt{3})+\cot^{-1}(-\sqrt{3})$$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right)+\cot^{-1}\left(-\cot\frac{\pi}{6}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right)+\cot^{-1}\left[\cot\left(\pi-\frac{\pi}{6}\right)\right] \quad \frac{1}{2}$$

$$[\because \tan^{-1}(\tan\theta) = \theta \, \forall \, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\& \ \cot^{-1}(\cot\theta) = \theta \, \forall \, \theta \in (0, \pi)]$$

$$= \frac{\pi}{3}+\frac{5\pi}{6} = \frac{7\pi}{6} \quad \frac{1}{2}$$

Q. 12. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$.

R&U **[O.D. Set I, 2013]**

Sol. $$\tan^{-1}\left[2\sin\left[2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]\right] = \frac{\pi}{3} \quad 1$$

[CBSE Marking Scheme 2013]

Alternative Method :

$$\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$= \tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right)\right] \quad \frac{1}{2}$$

$$= \tan^{-1}\left[2\sin\left(2\left(\frac{\pi}{6}\right)\right)\right] = \tan^{-1}\left[2\times\frac{\sqrt{3}}{2}\right]$$

$$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \quad \frac{1}{2}$$

Q. 13. Write the principal value of $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$.

R&U **[Foreign Set I 2013] [NCERT Exemplar]**

Sol. $$\tan^{-1}\left(\tan\frac{9\pi}{8}\right) = \frac{\pi}{8}. \quad 1$$

[CBSE Marking Scheme 2013]

Detailed Answer :

$$\tan^{-1}\left(\tan\frac{9\pi}{8}\right) = \tan^{-1}\left[\tan\left(\pi+\frac{\pi}{8}\right)\right]$$

$$= \tan^{-1}\left[\tan\frac{\pi}{8}\right] \quad \frac{1}{2}$$

$$\left[\because \tan^{-1}(\tan\theta) = \theta \, \forall \, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{8} \quad \frac{1}{2}$$

AI **Q.14. Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.**

R&U **[NCERT][S.Q.P. Comptt. 2018] [O.D. Set I, II, III Comptt. 2012] [NCERT Exemplar]**

Sol. $$\frac{\pi}{3}-\frac{2\pi}{3} = -\frac{\pi}{3}$$

$\frac{1}{2}$ for any one of $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ $\frac{1}{2}+\frac{1}{2}$ **1**

[CBSE Marking Scheme 2018]

Detailed Answer :

$$\tan^{-1}(\sqrt{3})-\sec^{-1}(-2)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{3}\right)\right]-\sec^{-1}\left[-\sec\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{3}-\left(\pi-\frac{\pi}{3}\right) \quad \frac{1}{2}$$

$$\left[\because \tan^{-1}(\tan\theta) = \theta \, \forall \, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$$

$$\left.\text{and } \sec^{-1}(\sec\theta) = \theta \, \forall \, \theta \in (0, \pi)-\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{2\pi}{3}-\pi = -\frac{\pi}{3} \quad \frac{1}{2}$$

Q. 15. Using principal values, write the value of

$$\left[\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)\right].$$ **[O.D. Set I Comptt. 2012]**

R&U **[NCERT]**

Sol. $\left[\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)\right] = \frac{2\pi}{3}$ **1**

[CBSE Marking Scheme 2012]

Alternative Method :

$$\left[\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \left[\cos^{-1}\left(\cos\frac{\pi}{3}\right)+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]$$

$$= \left[\frac{\pi}{3}+2\times\frac{\pi}{6}\right] \quad ½$$

[∵ $\cos^{-1}(\cos\theta) = \theta\ \forall\ \theta \in [0, \pi]$

and $\sin^{-1}(\sin\theta) = \theta\ \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$]

$$= \frac{\pi}{3}+\frac{\pi}{3}$$

$$= \frac{2\pi}{3} \quad ½$$

Q. 16. Write the principal value of

$$\left[\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)\right].$$ R&U **[Delhi Set I 2012]**

Sol. $\left[\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)\right] = \frac{2\pi}{3}$ **1**

[CBSE Marking Scheme 2012]

Detailed Answer :

$$\left[\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \left[\cos^{-1}\left(\cos\frac{\pi}{3}\right)-2\sin^{-1}\left(-\sin\frac{\pi}{6}\right)\right]$$

$$= \left[\frac{\pi}{3}-2\times\left(-\frac{\pi}{6}\right)\right] \quad ½$$

[∵ $\cos^{-1}(\cos\theta) = \theta\ \forall\ \theta \in [0, \pi]$

and $\sin^{-1}(\sin\theta) = \theta\ \forall \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$]

$$= \left[\frac{\pi}{3}+\frac{\pi}{3}\right] = \frac{2\pi}{3} \quad ½$$

Q. 17. Using principal values, write the value of $2\cos^{-1}\frac{1}{2} + 3\sin^{-1}\frac{1}{2}$. R&U **[S.Q.P. 2012]**

Sol. $2\cos^{-1}\frac{1}{2}+3\sin^{-1}\frac{1}{2}$

$$= 2\cos^{-1}\left(\cos\frac{\pi}{3}\right)+3\sin^{-1}\left[\sin\left(\frac{\pi}{6}\right)\right] \quad ½$$

$$= \frac{2\pi}{3}+\frac{3\pi}{6}=\frac{7\pi}{6} \quad ½$$

[∵ $\cos^{-1}(\cos\theta) = \theta\ \forall\ \theta \in [0, \pi]$

and $\sin^{-1}(\sin\theta) = \theta\ \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$]

Q. 18. Evaluate : $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$. **[S.Q.P. 2012]** R&U **[NCERT]**

Sol. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

$$= \tan^{-1}\left[2\cos\left(2\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right)\right] \quad ½$$

$$= \tan^{-1}\left[2\cos\left(\frac{2\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$= \frac{\pi}{4} \quad ½$$

Q. 19. Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $|x| > 1$ **in the simplest form.** A **[Foreign Set III, 2013] [NCERT]**

Sol. $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x$ **1**

[CBSE Marking Scheme 2013]

Detailed Answer :

Putting $x = \sec\theta$ or $\theta = \sec^{-1}x$

$$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right) \quad ½$$

$$= \cot^{-1}\left(\frac{1}{\tan\theta}\right)$$

$$= \cot^{-1}(\cot\theta)$$

$$= \theta$$

[∵ $\cot^{-1}(\cot\theta) = \theta\ \forall\ \theta \in (0, \pi)$]

$$= \sec^{-1}x \quad ½$$

Q. 20. Write the value of $\cot(\tan^{-1}a + \cot^{-1}a)$**.** A **[NCERT][Foreign Set I 2012]**

Sol. $\cot(\tan^{-1}a + \cot^{-1}a) = 0$ **1**

[CBSE Marking Scheme 2012]

Detailed Answer :

$$\cot(\tan^{-1}a + \cot^{-1}a) = \cot\left(\frac{\pi}{2}\right) \quad ½$$

$$\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$$

$$= 0 \quad ½$$

Q. 21. Evaluate : $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right)$. A **[SQP 2015]**

Sol. $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right)$

$= 2\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

$= 2\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)\left(-\frac{3}{5}\right)$ ½

$= -2\sin\left(\cos^{-1}\left(\frac{-3}{5}\right)\right)\left(\frac{-3}{5}\right)$

$= 2\sin\left(\pi - \cos^{-1}\frac{3}{5}\right)\left(\frac{-3}{5}\right)$ $[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$

$= 2\sin\left(\cos^{-1}\frac{3}{5}\right)\left(\frac{-3}{5}\right)$ $[\because \sin(\pi-\theta) = \sin\theta]$ ½

$= 2\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)\left(\frac{-3}{5}\right)$

$= 2\times\frac{4}{5}\times\frac{-3}{5}$

$= \frac{-24}{25}$

[CBSE Marking Scheme 2015]

Commonly Made Error

- Some students commit errors while interpret the formula $\sin 2x = 2\sin x\cos x$. in the given question.

Q. 22. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$.

A **[O.D. Set I, 2014] [Delhi Comptt. 2012]**

Sol. We have $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$

or $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4}$

or $\frac{x+y}{1-xy} = \tan\frac{\pi}{4}$ ½

or $x + y = 1 - xy$ ½

or $x + y + xy = 1$

Q. 23. Write the value of the following :

$$\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{a-b}{a+b}\right).$$

A **[Delhi Set I Comptt. 2013]**

Sol. $\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{a-b}{a+b}\right) = \frac{\pi}{4}$ 1

[CBSE Marking Scheme 2013]

Commonly Made Error

- Some students get the formula for $\tan^{-1}x - \tan^{-1}y$ wrong. Several candidates convert into $\sin^{-1}$ or $\cos^{-1}$ and get into more difficulties.

Detailed Answer :

$$\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{a-b}{a+b}\right) = \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{\frac{a}{b}-1}{1+\frac{a}{b}}\right)$$

By taking $\frac{a}{b} = \tan\theta$

$\tan^{-1}(\tan\theta) - \tan^{-1}\left(\frac{\tan\theta - \tan\left(\frac{\pi}{4}\right)}{1+\tan\theta\tan\left(\frac{\pi}{4}\right)}\right)$ ½

$= \tan^{-1}(\tan\theta) - \tan^{-1}\left[\tan\left(\theta - \frac{\pi}{4}\right)\right]$

$= \theta - \theta + \frac{\pi}{4}$ $\left[\because \tan^{-1}(\tan\theta) = \theta\ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

$= \frac{\pi}{4}$ ½

Q. 24. Write the value of $\sin\left(2\sin^{-1}\frac{3}{5}\right)$.

A **[Foreign Set I 2013]**

Sol. $\sin\left(2\sin^{-1}\frac{3}{5}\right) = \frac{24}{25}$ 1

[CBSE Marking Scheme 2013]

Detailed Answer :

$\sin\left(2\sin^{-1}\frac{3}{5}\right) = \sin\left(2\tan^{-1}\frac{3}{4}\right)$

$= \sin\left[\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right)\right]$ ½

$= \sin\left[\tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)\right]$

$= \sin\left[\tan^{-1}\frac{24}{7}\right]$

$= \sin\left[\sin^{-1}\frac{24}{25}\right]$

$= \frac{24}{25}$ ½

$\left[\because \sin^{-1}(\sin\theta) = \theta\ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Q. 25. Evaluate: $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$.

R&U **[Delhi 2011][NCERT]**

Sol. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$

$$= \sin\left[\frac{\pi}{3} - \sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)\right]$$

$$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$$

$$= \sin\frac{\pi}{2} = \mathbf{1}$$ **1**

Q. 26. Using principal values, write the value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$.

A [NCERT Exemplar][OD Comptt. 2011] [Delhi 2010]

Sol. $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

$$= \sin^{-1}\left(\sin\left(\frac{-\pi}{3}\right)\right)$$

$$= \frac{-\pi}{3},\ \frac{-\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$ **1**

Q. 27. Find the principal value of

$\sin^{-1}\left(\frac{-1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$. **A [Delhi 2010]**

Sol. $\sin^{-1}\left(\frac{-1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$.

$$= \sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right) + \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right) + \cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$$

$$= \frac{-\pi}{6} + \frac{2\pi}{3}$$

$$= \frac{\pi}{2}$$ **1**

Alternate Solution :

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{2}$$

Q. 28. Find the principal value of $\sec^{-1}(-2)$?

A [OD 2010]

Sol. Let $\sec^{-1}(-2) = \theta$
or $\sec\theta = -2$

$$\Rightarrow \sec\theta = -\sec\left(\frac{\pi}{3}\right)$$

$$= \sec\left(\pi - \frac{\pi}{3}\right)$$

$$= \sec\left(\frac{2\pi}{3}\right)$$

$$\therefore \sec^{-1}(-2) = \frac{2\pi}{3},\quad \frac{2\pi}{3} \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$$ **1**

Q. 29. Using principal values, evaluate the following: $\tan^{-1} 1 + \sin^{-1}\left(\frac{-1}{2}\right)$ **R&U [Delhi Comptt. 2009]**

Sol. $\tan^{-1}(1) + \sin^{-1}\left(\frac{-1}{2}\right)$

$$= \frac{\pi}{4} + \sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$ **1**

Q. 30. Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

R&U [NCERT][Delhi 2011]

Sol. Since $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$,

we write $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{-\pi}{4}\right)\right)$$

$$= \frac{-\pi}{4},\qquad \frac{-\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$ **1**

Q.31. What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$? **R&U [OD 2011, 2008]**

[OD Comptt. 2009]

Sol. Here

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3} \quad \text{as } \frac{2\pi}{3} \in [0,\pi]$$

$$\text{but } \frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$= \frac{2\pi}{3} + \sin^{-1}\left(\sin\left(\pi - \frac{2\pi}{3}\right)\right)$$

$$= \frac{2\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3} + \frac{\pi}{3} = \pi$$ **1**

Q. 32. What is the principal value of $\tan^{-1}(-1)$?

A [Foreign 2011]

[NCERT][Foreign Comptt. 2008]

Sol. $\tan^{-1}(-1) = \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$= \tan^{-1}\left(\tan\left(\frac{-\pi}{4}\right)\right)$

$\therefore \tan^{-1}(-1) = \frac{-\pi}{4}$, $\frac{-\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ 1

Q. 33. What is the domain of the function $\sin^{-1}x$?

A **[Foreign 2010]**

Sol. Domain of $\sin^{-1}x$ is $[-1, 1]$ **1**

Q. 34. If $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$, then find x.

R&U **[Delhi Comptt. 2010]**

Sol. $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$

or $\sin^{-1}\left(\frac{1}{3}\right) + \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2}$

or $\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}x$ **1**

$\therefore x = \frac{1}{3}$

Q. 35. If $\tan^{-1}(\sqrt{3}) + \cot^{-1}x = \frac{\pi}{2}$, then find x.

A **[OD Comptt. 2010]**

Sol. $\tan^{-1}(\sqrt{3}) + \cot^{-1}x = \frac{\pi}{2}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{2} - \cot^{-1}x$

$\tan^{-1}(\sqrt{3}) = \tan^{-1}x$

or $x = \sqrt{3}$ **1**

Commonly Made Error

- Most candidates attempt this question using long calculations instead of using simple fundamental concepts.

Answering Tips

- Make sufficient practice for conversion through diagram and by using formulae.

Short Answer Type Questions

(2 marks each)

Q. 1. Simplify : $\cot^{-1}\frac{1}{\sqrt{x^2-1}}$ for $x < -1$.

R&U **[S.Q.P. 2016-17]**

Sol. Let $\sec^{-1}x = \theta$, then $x = \sec\theta$ and for $x < -1$,

$\frac{\pi}{2} < \theta < \pi$ ½

Given expression $= \cot^{-1}(-\cot\theta)$ ½

$= \cot^{-1}[\cot(\pi-\theta)] = \pi - \sec^{-1}x$ as $0 < \pi - \theta < \frac{\pi}{2}$ **1**

Q. 2. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then find the value of x.

R&U **[S.Q.P. 2018]**

Sol. $4\sin^{-1}x + \cos^{-1}x = \pi$

or $4\sin^{-1}x + \left(\frac{\pi}{2} - \sin^{-1}x\right) = \pi$

or $4\sin^{-1}x - \sin^{-1}x = \pi - \frac{\pi}{2}$

or $3\sin^{-1}x = \frac{\pi}{2}$

or $\sin^{-1}x = \frac{\pi}{6}$

$\therefore x = \frac{1}{2}$ **1**

Q. 3. Prove that $3\cos^{-1}x = \cos^{-1}[4x^3 - 3x]$, $x \in \left[\frac{1}{2}, 1\right]$

A **[CBSE Comptt. Set I, II, III 2018]**

Sol. Put $x = \cos\theta$ in R.H.S. ½

as $\frac{1}{2} \le x \le 1$, RHS $= \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$= \cos^{-1}(\cos 3\theta) = 3\theta$ ½+½

$= 3\cos^{-1}x =$ LHS ½

[CBSE Marking Scheme 2018]

Q. 4. Prove that $3\sin^{-1}x = \sin^{-1}[3x - 4x^3]$ $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

A **[Delhi, O.D., 2018]**

Sol. In RHS, put $x = \sin\theta$ ½

RHS $= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$ **1**

$= 3\theta = 3\sin^{-1}x =$ LHS ½

[CBSE Marking Scheme 2018]

Long Answer Type Questions-I **(4 marks each)**

Q. 1. Solve the equation for x : $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$. R&U **[O.D. Set I, II, III, 2016]** **[NCERT Exemplar]**

Sol. Given that

$$\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x \quad 1$$

Taking sine of both sides,

$$\sin(\sin^{-1}x + \sin^{-1}(1-x)) = \sin(\cos^{-1}x)$$

or $\sin(\sin^{-1}x)\cos(\sin^{-1}(1-x)) + \cos(\sin^{-1}x)\sin(\sin^{-1}(1-x)) = \sin(\cos^{-1}x)$

or $x\sqrt{1-(1-x)^2} + (1-x)\sqrt{1-x^2} = \sqrt{1-x^2}$ 1

$$\left[\because \sin^{-1}(\sin\theta) = \theta \,\forall\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$$

and $\cos^{-1}(\cos\theta) = \theta \,\forall\, \theta \in [0, \pi]]$

$$[\text{Also, } \cos\{\sin^{-1}(1-x)\}] = \sqrt{1-\sin^2[\sin^{-1}(1-x)]}$$

$$= \sqrt{1-[\sin\{\sin^{-1}(1-x)\}]^2}$$

$$= \sqrt{1-(1-x)^2}$$

$$\cos(\sin^{-1}x) = \sqrt{1-\sin^2(\sin^{-1}x)}$$

$$= \sqrt{1-x^2} \quad 1$$

or $x\sqrt{2x-x^2} + \sqrt{1-x^2}(1-x-1) = 0$

or $x\left(\sqrt{2x-x^2} - \sqrt{1-x^2}\right) = 0$

or $x = 0$ or $2x - x^2 = 1 - x^2$

or $x = 0$ or $x = \frac{1}{2}$ 1

Q. 2. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, then prove that $\frac{x^2}{a^2} - 2\frac{xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$. R&U **[O.D. Set I, II, III, 2016]**

Sol. Given $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$

or $A + B = \alpha$,

where $A = \cos^{-1}\frac{x}{a}$, $B = \cos^{-1}\left(\frac{y}{b}\right)$

or $\cos A = \frac{x}{a}$, $\cos B = \frac{y}{b}$ ½

or $\sin A = \sqrt{1-\frac{x^2}{a^2}}$;

and $\sin B = \sqrt{1-\frac{y^2}{b^2}}$

$\therefore$ $\cos(A+B) = \cos\alpha$

or $\cos A\cos B - \sin A\sin B = \cos\alpha$ ½

or $\frac{x}{a}\times\frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha$

or $\frac{xy}{ab} - \cos\alpha = \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}$

or $\left(\frac{xy}{ab} - \cos\alpha\right)^2 = \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$ 1½

or $\frac{x^2y^2}{a^2b^2} + \cos^2\alpha - \frac{2xy}{ab}\cos\alpha = 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} + \frac{x^2y^2}{a^2b^2}$ ½

or $\frac{x^2}{a^2} - 2\frac{xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$ 1½

Hence proved.

Commonly Made Error

- Some errors are made by students in converting inverse trigonometric functions (one to another) and also algebraic calculations.

Q. 3. Prove that :

$$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right) = \frac{\pi}{4} - \frac{x}{2},$$

where $\pi < x < \frac{3\pi}{2}$. A **[S.Q.P. 2015]**

Sol. $\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right) \quad 1$$

$$= \tan^{-1}\left(\frac{-\sqrt{2}\cos\frac{x}{2}+\sqrt{2}\sin\frac{x}{2}}{-\sqrt{2}\cos\frac{x}{2}-\sqrt{2}\sin\frac{x}{2}}\right) \quad 1½$$

$$\left(\text{As } \pi < x < \frac{3\pi}{2} \text{ or } \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \text{ or } \cos\frac{x}{2} < 0, \ \sin\frac{x}{2} > 0\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) \quad ½$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) \quad 1$$

$$= \frac{\pi}{4} - \frac{x}{2} \qquad \left(\text{As, } -\frac{\pi}{4} > \frac{\pi}{4} - \frac{x}{2} > -\frac{\pi}{2}\right) \quad 1$$

[CBSE Marking Scheme 2015]

Q. 4. Prove that : $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$.

A **[Delhi Set I, II, III, 2016, 2013; Comptt. 2012]**

Sol. L.H.S. $= \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right)+\tan^{-1}\left(\frac{1}{8}\right) \quad \mathbf{1}$$

$$= \tan^{-1}\left(\frac{\frac{7}{10}}{\frac{9}{10}}\right)+\tan^{-1}\left(\frac{1}{8}\right) \quad \mathbf{1}$$

$$= \tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right) \quad \mathbf{1}$$

$$= \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \text{R.H.S.} \quad \mathbf{1}$$

Q. 5. Solve for x : $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)$.

A **[NCERT][Delhi Set I, II, III, 2016; Comptt. 2014]**
[NCERT Exemplar][OD 2009, Foreign 2015]

Sol. Given, $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)$

or $\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\,\text{cosec}\, x)$ **1½**

or $2\cot x\,\text{cosec}\, x = 2\,\text{cosec}\, x$ **1**

or $\cot x = 1$ **½**

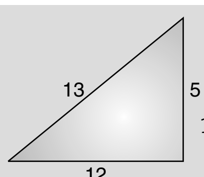
or $x = \cot^{-1}(1) = \frac{\pi}{4}$ **1**

[CBSE Marking Scheme 2014]

Q. 6. Prove that : $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$.

R&U **[NCERT][Delhi Set I, II, III Comptt. 2015]**
[NCERT Exemplar]

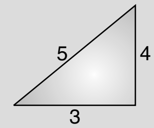
Sol. $\sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12}$



and $\cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3}$ **1**

R.H.S. $= \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

$$= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}$$



$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}}\right) \quad \mathbf{1½}$$

$$= \tan^{-1}\left(\frac{63}{16}\right) = \text{L.H.S.} \quad \mathbf{1½}$$

Hence proved.

[CBSE Marking Scheme 2015]

Q. 7. Solve for x : $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ x>0$.

R&U **[O.D. Set I, II, III Comptt. 2017]**
[NCERT][O.D. Set I, II, III Comptt. 2015, 2014]

Sol. Given

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$$

or $\tan^{-1}1-\tan^{-1}x = \frac{1}{2}\tan^{-1}x$ **1½**

or $\frac{3}{2}\tan^{-1}x = \frac{\pi}{4}$

or $\tan^{-1}x = \frac{\pi}{6}$ **½**

or $x = \tan\left(\frac{\pi}{6}\right)$ **1½**

$\therefore$ $x = \frac{1}{\sqrt{3}}$ **½**

[CBSE Marking Scheme 2015]

Q. 8. If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$, then find x.

R&U **[Delhi I, II, III 2015]**

Sol. Given that $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$ ...(i)

We know that,

$$\cot^{-1}(A) = \sin^{-1}\frac{1}{\sqrt{1+A^2}}$$

Here, $A = x+1$

Applying this identity in equation (i), we have

$$\sin\left[\sin^{-1}\frac{1}{\sqrt{1+(1+x)^2}}\right] = \cos(\tan^{-1}x) \quad \text{...(ii)} \ \mathbf{1½}$$

Also, we know that

$$\tan^{-1}A = \cos^{-1}\frac{1}{\sqrt{1+A^2}}$$

Here, $A = x$

Applying this identity in equation (ii), we have

$$\sin\left[\sin^{-1}\frac{1}{\sqrt{1+(1+x)^2}}\right] = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) \quad \mathbf{1½}$$

or $\frac{1}{\sqrt{1+(1+x)^2}} = \frac{1}{\sqrt{1+x^2}}$

$$\left[\because \sin^{-1}(\sin\theta) = \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$$

and $\cos^{-1}(\cos\theta) = \theta \; \forall \in \theta \; [0, \pi]]$

Squaring and Reciprocating on both sides, we have

$$1 + (1 + x)^2 = 1 + x^2 \qquad ½$$

or $1 + 1 + x^2 + 2x = 1 + x^2$

or $1 + 2x = 0$

or $x = -\dfrac{1}{2}$ ½

[CBSE Marking Scheme 2015]

Commonly Made Error

- Generally students get confused with trigonometric identities of inverse trigonometric functions. They substitute wrong identities.

Q. 9. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \dfrac{5\pi^2}{8}$, find x.

R&U **[Delhi I, II, III 2015]**

Sol. $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$

or $(\tan^{-1}x)^2 + \left(\dfrac{\pi}{2} - \tan^{-1}x\right)^2 = \dfrac{5\pi^2}{8}$ 1

or $2(\tan^{-1}x)^2 - \pi\tan^{-1}x + \dfrac{\pi^2}{4} - \dfrac{5\pi^2}{8} = 0$

or $2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \dfrac{3\pi^2}{8} = 0$ 1½

or $\tan^{-1}x = \dfrac{\pi \pm \sqrt{\pi^2 + 3\pi^2}}{4}$ 1

or $\tan^{-1}x = \dfrac{3\pi}{4}, \dfrac{-\pi}{4}$

$\therefore$ $x = -1$ ½

[CBSE Marking Scheme 2015]

Q.10. Solve for x : $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\dfrac{\pi}{2}$.

R&U **[NCERT Exemplar][S.Q.P. 2015]**

Sol. Given,

$$\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\frac{\pi}{2}$$

or $\sin^{-1}6x = \left(\dfrac{-\pi}{2} - \sin^{-1}6\sqrt{3}x\right)$ ½

or $6x = \sin\left(-\dfrac{\pi}{2} - \sin^{-1}6\sqrt{3}x\right)$

or $6x = -\sin\left(\dfrac{\pi}{2} + \sin^{-1}6\sqrt{3}x\right)$ ½

or $6x = -\cos\left[\sin^{-1}6\sqrt{3}x\right]$ ½

or $6x = -\sqrt{1-108x^2}$ 1

or $36x^2 = 1 - 108x^2$, squaring both the sides

or $144x^2 = 1$

or $x = \pm\dfrac{1}{12}$ ½

Since $x = \dfrac{1}{12}$ does not satisfy the given equation.

$\therefore$ $x = -\dfrac{1}{12}$ 1

Commonly Made Error

- Some students use formula,

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

With this formula the equation will be complicated and solution in the form of x^4 which will be difficult to solve.

Q. 11. Prove that : $2\sin^{-1}\dfrac{3}{5} - \tan^{-1}\dfrac{17}{31} = \dfrac{\pi}{4}$.

A **[OD Set II 2016][S.Q.P. 2015] [NCERT Exemplar]**

Sol. L.H.S. $= 2\sin^{-1}\dfrac{3}{5} - \tan^{-1}\dfrac{17}{31}$

$= 2\tan^{-1}\dfrac{3}{4} - \tan^{-1}\dfrac{17}{31}$

$= \tan^{-1}\left(\dfrac{2\left(\frac{3}{4}\right)}{1-\frac{9}{16}}\right) - \tan^{-1}\dfrac{17}{31}$ 1

$= \tan^{-1}\left(\dfrac{24}{7}\right) - \tan^{-1}\dfrac{17}{31}$ 1

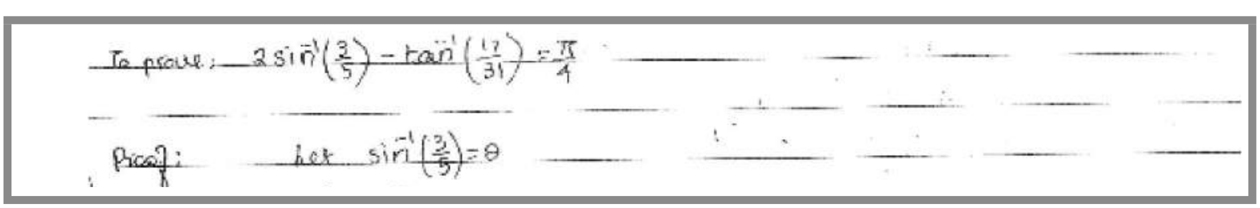
$= \tan^{-1}\left(\dfrac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7}\cdot\frac{17}{31}}\right)$ 1

$= \tan^{-1}\left(\dfrac{744-119}{217+408}\right)$

$= \tan^{-1}(1)$

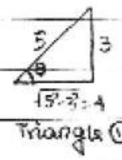
$= \dfrac{\pi}{4}$ = R.H.S. **Hence Proved.** 1

Detailed Answer :



$\therefore \sin\theta = \frac{3}{5}$

$\therefore \tan\theta = \frac{3}{4}$ (From triangle ①)



$\therefore \theta = \tan^{-1}\left(\frac{3}{4}\right)$

Now $2\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)$ $\left\{2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$

$= \tan^{-1}\left(\frac{\frac{3}{2}}{1-\frac{9}{16}}\right) = \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right) = \tan^{-1}\left(\frac{3\times 8}{7}\right) = \tan^{-1}\left(\frac{24}{7}\right)$

$\therefore 2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$ $\left\{\tan^{-1}a - \tan^{-1}b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)\right\}$

$= \tan^{-1}\left(\frac{24\times 31 - 17\times 7}{7\times 31 + 24\times 17}\right) = \tan^{-1}\left(\frac{625}{625}\right) = \tan^{-1} 1 = \frac{\pi}{4}$ Hence Proved

[Topper's Answer 2016]

Q. 12. Show that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

A [NCERT][Delhi Set III Comptt. 2017] [Delhi Set I Comptt. 2014] [Delhi Set I, 2014]

Sol. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Now, multiply and divide by the conjugation of denominator,

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) \times \frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})} \quad \mathbf{1}$$

$$= \cot^{-1}\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$= \cot^{-1}\left[\frac{2+2\cos x}{2\sin x}\right]$$

$$= \cot^{-1}\frac{1+2\cos^2\frac{x}{2}-1}{2\sin\frac{x}{2}.\cos\frac{x}{2}} \quad \mathbf{1}$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) \quad [\cot^{-1}(\cot\theta) = \theta \ \forall\ \theta \in (0, \pi)]$$

$$= \frac{x}{2} \quad \mathbf{1}$$

AI Q. 13. Prove that $\cos^{-1}(x) + \cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$.

R&U [SQP 2018][O.D. Set I Comptt. 2014]

Sol. Let, $\cos^{-1}x = \alpha$ or $x = \cos\alpha$

$$\text{LHS} = \alpha + \cos^{-1}\left[\cos\alpha\cos\left(\frac{\pi}{3}\right)+\frac{\sqrt{3}}{2}\sqrt{1-\cos^2\alpha}\right] \quad \mathbf{1\tfrac{1}{2}}$$

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3}\right)\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right] \quad \mathbf{1}$$

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3}-\alpha\right)\right] \quad \tfrac{1}{2}$$

$$= \alpha + \frac{\pi}{3} - \alpha$$

$[\because \cos^{-1}(\cos\theta) = \theta \ \forall\ \theta \in (0, \pi)]$ ½

$$= \frac{\pi}{3} = \text{RHS} \quad \tfrac{1}{2}$$

[CBSE Marking Scheme 2014]

Q. 14. Solve for x **:** $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$.

A [O.D. Set I Comptt. 2014]

Sol. Given, $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

or $\tan^{-1}x + 2\left(\frac{\pi}{2}-\tan^{-1}x\right) = \frac{2\pi}{3}$ **1**

or $-\tan^{-1}x = \frac{2\pi}{3} - \pi$ **1**

or $-\tan^{-1}x = -\frac{\pi}{3}$

or $\tan^{-1} x = \frac{\pi}{3}$ **1**

or $x = \tan\frac{\pi}{3}$

$= \sqrt{3}$ **1**

[CBSE Marking Scheme 2014]

Q. 15. Prove that $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$. R&U **[Delhi Set I, 2014]**

Sol. $\text{LHS} = 2\left[\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$ **1½ + ½**

$= 2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$ **1**

$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{28}} = \tan^{-1}\left(\frac{25}{25}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ **1**

[CBSE Marking Scheme 2014]

Commonly Made Error

- Some candidates could not convert $2\tan^{-1}\left(\frac{1}{5}\right)$ into $\tan^{-1}\left(\frac{3}{4}\right)$.
- Some use the formula $\tan^{-1}x - \tan^{-1}y$ which is wrong.

Answering Tips

- Use $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.
- Do not change into $\sin^{-1}$ or $\cos^{-1}$.

Q. 16. Prove that : $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le 1$. **[O.D. Set I, 2014]**

R&U **[NCERT][Delhi Set I, II, Comptt. 2017]**

Sol. Putting $x = \cos\theta$ and $\theta = \cos^{-1}x$ in L.H.S., we get

$\text{LHS} = \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right]$ **1**

$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]$ **1**

$= \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$ **½**

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$ **1**

$= \frac{\pi}{4}-\frac{\theta}{2}$

$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{R.H.S.}$ **½**

[CBSE Marking Scheme 2014]

Q. 17. If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, find the values of x. R&U **[O.D. Set I 2014]**

Sol. Given equation can be written as :

$\tan^{-1}\left(\frac{x-2}{x-4}\right) = \tan^{-1}1-\tan^{-1}\left(\frac{x+2}{x+4}\right)$ **½**

$= \tan^{-1}\left(\frac{1-\frac{x+2}{x+4}}{1+\frac{x+2}{x+4}}\right)$ **1**

$= \tan^{-1}\left(\frac{2}{2x+6}\right)$ **½**

$\therefore \frac{x-2}{x-4} = \frac{1}{x+3}$ **½**

or $x^2 + x - 6 = x - 4$ or $x^2 = 2$ **1**

$\therefore x = \pm\sqrt{2}$ **½**

[CBSE Marking Scheme 2014]

Detailed Answer :

The given equation is :

$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$

or $\tan^{-1}\left(\frac{x-2}{x-4}\right) = \tan^{-1}1-\tan^{-1}\left(\frac{x+2}{x+4}\right)$ **½**

or $\tan^{-1}\left(\frac{x-2}{x-4}\right) = \tan^{-1}\left(\frac{1-\frac{x+2}{x+4}}{1+\frac{x+2}{x+4}}\right)$ **½**

or $\frac{x-2}{x-4} = \frac{1-\frac{x+2}{x+4}}{1+\frac{x+2}{x+4}}$ **½** or $\dfrac{x-2}{x-4} = \dfrac{x+4-x-2}{x+4+x+2}$ ½

or $\dfrac{x-2}{x-4} = \dfrac{2}{2(x+3)}$ Or $\dfrac{x-2}{x-4} = \dfrac{1}{x+3}$ 1

or $x^2 + x - 6 = x - 4$

or $x^2 = 2$ Or $x = \pm\sqrt{2}$ 1

AI Q. 18. Solve for x : $\cos[\tan^{-1}(x)] = \sin\left[\cot^{-1}\left(\dfrac{3}{4}\right)\right]$.

R&U [Foreign Set I, II, III 2014]
[O.D. Set I, II, II 2013][O.D. Set II 2016]
[NCERT Exemplar]

Sol. $\cos[\tan^{-1}(x)] = \sin\left[\cot^{-1}\dfrac{3}{4}\right]$

or $\cos\left[\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)\right] = \sin\left[\sin^{-1}\left(\dfrac{4}{5}\right)\right]$ 2

or $\dfrac{1}{\sqrt{1+x^2}} = \dfrac{4}{5}$ 1

or $1 + x^2 = \dfrac{25}{16}$

or $x^2 = \dfrac{25}{16} - 1 = \dfrac{9}{16}$

or $x = \dfrac{3}{4}, -\dfrac{3}{4}$ ½

or $x = -\dfrac{3}{4}$ does not satisfy, so $x = \dfrac{3}{4}$ ½

[CBSE Marking Scheme 2014]

AI Q. 19. Prove that $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$.

A [Foreign Set I, II, III, 2014] [S.Q.P. 2013]
[NCERT Exemplar]

Sol. L.H.S.$= \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8} + \tan^{-1}\left(\dfrac{1}{18}\right)$ 1

$\left(\because \cot^{-1}\theta = \tan^{-1}\dfrac{1}{\theta}\right)$

$= \tan^{-1}\left(\dfrac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right) + \tan^{-1}\dfrac{1}{18}$ 1

$= \tan^{-1}\left(\dfrac{3}{11}\right) + \tan^{-1}\dfrac{1}{18}$ ½

$= \tan^{-1}\left(\dfrac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right)$ ½

$= \tan^{-1}\left(\dfrac{54+11}{198-3}\right) = \tan^{-1}\left(\dfrac{1}{3}\right)$ ½

$= \cot^{-1}(3) = $ RHS ½

[CBSE Marking Scheme 2014]

Commonly Made Error

- Many candidates apply the correct formula but make errors while solving further hence, couldn't get the required result.

Answering Tips

- Give adequate practice in solving problems on different types of inverse trigonometric functions.

Q. 20. Find the greatest and least values of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$. **R&U [S.Q.P. 2013]**
[NCERT Exemplar]

Sol. $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$

$= (\sin^{-1} x + \cos^{-1} x)^2 - 2\sin^{-1} x\cos^{-1} x$

$= \left(\dfrac{\pi}{2}\right)^2 - 2\sin^{-1} x\left(\dfrac{\pi}{2} - \sin^{-1} x\right)$ ½

$= \dfrac{\pi^2}{4} - \pi\sin^{-1} x + 2(\sin^{-1} x)^2$ ½

$= 2\left[(\sin^{-1} x)^2 - \dfrac{\pi}{2}\sin^{-1} x + \dfrac{\pi^2}{8}\right]$ ½

$= 2\left[\left(\sin^{-1} x - \dfrac{\pi}{4}\right)^2 + \dfrac{\pi^2}{16}\right]$ ½

$\therefore$ Least value

$= 2\left(\dfrac{\pi^2}{16}\right) = \dfrac{\pi^2}{8}$ 1

and Greatest value

$= 2\left[\left(-\dfrac{\pi}{2} - \dfrac{\pi}{4}\right)^2 + \dfrac{\pi^2}{16}\right] = \dfrac{5\pi^2}{4}$ 1

Q. 21. Prove that $\cos^{-1}\left(\dfrac{12}{13}\right) + \cos^{-1}\left(\dfrac{4}{5}\right) = \tan^{-1}\left(\dfrac{56}{33}\right)$.

A [O.D. Set I Comptt. 2013]

Sol. $\cos^{-1}\left(\dfrac{12}{13}\right) = \tan^{-1}\dfrac{5}{12}$,

and $\cos^{-1}\left(\dfrac{4}{5}\right) = \tan^{-1}\dfrac{3}{4}$ ½ + ½

L.H.S. $= \tan^{-1}\dfrac{5}{12} + \tan^{-1}\dfrac{3}{4}$ 1

$= \tan^{-1}\left(\dfrac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right)$ 1

$= \tan^{-1}\left(\dfrac{\frac{20+36}{48}}{\frac{48-15}{48}}\right)$

$= \tan^{-1}\left(\dfrac{56}{33}\right) = $ R.H.S. 1

Q. 22. Solve for x : $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$.

R&U **[NCERT][O.D. Set I Comptt. 2013]**

Given equation is :

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

or $\sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}(x)$

or $(1-x) = \sin\left[\frac{\pi}{2} + 2\sin^{-1}(x)\right]$ **1**

or $(1-x) = \cos[2\sin^{-1}(x)]$

Let, $x = \sin\theta$ Or $\theta = \sin^{-1}(x)$

or $(1-x) = \cos 2\theta$

or $(1-x) = 1 - 2\sin^2\theta$

or $(1-x) = 1 - 2x^2$ **1**

or $-x = -2x^2$

or $2x^2 - x = 0$

or $x(2x-1) = 0$ **1**

$\therefore$ $x = 0$ or $x = \frac{1}{2}$

If $x = 0$, then

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \sin^{-1}1 - 2\sin^{-1}0 = \frac{\pi}{2}$$ ½

If $x = \frac{1}{2}$, then

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \sin^{-1}\left(1-\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= -\frac{\pi}{6} \neq \frac{\pi}{2}$$

$\therefore$ The solution is $x = 0$ ½

Commonly Made Error

- Errors are made by students while squaring, simplifying and solving higher degree algebraic equations.

Q. 23. Find the value of the following :

$$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right], |x| < 1, y > 0$$

and $xy < 1$. A **[NCERT][Delhi Set I, 2013]**

Sol. Let, $Z = \tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$

By putting $x = \tan A$ or $A = \tan^{-1}x$ and $y = \tan B$ or $B = \tan^{-1}y$ ½

We get,

$$Z = \tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan A}{1+\tan^2 A}\right) + \cos^{-1}\left(\frac{1-\tan^2 B}{1+\tan^2 B}\right)\right]$$ **1**

$$= \tan\frac{1}{2}\left[\sin^{-1}(\sin 2A) + \cos^{-1}(\cos 2B)\right]$$ **1**

$$= \tan\frac{1}{2}[2A + 2B]$$

$= \tan[A + B]$ ½

$= \tan[\tan^{-1}x + \tan^{-1}y]$

$= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$ ½

$= \frac{x+y}{-xy}$ $\left[\because \tan^{-1}(\tan\theta) = \theta \ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$ ½

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right] = \frac{x+y}{1-xy}$$

Q. 24. Find the value of $\sin\left(2\tan^{-1}\frac{1}{4}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$.

[CBSE SQP, 2018]

Sol. $\sin\left(2\tan^{-1}\frac{1}{4}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$

Lets evaluated, $\left(2\tan^{-1}\frac{1}{4}\right)$

Put $\tan^{-1}\frac{1}{4} = \theta$

$\Rightarrow$ $\tan\theta = \frac{1}{4}$

Now, $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$$= \frac{2\times\frac{1}{4}}{1+\left(\frac{1}{4}\right)^2} = \frac{8}{17}$$ **1½**

To evaluate $\cos(\tan^{-1}2\sqrt{2})$, put $\tan^{-1}2\sqrt{2} = \phi$

$\Rightarrow$ $\tan\phi = 2\sqrt{2}$

$\Rightarrow$ $\cos\phi = \frac{1}{3}$ **1½**

$$\sin\left(2\tan^{-1}\frac{1}{4}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right) = \frac{8}{17} + \frac{1}{3} = \frac{41}{51}$$ **1**

[CBSE Marking Scheme 2018]

Commonly Made Error

- Some errors are made by students in converting inverse trigonometric functions into simple trigonometric form and also algebraic calculations.

Q. 25. Show that : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$.

A **[NCERT Exemplar][O.D. Set I, 2013]**

Sol. Given, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

Let, $\theta = \sin^{-1}\frac{3}{4}$ or $\sin\theta = \frac{3}{4}$ and $\cos\theta = \frac{\sqrt{7}}{4}$ ½

$$\text{LHS} = \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$$

$$= \tan\left(\frac{\theta}{2}\right) \qquad \tfrac{1}{2}$$

$$= \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$= \frac{\sqrt{1-\frac{\sqrt{7}}{4}}}{\sqrt{1+\frac{\sqrt{7}}{4}}} \qquad \tfrac{1}{2}$$

$$= \frac{\sqrt{\frac{4-\sqrt{7}}{4}}}{\sqrt{\frac{4+\sqrt{7}}{4}}} \qquad \tfrac{1}{2}$$

$$= \frac{\sqrt{4-\sqrt{7}}}{\sqrt{4+\sqrt{7}}} \times \frac{\sqrt{4-\sqrt{7}}}{\sqrt{4-\sqrt{7}}} \qquad 1$$

$$= \frac{4-\sqrt{7}}{\sqrt{16-7}} = \frac{4-\sqrt{7}}{\sqrt{9}}$$

$$= = \frac{4-\sqrt{7}}{3} = \text{RHS} \qquad 1$$

$$\therefore \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$

Q. 26. If $y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$, then prove that $\sin y = \tan^2\left(\frac{x}{2}\right)$.

R&U **[Foreign Set I, II, III 2013]**

Sol. $y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$

or $y = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$

or $y = \frac{\pi}{2} - 2\tan^{-1}\left(\sqrt{\cos x}\right)$

or $y = \frac{\pi}{2} - \left[\tan^{-1}\left(\frac{\left(2\sqrt{\cos x}\right)}{1-\left(\sqrt{\cos x}\right)^2}\right)\right]$ 1

or $\left[\tan^{-1}\left(\frac{\left(2\sqrt{\cos x}\right)}{1-\left(\sqrt{\cos x}\right)^2}\right)\right] = \frac{\pi}{2} - y$

or $\frac{\left(2\sqrt{\cos x}\right)}{1-\cos x} = \tan\left(\frac{\pi}{2} - y\right)$ 1

or $\cot y = \frac{\left(2\sqrt{\cos x}\right)}{1-\cos x}$

$$\cot y = \frac{2\sqrt{\cos x}}{1-\cos x}$$

$$\cot^2 y = \frac{4\cos x}{(1-\cos x)^2}$$

$$\operatorname{cosec}^2 y - 1 = \frac{4\cos x}{(1-\cos x)^2}$$

$$\operatorname{cosec}^2 y = \frac{4\cos x}{(1-\cos)^2} + 1$$

$$\operatorname{cosec}^2 y = \frac{4\cos x + (1-\cos x)^2}{(1-\cos x)^2}$$

$$\operatorname{cosec}^2 y = \frac{(1+\cos x)^2}{(1-\cos x)^2}$$

$$\sin^2 y = \frac{(1-\cos x)^2}{(1+\cos x)^2}$$

$$\sin y = \frac{1-\cos x}{1+\cos x}$$

$$\sin y = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\sin y = \tan^2\frac{x}{2}$$

Q. 27. Solve for x : $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$.

A **[OD Comptt. 2009]**
[Delhi Set I, II, III Comptt. 2015]
[NCERT][Delhi Set I Comptt. 2012]

Sol. $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$

or $\tan^{-1}\left(\frac{3x+2x}{1-(3x)(2x)}\right) = \frac{\pi}{4}$ 1

or $\frac{5x}{1-6x^2} = \tan\left(\frac{\pi}{4}\right)$

or $\frac{5x}{1-6x^2} = 1$ 1

or $5x = 1 - 6x^2$

or $6x^2 + 5x - 1 = 0$

or $6x^2 + 6x - x - 1 = 0$

or $6x(x + 1) - 1(x + 1) = 0$

or $(6x - 1)(x + 1) = 0$ 1

or $x = -1$ or $\frac{1}{6}$ 1

$x \neq -1 \qquad \therefore\ x = \frac{1}{6}$

Q. 28. Prove that :

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2},\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

R&U **[Delhi Set I, 2012]**

Sol. $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right)$ 1

$$= \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)$$ **1**

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)$$ **1**

$$= \frac{\pi}{4}-\frac{x}{2}$$ **1**

[CBSE Marking Scheme 2012]

Q. 29. Prove the following :

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}.$$

A **[O.D. Set I, 2012]**

Sol. $\sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{3}{4}\right)$

and $\cot^{-1}\left(\frac{3}{2}\right)=\tan^{-1}\left(\frac{2}{3}\right)$ **1**

$$\therefore \cos\left[\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right]$$ **1**

$$= \cos\left[\tan^{-1}\left(\frac{17}{}\right)\right] = \cos\left[\cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right]$$

$[\because \cos^{-1}(\cos\theta) = \theta\ \forall\ \theta \in (0, \pi)]$

$$= \frac{6}{5\sqrt{13}} = \text{RHS}$$ **1**

[CBSE Marking Scheme 2012]

Q. 30. Prove the following :

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right).$$

A **[NCERT][O.D. Set II, 2012]**

Sol. Getting, $\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$ **1**

and $\cos^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

$$\text{L.H.S.} = \cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{5}{12}\right)$$ **1**

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}\right)$$ **1**

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left[\frac{x+y}{1-xy}\right]\right]$$

$$= \tan^{-1}\left(\frac{\frac{36+20}{48}}{\frac{48-15}{48}}\right)$$

$$= \tan^{-1}\left(\frac{56}{33}\right)$$

$$= \cos^{-1}\left(\frac{33}{65}\right) = \text{R.H.S.}$$ **1**

Q. 31. Solve for x : $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x), x \neq \frac{\pi}{2}$.

R&U **[Foreign Set III, 2012]**

Sol. $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$

or $\tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}(2\sec x)$

or $\frac{2\sin x}{\cos^2 x} = 2\sec x$ **1**

or $\frac{2\sin x}{\cos^2 x}-\frac{2}{\cos x} = 0$

or $\frac{2\sin x}{\cos^2 x}-\frac{2\cos x}{\cos^2 x} = 0$ **1**

or $\sin x - \cos x = 0$ $\left[\because x \neq \frac{\pi}{2} \text{ or } \cos x \neq 0\right]$ **1**

or $\sin x = \cos x$

or $\tan x = 1$

or $x = \frac{\pi}{4}$ **1**

Q. 32. Show that $\sin[\cot^{-1}(\cos(\tan^{-1}x))] = \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$.

R&U **[S.Q.P. 2012]**

Sol. $\cos(\tan^{-1}x) = \cos\left[\cos^{-1}\frac{1}{\sqrt{x^2+1}}\right]$

$$= \frac{1}{\sqrt{x^2+1}}$$

$[\because \cos(\cos^{-1}x) = x\ \forall\ x \in [-1, 1]]$ **1**

$$\cot^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) = \sin^{-1}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}\right)$$ **1**

$$\therefore \sin\left[\cot^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right]$$

$$= \sin\left[\sin^{-1}\left(\sqrt{\frac{x^2+1}{x^2+2}}\right)\right]$$ **1**

$$= \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$$

$[\because \sin(\sin^{-1}x) = x\ \forall\ x \in [-1, 1]]$ **1**

Q. 33. Solve for x : $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$.

A **[NCERT][Delhi Set I Comptt. 2012]**
[OD Set I, II, III Comptt. 2016]

Sol. The given equation can be written as :

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}1 - \tan^{-1}\left(\frac{x+1}{x+2}\right) \quad \tfrac{1}{2}$$

$$= \tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$$

$$= \tan^{-1}\left(\frac{x+2-x-1}{2x+3}\right) \quad 1$$

$$\therefore \quad \frac{x-1}{x-2} = \frac{1}{2x+3}$$

or $(x-1)(2x+3) = x-2$ ½

or $2x^2+x-3 = x-2$

or $x^2 = \frac{1}{2}$ 1

or $x = \pm\frac{1}{\sqrt{2}}$ 1

[CBSE Marking Scheme 2012]

Commonly Made Error

- Generally students do not apply the correct formulae thus make errors while solving further hence couldn't get the required result.

Answering Tips

- Give adequate practice in solving problems on different types of inverse trigonometric functions.

Q. 34. Prove that $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$

$= \frac{\pi}{4}+\frac{x}{2}$, if $0 < x < \frac{\pi}{2}$.

A **[Delhi Set I, II, III, Comptt. 2016]**

Sol. $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$

$$= \tan^{-1}\left\{\frac{\sqrt{2}\cdot\cos\frac{x}{2}+\sqrt{2}\sin\frac{x}{2}}{\sqrt{2}\cdot\cos\frac{x}{2}-\sqrt{2}\cdot\sin\frac{x}{2}}\right\} \quad 2$$

$$= \tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] \quad 1\tfrac{1}{2}$$

$$= \frac{\pi}{4}+\frac{x}{2} \quad \tfrac{1}{2}$$

[CBSE Marking Scheme 2016]

Q. 35. If $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x-4}=\frac{\pi}{4}$, then find the value of x. A **[OD Set I, II, III, 2017]**

Sol.

$$\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}.$$

$$\tan^{-1}\left(\frac{\frac{x-3}{x-4}+\frac{x+3}{x+4}}{1-\left(\frac{x-3}{x-4}\right)\left(\frac{x+3}{x+4}\right)}\right) = \tan^{-1}(1)$$

$\left[\because \tan^{-1}\left(\frac{x-3}{x-4}\right) \le \frac{\pi}{4}, \quad \tan^{-1}\left(\frac{x+3}{x+4}\right) \le \frac{\pi}{4} \quad \therefore \tan^{-1}(\alpha)+\tan^{-1}(\beta) = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\right]$

$$\tan^{-1}\left(\frac{\frac{(x-3)(x+4)+(x+3)(x-4)}{x^2-16}}{\frac{x^2-16-(x^2-9)}{x^2-16}}\right) = \tan^{-1}(1)$$

$$\tan^{-1}\left(\frac{x^2-3x+4x-12+x^2+3x-4x-12}{x^2-16-x^2+9}\right) = \tan^{-1}(1)$$

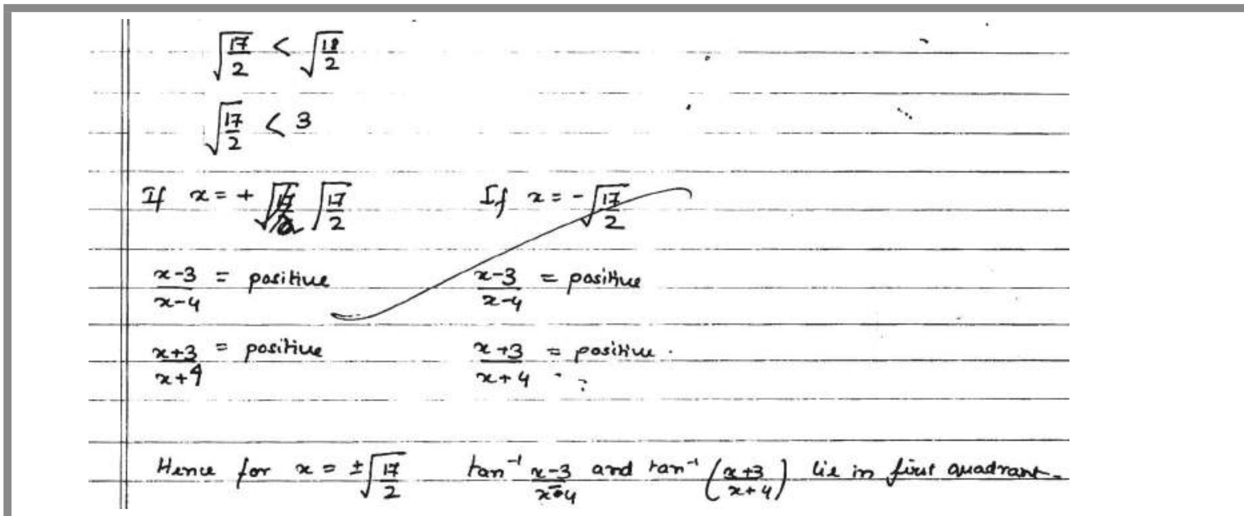
$$\tan^{-1}\left(\frac{2x^2-24}{-7}\right) = \tan^{-1}(1)$$

$$\frac{2x^2-24}{-7} = 1.$$

$$2x^2-24 = -7$$

$$2x^2 = -7+24 = 17$$

$$x^2 = \frac{17}{2} \Rightarrow x = \pm\sqrt{\frac{17}{2}}$$



$\sqrt{\frac{17}{2}} < \sqrt{\frac{18}{2}}$

$\sqrt{\frac{17}{2}} < 3$

If $x = +\sqrt{\frac{17}{2}}$ If $x = -\sqrt{\frac{17}{2}}$

$\frac{x-3}{x-4}$ = positive $\frac{x-3}{x-4}$ = positive

$\frac{x+3}{x+4}$ = positive $\frac{x+3}{x+4}$ = positive.

Hence for $x = \pm\sqrt{\frac{17}{2}}$ $\tan^{-1}\frac{x-3}{x-4}$ and $\tan^{-1}\left(\frac{x+3}{x+4}\right)$ lie in first quadrant.

[Topper's Answer 2017]

Q. 36. Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$.

A [Outside Delhi Set I, II, III, Comptt. 2016]
[NCERT][OD 2011, Delhi 2009]

Sol.
$$2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\frac{1}{4}}$$
$$= \tan^{-1}\frac{4}{3} \qquad 1$$
$$\therefore \quad \text{LHS} = \tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7} \qquad 1$$
$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad 1$$
$$= \tan^{-1}\frac{31}{17} = \text{RHS} \qquad 1$$

[CBSE Marking Scheme 2016]

Q. 37. Prove that $\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right)$

$= \tan^{-1}2x; |2x| < \frac{1}{\sqrt{3}}$.

R&U [Outside Delhi Set I, II, III, Comptt. 2016]

Sol. Let, $2x = \tan\theta$ 1

$$\text{LHS} = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)-\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \qquad 1$$
$= \tan^{-1}(\tan 3\theta) - \tan^{-1}(\tan 2\theta)$ 1
$= 3\theta - 2\theta$
$= \theta$ or $\tan^{-1} 2x$ 1
$\therefore$ LHS = RHS

[CBSE Marking Scheme 2016]

Q. 38. Solve for x : $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$. **A [Outside Delhi 2016]**

Sol. $\tan^{-1}(x-1) + \tan^{-1}(x+1)$
$= \tan^{-1}3x - \tan^{-1}x$ ½

or $\tan^{-1}\left(\frac{2x}{2-x^2}\right) = \tan^{-1}\left(\frac{2x}{1+3x^2}\right)$ 1½

or $\frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$ ½

or $2x(1 + 3x^2 - 2 + x^2) = 0$ ½

or $x = 0, \frac{1}{2}, -\frac{1}{2}$ 1

[CBSE Marking Scheme 2016]

Q. 39. Find the value of

$$\cot\frac{1}{2}\left[\cos^{-1}\frac{2x}{1+x^2}+\sin^{-1}\frac{1-y^2}{1+y^2}\right], \; |x| < 1, y > 0$$

and $xy < 1$ **R&U [Foreign set I 2017]**

Sol. $\cot\frac{1}{2}\left[\cos^{-1}\left(\frac{2x}{1+x^2}\right)+\sin^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$ 1

$= \cot\frac{1}{2}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{\pi}{2}-\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$

$= \cot\frac{1}{2}\left[\pi-2\tan^{-1}x-2\tan^{-1}y\right]$ 1

$= \cot\left[\frac{\pi}{2}-(\tan^{-1}x+\tan^{-1}y)\right]$ 1

$= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] = \frac{x+y}{1-xy}$ 1

[CBSE Marking Scheme 2017]

Q. 40. Prove that $\tan^{-1}2x+\tan^{-1}\frac{4x}{1-4x^2}$

$= \tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right); |x| < \frac{1}{2\sqrt{3}}$

R&U [Foreign Set II 2017]

Sol. $\text{LHS} = \tan^{-1}\left[\dfrac{2x+\dfrac{4x}{1-4x^2}}{1-2x\left(\dfrac{4x}{1-4x^2}\right)}\right]$ **2**

$= \tan^{-1}\left[\dfrac{2x-8x^3+4x}{1-4x^2-8x^2}\right]$ **1**

$= \tan^{-1}\left[\dfrac{6x-8x^3}{1-12x^2}\right] = \text{RHS}$ **1**

[CBSE Marking Scheme 2017]

Q. 41. Prove that $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2; -1 < x < 1$ **[Foreign set III 2017]**

R&U **[NCERT Exemplar]**

Sol. Put $x^2 = \cos 2\theta$ or $\theta = \dfrac{1}{2}\cos^{-1}x^2$

$\text{LHS} = \tan^{-1}\left(\dfrac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$ **1**

$= \tan^{-1}\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$ **1½**

$= \tan^{-1}\left(\tan\left(\dfrac{\pi}{4}+\theta\right)\right) = \dfrac{\pi}{4}+\theta$ **1**

$= \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2, -1 < x < 1 = \text{RHS}$ **½**

[CBSE Marking Scheme 2017]

Q. 42. Show that $\tan^{-1}\dfrac{3}{4}+\tan^{-1}\dfrac{3}{5}-\tan^{-1}\dfrac{8}{19} = \dfrac{\pi}{4}$.

R&U **[Delhi Comptt. 2012]**
[OD Comptt. 2009]

Sol. $\tan^{-1}\dfrac{3}{4}+\tan^{-1}\dfrac{3}{5}-\tan^{-1}\dfrac{8}{19}$

$= \tan^{-1}\left(\dfrac{\dfrac{15+12}{20}}{\dfrac{20-9}{20}}\right)-\tan^{-1}\dfrac{8}{19}$

$\left[\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)\right]$ **1**

$= \tan^{-1}\dfrac{27}{11}-\tan^{-1}\dfrac{8}{19}$

$\left[\because \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)\right]$ **1**

$= \tan^{-1}\left(\dfrac{\dfrac{27}{11}-\dfrac{8}{19}}{1+\dfrac{27\times 8}{11\times 19}}\right)$

$= \tan^{-1}\left(\dfrac{\dfrac{513-88}{209}}{\dfrac{209+216}{209}}\right)$ **1**

$= \tan^{-1}\left(\dfrac{425}{425}\right)$

$= \tan^{-1}1 = \dfrac{\pi}{4}$ **1**

Q. 43. Show that $\tan^{-1}\dfrac{x}{y}-\tan^{-1}\dfrac{x-y}{x+y} = \dfrac{\pi}{4}$

R&U **[Delhi 2011][NCERT]**

Sol. $\tan^{-1}\dfrac{x}{y}-\tan^{-1}\dfrac{x-y}{x+y}$

$= \tan^{-1}\left[\dfrac{\dfrac{x}{y}-\dfrac{x-y}{x+y}}{1+\left(\dfrac{x}{y}\right)\left(\dfrac{x-y}{x+y}\right)}\right]$

$\left[\because \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)\right]$ **1**

$= \tan^{-1}\left[\dfrac{\dfrac{x(x+y)-y(x-y)}{y(x+y)}}{\dfrac{y(x+y)+x(x-y)}{y(x+y)}}\right]$ **1**

$= \tan^{-1}\left[\dfrac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right]$ **1**

$= \tan^{-1}\left(\dfrac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1}1 = \dfrac{\pi}{4}$ **Hence proved. 1**

Q. 44. Show that $\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9} = \dfrac{1}{2}\tan^{-1}\dfrac{4}{3}$.

R&U **[OD Comptt. 2011]**

Sol. $\text{LHS} = \tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9}$

$= \tan^{-1}\left[\dfrac{\dfrac{1}{4}+\dfrac{2}{9}}{1-\dfrac{1\times 2}{4\times 9}}\right]$

$\left[\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}\right]$

$= \tan^{-1}\left[\dfrac{\dfrac{9+8}{36}}{\dfrac{36-2}{36}}\right]$ **1½**

$= \tan^{-1}\left(\dfrac{17}{34}\right) = \tan^{-1}\left(\dfrac{1}{2}\right)$

$$= \frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}\right)$$

$$\left[\because\ 2\tan^{-1} = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{3/4}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)$$ **1½**

Q. 45. Prove that

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

R&U **[NCERT][Delhi 2016]**

[OD Comptt. 2009]

Sol. $\text{LHS} = \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$ **1**

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$ **1**

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{138+187}{391}}{\frac{391-66}{391}}\right)$$ **1**

$$= \tan^{-1}\left(\frac{325}{325}\right)$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4} = \text{RHS}$$ **1**

Q. 46. Prove that $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

R&U **[Delhi Comptt. 2014]**

Sol. $\text{LHS} = 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$ **1**

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$$ **1**

$$= \tan^{-1}\left(\frac{24\times31-17\times7}{31\times7+24\times17}\right)$$

$$= \tan^{-1}\left(\frac{744-119}{217+408}\right)$$

$$= \tan^{-1}\left(\frac{625}{625}\right)$$ **1**

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4} = \text{RHS}$$ **1**

Q. 47. Prove that $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

R&U **[Foreign 2011][NCERT]**

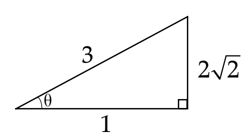
Sol. $\text{LHS} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$

$$= \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right]$$ **1**

$$= \frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right)$$ **1**

$$= \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$ **1**

= RHS **1**



Here, by right angle triangle

$$\cos^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Q. 48. Prove that $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

A **[Delhi 2010]**

Sol. $\text{LHS} = \tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

$$= \frac{\pi}{4} + \left[\frac{\pi}{2} - \cot^{-1}(2)\right] + \frac{\pi}{2} - \cot^{-1}3$$

$$= \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2} - \left[\cot^{-1}(2) + \cot^{-1}(3)\right]$$ **1**

$$= \frac{5\pi}{4} - \left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{5\pi}{4} - \left[\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)\right] \quad \mathbf{1}$$

$$= \frac{5\pi}{4} - \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)$$

$$= \frac{5\pi}{4} - \tan^{-1}(1) \quad \mathbf{1}$$

$$= \frac{5\pi}{4} - \frac{\pi}{4}$$

$$= \pi = \text{RHS} \quad \mathbf{1}$$

Q. 49. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0,1)$

A **[NCERT][Delhi 2010]**

Sol. $\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

Let $\sqrt{x} = \tan\theta$ **1**

$$\therefore \text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right)$$

$$\therefore \text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-(\ x)^2}{1+(\ x)^2}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \quad \mathbf{1}$$

$$\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) \quad \mathbf{1}$$

$$= \frac{1}{2}(2\theta)$$

$$= \theta = \tan^{-1}\sqrt{x}$$

$$= \text{LHS}$$

$\left(\text{since } \sqrt{x}\tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}\right)$ **1**

Q. 50. Solve for x**:** $\cos(2\sin^{-1}x) = \frac{1}{9}; x > 0$

R&U **[OD Comptt. 2011]**

Sol. Given, $\cos\left(2\sin^{-1}x\right) = \frac{1}{9}$

Let $\sin^{-1}x = \theta$

or $x = \sin\theta$ ½

$\Rightarrow x = \sin\theta$

$$\therefore \cos 2\theta = \frac{1}{9}$$

$$1 - 2\sin^2\theta = \frac{1}{9} \quad \mathbf{1}$$

$$= \frac{8}{9}$$

$$2\sin^2\theta = \frac{8}{9} \quad ½$$

$$\sin^2\theta = \frac{4}{9}$$

or $x^2 = \frac{4}{9}$

or $x = \pm\frac{2}{3}$ [From (i)] **1**

But $x > 0$

$\therefore x = \frac{2}{3}$ **1**

Q. 51. Solve for x **:** $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}; -1 < x < 1$

R&U **[Delhi Comptt. 2011]**

Sol. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$

or $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$ **1**

or $2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

or $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$

or $\frac{2x}{1-x^2} = \tan\left(\frac{\pi}{6}\right)$ 1

$$= \frac{1}{\sqrt{3}}$$

or $2\sqrt{3}x = 1 - x^2$ ½

or $x^2 + 2\sqrt{3}x - 1 = 0$

or $x = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$ ½

$$= \frac{-2\sqrt{3} \pm 4}{2}$$

$$\therefore x = \frac{-2\sqrt{3}-4}{2} = -\left(2+\sqrt{3}\right)$$

$$x = \frac{-2\sqrt{3}+4}{2} = 2 - \sqrt{3}$$

Since $-1 < x < 1$ ½

$\therefore x = 2 - \sqrt{3}$

as $x = -\left(2+\sqrt{3}\right)$ is not possible. ½

Q. 52. Prove that $\cos\left[\tan^{-1}\left\{\sin(\cot^{-1}x)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}$

R&U **[OD 2010]**

Sol. Let $\cot^{-1} x = \theta$

$\Rightarrow x = \cot\theta$

$\therefore \text{LHS} = \cos\left[\tan^{-1}(\sin\theta)\right]$

$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$

$\left[\begin{array}{l}\text{Here, } x = \cot\theta \\ \text{cosec}^2\theta - 1 = \cot^2\theta \\ \Rightarrow \text{cosec}^2\theta = 1+\cot^2\theta \\ \Rightarrow \text{cosec}\theta = \sqrt{1+x^2} \\ \Rightarrow \sin\theta = \frac{1}{\sqrt{1+x^2}}\end{array}\right]$ **1½**

$\therefore \text{LHS} = \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$

Let $\tan^{-1}\frac{1}{\sqrt{1+x^2}} = \alpha$

or $\frac{1}{\sqrt{1+x^2}} = \tan\alpha$

$[\because 1+\tan^2\alpha = \sec^2\alpha]$

or $1+\frac{1}{1+x^2} = \sec^2\alpha$

or $\frac{1+x^2+1}{1+x^2} = \sec^2\alpha$

or $\frac{2+x^2}{1+x^2} = \sec^2\alpha$

or $\sqrt{\frac{2+x^2}{1+x^2}} = \sec\alpha$ **1½**

$\therefore \quad \text{LHS} = \cos\alpha$

$= \frac{1}{\sec\alpha}$

$= \sqrt{\frac{1+x^2}{2+x^2}}$ 1

$= \text{RHS}$

Q. 53. Solve, for x $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$

R&U **[OD Comptt. 2010]**

Sol. $\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$

or $\cos^{-1}x = \frac{\pi}{6} - \sin^{-1}\frac{x}{2}$

or $x = \cos\left(\frac{\pi}{6} - \sin^{-1}\frac{x}{2}\right)$

$\therefore x = \cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right) + \sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right)$ **1**

or $x = \frac{\sqrt{3}}{2}\cos\left(\sin^{-1}\frac{x}{2}\right) + \frac{1}{2}\cdot\frac{x}{2}$

$\left[\begin{array}{l}\text{Here let } \sin^{-1}\frac{x}{2} = \theta \\ \text{or } \frac{x}{2} = \sin\theta \\ \text{or } \cos\theta = \sqrt{1-\frac{x^2}{4}} \\ \text{or } \theta = \cos^{-1}\left(\sqrt{1-\frac{x^2}{4}}\right)\end{array}\right]$

$\therefore x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$

$= \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$ **1**

$x - \frac{x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$

$\Rightarrow \frac{3x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$

$\Rightarrow \frac{9x^2}{16} = \frac{3}{4}\left(1-\frac{x^2}{4}\right)$

$\Rightarrow \frac{9x^2}{16} + \frac{3x^2}{16} = \frac{3}{4}$

$\Rightarrow \frac{3}{4}x^2 = \frac{3}{4}$

$\Rightarrow x^2 = 1$

$\Rightarrow x^2 = \pm 1$ **1**

But $x = -1$, does not satisfy the given equation. ½

$\therefore x = 1$ is the solution ½

Q. 54. Solve for x, $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

R&U **[OD 2015]**

Sol. We have

$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

$\Rightarrow \tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x^2-1)}\right) = \tan^{-1}\frac{8}{31}$ **1**

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 62x = 16 - 8x^2$ **1**

$\Rightarrow 8x^2 + 62x - 16 = 0$

$\Rightarrow 4x^2 + 31x - 8 = 0$

$\Rightarrow (4x - 1)(x + 8) = 0$

$\Rightarrow x = \frac{1}{4}$ or $x = -8$

1

For $x = -8$, LHS < 0 and RHS > 0

$\therefore$ not possible

$\therefore x = \frac{1}{4}$

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