# CHAPTER VII.

## SCALES OF NOTATION.

76. The ordinary numbers with which we are acquainted in Arithmetic are expressed by means of multiples of powers of 10; for instance

$$25 = 2 \times 10 + 5;$$

$$4705 = 4 \times 10^3 + 7 \times 10^2 + 0 \times 10 + 5.$$

This method of representing numbers is called the **common** or **denary scale of notation**, and ten is said to be the **radix** of the scale. The symbols employed in this system of notation are the nine digits and zero.

In like manner any number other than ten may be taken as the radix of a scale of notation; thus if 7 is the radix, a number expressed by 2453 represents $2 \times 7^3 + 4 \times 7^2 + 5 \times 7 + 3$; and in this scale no digit higher than 6 can occur.

Again in a scale whose radix is denoted by $r$ the above number 2453 stands for $2r^3 + 4r^2 + 5r + 3$. More generally, if in the scale whose radix is $r$ we denote the digits, beginning with that in the units' place, by $a_0$, $a_1$, $a_2, \ldots a_n$; then the number so formed will be represented by

$$a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \ldots + a_2 r^2 + a_1 r + a_0,$$

where the coefficients $a_n$, $a_{n-1}, \ldots a_0$ are integers, all less than $r$, of which any one or more after the first may be zero.

Hence in this scale the digits are $r$ in number, their values ranging from 0 to $r - 1$.

77. The names Binary, Ternary, Quaternary, Quinary, Senary, Septenary, Octenary, Nonary, Denary, Undenary, and Duodenary are used to denote the scales corresponding to the values *two*, *three*,...*twelve* of the radix.

In the undenary, duodenary,... scales we shall require symbols to represent the digits which are greater than nine. It is unusual to consider any scale higher than that with radix twelve; when necessary we shall employ the symbols $t$, $e$, $T$ as digits to denote 'ten', 'eleven' and 'twelve'.

It is especially worthy of notice that in every scale 10 is the symbol not for 'ten', but for the radix itself.

78. The ordinary operations of Arithmetic may be performed in any scale; but, bearing in mind that the successive powers of the radix are no longer powers of ten, in determining the *carrying figures* we must not divide by ten, but by the radix of the scale in question.

*Example* 1. In the scale of eight subtract 371532 from 530225, and multiply the difference by 27.

$$\begin{array}{r} 530225 \\ 371532 \\ \hline 136473 \end{array} \qquad \qquad \begin{array}{r} 136473 \\ 27 \\ \hline 1226235 \\ 275166\phantom{0} \\ \hline 4200115 \end{array}$$

*Explanation.* After the first figure of the subtraction, since we cannot take 3 from 2 we add 8; thus we have to take 3 from ten, which leaves 7; then 6 from ten, which leaves 4; then 2 from eight which leaves 6; and so on.

Again, in multiplying by 7, we have

$$3 \times 7 = \text{twenty one} = 2 \times 8 + 5;$$

we therefore put down 5 and carry 2.

Next $\qquad 7 \times 7 + 2 = \text{fifty one} = 6 \times 8 + 3;$

put down 3 and carry 6; and so on, until the multiplication is completed.

In the addition,

$$3 + 6 = \text{nine} = 1 \times 8 + 1;$$

we therefore put down 1 and carry 1.

Similarly $\qquad 2 + 6 + 1 = \text{nine} = 1 \times 8 + 1;$

and $\qquad 6 + 1 + 1 = \text{eight} = 1 \times 8 + 0;$

and so on.

*Example* 2. Divide $15et20$ by 9 in the scale of twelve.

$$\begin{array}{r} 9)\underline{15et20} \\ 1ee96 \ldots 6. \end{array}$$

*Explanation.* Since $15 = 1 \times T + 5 = \text{seventeen} = 1 \times 9 + 8$, we put down 1 and carry 8.

Also $8 \times T + e = \text{one hundred and seven} = e \times 9 + 8$; we therefore put down $e$ and carry 8; and so on.

*Example* 3. Find the square root of 442641 in the scale of seven.

```
        442641(546
        34
   134 | 1026
       |  602
  1416 | 12441
       | 12441
```

## EXAMPLES. VII. a.

1. Add together 23241, 4032, 300421 in the scale of five.

2. Find the sum of the nonary numbers 303478, 150732, 264305.

3. Subtract 1732765 from 3673124 in the scale of eight.

4. From 3*te*756 take 2*e*46*t*2 in the duodenary scale.

5. Divide the difference between 1131315 and 235143 by 4 in the scale of six.

6. Multiply 6431 by 35 in the scale of seven.

7. Find the product of the nonary numbers 4685, 3483.

8. Divide 102432 by 36 in the scale of seven.

9. In the ternary scale subtract 121012 from 11022201, and divide the result by 1201.

10. Find the square root of 300114 in the quinary scale.

11. Find the square of *tttt* in the scale of eleven.

12. Find the G. C. M. of 2541 and 3102 in the scale of seven.

13. Divide 14332216 by 6541 in the septenary scale.

14. Subtract 20404020 from 103050301 and find the square root of the result in the octenary scale.

15. Find the square root of *eet*001 in the scale of twelve.

16. The following numbers are in the scale of six, find by the ordinary rules, without transforming to the denary scale:

(1) the G. C. M. of 31141 and 3102;

(2) the L. C. M. of 23, 24, 30, 32, 40, 41, 43, 50.

79. *To express a given integral number in any proposed scale.*

Let $N$ be the given number, and $r$ the radix of the proposed scale.

Let $a_0, a_1, a_2, \ldots a_n$ be the required digits by which $N$ is to be expressed, beginning with that in the units' place; then

$$N = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0.$$

We have now to find the values of $a_0, a_1, a_2, \ldots a_n$

Divide $N$ by $r$, then the remainder is $a_0$, and the quotient is

$$a_n r^{n-1} + a_{n-1} r^{n-2} + \dots + a_2 r + a_1.$$

If this quotient is divided by $r$, the remainder is $a_1$;
if the next quotient .......................................... $a_2$;
and so on, until there is no further quotient.

Thus all the required digits $a_0, a_1, a_2, \dots a_n$ are determined by successive divisions by the radix of the proposed scale.

*Example* 1. Express the denary number 5213 in the scale of seven.

```
7)5213
 7)744......5
  7)106......2
   7)15......1
       2......1
```

Thus $5213 = 2 \times 7^4 + 1 \times 7^3 + 1 \times 7^2 + 2 \times 7 + 5$;
and the number required is 21125.

*Example* 2. Transform 21125 from scale seven to scale eleven.

```
e)21125
 e)1244......t
   e)61......0
      3......t
```

∴ the required number is $3t0t$.

*Explanation.* In the first line of work

$$21 = 2 \times 7 + 1 = \text{fifteen} = 1 \times e + 4;$$

therefore on dividing by $e$ we put down 1 and carry 4.

Next $4 \times 7 + 1 = \text{twenty nine} = 2 \times e + 7$;
therefore we put down 2 and carry 7; and so on.

*Example* 3. Reduce 7215 from scale twelve to scale ten by working in scale ten, and verify the result by working in the scale twelve.

```
             7215                     t)7215
              12                       t)874......1
              86                        t)t4......0     In scale
In scale      12                        t)10......4     of twelve
of ten      1033                            1......2
              12
           12401
```

Thus the result is 12401 in each case.

*Explanation.* 7215 in scale twelve means $7 \times 12^3 + 2 \times 12^2 + 1 \times 12 + 5$ in scale ten. The calculation is most readily effected by writing this expression in the form $[\{(7 \times 12 + 2)\} \times 12 + 1] \times 12 + 5$; thus we multiply 7 by 12, and add 2 to the product; then we multiply 86 by 12 and add 1 to the product; then 1033 by 12 and add 5 to the product.

80. Hitherto we have only discussed whole numbers; but fractions may also be expressed in any scale of notation; thus

$$\cdot 25 \text{ in scale ten denotes } \frac{2}{10} + \frac{5}{10^2};$$

$$\cdot 25 \text{ in scale six denotes } \frac{2}{6} + \frac{5}{6^2};$$

$$\cdot 25 \text{ in scale } r \text{ denotes } \frac{2}{r} + \frac{5}{r^2}.$$

Fractions thus expressed in a form analogous to that of ordinary decimal fractions are called **radix-fractions**, and the point is called the **radix-point.** The general type of such fractions in scale $r$ is

$$\frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots\dots;$$

where $b_1$, $b_2$, $b_3$,... are integers, all less than $r$, of which any one or more may be zero.

81. *To express a given radix fraction in any proposed scale.*

Let $F$ be the given fraction, and $r$ the radix of the proposed scale.

Let $b_1$, $b_2$, $b_3$,... be the required digits beginning from the left; then

$$F = \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots\dots$$

We have now to find the values of $b_1$, $b_2$, $b_3$,......

Multiply both sides of the equation by $r$; then

$$rF = b_1 + \frac{b_2}{r} + \frac{b_3}{r^2} + \dots\dots;$$

Hence $b_1$ is equal to the integral part of $rF$; and, if we denote the fractional part by $F_1$, we have

$$F_1 = \frac{b_2}{r} + \frac{b_3}{r^2} + \dots\dots$$

Multiply again by $r$; then, as before, $b_2$ is the integral part of $rF_1$; and similarly by successive multiplications by $r$, each of the digits may be found, and the fraction expressed in the proposed scale.

If in the successive multiplications by $r$ any one of the products is an integer the process terminates at this stage, and the given fraction can be expressed by a finite number of digits. But if none of the products is an integer the process will never terminate, and in this case the digits recur, forming a radix-fraction analogous to a recurring decimal.

*Example* 1. Express $\frac{13}{16}$ as a radix fraction in scale six.

$$\frac{13}{16} \times 6 = \frac{13 \times 3}{8} = 4 + \frac{7}{8};$$

$$\frac{7}{8} \times 6 = \frac{7 \times 3}{4} = 5 + \frac{1}{4};$$

$$\frac{1}{4} \times 6 = \frac{1 \times 3}{2} = 1 + \frac{1}{2};$$

$$\frac{1}{2} \times 6 = 3.$$

$$\therefore \text{ the required fraction} = \frac{4}{6} + \frac{5}{6^2} + \frac{1}{6^3} + \frac{3}{6^4}$$

$$= \cdot 4513.$$

*Example* 2. Transform 16064·24 from scale eight to scale five.

We must treat the integral and the fractional parts separately,

```
5)16064                  ·24
 5)2644...0                5
  5)440...4             1·44
    5)71...3               5
    5)13...2            2·64
       2...1               5
                        4·04
                           5
                        0·24
```

After this the digits in the fractional part recur; hence the required number is $212340 \cdot \dot{1}24\dot{0}$.

82. *In any scale of notation of which the radix is* r, *the sum of the digits of any whole number divided by* r – 1 *will leave the same remainder as the whole number divided by* r – 1.

Let $N$ denote the number, $a_0, a_1, a_2, \ldots\ldots a_n$ the digits beginning with that in the units' place, and $S$ the sum of the digits; then

$$N = a_0 + a_1 r + a_2 r^2 + \ldots\ldots + a_{n-1} r^{n-1} + a_n r^n;$$

$$S = a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n$$

$$\therefore N - S = a_1 (r-1) + a_2 (r^2 - 1) + \ldots\ldots + a_{n-1} (r^{n-1} - 1) + a_n (r^n - 1).$$

Now every term on the right hand side is divisible by $r-1$;

$$\therefore \frac{N-S}{r-1} = \text{an integer};$$

that is,
$$\frac{N}{r-1} = I + \frac{S}{r-1},$$

where $I$ is some integer; which proves the proposition.

Hence a number in scale $r$ will be divisible by $r-1$ when the sum of its digits is divisible by $r-1$.

83. By taking $r=10$ we learn from the above proposition that a number divided by 9 will leave the same remainder as the sum of its digits divided by 9. The rule known as "casting out the nines" for testing the accuracy of multiplication is founded on this property.

The rule may be thus explained:

Let two numbers be represented by $9a+b$ and $9c+d$, and their product by $P$; then

$$P = 81ac + 9bc + 9ad + bd.$$

Hence $\frac{P}{9}$ has the same remainder as $\frac{bd}{9}$; and therefore the *sum of the digits* of $P$, when divided by 9, gives the same remainder as the *sum of the digits* of $bd$, when divided by 9. If on trial this should not be the case, the multiplication must have been incorrectly performed. In practice $b$ and $d$ are readily found from the sums of the digits of the two numbers to be multiplied together.

*Example.* Can the product of 31256 and 8427 be 263395312?

The sums of the digits of the multiplicand, multiplier, and product are 17, 21, and 34 respectively; again, the sums of the digits of these three numbers are 8, 3, and 7, whence $bd = 8 \times 3 = 24$, which has 6 for the sum of the digits; thus we have two different remainders, 6 and 7, and the multiplication is incorrect.

84. *If* N *denote any number in the scale of* r, *and* D *denote the difference, supposed positive, between the sums of the digits in the odd and the even places; then* N − D *or* N + D *is a multiple of* r + 1.

Let $a_0, a_1, a_2, \ldots\ldots a_n$ denote the digits beginning with that in the units' place; then

$$N = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots\ldots + a_{n-1} r^{n-1} + a_n r^n.$$

$\therefore\ N - a_0 + a_1 - a_2 + a_3 - \ldots = a_1 (r+1) + a_2 (r^2 - 1) + a_3 (r^3 + 1) + \ldots$; and the last term on the right will be $a_n (r^n + 1)$ or $a_n (r^n - 1)$ according as $n$ is odd or even. Thus every term on the right is divisible by $r + 1$; hence

$$\frac{N - (a_0 - a_1 + a_2 - a_3 + \ldots\ldots)}{r+1} = \text{an integer}.$$

Now $$a_0 - a_1 + a_2 - a_3 + \ldots\ldots = \pm D;$$

$$\therefore\ \frac{N \mp D}{r+1} \text{ is an integer;}$$

which proves the proposition.

Cor. If the sum of the digits in the even places is equal to the sum of the digits in the odd places, $D = 0$, and $N$ is divisible by $r + 1$.

*Example* 1. Prove that 4·41 is a square number in any scale of notation whose radix is greater than 4.

Let $r$ be the radix; then

$$4{\cdot}41 = 4 + \frac{4}{r} + \frac{1}{r^2} = \left(2 + \frac{1}{r}\right)^2;$$

thus the given number is the square of 2·1.

*Example* 2. In what scale is the denary number 2·4375 represented by 2·13?

Let $r$ be the scale; then

$$2 + \frac{1}{r} + \frac{3}{r^2} = 2{\cdot}4375 = 2\frac{7}{16};$$

whence $$7r^2 - 16r - 48 = 0;$$

that is, $$(7r + 12)(r - 4) = 0.$$

Hence the radix is 4.

Sometimes it is best to use the following method.

*Example* 3. In what scale will the nonary number 25607 be expressed by 101215?

The required scale must be less than 9, since the new number *appears* the greater; also it must be greater than 5; therefore the required scale must be 6, 7, or 8; and *by trial* we find that it is 7.

*Example* 4. By working in the duodenary scale, find the height of a rectangular solid whose volume is 364 cub. ft. 1048 cub. in., and the area of whose base is 46 sq. ft. 8 sq. in.

The volume is $364\frac{1048}{1728}$ cub. ft., which expressed in the scale of twelve is 264·734 cub. ft.

The area is $46\frac{8}{144}$ sq. ft., which expressed in the scale of twelve is $3t$·08.

We have therefore to divide 264·734 by $3t$·08 in the scale of twelve.

$$\begin{array}{r} 3t08)26473\cdot4(7\cdot e \\ 22t48 \\ \hline 36274 \\ 36274 \\ \hline \end{array}$$

Thus the height is 7ft. 11in.

## EXAMPLES. VII. b.

1. Express 4954 in the scale of seven.
2. Express 624 in the scale of five.
3. Express 206 in the binary scale.
4. Express 1458 in the scale of three.
5. Express 5381 in powers of nine.
6. Transform 212231 from scale four to scale five.
7. Express the duodenary number $398e$ in powers of 10.
8. Transform $6t12$ from scale twelve to scale eleven.
9. Transform 213014 from the senary to the nonary scale.
10. Transform 23861 from scale nine to scale eight.
11. Transform 400803 from the nonary to the quinary scale.
12. Express the septenary number 20665152 in powers of 12.
13. Transform *tttcee* from scale twelve to the common scale.
14. Express $\frac{3}{10}$ as a radix fraction in the septenary scale.
15. Transform 17·15625 from scale ten to scale twelve.
16. Transform 200·211 from the ternary to the nonary scale.
17. Transform 71·03 from the duodenary to the octenary scale.
18. Express the septenary fraction $\frac{1552}{2626}$ as a denary vulgar fraction in its lowest terms.
19. Find the value of $\cdot\dot{4}$ and of $\cdot\dot{4}\dot{2}$ in the scale of seven.
20. In what scale is the denary number 182 denoted by 222?
21. In what scale is the denary fraction $\frac{25}{128}$ denoted by ·0302?

22. Find the radix of the scale in which 554 represents the square of 24.

23. In what scale is 511197 denoted by 1746335?

24. Find the radix of the scale in which the numbers denoted by 479, 698, 907 are in arithmetical progression.

25. In what scale are the radix-fractions ·16, ·20, ·28 in geometric progression?

26. The number 212542 is in the scale of six; in what scale will it be denoted by 17486?

27. Shew that 148·84 is a perfect square in every scale in which the radix is greater than eight.

28. Shew that 1234321 is a perfect square in any scale whose radix is greater than 4; and that the square root is always expressed by the same four digits.

29. Prove that 1·331 is a perfect cube in any scale whose radix is greater than three.

30. Find which of the weights 1, 2, 4, 8, 16,... lbs. must be used to weigh one ton.

31. Find which of the weights 1, 3, 9, 27, 81,... lbs. must be used to weigh ten thousand lbs., not more than one of each kind being used but in either scale that is necessary.

32. Shew that 1367631 is a perfect cube in every scale in which the radix is greater than seven.

33. Prove that in the ordinary scale a number will be divisible by 8 if the number formed by its last three digits is divisible by eight.

34. Prove that the square of $rrrr$ in the scale of $s$ is $rrrq0001$, where $q$, $r$, $s$ are any three consecutive integers.

35. If any number $N$ be taken in the scale $r$, and a new number $N'$ be formed by altering the order of its digits in any way, shew that the difference between $N$ and $N'$ is divisible by $r-1$.

36. If a number has an even number of digits, shew that it is divisible by $r+1$ if the digits equidistant from each end are the same.

37. If in the ordinary scale $S_1$ be the sum of the digits of a number $N$, and $3S_2$ be the sum of the digits of the number $3N$, prove that the difference between $S_1$ and $S_2$ is a multiple of 3.

38. Shew that in the ordinary scale any number formed by writing down three digits and then repeating them in the same order is a multiple of 7, 11, and 13.

39. In a scale whose radix is odd, shew that the sum of the digits of any number will be odd if the number be odd, and even if the number be even.

40. If $n$ be odd, and a number in the denary scale be formed by writing down $n$ digits and then repeating them in the same order, shew that it will be divisible by the number formed by the $n$ digits, and also by 9090...9091 containing $n-1$ digits.