Differential Equation **Chapter 23**

DIFFERENTIAL EQUATION

An equation involving an independent variable, a dependent variable and the derivatives of the dependent variable and their power are called *differential equation*.

ORDER OF A DIFFERENTIAL EQUATION

The order of highest order derivative appearing in a differential equation is called the *order* of the differential equation.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of an algebraic differential equation is the degree of the derivative (or differential) of the highest order in the equation, after the equation is freed from radicals and fractions in its derivatives.

FORMATION OF A DIFFERENTIAL EQUATION

An equation involving-independent variable x, the dependent variable y and 'n' independent arbitrary constants, to form the differential equation of such family of curves we have to eliminate the 'n' independent arbitrary constants form the given equation.

This can be achieved by differentiating given equation n times and, we get a differential equation of nth order corresponding to given family of curves.

SOLUTION OF A DIFFERENTIAL EQUATION

Any relation between the dependent and independent variables (not involving the derivative) which, when substituted in the differential equation reduces it to an identity is called a *solution* of the differential equation.

GENERAL SOLUTION

The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the *general solution*. Thus, the general solution of a differential equation of the nth order has n arbitrary constants.

PARTICULAR SOLUTION

A solution obtained by giving particular values to arbitrary constant in the general solution is called a *particular solution*.

SINGULAR SOLUTION

This solution does not contain any arbitrary constant nor can it be derived from the complete solution by giving any particular value to the arbitrary constant, it is called the *singular solution* of the differential equation.

The singular solution represents the envelope of the family of straight lines represented by the complete solution.

SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

The following methods may be used to solve first order and first degree differential equations.

1. **VARIABLE SEPARABLE DIFFERENTIABLE EQUATIONS**

A differential equation of the form $f(x)+g(y)\frac{dy}{dx}=0$ …(1)

or $f(x)dx+g(y)dy=0$, is said to have separated variables.

Integrating (1), we obtain $\int f(x)\,dx+\int g(y)\frac{dy}{dx}dx=C$, where C is an arbitrary constant.

Hence, $\int f(x)\,dx+\int g(y)\,dy=C$ is the solution of (1).

2. EQUATIONS REDUCIBLE TO SEPARABLE FORM

Sometimes in a given differential equation, the variables are not separable. But, some suitable substitution reduces to a form in which the variables are separable *i.e.*, the differential equation of the type $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to variable separable form by substitution $ax+by+c=t$. The reduced variable separable form is :

$$\frac{dt}{bf(t)+a} = dx.$$

Integrate both sides to obtain the solution of this differential equation.

3. HOMOGENEOUS DIFFERENTIAL EQUATION

A differential equation $P(x,y)dx + Q(x,y)dy = 0$ ….(*)

is called **homogenous,** if $P(x,y)$ *and* $Q(x,y)$ are homogenous functions of the same degree in x *and* y. Equation (*) may be reduced to the from $y' = f\left(\frac{y}{x}\right)$. By means of the substitution $y = vx$, where v is some unknown function, the equation is transformed to an equation with variable separable. (We can also use substitution $x = vy$).

4. EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM

TYPE I: Consider a differential equation of the form :

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \quad \ldots(1)$$

This is clearly non-homogeneous. In order to make it homogeneous, we proceed as follows :
We substitute $x = X + h$ and $y = Y + k$ in (1), where h, k are constants to be determined suitably.

We have $\frac{dx}{dX} = 1$ and $\frac{dy}{dY} = 1$, so that

$$\frac{dy}{dx} = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx} = \frac{dY}{dX}.$$

Now (1) becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \quad \ldots(2)$$

Choose h and k so that

$$a_1h + b_1k + c_1 = 0,$$
$$a_2h + b_2k + c_2 = 0.$$

These equations give

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \quad \ldots(3)$$

Now equation (2) becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y},$$

which being a homogeneous equation can be solved by means of the substitution $Y = VX$.

TYPE II : Consider a differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \text{ where } \frac{a_1}{a_2} = \frac{b_1}{b_2} = k \text{ (say)}$$

Since $a_1b_2 - a_2b_1 = 0$, the above method fails in view of (3).

We have $$\frac{dy}{dx} = \frac{k\left(a_2x + b_2y\right) + c_1}{a_2x + b_2y + c_2} \qquad \ldots(4)$$

Substituting $a_2x + b_2y = z$ so that $a_2 + b_2\frac{dy}{dx} = \frac{dz}{dx}$.

Now (4) becomes $\frac{dz}{dx} = b_2 \cdot \frac{kz + c_1}{z + c_2} + a_2$, which is an equation with variables separable.

EXACT DIFFERENTIAL EQUATION

If *M* and *N* are functions of *x* and *y*, the equation $Mdx + Ndy = 0$ is called exact when there exists a functions, $f\left(x, y\right)$ of *x* and *y* such that

$$d\left[f\left(x, y\right)\right] = Mdx + Ndy \; i.e. \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

where $\frac{\partial f}{\partial x}$ = Partial derivative of $f\left(x, y\right)$ with respect to x (keeping *y* constant).

$\frac{\partial f}{\partial y}$ = Partial derivative of $f\left(x, y\right)$ with respect to *y* (treating *x* as constant).

The necessary and sufficient condition for the differential equation

$Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

An exact differential equation can always be derived from its general solution directly by differentiation without any subsequent multiplication elimination *etc*.

Integrating factor

If an equation of the form $Mdx + Ndy = 0$ is not exact, it can always be made exact by multiplying by some function of *x* and *y*. Such a multiplier is called an *integrating factor*.

Working rule for solving an exact differential equation

Step (i) Compare the given equation with $Mdx + Ndy = 0$ and find out *M* and *N*. Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given equation is exact.

Step (ii) Integrate *M* with respect to *x* treating *y* as a constant.

Step (iii) Integrate *N* with respect to *y* treating *x* as constant and omit those terms which have been already obtained by integrating *M*.

Step (iv) On adding the terms obtained in steps (ii) and (iii) and equating to an arbitrary constant, we get the required solution. In other words, solution of an exact differential equation is

$$\underset{\substack{\text{Regarding } y \text{ as} \\ \text{constant}}}{\int} Mdx + \underset{\substack{\text{Only those terms} \\ \text{not containing } x}}{\int} Ndy = c$$

METHODS FOR SOLVING EXACT DIFFERENTIAL EQUATIONS

1. **Solution by inspection**

If we can write the differential equation in the form

$$f\left(f_1\left(x, y\right) df_1\left(x, y\right)\right) + \phi\left(f_2\left(x, y\right) d\left(f_2\left(x, y\right)\right)\right) + \ldots = 0,$$

then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d\left(x + y\right) = dx + dy$ (ii) $d\left(xy\right) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right)=\frac{2xy\,dx-x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-xdy}{x^2+y^2}$

(x) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$

(xi) $d\left[\ln(xy)\right]=\frac{xdy+ydx}{xy}$

(xii) $d\left(\ln\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\ln\left(x^2+y^2\right)\right]=\frac{xdx+ydy}{x^2+y^2}$

(xiv) $d\left[\ln\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$

(xvi) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(xvii) $d\left(\frac{e^v}{x}\right)=\frac{xe^vdy-e^ydx}{x^2}$

(xviii) $d\left(x^my^n\right)=x^{m-1}\,y^{n-1}\left(my\,dx+nx\,dy\right)$

(xix) $d\left(\sqrt{x^2+y^2}\right)=\frac{xdx+ydy}{\sqrt{x^2+y^2}}$

2. If the given equation $Mdx+Ndy=0$ is homogeneous and $\left(Mx+Ny\right)\neq 0$, then $1/\left(Mx+Ny\right)$ is an integrating factor.
3. If the equation $Mdx+Ndy=0$ is of the form $f_1\left(xy\right)y\ dx+f_2\left(xy\right)x\ dy=0$, then $1/\left(Mx-Ny\right)$ is an integrating factor of $Mdx+Ndy=0$ provided $\left(Mx-Ny\right)\neq 0$.
4. If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone say $f\left(x\right)$, then $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.
5. If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is function of y alone say $f\left(y\right)$, then $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.
6. If the given equation $Mdx+Nay=0$ is of the form $x^{\alpha}y^{\beta}\ \left(my\ dx+nx\,dy\right)=0$, then its I.F. is $x^{km-1-\alpha}y^{kn-1-\beta}$, where k can have any value.

LINEAR EQUATION

An equation of the form $\frac{dy}{dx}+Py=Q$

In which P and Q are functions of x alone or constant is called a linear equation of the first order. The general solution of the above equation can be found as follows.

Multiply both sides of the equation by $e^{\int Pdx}$.

$$\therefore \qquad \frac{dy}{dx}e^{\int Pdx}+Pye^{\int Pdx}=Qe^{\int Pdx}$$

i.e., $\frac{d}{dx}\left(ye^{\int P\,dx}\right)=Qe^{\int P\,dx}.$

$\therefore$ integrating $ye^{\int P\,dx}=\int Qe^{\int P\,dx}dx+c$

or $\mathrm{y}=e^{-\int \mathrm{P}\,dx}\left[\int Qe^{\int \mathrm{P}\,dx}\mathrm{dx}+\mathrm{c}\right]$ is the required solution.

Cor.1 If in the above equation if Q is zero, the general solution is $y = Ce^{-\int P\,dx}$.

Cor.2 If P be a constant and equal to $-m$, then the solution is $y = e^{mx}\left[\int e^{-mx}Q\,dx+C\right]$.

Linear differential equations of the form $\frac{dx}{dy}+Rx=s.$

Sometimes a linear differential equation can be put in the form

$$\frac{dx}{dy}+Rx=s$$

where R and S are functions of y alone or constants.

Note : y is independent variable and x is a dependent variable.

Bernoulli's Equation

An equation of the form $\frac{dy}{dx}+Py=Qy^{n}$,

Where P and Q are functions of x alone, is known as Bernoulli's equation. It is easily reduced to the linear form.

Dividing both sides by y^{n}, we get $y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q.$

Putting $y^{-n+1}=v,$ and hence $\left(-n+1\right)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$ the equation reduces to $\frac{dv}{dx}+\left(1-n\right)Pv=\left(1-n\right)Q.$

This being linear in v can be solved by the method of the previous article.

ORTHOGONAL TRAJECTORY

Any curve which cuts every member of a given family of curves at right angle is called an *orthogonal trajectory* of the family. *For example*, each straight line $y=mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^{2}+y^{2}=a^{2}$.

Procedure for finding the orthogonal trajectory

(i) Let $f\left(x,\ y,\ c\right)=0$ be the equation, where c is an arbitrary parameter.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\frac{dy}{dx} by -\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation in (iii).

DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

The typical equation of the first order and the n^{th} degree can be written as

$$P^{n}+P_{1}p^{n-1}+P_{2}p^{n-2}+......+P_{n-1}p+P_{n}=0 \qquad \text{… (i)}$$

where p stands for $\frac{dy}{dx}$ and $P_{1},\ P_{2},.....,P_{n}$ are function of x and y.

The complete solution of such an equation would involve only one arbitrary constant.

The equations which are of first order but not of the first degree, the following types of equations are discussed.

(a) Equations solvable for $p = dy/dx$ (b) Equations solvable for y

(c) Equations solvable for x (d) Clairut's equations

(i) Resolving left side of equation (i) into factors we have

$$\left(p - f_1(x,y)\right)\left(p - f_2(x,y)\right)......\left(p - f_n(x,y)\right) = 0 \quad \text{... (ii)}$$

It is evident from above that a solution of any one of the equations.

$$p - f_1(x,y) = 0,\ p - f_2(x,y) = 0,\ ..., p - f_n(x,\ y) = 0 \quad \text{... (iii)}$$

is also a solution of (i).

Let the solution of equation (iii) be $g_1(x,y,c_1) = 0,\ g_2(x,y,c_2) = 0,.....g_n(x,y,c_n) = 0$

Where $c_1, c_2, ...c_n$ are arbitrary constant of the integration.

These solutions are evidently just as general, if $c_1 = c_2 = ... = c_n$, since the individual solutions are all independent of one another and all the $c's$ can have any one of an infinite number of values. All the solution can thus without loss of generality be obtained into the single equation.

(ii) **Equation solvable for y**

Suppose the equation is put in the from $y = f(x,p)$

Differentiating this w.r.t x, we shall get an equation in two variables x and p; suppose the solution of the latter equation is $\phi(x,p,c) = 0$.

The p eliminated between this equation and original equation gives a equation between x and y and c, which is the required solution.

(iii) **Equation solvable for x**

Differentiating this w.r.t y and noting that $\frac{dx}{dy} = \frac{1}{p}$, we shall get an equation in two variables y and p. If p be eliminated between the solution of the latter equation (which contains an arbitrary constant) and the original equation we shall get the required solution.

(iv) **Clairut's equation :** The equation of the form $y = px + f(p)$ is known as **Clairut's equation**.

$$y = px + f(p) \quad \text{...(1)}$$

Differentiating w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0 \quad \Rightarrow \quad \frac{dp}{dx} = 0, \quad \text{or} \quad x + f'(p) = 0$$

If $\frac{dp}{dx} = 0$, we have $p = \text{constant} = c$ (say).

Eliminating p from (1) we have $y = cx + f(c)$ as a solution.

If $x + f'(p) = 0$, then by eliminating p, we will obtain another solution. This solution is called *singular solution*.

Note : Sometime transformation to the polar co-ordinates facilitates separation of variables. In this connections it is convenient to remember the following differentials.

If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $x\,dx + y\,dy = r\,dr$

(ii) $x\,dy - y\,dx = r^2 d\theta$

(iii) $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

If $x = r\sec\theta$; $y = r\tan\theta$ then,

(i) $x\ dx - y\ dy = r\,dr$

(ii) $x\ dy - y\ dx = r^2 \sec\theta\, d\theta$.