Session 4

Intersection of a Line and a Circle, Product of the Algebraical Distances *PA* and *PB* is Constant when from *P, A* Secant be Drawn to Cut the Circle in the Points *A* and *B*, The Length of Intercept Cut-off from a Line by a Circle, Tangent to a Circle at a Given Point, Normal to a Circle at a Given Point

Intersection of a Line and a Circle

Let the equation of the circle be

$$x^2 + y^2 = a^2 \quad \text{...(i)}$$

and the equation of the line be

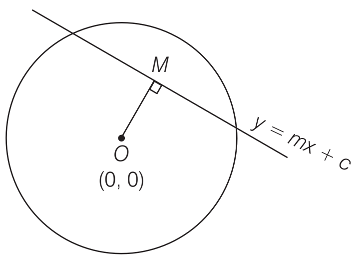
$$y = mx + c \quad \text{...(ii)}$$

From Eq. (i) and Eq. (ii)

$$x^2 + (mx + c)^2 = a^2$$

or $$(1 + m^2)\,x^2 + 2mcx + c^2 - a^2 = 0 \quad \text{...(iii)}$$

Case I When points of intersection are real and distinct, then Eq. (iii) has two distinct roots.



$\therefore$ $$B^2 - 4AC > 0$$

or $$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

or $$a^2 > \frac{c^2}{1 + m^2}$$

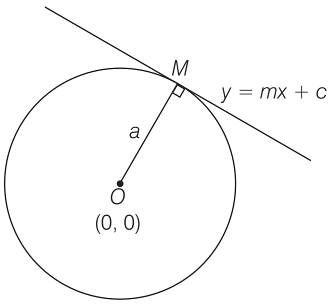
or $$a > \frac{|c|}{\sqrt{(1 + m^2)}} = \text{length of perpendicular}$$

from $(0, 0)$ to $y = mx + c$

$\Rightarrow$ $a >$ length of perpendicular from $(0, 0)$ to $y = mx + c$

Thus, a line intersects a given circle at two distinct points if radius of circle is greater than the length of perpendicular from centre of the circle to the line.

Case II When the points of intersection are coincident, then Eq. (iii) has two equal roots



$\therefore$ $$B^2 - 4AC = 0$$

$\Rightarrow$ $$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

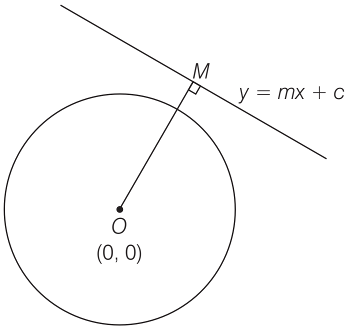
$\therefore$ $$a^2 = \frac{c^2}{(1 + m^2)}$$

or $$a = \frac{|c|}{\sqrt{(1 + m^2)}}$$

$a =$ length of the perpendicular from the point $(0, 0)$ to $y = mx + c$

Thus, a line touches the circle if radius of circle is equal to the length of perpendicular from centre of the circle to the line.

Case III When the points of intersection are imaginary. In this case (iii) has imaginary roots



$\therefore \qquad B^2 - 4AC < 0$

$$4m^2c^2 - 4(1+m^2)(c^2 - a^2) = 0$$

$\therefore \qquad a^2 < \dfrac{c^2}{1+m^2}$

or $a < \dfrac{|c|}{\sqrt{1+m^2}}$ = length of perpendicular from $(0, 0)$ to $y = mx + c$

or $a <$ length of perpendicular from $(0, 0)$ to $y = mx + c$

Thus, a line does not intersect a circle if the radius of circle is less than the length of perpendicular from centre of the circle to the line.

Example 26. ***Find the points of intersection of the line*** $2x + 3y = 18$ ***and the circle*** $x^2 + y^2 = 25$.

Sol. We have, $\qquad 2x + 3y = 18 \qquad$ ....(i)

and $\qquad x^2 + y^2 = 25 \qquad$ ...(ii)

From Eq. (i), $\qquad y = \dfrac{18 - 2x}{3}$

Substituting in Eq. (ii), then $\quad x^2 + \left(\dfrac{18-2x}{3}\right)^2 = 25$

$\Rightarrow \qquad 9x^2 + 4(9 - x)^2 = 225$

$\Rightarrow \qquad 9x^2 + 4(81 - 18x + x^2) = 225$

$\Rightarrow \qquad 13x^2 - 72x + 324 - 225 = 0$

$\Rightarrow \qquad 13x^2 - 72x + 99 = 0$

$\Rightarrow \qquad (x - 3)(13x - 33) = 0$

$\Rightarrow \qquad x = 3 \quad \text{or} \quad x = \dfrac{33}{13}$

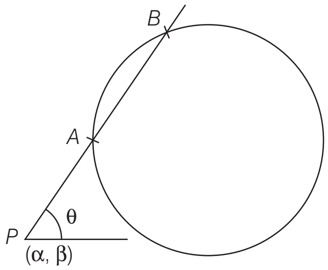
From Eq. (i), $\qquad y = 4 \quad \text{or} \quad y = \dfrac{56}{13}$

Hence, the points of intersection of the given line and the given circle are $(3, 4)$ and $\left(\dfrac{33}{13}, \dfrac{56}{13}\right)$.

Product of the Algebraical Distances *PA* and *PB* is Constant when from *P*, A Secant be Drawn to Cut the Circle in the Points *A* and *B*

If a straight line through $P(\alpha, \beta)$ makes an angle θ with the positive direction of X-axis, then its equation is

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r$$



where, r is the algebraical distance of the point (x, y) from the point $P(\alpha, \beta)$.

$\therefore \quad (x, y) = (\alpha + r\cos\theta, \beta + r\sin\theta)$

If this point lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

or $\qquad (\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 + 2g(\alpha + r\cos\theta) + 2f(\beta + r\sin\theta) + c = 0$

$\Rightarrow \qquad r^2 + 2r(\alpha\cos\theta + \beta\sin\theta + g\cos\theta + f\sin\theta) + (\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c) = 0$

This is quadratic equation in r, then PA and PB are the roots of this equation.

$\therefore \qquad PA \cdot PB = \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c = \text{constant}$

Since, RHS is independent of θ.

Remark

Secants are drawn from a given point A to cut a given circle at the pairs of points $P_1, Q_1; P_2, Q_2; \ldots; P_n, Q_n$, then

$$AP_1 \cdot AQ_1 = AP_2 \cdot AQ_2 = \ldots = AP_n \cdot AQ_n$$

The Length of Intercept Cut-off from a Line by a Circle

Theorem : The length of the intercept cut-off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is

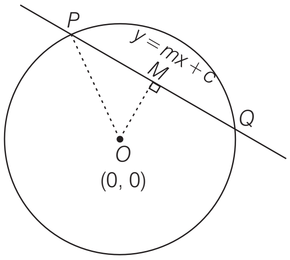
$$2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{(1+m^2)}\right\}}$$

Proof : Draw OM perpendicular to PQ

Now, OM = length of perpendicular from $O\,(0, 0)$ to

$$(y = mx + c) = \frac{|c|}{\sqrt{(1+m^2)}}$$

and $\quad OP$ = radius of the circle $= a$



In ΔOPM, $\quad PM = \sqrt{(OP)^2 - (OM)^2}$

$$= \sqrt{a^2 - \frac{c^2}{(1+m^2)}} = \sqrt{\left\{\frac{a^2\,(1+m^2) - c^2}{1+m^2}\right\}}$$

$\therefore$ $$PQ = 2PM = 2\sqrt{\left\{\frac{a^2\,(1+m^2) - c^2}{1+m^2}\right\}}$$

Remarks

1. If the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$, then intercepted length is zero

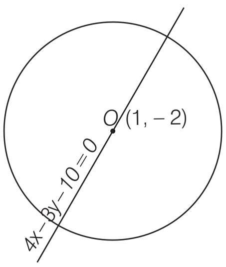
   i.e. $\quad PQ = 0 \quad \Rightarrow 2\sqrt{\left\{\frac{a^2\,(1+m^2) - c^2}{1+m^2}\right\}} = 0$

   $\therefore \quad c^2 = a^2\,(1+m^2)$

   which is the required condition for tangency.
2. If a line touches the circle, then length of perpendicular from the centre upon the line is equal to the radius of the circle.

Example 27. ***Find the length of the intercept on the straight line*** $4x - 3y - 10 = 0$ ***by the circle*** $x^2 + y^2 - 2x + 4y - 20 = 0$.

Sol. Centre and radius of the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are $(1, -2)$ and $\sqrt{1 + 4 + 20} = 5$ respectively.



Let OM be the perpendicular from O on the line

$$4x - 3y - 10 = 0$$

then $$OM = \frac{|\,4\times 1 - 3\times(-2) - 10\,|}{\sqrt{4^2 + (-3)^2}} = 0$$

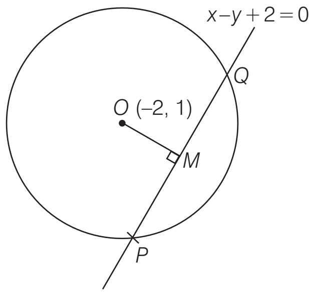
Hence, line $4x - 3y - 10 = 0$ passes through the centre of the circle.

Hence, intercepted length = diameter of the circle

$$= 2 \times 5 = 10$$

Example 28. Find the coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts-off the line $x - y + 2 = 0$.

Sol. Centre and radius of the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ are $(-2, 1)$ and $\sqrt{4 + 1 + 3} = 2\sqrt{2}$ respectively.



Draw perpendicular from O upon $x - y + 2 = 0$ is OM.

Equation of OM which is perpendicular to $x - y + 2 = 0$ is $x + y = \lambda$, it passes through $(-2, 1)$

Then, $\quad -2 + 1 = \lambda$

$\therefore \quad \lambda = -1$

then equation of OM is $x + y + 1 = 0$

Since, M is the mid-point of PQ which is point of intersection of $x - y + 2 = 0$ and $x + y + 1 = 0$, coordinates of M is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Aliter : Let $M \equiv (\alpha, \beta)$, then

$$\frac{\alpha + 2}{1} = \frac{\beta - 1}{-1} = -\frac{(-2 - 1 + 2)}{1 + 1}$$

(Here, M is foot of perpendicular)

$\Rightarrow$ $$\frac{\alpha + 2}{1} = \frac{\beta - 1}{-1} = \frac{1}{2}$$

or $$\alpha = -\frac{3}{2} \text{ and } \beta = \frac{1}{2}$$

$\therefore$ $$M \equiv \left(-\frac{3}{2}, \frac{1}{2}\right)$$

Example 29. For what value of λ will the line $y = 2x + \lambda$ be a tangent to the circle $x^2 + y^2 = 5$?

Sol. Comparing the given line with $y = mx + c$, we get

$m = 2$, $c = \lambda$ and given circle with $x^2 + y^2 = a^2$

then $\quad a^2 = 5$

$\because$ Condition for tangency is

$$c^2 = a^2(1+m^2)$$

$\Rightarrow$ $$\lambda^2 = 5(1+4)$$

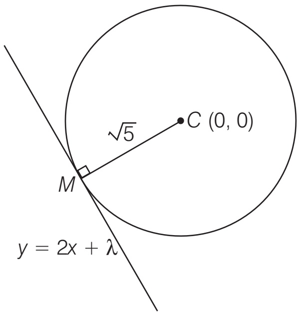
$$\lambda^2 = 25$$

$\therefore$ $$\lambda = \pm 5$$

Aliter : Since, line $y = 2x + \lambda$

or $2x - y + \lambda = 0$

is the tangent to the circle $x^2 + y^2 = 5$ then length of perpendicular from centre upon the line is equal to the radius of the circle



$\therefore$ $$|CM| = \sqrt{5}$$

or $$\frac{|0-0+\lambda|}{\sqrt{4+1}} = \sqrt{5}$$

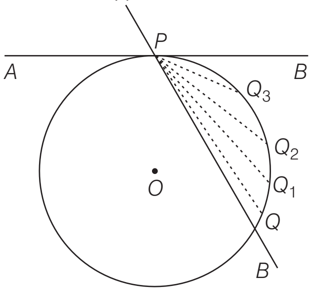
$\Rightarrow$ $$\frac{|\lambda|}{\sqrt{5}} = \sqrt{5}$$

$\Rightarrow$ $$|\lambda| = 5$$

or $$\lambda = \pm 5$$

Tangent to a Circle at a Given Point

Let PQ be a chord and AB be a secant passing through P. Let P be the fixed point and move along the circle towards P, then the secant PQ turns about P. In the limit, when Q coincides with P, then the secant AB becomes a tangent to the circle at the point P.



Different forms of the equations of tangents

1. Point form :

Theorem : The equation of tangent at the point $P(x_1, y_1)$ to a circle

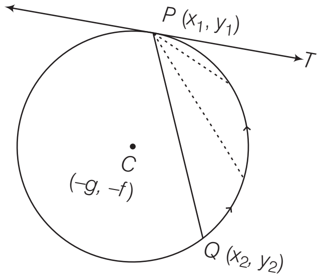
$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Proof : Since, $P(x_1, y_1)$ be a point on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Let $Q(x_2, y_2)$ be any other point on the circle Eq. (i). Since, points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle, therefore



$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(ii)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{...(iii)}$$

On subtracting Eq. (ii) from Eq. (iii), we have

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g$$
$$(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$\Rightarrow$ $$(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)$$
$$(y_2 + y_1 + 2f) = 0$$

$\Rightarrow$ $$\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -\left(\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}\right) \quad \text{...(iv)}$$

Now, the equation of the chord PQ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \quad \text{...(v)}$$

Putting the value of $\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ from Eq. (iv) in Eq. (v), then equation PQ becomes

$$y - y_1 = -\left(\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}\right)(x - x_1) \quad \text{...(vi)}$$

Now, when $Q \to P$ (along the circle), line PQ becomes tangent at P, we have $x_2 \to x_1, y_2 \to y_1$. So, the equation of tangent at $P(x_1, y_1)$ is :

$$y - y_1 = -\left(\frac{x_1 + x_1 + 2g}{y_1 + y_1 + 2f}\right)(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow \quad (y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

On adding $gx_1 + fy_1 + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{[from Eq. (ii)]}$$

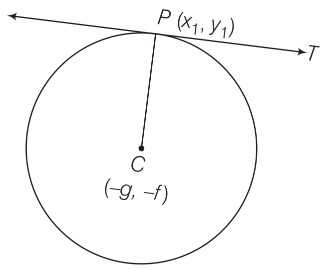
$$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

This is the required equation of the tangent PT to the circle at the point (x_1, y_1).

Aliter : Since, circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$P(x_1, y_1)$ lie on the circle

$$\therefore \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(i)}$$



Its centre is $C(-g, -f)$

$$\therefore \quad \text{Slope of } CP = \frac{y_1 - (-f)}{x_1 - (-g)} = \frac{y_1 + f}{x_1 + g}$$

Since, tangent PT is perpendicular to CP.

$$\therefore \quad \text{Slope of tangent} = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

$\therefore$ Equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$\Rightarrow \quad (y - y_1)(y_1 + f) + (x_1 + g)(x - x_1) = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

On adding $gx_1 + fy_1 + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{[from Eq. (i)]}$$

or $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

This is the required equation of the tangent PT to the circle at the point $P(x_1, y_1)$.

Remarks

1. For equation of tangent of circle at (x_1, y_1), substitute xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1y}{2}$ for xy and keep the constant as such.
2. This method of tangent at (x_1, y_1) is applied only for any conics of second degree. i.e. equation of tangent of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Wrong process : Mostly students use wrong process

Suppose any curve

$$ax^3 + by^3 = c$$

or $\quad a(x)(x^2) + b(y)(y^2) = c$

Equation of tangent at (x_1, y_1)

$$\Rightarrow \quad a\left(\frac{x + x_1}{2}\right)xx_1 + b\left(\frac{y + y_1}{2}\right)yy_1 = c^2$$

which is a second degree conic not the equation of tangent.

Reason : This method is applicable only for second degree conic, its a third degree conic. (find its tangent only by calculus)

Example 30. ***Prove that the tangents to the circle $x^2 + y^2 = 25$ at $(3, 4)$ and $(4, -3)$ are perpendicular to each other.***

Sol. The equations of tangents to $x^2 + y^2 = 25$ at (3, 4) and (4, – 3) are

$$3x + 4y = 25 \quad \text{...(i)}$$

and $\quad 4x - 3y = 25 \quad$ ...(ii)

respectively.

Now, $\quad$ slope of Eq. (i) $= -\frac{3}{4} = m_1 \quad$ (say)

and $\quad$ slope of Eq. (ii) $= \frac{4}{3} = m_2 \quad$ (say)

Clearly, $\quad m_1m_2 = -1$

Hence, Eq. (i) and Eq. (ii) are perpendicular to each other.

Example 31. Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $[a(1 + \cos\alpha), a\sin\alpha]$.

Sol. The equation of tangent of $x^2 + y^2 - 2ax = 0$ at $[a(1 + \cos\alpha), a\sin\alpha]$ is

$$x \cdot a(1 + \cos\alpha) + y \,.\, a\sin\alpha - a[x + a(1 + \cos\alpha)] = 0$$

$$\Rightarrow \quad ax\cos\alpha + ay\sin\alpha - a^2(1 + \cos\alpha) = 0$$

or $\quad x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$

Example 32. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ touch at $(3,-1)$.

Sol. Equation of tangent at $(3,-1)$ of the circle $x^2+y^2-4x+6y+8=0$ is

$$3x+(-1)y-2(x+3)+3(y-1)+8=0$$

or $\quad x+2y-1=0 \quad$ ...(i)

and equation of tangent at $(3,-1)$ of the circle $x^2+y^2-10x-6y+14=0$ is

$$3\cdot x+(-1)\cdot y-5(x+3)-3(y-1)+14=0$$

or $\quad -2x-4y+2=0$

or $\quad x+2y-1=0 \quad$ ...(ii)

which is the same as Eq (i).

Hence, the given circles touch at $(3,-1)$.

2. Parametric form :

Theorem : The equation of tangent to the circle $x^2+y^2=a^2$ at the point $(a\cos\theta, a\sin\theta)$ is

$$x\cos\theta+y\sin\theta=a$$

Proof : The equation of tangent of $x^2+y^2=a^2$ at (x_1,y_1) is $xx_1+yy_1=a^2$ (using point form of the tangent)

Putting $\quad x_1=a\cos\theta,\ y_1=a\sin\theta$

then, we get $\quad x\cos\theta+y\sin\theta=a$

***Corollary* 1 :** Equation of chord joining $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$ is

$$x\cos\left(\frac{\theta+\phi}{2}\right)+y\sin\left(\frac{\theta+\phi}{2}\right)=a\cos\left(\frac{\theta-\phi}{2}\right)$$

***Corollary* 2 :** Point of intersection of tangents at $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$ is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

Remembering method :

$\because \quad x\cos\left(\frac{\theta+\phi}{2}\right)+y\sin\left(\frac{\theta+\phi}{2}\right)=a\cos\left(\frac{\theta-\phi}{2}\right)$

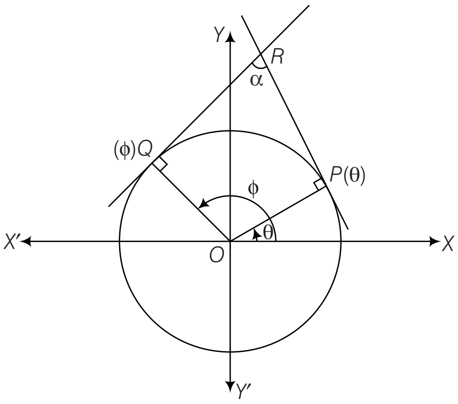
or $\quad x\left\{\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}+y\left\{\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}=a^2$

We get $\quad \left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$

***Corollary* 3 :** The angle between a pair of tangents from a point P to the circle $x^2+y^2=a^2$ is α. Then, the locus of the point P is

$$x^2+y^2=\frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$$

Proof



$\because \quad \phi-\theta+\alpha=180^\circ$

$\therefore \quad \frac{\theta-\phi}{2}=-\left(90^\circ-\frac{\alpha}{2}\right)$

or $\quad \cos\left(\frac{\theta-\phi}{2}\right)=\sin\left(\frac{\alpha}{2}\right)$

Now, point of intersection is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)},\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right)$$

Let $\quad x=\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$ and $y=\frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$

$\therefore \quad x^2+y^2=\frac{a^2}{\sin^2\left(\frac{\alpha}{2}\right)}$

Remarks

1. The angle between a pair of tangents from a point P to the circle $x^2+y^2+2gx+2fy+c=0$ is 2θ, then the locus of P is $x^2+y^2+2gx+2fy+c=(g^2+f^2-c)\cot^2\theta$
2. If angle between a pair of tangents from a point P to the circle $x^2+y^2=a^2$ is $\frac{\pi}{2}$, then the locus of P is
$$x^2+y^2=2a^2 \qquad \left(\text{Here, } \alpha=\frac{\pi}{2}\right)$$
which is **director circle** of $x^2+y^2=a^2$.
($\because$ locus of point of intersection of perpendicular tangents is **director circle**)
3. The equation of the tangent to the circle $(x-a)^2+(y-b)^2=r^2$ at the point $(a+r\cos\theta, b+r\sin\theta)$ is
$$(x-a)\cos\theta+(y-b)\sin\theta=r.$$

Example 33. ***The angle between a pair of tangents from a point P to the circle $x^2+y^2=25$ is $\frac{\pi}{3}$. Find the equation of the locus of the point P.***

Sol. Here, $\alpha=\frac{\pi}{3}$

$\therefore$ Required locus is $x^2+y^2=\dfrac{25}{\sin^2\left(\frac{\pi}{6}\right)}=100$

Example 34. ***The angle between a pair of tangents from a point P to the circle $x^2+y^2-6x-8y+9=0$ is $\frac{\pi}{3}$. Find the equation of the locus of the point P.***

Sol. Here, $2\theta=\frac{\pi}{3}$ or $\theta=\frac{\pi}{6}$

$\therefore$ Required locus is
$$x^2+y^2-6x-8y+9=(9+16-9)\cot^2\frac{\pi}{6}$$
or $$x^2+y^2-6x-8y+9=16\times3$$
or $$x^2+y^2-6x-8y-39=0$$

3. Slope form :

Theorem : The equation of a tangent of slope m to the circle $x^2+y^2=a^2$ is $y=mx\pm a\sqrt{(1+m^2)}$ and the coordinates of the point of contact are
$$\left(\pm\frac{am}{\sqrt{(1+m^2)}}, \mp\frac{a}{\sqrt{(1+m^2)}}\right)$$

Proof : Let $y=mx+c$ is the tangent of the circle $x^2+y^2=a^2$.

$\therefore$ Length of perpendicular from centre of circle $(0,0)$ on $(y=mx+c)$ = radius of circle

$$\therefore \quad \frac{|c|}{\sqrt{(1+m^2)}}=a \Rightarrow c=\pm a\sqrt{(1+m^2)}$$

On substituting this value of c in $y=mx+c$, we get
$$\boldsymbol{y=mx\pm a\sqrt{(1+m^2)}} \qquad \text{...(i)}$$
which are the required equations of tangents.

Also, let (x_1,y_1) be the point of contact, then equation of tangent at (x_1,y_1) to the circle $x^2+y^2=a^2$ is
$$xx_1+yy_1=a^2 \qquad \text{...(ii)}$$
On comparing Eq. (i) and Eq. (ii), we get
$$\frac{x_1}{m}=\frac{y_1}{-1}=\frac{a^2}{\pm a\sqrt{(1+m^2)}}$$
$$\Rightarrow \quad \frac{x_1}{m}=-\frac{y_1}{1}=\pm\frac{a}{\sqrt{(1+m^2)}}$$
$$\Rightarrow \quad x_1=\pm\frac{am}{\sqrt{1+m^2}} \quad \text{and} \quad y_1=\mp\frac{a}{\sqrt{(1+m^2)}}$$
Hence, $(x_1,y_1)=\left(\pm\dfrac{am}{\sqrt{(1+m^2)}}, \mp\dfrac{a}{\sqrt{(1+m^2)}}\right)$

Corollary : It also follows that $y=mx+c$ is a tangent to $x^2+y^2=a^2$, if $c^2=a^2(1+m^2)$ which is condition of tangency.

Remarks

1. The reason why there are two equations $y=mx\pm a\sqrt{1+m^2}$, there are two tangents, both are parallel and at the ends of diameter.
2. The line $ax+by+c=0$ is tangent to the circle $x^2+y^2=r^2$ if and only if $c^2=r^2(a^2+b^2)$.
3. If the line $y=mx+c$ is the tangent to the circle $x^2+y^2=r^2$, then point of contact is given by $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$
4. If the line $ax+by+c=0$ is the tangent to the circle $x^2+y^2=r^2$, then point of contact is given by $\left(-\frac{ar^2}{c}, -\frac{br^2}{c}\right)$.
5. The condition that the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$ is $(lg+mf-n)^2=(l^2+m^2)(g^2+f^2-c)$.
6. Equation of tangent of the circle $x^2+y^2+2gx+2fy+c=0$ in terms of slope is
$y+f=m(x+g)\pm\sqrt{(g^2+f^2-c)(1+m^2)}$
7. The equation of tangents of slope m to the circle $(x-a)^2+(y-b)^2=r^2$ are given by
$(y-b)=m(x-a)\pm r\sqrt{(1+m^2)}$
and the coordinates of the points of contact are
$$\left(a\pm\frac{mr}{\sqrt{(1+m^2)}}, b\mp\frac{r}{\sqrt{(1+m^2)}}\right)$$

Example 35. Find the equations of the tangents to the circle $x^2+y^2=9$, which

(i) are parallel to the line $3x+4y-5=0$

(ii) are perpendicular to the line $2x+3y+7=0$

(iii) make an angle of 60° with the X-axis

Sol. (i) Slope of $3x+4y-5=0$ is $-\frac{3}{4}$

Let $m=-\frac{3}{4}$

and equation of circle is $x^2+y^2=9$

$\therefore$ Equations of tangents

$$y=-\frac{3}{4}x\pm 3\sqrt{\left(1+\left(-\frac{3}{4}\right)^2\right)}$$

$\Rightarrow \quad 4y=-3x\pm 15$ or $3x+4y\pm 15=0$

(ii) Slope of $2x+3y+7=0$ is $-\frac{2}{3}$

$\therefore$ Slope of perpendicular to $2x+3y+7=0$ is $\frac{3}{2}=m$ (say)

and given circle is $\quad x^2+y^2=9$

$\therefore$ Equations of tangents perpendicular to $2x+3y+7=0$ is

$$y=\frac{3}{2}x\pm 3\sqrt{1+\left(\frac{3}{2}\right)^2}$$

$\Rightarrow \quad 2y=3x\pm 3\sqrt{13}$

or $\quad 3x-2y\pm 3\sqrt{13}=0$

(iii) Since, tangent make an angle 60° with the X-axis

$\therefore \quad m=\tan 60°=\sqrt{3}$

and given circle $\quad x^2+y^2=9$

$\therefore$ Equation of tangents $y=\sqrt{3}x\pm 3\sqrt{1+(\sqrt{3})^2}$

or $\quad \sqrt{3}\,x-y\pm 6=0$

Aliter :

(i) Let tangent parallel to $3x+4y-5=0$ is

$$3x+4y+\lambda=0 \quad \text{...(i)}$$

and circle $\quad x^2+y^2=9$

then perpendicular distance from $(0,0)$ on Eq. (i) = radius

$$\frac{|\lambda|}{\sqrt{(3^2+4^2)}}=3$$

or $\quad |\lambda|=15$

$\therefore \quad \lambda=\pm 15$

From Eq. (i), equations of tangents are

$$3x+4y\pm 15=0$$

(ii) Let tangent perpendicular to $2x+3y+7=0$ is

$$3x-2y+\lambda=0 \quad \text{...(ii)}$$

and circle $\quad x^2+y^2=9$

then, perpendicular distance from $(0,0)$ on Eq. (ii) = radius

$$\frac{|\lambda|}{\sqrt{3^2+(-2)^2}}=3$$

or $\quad |\lambda|=3\sqrt{13}$

or $\quad \lambda=\pm 3\sqrt{13}$

From Eq. (ii), equations of tangents are

$$3x-2y\pm 3\sqrt{13}=0$$

(iii) Let equation of tangent which makes an angle of 60° with the X-axis is

$$y=\sqrt{3}x+c \quad \text{...(iii)}$$

or $\quad \sqrt{3}x-y+c=0$

and circle $\quad x^2+y^2=9$

then, perpendicular distance from $(0,0)$ to Eq. (iii) = radius

$$\frac{|c|}{\sqrt{(\sqrt{3})^2+(-1)^2}}=3$$

or $\quad |c|=6$

or $\quad c=\pm 6$

From Eq. (iii), equations of tangents are

$$\sqrt{3}x-y\pm 6=0$$

Example 36. Prove that the line $lx+my+n=0$ touches the circle $(x-a)^2+(y-b)^2=r^2$ if $(al+bm+n)^2=r^2(l^2+m^2)$.

Sol. If the line $lx+my+n=0$ touches the circle $(x-a)^2+(y-b)^2=r^2$, then length of the perpendicular from the centre = radius

$$\frac{|la+mb+n|}{\sqrt{(l^2+m^2)}}=r$$

$$\Rightarrow \quad (la+mb+n)^2=r^2(l^2+m^2)$$

Aliter :

Here, line is $lx+my+n=0$ and circle is $(x-a)^2+(y-b)^2=r^2$. Here, centre of circle (a,b) shift at $(0,0)$, then replacing x by $x+a$ and y by $y+b$ in the equation of straight line $lx+my+n=0$ and circle $(x-a)^2+(y-b)^2=r^2$, the new form of straight line and circle are

$$l(x+a)+m(y+b)+n=0$$

or $\quad lx+my+(al+mb+n)=0 \quad \text{...(i)}$

and $\quad x^2+y^2=r^2 \quad \text{...(ii)}$

respectively.

On comparing Eq. (i) with $y=Mx+C$

then $\quad M=-\frac{l}{m}$

and $\quad C=-\frac{(al+bm+n)}{m}$

Since, Eq. (i) is the tangent of Eq. (ii), then

$$C^2 = r^2 (1 + M^2)$$

or $$\frac{(al + bm + n)^2}{m^2} = r^2 \left(1 + \frac{l^2}{m^2}\right)$$

or $$(al + bm + n)^2 = r^2 (l^2 + m^2)$$

Example 37. Show that the line $3x - 4y = 1$ touches the circle $x^2 + y^2 - 2x + 4y + 1 = 0$. Find the coordinates of the point of contact.

Sol. The centre and radius of the circle

$$x^2 + y^2 - 2x + 4y + 1 = 0 \text{ are } (1, -2)$$

and $\sqrt{(-1) + (2)^2 - 1} = 2$ respectively.

Since, length of perpendicular from centre $(1, -2)$ on $3x - 4y = 1$ is

$$\frac{|3 \times 1 - 4 \times (-2) - 1|}{\sqrt{(3)^2 + (-4)^2}} = \frac{10}{5}$$

$$= 2 = \text{radius of the circle}$$

Hence, $3x - 4y = 1$ touches the circle

$$x^2 + y^2 - 2x + 4y + 1 = 0$$

Second part : Let point of contact is (x_1, y_1), then tangent at (x_1, y_1) on $x^2 + y^2 - 2x + 4y + 1 = 0$ is

$$xx_1 + yy_1 - (x + x_1) + 2(y + y_1) + 1 = 0$$

$\Rightarrow$ $$x(x_1 - 1) + y(y_1 + 2) - x_1 + 2y_1 + 1 = 0 \quad \text{...(i)}$$

and given line $$3x - 4y - 1 = 0 \quad \text{...(ii)}$$

Since, Eq. (i) and Eq. (ii) are identical, then comparing Eq. (i) and Eq. (ii), we get

$$\frac{x_1 - 1}{3} = \frac{y_1 + 2}{-4} = \frac{-x_1 + 2y_1 + 1}{-1}$$

or $$x_1 = -\frac{1}{5} \text{ and } y_1 = -\frac{2}{5}$$

$\therefore$ Point of contact is $\left(-\frac{1}{5}, -\frac{2}{5}\right)$.

Aliter for second part : Since, perpendicular line to tangent always passes through the centre of the circle, perpendicular line to

$$3x - 4y = 1 \quad \text{...(i)}$$

is $$4x + 3y = \lambda \quad \text{...(ii)}$$

which passes through $(1, -2)$, then

$$4 - 6 = \lambda$$

$\therefore$ $$\lambda = -2$$

From Eq. (ii), $$4x + 3y = -2 \quad \text{...(iii)}$$

Solving Eq. (i) and Eq. (iii), we get the point of contact i.e. $x = -\frac{1}{5}$ and $y = -\frac{2}{5}$

Hence, point of contact is $\left(-\frac{1}{5}, -\frac{2}{5}\right)$.

Example 38. If $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$, prove that the point (l, m) lies on the circle $x^2 + y^2 = a^{-2}$.

Sol. Since, $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$.

Then, length of perpendicular from $(0, 0)$ on $lx + my = 1$ is equal to radius

then, $\frac{|-1|}{\sqrt{l^2 + m^2}} = a$ or $l^2 + m^2 = a^{-2}$

Hence, locus of (l, m) is $x^2 + y^2 = a^{-2}$

Aliter : Let the point of contact of line $lx + my = 1$ and circle $x^2 + y^2 = a^2$ is (x_1, y_1), then tangent of circle at (x_1, y_1) is $xx_1 + yy_1 = a^2$

Since, $xx_1 + yy_1 = a^2$ and $lx + my = 1$

are identical, then $\frac{x_1}{l} = \frac{y_1}{m} = \frac{a^2}{1}$

$\therefore$ $$x_1 = la^2, y_1 = ma^2$$

but (x_1, y_1) lie on $x^2 + y^2 = a^2$

then, $$l^2 a^4 + m^2 a^4 = a^2$$

$\therefore$ $$l^2 + m^2 = a^{-2}$$

$\therefore$ Locus of (l, m) is $x^2 + y^2 = a^{-2}$

Example 39. Show that the line $(x - 2)\cos\theta + (y - 2)\sin\theta = 1$ touches a circle for all values of θ. Find the circle.

Sol. Given line is $(x - 2)\cos\theta + (y - 2)\sin\theta$

$$1 = \cos^2\theta + \sin^2\theta$$

On comparing

$$x - 2 = \cos\theta \quad \text{...(i)}$$

and $$y - 2 = \sin\theta \quad \text{...(ii)}$$

Squaring and adding Eq. (i) and Eq. (ii), then

$$(x - 2)^2 + (y - 2)^2 = \cos^2\theta + \sin^2\theta$$

$\Rightarrow$ $$(x - 2)^2 + (y - 2)^2 = 1$$

or $$x^2 + y^2 - 4x - 4y + 7 = 0$$

Aliter : Since, tangent at $(\cos\theta, \sin\theta)$ of

$$x^2 + y^2 = 1 \quad \text{...(i)}$$

is $$x\cos\theta + y\sin\theta = 1 \quad \text{...(ii)}$$

replacing x by $x - 2$ and y by $y - 2$ in Eqs. (i) and (ii), then

$$(x - 2)^2 + (y - 2)^2 = 1 \quad \text{...(iii)}$$

$$(x - 2)\cos\theta + (y - 2)\sin\theta = 1 \quad \text{...(iv)}$$

Hence, Eq. (iv) touches the circle Eq. (iii).

$\therefore$ Equation of circle is

$$(x - 2)^2 + (y - 2)^2 = 1$$

or $$x^2 + y^2 - 4x - 4y + 7 = 0$$

Normal to a Circle at a Given Point

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point and always passes through the centre of the circle.

Different form of the Equation of Normals

1. Point form :

Theorem : The equation of normal at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

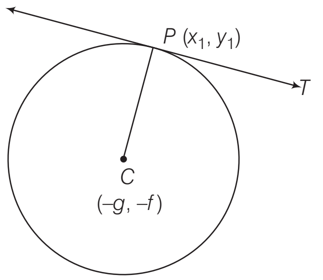
$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

Proof :

Equation of the given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Its centre C is $(-g, -f)$



Let $P(x_1, y_1)$ be the given point.

$\because$ Normal of the circle at $P(x_1, y_1)$ passes through centre $C(-g, -f)$ of the circle.

Then, equation of normal CP passes through the points $C(-g, -f)$ and $P(x_1, y_1)$ is

$$y - y_1 = \frac{(y_1 + f)}{(x_1 + g)}(x - x_1)$$

or $$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

This is the required equation of normal at $P(x_1, y_1)$ of the given circle.

Remark

Easy method to find normal at (x_1, y_1) of second degree conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

then, according to determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

write first two rows as $ax_1 + hy_1 + g$ and $hx_1 + by_1 + f$

Then, normal at (x_1, y_1) of conic (i)

$$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$$

***Corollary* 1 :** Equation of normal of $x^2 + y^2 = a^2$ at (x_1, y_1) is

$$\frac{x - x_1}{1 \cdot x_1 + 0 + 0} = \frac{y - y_1}{0 + 1 \cdot y_1 + 0}$$

(Here, $g, f = 0$ and $a = b = 1$)

$\Rightarrow$ $$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1}$$

or $$\frac{x}{x_1} = \frac{y}{y_1}$$

***Corollary* 2 :** Equation of normal of $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f} \qquad \text{(Here, } a = b = 1 \text{ and } h = 0)$$

Remarks

1. Normal always passes through the centre of the circle. Just write the equation of the line joining (x_1, y_1) and the centre of the circle.
2. The equations of the normals show that they pass through the centre i.e. the normals are the radii which we know from *Euclidean geometry*.

Example 40. ***Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.***

Sol. Given circle is $x^2 + y^2 - 2x = 0$

Centre of given circle is $(1, 0)$

Since, normal is parallel to $x + 2y = 3$

let the equation of normal is $x + 2y = \lambda$

Since, normal passes through the centre of the circle i.e. $(1, 0)$

then $1 + 0 = \lambda$

$\therefore$ $\lambda = 1$

then, equation of normal is $x + 2y = 1$

or $x + 2y - 1 = 0$

Aliter Equation of normal at (x_1, y_1) of $x^2 + y^2 - 2x = 0$ is

$$\frac{x - x_1}{x_1 - 1} = \frac{y - y_1}{y_1 - 0}$$

or Slope $= \dfrac{y_1}{x_1 - 1} = m_1$ (say)

Since normal is parallel to $x + 2y = 3$

$\therefore$ Slope $= -\dfrac{1}{2} = m_2$ (say)

but given $m_1 = m_2$

$$\frac{y_1}{x_1 - 1} = -\frac{1}{2} \text{ or } x_1 + 2y_1 - 1 = 0$$

$\therefore$ Locus of (x_1, y_1) is $x + 2y - 1 = 0$

Example 41. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point $(5, 6)$.

Sol. Equation of the normal at $(5, 6)$ is

$$\frac{x-5}{5-\frac{5}{2}} = \frac{y-6}{6+1} \Rightarrow \frac{x-5}{\frac{5}{2}} = \frac{y-6}{7} \Rightarrow \frac{2x-10}{5} = \frac{y-6}{7}$$

$\Rightarrow \quad 14x - 70 = 5y - 30$

$\therefore \quad 14x - 5y - 40 = 0$

Aliter I : Since, centre of the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ is $\left(\frac{5}{2}, -1\right)$, normal at $(5,6)$ is the equation of a line, which passes through $\left(\frac{5}{2}, -1\right)$ and $(5, 6)$ is

$$y + 1 = \frac{6+1}{5-\frac{5}{2}}\left(x - \frac{5}{2}\right) \Rightarrow y + 1 = \frac{14}{5}\left(x - \frac{5}{2}\right)$$

$\Rightarrow \quad y + 1 = \dfrac{7}{5}(2x - 5)$

$\Rightarrow \quad 5y + 5 = 14x - 35 \quad$ or $\quad 14x - 5y - 40 = 0$

Aliter II : Equation of tangent at $(5, 6)$ is

$$5 \cdot x + 6 \cdot y - \frac{5}{2}(x + 5) + (y + 6) - 48 = 0$$

$\Rightarrow \quad 10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$

$\Rightarrow \quad 5x + 14y - 109 = 0$

Slope of tangent $= -\dfrac{5}{14}$

$\therefore$ Slope of normal $= \dfrac{14}{5}$

$\therefore$ Equation of normal at $(5, 6)$ with slope $\dfrac{14}{5}$ is

$$y - 6 = \frac{14}{5}(x - 5)$$

$\Rightarrow \quad 5y - 30 = 14x - 70$

or $\quad 14x - 5y - 40 = 0$

2. Parametric form

Since, parametric coordinates of circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.

$\therefore$ Equation of normal at $(a\cos\theta, a\sin\theta)$ is

$$\frac{x}{a\cos\theta} = \frac{y}{a\sin\theta}$$

or $\quad \dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta}$ or $y = x\tan\theta$

or $\quad y = mx$, where $m = \tan\theta$

which is slope form of normal.

Exercise for Session 4

1. The length of the chord cut-off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$ is

(a) $\dfrac{5}{6}$ (b) $\dfrac{6}{5}$ (c) $\dfrac{6}{\sqrt{5}}$ (d) $\dfrac{\sqrt{5}}{6}$

2. Circle $x^2 + y^2 - 4x - 8y - 5 = 0$ will intersect the line $3x - 4y = \lambda$ in two distinct points, if

(a) $-10 < \lambda < 5$ (b) $9 < \lambda < 20$ (c) $-35 < \lambda < 15$ (d) $-16 < \lambda < 30$

3. If the line $3x - 4y + \lambda = 0$, $(\lambda > 0)$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $\lambda + a + b$ is equal to

(a) -22 (b) -20 (c) 20 (d) 22

4. Tangent which is parallel to the line $x - 3y - 2 = 0$ of the circle $x^2 + y^2 - 4x + 2y - 5 = 0$, has point/points of contact

(a) $(1, -2)$ (b) $(-1, 2)$ (c) $(3, 4)$ (d) $(3, -4)$

5. If a circle, whose centre is $(-1, 1)$ touches the straight line $x + 2y = 12$, then the co-ordinates of the point of contact are

(a) $\left(-\dfrac{7}{2}, -4\right)$ (b) $\left(-\dfrac{18}{5}, -\dfrac{21}{5}\right)$ (c) $(2, -7)$ (d) $(-2, -5)$

6. The area of the triangle formed by the tangent at the point (a, b) to the circle $x^2 + y^2 = r^2$ and the coordinate axes is

(a) $\frac{r^4}{2ab}$ (b) $\frac{r^4}{2|ab|}$ (c) $\frac{r^4}{ab}$ (d) $\frac{r^4}{|ab|}$

7. The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which make equal intercepts on the positive coordinate axes is

(a) $x + y = 2$ (b) $x + y = 2\sqrt{2}$ (c) $x + y = 4$ (d) $x + y = 8$

8. If $a > 2b > 0$, then the positive value of m for which $y = mx - b\sqrt{(1+m^2)}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

(a) $\frac{2b}{\sqrt{(a^2 - 4b^2)}}$ (b) $\frac{\sqrt{(a^2 - 4b^2)}}{2b}$ (c) $\frac{2b}{a - 2b}$ (d) $\frac{b}{a - 2b}$

9. The angle between a pair of tangents from a point P to the circle $x^2 + y^2 = 16$ is $\frac{\pi}{3}$ and locus of P is $x^2 + y^2 = r^2$, then value of r is

(a) 5 (b) 6 (c) 7 (d) 8

10. The normal at the point $(3, 4)$ on a circle cuts the circle at the point $(-1, -2)$. Then, the equation of the circle is

(a) $x^2 + y^2 + 2x - 2y - 13 = 0$ (b) $x^2 + y^2 - 2x - 2y - 11 = 0$
(c) $x^2 + y^2 - 2x + 2y + 12 = 0$ (d) $x^2 + y^2 - 2x - 2y + 14 = 0$

11. The line $ax + by + c = 0$ is a normal to the circle $x^2 + y^2 = r^2$. The portion of the line $ax + by + c = 0$ intercepted by this circle is of length

(a) $\sqrt{r}$ (b) r (c) r^2 (d) $2r$

12. If the line $ax + by + c = 0$ touches the circle $x^2 + y^2 - 2x = \frac{3}{5}$ and is normal to the circle $x^2 + y^2 + 2x - 4y + 1 = 0$, then (a, b) are

(a) (1, 3) (b) (3, 1) (c) (1, 2) (d) (2, 1)

13. Show that for all values of θ, $x \sin\theta - y \cos\theta = a$ touches the circle $x^2 + y^2 = a^2$.

14. Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$
which are (i) parallel (ii) perpendicular to the line $3x - 4y - 1 = 0$.

15. Find the equation of the family of circle which touch the pair of straight lines $x^2 - y^2 + 2y - 1 = 0$.

16. Find the value of λ so that the line $3x - 4y = \lambda$ may touch the circle $x^2 + y^2 - 4x - 8y - 5 = 0$.

17. Show that the area of the triangle formed by the positive X-axis, the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $2\sqrt{3}$.

Answers

Exercise for Session 4

1. (c) **2.** (c) **3.** (c) **4.** (d) **5.** (b)
6. (b) **7.** (b) **8.** (a) **9.** (d) **10.** (b)
11. (d) **12.** (a)

14. (i) $3x - 4y + 20 = 0$ and $3x - 4y - 10 = 0$ (ii) $4x + 3y + 5 = 0$ and $4x + 3y - 25 = 0$,

15. centre of the circle $(0, 1, \pm r\sqrt{2})$, where r is radius

16. 15, -35