Chapter 02

RELATIONS, FUNCTIONS & INVERSE TRIGONOMETRIC FUNCTIONS

RELATIONS

1. CARTESIAN PRODUCT OF SETS

Definition : Given two non-empty sets P & Q. The cartesian product $P \times Q$ is the set of all ordered pairs of elements from P & Q i.e.

$$P \times Q = \{(p, q); p \in P; q \in Q\}$$

2. RELATIONS

2.1 Definition : Let A & B be two non-empty sets. Then any subset 'R' of $A \times B$ is a relation from A to B.

If $(a, b) \in R$, then we write it as a R b which is read as 'a is related to b' by the relation R, 'b' is also called image of 'a' under R.

2.2 Domain and range of a relation : If R is a relation from A to B, then the set of first elements in R is called domain & the set of second elements in R is called range of R. symbolically.

$$\text{Domain of } R = \{x : (x, y) \in R\}$$

$$\text{Range of } R = \{y : (x, y) \in R\}$$

The set B is called co-domain of relation R.

Note that range $\subset$ co-domain.

NOTES :

Total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and total number of relations is 2^{pq}.

2.3 Inverse of a relation : Let A, B be two sets and let R be a relation from a set A to set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Domain (R) = Range (R^{-1}) and Range (R) = Domain (R^{-1}).

3. TYPES OF RELATION

(a) Void Relation : Let A be a non-empty set. Then $\phi \subseteq A \times A$ and so it is a relation on set A. This relation is called the void or empty relation on set A.

(b) Universal Relation : Let A be a non-empty set. Then, $A \times A$ is known as the universal relation set A.

(c) Identity Relation : Let A be a non-empty set. Then, $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

(d) Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

(e) Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

(f) Antisymmetric Relation : A relation R on set A is said to be antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

(g) Transitive Relation :

We say that a relation R on a set A is transitive if whenever a R b and b R c, then a R c.

It means that if a related to b and b related to c then a related to c for all $(a,b,c) \in A$.

(h) Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive

(ii) it is symmetric and

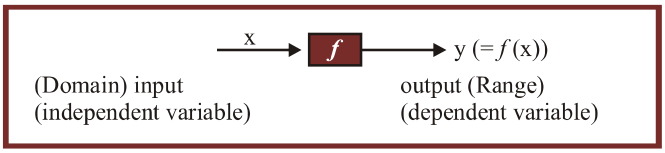
(iii) it is transitive

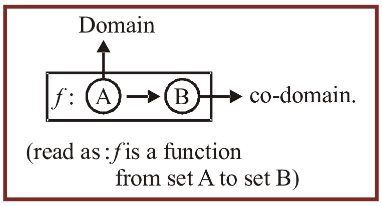
FUNCTIONS

1. DEFINITION

A relation 'f' from a set A to set B is said to be a function if every element of set A has one and only one image in set B.

Notations





2. DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Domain : When we define $y = f(x)$ with a formula and the domain is not stated explicitly, the domain is assumed to be the largest set of x–values for which the formula gives real y–values.

The domain of $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Rules for finding Domain :

(i) Expression under even root (i.e. square root, fourth root etc.) should be non–negative.

(ii) Denominator $\neq 0$.

(iii) $\log_a x$ is defined when $x > 0$, $a > 0$ and $a \neq 1$.

(iv) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) . g(x)$ is $D_1 \cap D_2$. While domain of

$$\frac{f(x)}{g(x)} \text{ is } D_1 \cap D_2 - \{x : g(x) = 0\}.$$

Range : The set of all f-images of elements of A is known as the range of f & denoted by $f(A)$.

Range $= f(A) = \{f(x) : x \in A\}$;

$f(A) \subseteq B$ {Range $\subseteq$ Co-domain}.

Rule for finding range :

First of all find the domain of $y = f(x)$

(i) If domain $\in$ finite number of points

$\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

(ii) If domain $\in$ R or R – {some finite points}

Put $y = f(x)$

Then express x in terms of y. From this find y for x to be defined. (i.e., find the values of y for which x exists).

(iii) If domain $\in$ a finite interval, find the least and greater value for range using monotonocity.

NOTES :

Two functions f & g are said to be equal (identical) iff

1. Domain of f = Domain of g

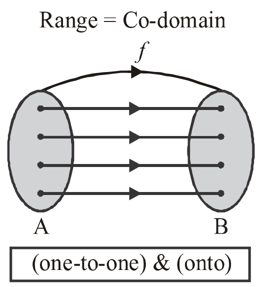
2. Co-Domain of f = Co-domain of g

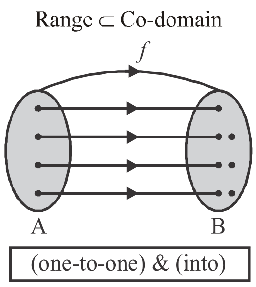
3. $f(x) = g(x)$ $\forall$ $x \in$ Domain.

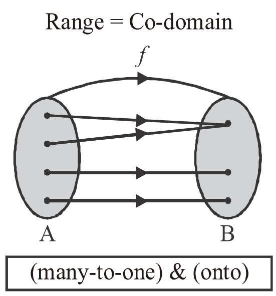
3. CLASSIFICATION OF FUNCTION

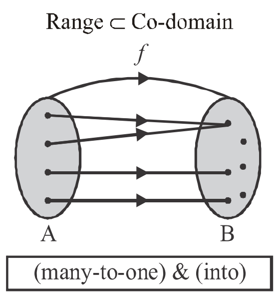
Definition 1 : A function $f : X \to Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called many-one.

Definition 2 : A function $f : X \to Y$ is said to be onto (or surjective, if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.









Methods to check one-one mapping

1. **Theoretically :** If $f(x_1) = f(x_2)$ $\Rightarrow x_1 = x_2$ only, then $f(x)$ is one-one.
2. **Graphically :** A function is one-one, iff no line parallel to x-axis meets the graph of function at more than one point.
3. **By Calculus :** For checking whether $f(x)$ is One-One, find whether function is only increasing or only decreasing in their domain. If yes, then function is one-one,

   i.e. if $f'(x) \geq 0,\ \forall\ x \in$ domain

   or if $f'(x) \leq 0,\ \forall\ x \in$ domain,

   then function is one-one.

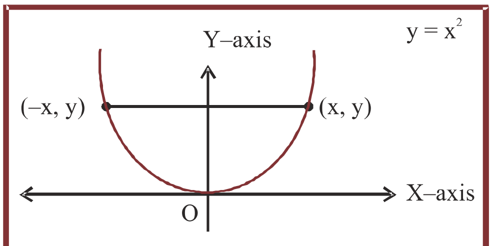
Methods to check into/onto mapping

Find the range of f(x) and compare with co-domain. If range equals co-domain then function is onto, otherwise it is into.

4. EVEN AND ODD FUNCTIONS

1. Even Function : $f(-x) = f(x),\ \forall\ x \in$ Domain

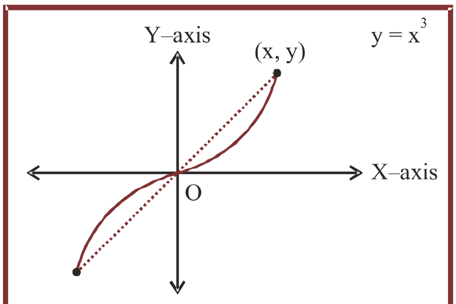
The graph of an even function $y = f(x)$ is symmetric about the y–axis. i.e., (x, y) lies on the graph $\Leftrightarrow$ (–x, y) lies on the graph.



Graph of an even function

2. Odd Function : $f(-x) = -f(x),\ \forall\ x \in$ Domain

The graph of an odd function $y = f(x)$ is symmetric about origin i.e. if point (x, y) is on the graph of an odd function, then (–x, –y) will also lie on the graph.

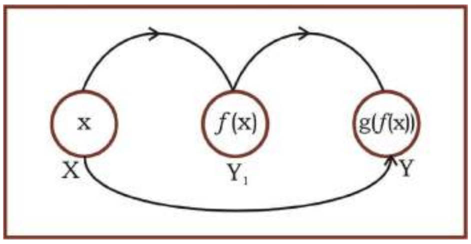


Graph of an odd function

5. COMPOSITE FUNCTIONS

Let us consider two functions, $f : X \to Y_1$ and $g : Y_1 \to Y$. We define function $h : X \to Y$; such that

$h(x) = g(f(x)) = (gof)(x)$.



To obtain h (x), we first take f–image of an element $x \in X$ so that $f(x) \in Y_1$, which is the domain of $g(x)$. Then take g–image of $f(x)$, i.e., $g(f(x))$ which would be an element of Y.

NOTES :

It should be noted that gof exists iff; the range of $f \subseteq$ domain of g. Similarly, fog exists; iff; the range of $g \subseteq$ domain of f.

6. INVERSE OF FUNCTION

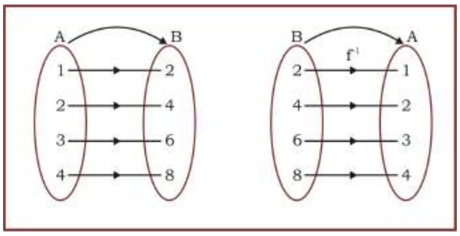
6.1 Definition : Let $f : A \to B$ be a one–one and onto function, then there exists a unique function, $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A$ and $y \in B$. Then g is said to be inverse of f.

Thus, $g = f^{-1} : B \to A = \{(f(x), x) | (x, f(x)) \in f\}$

Let us consider one–one function with domain A and range B.

where A = {1, 2, 3, 4} and B = {2, 4, 6, 8} and $f: A \to B$ is given by $f(x) = 2x$, then write f and f^{-1} as a set of ordered pairs.

Here, member $y \in B$ arises from one and only one member $x \in A$.



So, $f = \{(1, 2)(2, 4)(3, 6)(4, 8)\}$

and $f^{-1} = \{(2, 1)(4, 2)(6, 3)(8. 4)\}$

NOTES :

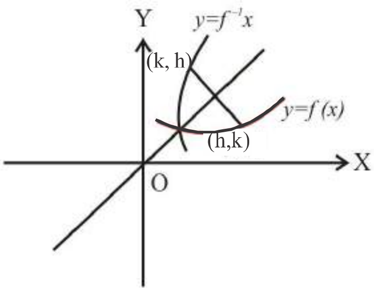
In above function, Domain of $f = \{1, 2, 3, 4\}$ = range of f^{-1}

Range of $f = \{2, 4, 6, 8\}$ = domain of f^{-1}

Which represents for a function to have its inverse, it must be one–one onto or bijective.

6.2 Graph of the inverse of an invertible function :

Let (h, k) be a point on the graph of the function f. Then (k, h) is the corresponding point on the graph of inverse of f i.e., f^{-1}.



The line segment joining the points (h, k) and (k, h) is bisected at right angle by the line $y = x$.

So that the two points play object–image role in the line $y = x$ as plane mirror.

It follows that the graph of $y = f(x)$ and its inverse written in form $y = g(x)$ are symmetrical about the line $y = x$.

6.3 Properties of inverse of a function

(i) The inverse of bijection is unique.

(ii) The inverse of bijection is also bijection.

(iii) If $f: A \to B$ is bijection and $g: B \to A$ is inverse of f, then $fog = I_B$ and $gof = I_A$.

Where, I_A and I_B are identity function on the sets A and B respectively.

(iv) If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijection and $(gof)^{-1} = (f^{-1}og^{-1})$.

(v) In general, $fog \neq gof$ but if, $(fog)(x) = x$ and $(gof)(x) = x$. then $f^{-1} = g$ and $g^{-1} = f$.

7. FUNCTIONAL EQUATION

If x, y are independent variable then ;

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\, ln\, x$ or $f(x) = 0$

(ii) $f(xy) = f(x) . f(y) \Rightarrow f(x) = x^n, n \in R$ or $f(x) = 0$

(iii) $f(x + y) = f(x) . f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(iv) $f(x)$ is continuous and takes rational values for all x $\Rightarrow f(x)$ is constant function.

(v) By considering a general n^{th} degree polynomial and writting the expression.

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\Rightarrow f(x) = \pm x^n + 1 = 1 \pm x^n$$

BINARY OPERATIONS

Definition 1 : A binary operations * on a set A is a function* : $A \times A \to A$. We denote * (a, b) by a * b.

Definition 2 : A binary operation * on a set A is called commutative, if a * b = b * a, for every $a, b \in A$.

Definition 3 : A binary operation * : $A \times A \to A$ is said to be associative if (a * b) * c = a * (b * c), $\forall$ a, b, c, $\in A$.

Definition 4 : Given a binary operation $* : A \times A \to A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a, \ \forall a \in A$.

INVERSE,TRIGONOMETRIC,FUNCTIONS

1. INTRODUCTION

	Function	Domain	Range
1.	$y = \sin^{-1} x$ iff $x = \sin y$	$-1 \le x \le 1$,	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$ iff $x = \cos y$	$-1 \le x \le 1$	$[0, \pi]$
3.	$y = \tan^{-1} x$ iff $x = \tan y$	$-\infty < x < \infty$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$ iff $x = \cot y$	$-\infty < x < \infty$	$(0, \pi)$
5.	$y = \text{cosec}^{-1} x$ iff $x = \text{cosec } y$	$(-\infty, -1] \cup [1, \infty]$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
6.	$y = \sec^{-1} x$ iff $x = \sec y$	$(-\infty, -1] \cup [1, \infty]$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

NOTES :

(i) $\sin^{-1} x$ & $\tan^{-1} x$ are increasing functions in their domain.

(ii) $\cos^{-1} x$ & $\cot^{-1} x$ are decreasing functions in their domain.

2. COMPOSITION OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

2.1 (i) $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$, for all $x \in R$

(iv) $\text{cosec}(\text{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1} x) = x$, for all $x \in R$

Proof. We know that, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ exists such that $f o f^{-1}(y) = f(f^{-1}(y)) = y$ for all $y \in B$.

Clearly, all these results are direct consequences of this property.

Aliter : Let $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$ such that $\sin \theta = x$ then, $\theta = \sin^{-1} x$

$\therefore \quad x = \sin \theta = \sin(\sin^{-1} x)$

Hence, $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$

Similarly, we can prove other results.

2.2 $T^{-1}(T(x)) \neq x$ always

$T^{-1}(T(x)) = x$ when x lies in principal domain of T.

eg: $\sin^{-1}(\sin \theta) \neq \theta$, if $\theta \notin [-\pi/2, \pi/2]$. Infact, we have

$$\sin^{-1}(\sin \theta) = \begin{cases} -\pi - \theta, & \text{if } \theta \in [-3\pi/2, -\pi/2] \\ \theta, & \text{if } \theta \in [-\pi/2, \pi/2] \\ \pi - \theta, & \text{if } \theta \in [\pi/2, 3\pi/2] \\ -2\pi + \theta, & \text{if } \theta \in [3\pi/2, 5\pi/2] \end{cases}$$ and so on.

Similarly,

$$\cos^{-1}(\cos \theta) = \begin{cases} -\theta, & \text{if } \theta \in [-\pi, 0] \\ \theta, & \text{if } \theta \in [0, \pi] \\ 2\pi - \theta, & \text{if } \theta \in [\pi, 2\pi] \\ -2\pi + \theta, & \text{if } \theta \in [2\pi, 3\pi] \end{cases}$$ and so on.

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi+\theta, & \text{if } \theta \in (-3\pi/2, -\pi/2) \\ \theta, & \text{if } \theta \in (-\pi/2, \pi/2) \\ \theta-\pi, & \text{if } \theta \in (\pi/2, 3\pi/2) \\ \theta-2\pi, & \text{if } \theta \in (3\pi/2, 5\pi/2) \end{cases} \text{ and so on.}$$

Graph for $y = \sin^{-1}(\sin x)$

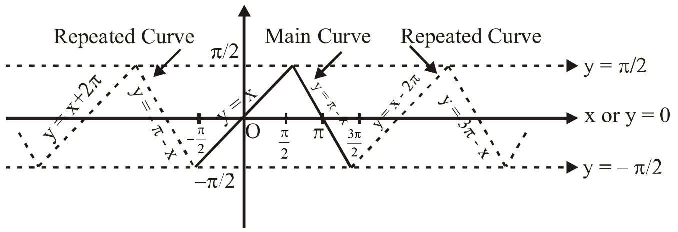
As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π.

∴ to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x.

As we know,

$$\sin^{-1}(\sin x) = \begin{cases} x, & -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}, \end{cases}$$

which is defined for the interval of length 2π, plotted as ;



Thus, the graph for $y = \sin^{-1}(\sin x)$, is a straight line up and a straight line down with slopes 1 and –1 respectively lying between $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Graph for $y = \cos^{-1}(\cos x)$

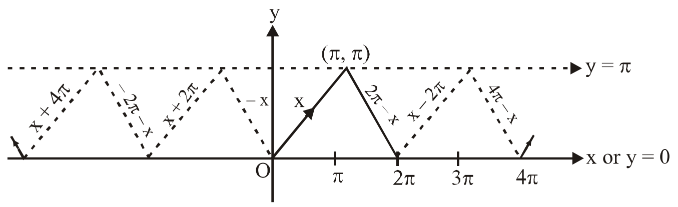
As, $y = \cos^{-1}(\cos x)$ is periodic with period 2π.

∴ to draw this graph we should draw the graph for one interval of length 2π and repear for entire values of x of length 2π.

As we know;

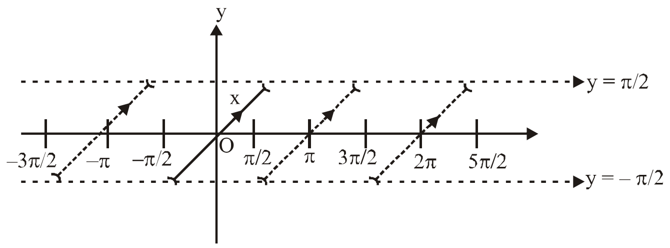
$$\cos^{-1}(\cos x) = \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & \pi \le x \le 2\pi, \end{cases}$$

Thus, it has been defined for $0 \le x \le 2\pi$ that has length 2π. So, its graph could be plotted as;



Thus, the curve $y = \cos^{-1}(\cos x)$.

Graph for $y = \tan^{-1}(\tan x)$



This is the curve for $y = \tan^{-1}(\tan x)$, where y is not defined for $x \in (2n+1)\dfrac{\pi}{2}$.

3. PROPERTIES

3.1 PROPERTY – I

(i) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, for all $x \in R$

(iv) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(v) $\tan^{-1}(-x) = -\tan^{-1}x$, for all $x \in R$

(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Proof. (i) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

let $\cos^{-1}(-x) = \theta$ ... (i)

then, $-x = \cos\theta$

$\Rightarrow x = -\cos\theta$

$\Rightarrow x = \cos(\pi - \theta)$

$\Rightarrow \cos^{-1}x = \cos^{-1}(\cos(\pi-\theta))$

$\cos^{-1}x = \pi - \theta$ {$\because x \in (-1, 1)$ and $\pi - \theta \in [0, \pi]$ for all $\theta \in [0, \pi]$}

$\Rightarrow \theta = \pi - \cos^{-1}x$ ... (ii)

from (i) and (ii), we get

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

Similarly, we can prove other results.

(iv) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

let $\sin^{-1}(-x) = \theta$

then, $-x = \sin\theta$ ... (i)

$\Rightarrow x = -\sin\theta$

$\Rightarrow x = \sin(-\theta) \Rightarrow \sin^{-1}(x) = \sin^{-1}(\sin(-\theta))$

$\Rightarrow \quad -\theta = \sin^{-1} x \qquad \text{... (ii)}$

$\{\because x \in [-1, 1] \text{ and } -\theta \in [-\pi/2, \pi/2] \text{ for all } \theta \in [-\pi/2, \pi/2]$

from (i) and (ii), we get

$\sin^{-1}(-x) = -\sin^{-1}(x)$

3.2 PROPERTY – II

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

Proof. Let, $\text{cosec}^{-1} x = \theta \qquad \text{... (i)}$

then, $x = \text{cosec}\,\theta$

$\Rightarrow \quad \frac{1}{x} = \sin\theta \Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \sin^{-1}(\sin\theta) = \theta$

$\{\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow \frac{1}{x} \in [-1, 1] - \{0\}\}$

$\text{cosec}^{-1} x = \theta \Rightarrow \theta \in [-\pi/2, \pi/2] - \{0\}$

$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{x}\right) \qquad \text{... (ii)}$

from (i) and (ii); we get

$\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x \;, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

Proof. Let $\cot^{-1} x = \theta$. Then $x \in R$, $x \neq 0$ and $\theta \in (0, \pi) \qquad \text{... (i)}$

Now two cases arises :

Case I : When $x > 0$

In this case, $\theta \in (0, \pi/2)$

$\therefore \quad \cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta \Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(\tan\theta)$

$\theta = \tan^{-1}\left(\frac{1}{x}\right) \qquad \text{... (ii)} \qquad \{\because \theta \in (0, \pi/2)\}$

from (i) and (ii), we get

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$, for all $x > 0$.

Case II : When $x < 0$

In this case $\theta \in (\pi/2, \pi) \qquad \{\because x = \cot\theta < 0)$

Now, $\frac{\pi}{2} < \theta < \pi$

$\Rightarrow \quad -\frac{\pi}{2} < \theta - \pi < 0$

$\Rightarrow \quad \theta - \pi \in (-\pi/2, 0)$

$\therefore \quad \cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \frac{1}{x} = -\tan(\pi - \theta)$

$\Rightarrow \quad \frac{1}{x} = \tan(\theta - \pi) \qquad \{\because \tan(\pi - \theta) = -\tan\theta\}$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(\tan(\theta - \pi))$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = -\pi + \theta \qquad \text{... (iii)} \; \{\because \theta - \pi \in (-\pi/2, 0)\}$

from (i) and (iii), we get

$\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1} x$, if $x < 0$

Hence,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$$

3.3 PROPERTY – III

(i) $\sin^{-1}x + \cos^{1}x = \pi/2$, for all $x \in [-1, 1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \pi/2$, for all $x \in R$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Proof. Let, $\sin^{-1}x = \theta$ ... (i)

then, $\theta \in [-\pi/2, \pi/2]$ $(\because x \in [-1, 1])$

$\Rightarrow$ $-\pi/2 \le \theta \le \pi/2$

$\Rightarrow$ $-\pi/2 \le -\theta \le \pi/2$

$\Rightarrow$ $0 \le \frac{\pi}{2} - \theta \le \pi$

$\Rightarrow$ $\frac{\pi}{2} - \theta \in [0, \pi]$

Now, $\sin^{-1}x = \theta$

$\Rightarrow$ $x = \sin\theta$

$\Rightarrow$ $x = \cos\left(\frac{\pi}{2} - \theta\right) \Rightarrow \cos^{-1}x = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - \theta\right)\right)$

$\Rightarrow$ $\cos^{-1}x = \frac{\pi}{2} - \theta$

$\{\because x \in [-1, 1]$ and $(\pi/2 - \theta) \in [0, \pi]\}$

$\Rightarrow$ $\theta + \cos^{-1}x = \pi/2$ ... (ii)

from (i) and (ii), we get

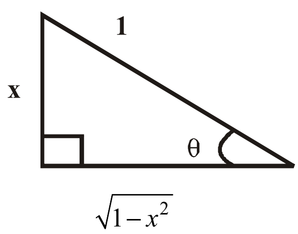
$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

4. INTERCONVERSION

4.1 If x > 0

If $\sin^{-1}x = \theta$ {θ lies in I quadrant}

Then to convert $\sin^{-1}x$ to other inverse trigonometric functions, $\sin\theta = \frac{x}{1} = \frac{p}{h}$



$$\text{So, } \cos\theta = \sqrt{1-x^2} \Rightarrow \theta = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

$$\text{and } \tan\theta = \frac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Similarly, $\sin^{-1}x$ canbe converted to any other inverse trigonometric function.

Similar procedure can be applied to convert any inverse trigonometric ratio to any other inverse trigonometric ratio.

4.2 when x < 0

We can convert these to positive number first.

eg. $\sin^{-1}x = \sin^{-1}(-(-x)) = -\sin^{-1}(-x)$ and

$\cos^{-1}(x) = \cos^{-1}(-(-x)) = \pi - \cos^{-1}(-x)$.

Now (-x) is positive and so procedure learnt in 4.1 can be applied to it.

5. SUM AND DIFFERENCE FORMULAE

(i) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & x \ge 0, y \ge 0, xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & x \ge 0, y \ge 0, xy > 1 \\ \frac{\pi}{2}, & x \ge 0, y \ge 0, xy = 1 \end{cases}$$

(ii) $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \quad x \ge 0,\ y \ge 0$$

(iii) $$\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{ if } x \ge 0, y \ge 0, x^2 + y^2 \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{ if } x \ge 0, y \ge 0, x^2 + y^2 > 1 \end{cases}$$

(iv) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$, if $0 \le x, y \le 1$

(v) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{\left(1-x^2\right)\left(1-y^2\right)}\right], 0 \le x, y \le 1$

(vi) $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left[xy+\sqrt{(1-x^2)(1-y^2)}\right], & 0 \le x < y \le 1 \\ -\cos^{-1}\left[xy+\sqrt{(1-x^2)(1-y^2)}\right], & 0 \le y < x \le 1 \end{cases}$

6. SUMMATION OF SERIES

The formula to be used in such problems is

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y$$

So first convert tan inverse term to form given in L.H.S.

If series given is in some other inverse trigonometric function, then first convert it to tan inverse using interconversion.

7. SIMPLIFICATION

Terms involving inverse trigonometric ratios can be simplified using proper trigonometric substitutions. For example,

1. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan x, \quad |x| < 1.$

   For this we use substitution x = tan θ in LHS.

2. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x, \quad |x| \le 1$

   Substitution used : x = tan θ

3. $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x, \quad |x| \le \frac{1}{\sqrt{2}}$

   Substitution used : x = sin θ

SOLVED EXAMPLES

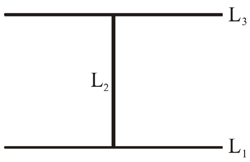
RELATIONS AND FUNCTIONS–II

Example – 1

Let L be the set of all lines in a plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$. Show that R is symmetric but neither reflexive nor transitive.

Sol. **R is not reflexive:** As a line L_1 can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$,

R is symmetric : As $(L_1, L_2) \in R$.



$\Rightarrow$ L_1 is perpendicular to L_2

$\Rightarrow$ L_2 is perpendicular to L_1

$\Rightarrow$ $(L_2, L_1) \in R$.

R is not transitive: If $(L_1, L_2) \in R$ & $(L_2, L_3) \in R$

$\Rightarrow$ L_1 is perpendicular to L_2 & L_2 is perpendicular to L_3. Then L_1 can not be perpendicular to L_3 as L_1 is parallel to L_3.

$\Rightarrow$ $(L_1, L_3) \notin R$.

Example – 2

Show that the relation R in the set Z of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

Sol. **R is reflexive:** As $2|a\text{-}a \ \forall\ a \in Z$ (| this symbol mean divide)

$\Rightarrow$ $(a, a) \in R$,

R is symmetric : Let $(a, b) \in R$

$\Rightarrow$ $2|a\text{-}b$

$\Rightarrow$ $2|\text{-}(b\text{-}a)$

$\Rightarrow$ $(b, a) \in R$

R is transitive : Let $(a, b) \in R$ & $(b, c) \in R$

$\Rightarrow$ $2|a\text{-}b$ & $2|b\text{-}c$

$\Rightarrow$ $2|a\text{-}b + b\text{-}c$

$\Rightarrow$ $2|a\text{-}c \Rightarrow (a, c) \in R$

Example – 3

Let $f : X \to Y$ be a function. Define a relation R in X given by $R = \{(a, b) : f(a) = f(b)\}$. Examine whether R is an equivalence relation or not.

Sol. **R is reflexive:** Since $f(a) = f(a)\ \forall\ a \in X$

$\Rightarrow$ $(a, a) \in R \Rightarrow R$ is reflexible

R is symmetric : Let $(a, b) \in R$

$\Rightarrow$ $f(a) = f(b)$

$\Rightarrow$ $f(b) = f(a)$

$\Rightarrow$ $(b, a) \in R$

$\Rightarrow$ R is symmetric

R is transitive : Let (a, b) & $(b,c) \in R$

$\Rightarrow$ $f(a) = f(b)$ & $f(b) = f(c)$

$\Rightarrow$ $f(a) = f(c)$

$\Rightarrow$ $(a, c) \in R$

$\Rightarrow$ R is transitive

Example – 4

Let $A = \{2, 3, 4, 5, 6, 7, 8, 9\}$. Let R be the relation on A defined by

$$R = \{(x, y) : x \in A, y \in A \text{ and } x \text{ divides } y\}.$$

Find (i) R; (ii) domain of R; (iii) range of R; (iv) R^{-1}

State whether or not R is (a) reflexive (b) symmetric (c) transitive.

Sol. Here, x R y iff x divides y, therefore,

(i) $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5) (6, 6) (7, 7), (8, 8) (9, 9)\}$

(ii) Domain of $R = \{2, 3, 4, 5, 6, 7, 8, 9\} = A$

(iii) Range of $R = \{2, 3, 4, 5, 6, 7, 8, 9\} = A$

(iv) $R^{-1} = \{(y, x) : (x, y) \in R\}$

$= \{(2, 2), (4, 2), (6, 2), (8, 2) (3, 3), (6, 3), (9, 3), (4, 4), (8, 4) (5, 5) (6, 6), (7, 7) (8, 8), (9, 9)\}$

Infact R^{-1} is $\{(y, x) : x, y \in A, y \text{ is divisible by } x\}$.

(a) As (2, 2) (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9) belong to R, therefore, (R) is reflexive.

(b) Here, R is not symmetric. We may observe that $(2, 4) \in R$ but $(4, 2) \notin R$. Infact, 'x divides y' does not imply 'y divides x' when $x \neq y$.

(c) As x' divides y' and 'y divides z' imply 'x divides z', therefore, the relation R is transitive.

Example – 5

Determine which of the following binary operations on the set R are associative and which are commutative.

(a) $a * b = 1 \ \forall\ a, b \in R$

(b) $a * b = \dfrac{(a+b)}{2} \ \forall\ a, b \in R$

Sol. (a) Clearly, by definition $a * b = b * a = 1$, $\forall\ a, b \in R$. Also $(a * b) * c = (1) * c = 1$ and $a * (b * c) = a * (1) = 1$, $\forall\ a, b\ c \in R$. Hence R is both associative and commutative.

(b) $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2} = b * a$, shows that * is commutative. Further,

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c$$

$$= \frac{\left(\frac{a+b}{2}\right)+c}{2} = \frac{a+b+2c}{4}.$$

But $a * (b * c) = a * \left(\dfrac{b+c}{2}\right)$

$$= \frac{a+\frac{b+c}{2}}{2} = \frac{2a+b+c}{4} \neq \frac{a+b+2c}{4}$$

in general.

Hence * is not associative.

Example – 6

Find the domain of

$$f(x) = \frac{1}{\sqrt{\log_{1/2}\left(x^2-7x+13\right)}}.$$

Sol. $f(x) = \dfrac{1}{\sqrt{\log_{1/2}\left(x^2-7x+13\right)}}$. exists if;

$\log_{1/2}(x^2 - 7x + 13) > 0$

$\Rightarrow \quad (x^2 - 7x + 13) < 1$ ...(i)

and $\quad x^2 - 7x + 13 > 0$ ...(ii)

considering equation (ii), $x^2 - 7x + 13 > 0$, we have

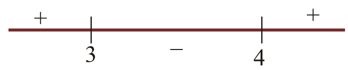
$$\left(x^2 - 7x + \frac{49}{4}\right) + 13 - \frac{49}{4} > 0$$

$$\Rightarrow \left(x - \frac{7}{2}\right)^2 + \frac{3}{4} > 0$$

which is true for all $x \in R$

as $\left(x - \dfrac{7}{2}\right)^2 \geq 0$ for all x. ....(a)

again taking (i), $x^2 - 7x + 13 < 1$



$\Rightarrow \quad x^2 - 7x + 12 < 0$

$\Rightarrow \quad (x-3)(x-4) < 0$

$\Rightarrow \quad 3 < x < 4$ ....(b)

combining (a) and (b), we have

Hence domain of $f(x) \in (3, 4)$ or $]3, 4[$

Example – 7

Find domain for $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

Sol. $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined for ;

$$-1 \leq \frac{1+x^2}{2x} \leq 1 \quad \text{or} \quad \left|\frac{1+x^2}{2x}\right| \leq 1$$

(since domain of $\sin^{-1} x = [-1, 1]$)

$\Rightarrow \quad |1 + x^2| \leq |2x|$

$\Rightarrow \quad 1 + x^2 \leq |2x|,$ $\{as\ 1 + x^2 > 0\}$

$\Rightarrow \quad x^2 - 2|x| + 1 \leq 0$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \leq 0$ $\{as\ x^2 = |x|^2\}$

$\Rightarrow \quad (|x| - 1)^2 \leq 0$

But $(|x|-1)^2$ is either always positive or zero.

Thus, $(|x|-1)^2 = 0$

$\Rightarrow \quad |x| = 1$

$\Rightarrow \quad x = \pm 1$

Thus, domain for $f(x)$ is $\{-1, 1\}$

Example – 8

Find the range of the function :

$$f(x) = 3\sin x + 8\cos\left(x - \frac{\pi}{3}\right) + 5$$

Sol. Here $f(x) = 3\sin x + 8\cos\left(x - \frac{\pi}{3}\right) + 5$

$= 3\sin x + 4(\cos x + \sqrt{3}\sin x) + 5$

$= (3 + 4\sqrt{3})\sin x + 4\cos x + 5.$

Put $3 + 4\sqrt{3} = r\cos\theta$ ..(i) and $4 = r\sin\theta$ ... (ii)

squaring and adding (1) & (2), dividing (i) and (2)

$$r = \sqrt{73 + 24\sqrt{3}} \text{ and } \theta = \tan^{-1}\frac{4}{3 + 4\sqrt{3}}$$

$\Rightarrow \quad f(x) = \sqrt{73 + 24\sqrt{3}}\,\sin(x + \theta) + 5$

$\Rightarrow$ Range of $f(x)$ is

$$\left[5 - \sqrt{73 + 24\sqrt{3}},\ 5 + \sqrt{73 + 24\sqrt{3}}\right].$$

Example – 9

The range of the function $\sin^2 x - 5\sin x - 6$ is

Sol. Here, $f(x) = \sin^2 x - 5\sin x - 6$

$= \left(\sin^2 x - 5\sin x + \frac{25}{4}\right) - 6 - \frac{25}{4}$

$= \left(\sin x - \frac{5}{2}\right)^2 - \frac{49}{4}$ ...(i)

where $\frac{9}{4} \le \left(\sin x - \frac{5}{2}\right)^2 \le \frac{49}{4}$ ... (ii)

$\left(\text{since } -1 \le \sin x \le 1 \Rightarrow \frac{-7}{2} \le \sin x - \frac{5}{2} \le \frac{-3}{2}\right.$

Squaring Both 2 sides 2

$$\Rightarrow \frac{9}{4} \le \left(\sin x - \frac{5}{2}\right)^2 \le \frac{49}{4}\Bigg)$$

squaring both sides ...(ii)

$\therefore$ From Eqs (i) and (ii),

$-10 \le f(x) \le 0$

$\therefore$ Range of $f(x) \in [-10, 0]$.

Example – 10

Find the range of the function :

$$f(x) = \ln\sqrt{x^2 + 4x + 5}$$

Sol. Here $f(x) = \ln\sqrt{x^2 + 4x + 5} = \ln\sqrt{(x+2)^2 + 1}$

i.e. $x^2 + 4x + 5$ takes all values in $[1, \infty)$

since $(x+2)^2 + 1 \ge 1$

$\Rightarrow \quad f(x)$ will take all values in $[0, \infty)$.

Hence range of $f(x)$ is $[0, \infty)$.

Example – 11

Find the range of the function

$$f(x) = \tan\sqrt{\frac{\pi^2}{9} - x^2} \text{ is}$$

Sol. For $f(x)$ to be defined, $\frac{\pi^2}{9} - x^2 \ge 0$

$\Rightarrow \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

$\therefore$ Domain of $f = \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

The greatest value of $f(x) = \tan\sqrt{\frac{\pi^2}{9} - 0}$, when $x = 0$

$= \tan\frac{\pi}{3}$

$= \sqrt{3}$

and the least value of $f(x) = \tan\sqrt{\frac{\pi^2}{9} - \frac{\pi^2}{9}}$, when $x = \frac{\pi}{3}$.

$= \tan 0$

$= 0$

$\therefore$ The greatest value of $f(x) = \sqrt{3}$ and the least value of $f(x) = 0$.

$\therefore$ Range of $f = [0, \sqrt{3}]$

Example – 12

Find the period of the function.

$$f(x) = \sin x + \{x\}$$

Sol. Here $f(x) = \sin x + \{x\}$

Period of sin x is 2π and that of $\{x\}$ is 1.

But the L.C.M. of 2π and 1 does not exist.

Hence $\sin x + \{x\}$ is not periodic.

Example – 13

Find the period of the function

$$f(x) = \tan\frac{x}{3} + \sin 2x$$

Sol. Here $f(x) = \tan x/3 + \sin 2x$.

Here $\tan(x/3)$ is periodic with period 3π and $\sin 2x$ is periodic with period π.

Hence $f(x)$ will be periodic with period 3π.

(Since L.C.M of 3π & π is 3π)

Example – 14

Find the period of the function

$f(x) = |\sin x| + |\cos x|$.

Sol. Here $f(x) = |\sin x| + |\cos x|$

Now, $|\sin x| = \sqrt{\sin^2 x} = \sqrt{\frac{1-\cos 2x}{2}}$, which is periodic with period π.

Similarly, $|\cos x|$ is periodic with period π.

Hence, according to rule of LCM, period of $f(x)$ must be π.

But $\left|\sin\left(\frac{\pi}{2}+x\right)\right| = |\cos x|$ and $\left|\cos\left(\frac{\pi}{2}+x\right)\right| = |\sin x|$

Since $\pi/2 < \pi$, period of $f(x)$ is $\pi/2$.

Example – 15

Let $f: R \to R$ be defined by $f(x) = 3x - 2$ and $g: R \to R$ be defined by $g(x) = \frac{x+2}{3}$. Show that $fog = I_R = gof$.

Sol. For all $x \in R$, $(fog)(x) = f(g(x))$

$= f\left(\frac{x+2}{3}\right) = 3\left(\frac{x+2}{3}\right) - 2 = x = I_R(x)$

Hence $fog = I_R$

Again,

$(gof)(x) = g(f(x)) = g(3x-2) = \frac{(3x-2)+2}{3} = x = I_R(x)$

$\therefore gof = I_R$.

Example – 16

Let $X = \{-2, -1, 0, 1, 2, 3\}$ and $Y = \{0, 1, 2, ..., 10\}$ and $f: X \to Y$ be a function defined by $f(x) = x^2$ for all $x \in X$, find $f^{-1}(A)$ where $(A) = \{0, 1, 2, 4\}$.

Sol. Here, we have to find $f^{-1}(0), f^{-1}(1), f^{-1}(2)$ and $f^{-1}(4)$.

Now $f(x) = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \Rightarrow f^{-1}(0) = \{0\}$,

$f(x) = 1 \Rightarrow x^2 = 1 \Rightarrow x = -1, 1$

$\Rightarrow f^{-1}(1) = \{-1, 1\}, f(x) = 2 \Rightarrow x^2 = 2$

$\Rightarrow x = -\sqrt{2}, \sqrt{2}$ but none of these is in X.

$\Rightarrow f^{-1}(2) = \phi, f(x) = 4 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$

$\Rightarrow f^{-1}(4) = \{-2, 2\}$.

Hence, $f^{-1}(A) = \{0, -1, 1, -2, 2\}$.

Example – 17

Let A be a non-empty set and $f: A \to A$, $g: A \to A$ be two functions such that $fog = I_A = gof$, show that f and g are bijections and that $g = f^{-1}$.

Sol. Consider $f: A \to A$, Let $y \in A$ be arbitrary. Since $fog = I_A$, therefore, $(fog)(y) = y$

$\Rightarrow f(g(y)) = y$

$\Rightarrow f(t) = y$, where $t = g(y) \in A$.

This means that for $y \in A$, there exists $t \in A$ such that $f(t) = y$. Hence f is onto.

Let $x, y \in A$ such that $f(x) = f(y)$

$\Rightarrow g(f(x)) = g(f(y))$

($\because$ g is a function)

$\Rightarrow (gof)(x) = (gof)(y)$

$\Rightarrow I_A(x) = I_A(y)$

$\Rightarrow \quad x = y.$

So, $f(x) = f(y) \Rightarrow x = y \Rightarrow f$ is one-one. Thus, we see that f is both one-one and onto i.e. f is a bijection. Similarly, we can show that g is a bijection.

Moreoever, for all $x \in A$,

$x = I_A(x) = (fog)(x) = f(g(x))$

$\Rightarrow \quad x = f(g(x)) \Rightarrow f^{-1}(x) = g(x)$

$\Rightarrow \quad f^{-1} = g.$

Example – 18

Let $f : R \to R$ be defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$. Is $f(x)$ invertible ? If so, find its inverse.

Sol. Let us check invertibility of $f(x)$:

(a) **One-one :** Here, $f'(x) = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow \quad f'(x) = \dfrac{e^{2x}+1}{2e^x}$ which is strictly increasing as

$e^{2x} > 0$ for all x.

Thus, one-one.

(b) **Onto :** Let $y = f(x)$

$\Rightarrow \quad y = \dfrac{e^x - e^{-x}}{2}$ where y is strictly monotonic.

Hence, range of $f(x) = (f(-\infty)), f(\infty))$

(Since domain of $f = (-\infty, \infty)$)

$\Rightarrow$ range of $f(x) = (-\infty, \infty)$

So range of $f(x) =$ co-domain.

Hence, $f(x)$ is one-one and onto.

$\Rightarrow$ f is invertiable

(c) **To find f^{-1} :** $y = \dfrac{e^{2x}-1}{2e^x}$

$\Rightarrow \quad e^{2x} - 2e^x y - 1 = 0$

$\Rightarrow \quad e^x = \dfrac{2y \pm \sqrt{4y^2+4}}{2}$

$\Rightarrow \quad x = \log\left(y \pm \sqrt{y^2+1}\right)$

$\Rightarrow \quad f^{-1}(y) = \log\left(y \pm \sqrt{y^2+1}\right)$

[as $f(x) = y \Rightarrow x = f^{-1}(y)$]

Since, $e^{f^{-1}(x)}$ is always positive.

So, neglecting negative sign.

Hence, $f^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

Example – 19

Let $f : [1/2, \infty) \to [3/4, \infty)$, where $f(x) = x^2 - x + 1$. Find the inverse of $f(x)$. Hence, solve the equation $x^2 - x + 1 = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$.

Sol. (a) $\quad f(x) = x^2 - x + 1$

$\Rightarrow \quad f(x) = \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$, which is clearly one-one and onto in given domain and co-domain.

$\Rightarrow$ f(x) is strictly increasing.

$\Rightarrow$ f(x) is one-one.

Also f(x) is onto.

(b) Thus, its inverse can be obtained.

Let $f(x) = y$

$\Rightarrow \quad y = \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

$\Rightarrow \quad x - \dfrac{1}{2} = \pm\sqrt{y - \dfrac{3}{4}}$

$\Rightarrow \quad x = \dfrac{1}{2} \pm \sqrt{y - \dfrac{3}{4}} \qquad [f(x) = y \Rightarrow x = f^{-1}(y)]$

$\Rightarrow \quad f^{-1}(y) = \dfrac{1}{2} + \sqrt{y - \dfrac{3}{4}}$

[neglecting –ve sign as $x > 0$]

(as x is always +ve)

$\Rightarrow \quad f^{-1}(x) = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$

(c) To solve $x^2 - x + 1 = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$, as $f(x) = f^{-1}(x)$ has only one solution in this case.

ie, $\quad f(x) = x$

$\Rightarrow \quad x^2 - x + 1 = x$

$\Rightarrow \quad x^2 - 2x + 1 = 0$

$\Rightarrow \quad (x-1)^2 = 0$

x = 1 is the required solution.

Example – 20

If $f(x) = x^2 - 3x + 2$ be a real valued function of the real variable, find *fof*.

Sol. We are given that the function $f : R \to R$, defined by $f(x) = x^2 - 3x + 2$ for all $x \in R$.

Now, $(fof)(x) = f(f(x)) = (f(x))^2 - 3f(x) + 2$

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

$= x^4 - 6x^3 + 10x^2 - 3x.$

Example – 21

Two functions are defined as under :

$$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases},$$

$$g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$$

Find *fog* and *gof*.

Sol. $(fog)(x) = f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Let us consider $-1 \le f(x) < 2$

(i) $x^2 \le 1, -1 \le x < 2$

$\Rightarrow$ $-1 \le x \le 1,\ -1 \le x < 2$

$\Rightarrow$ $-1 \le x \le 1$

(ii) $x + 2 \le 1, 2 \le x \le 3$

$\Rightarrow$ $x \le -1, 2 \le x \le 3$

$\Rightarrow$ $x = \phi.$

(so no value of x is possible)

Let us consider, $1 < g(x) \le 2.$

(iii) $1 < x^2 \le 2,\ -1 \le x < 2$

$\Rightarrow$ $x \in \left[-\sqrt{2}, -1\right) \cup \left(1, \sqrt{2}\right],\ -1 \le x < 2$

$\Rightarrow$ $1 < x \le \sqrt{2}.$

(iv) $1 < x + 2 \le 2,\ 2 \le x \le 3$

$\Rightarrow$ $-1 < x \le 0,\ 2 \le x \le 3,\ x = \phi$

Thus $f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$

Now, Let us consider *gof* :

$$gof = g(f(x)) = \begin{cases} f^2(x), & -1 \le f(x) < 2 \\ f(x)+2, & 2 \le f(x) \le 3 \end{cases}$$

Let us consider $-1 \le f(x) < 2$

(i) $-1 \le x + 1 < 2,\ x \le 1$

$\Rightarrow$ $-2 \le x < 1,\ x \le 1$

$\Rightarrow$ $-2 \le x < 1$

(ii) $-1 \le 2x + 1 < 2,\ 1 < x \le 2$

$\Rightarrow$ $-1 \le x < 1/2,\ 1 < x \le 2$

$\Rightarrow$ $x = \phi.$

Let us consider $2 \le f(x) \le 3$

(iii) $2 \le x + 1 \le 3,\ x \le 1$

$\Rightarrow$ $1 \le x \le 2,\ x \le 1$

$\Rightarrow$ $x = 1$

(iv) $2 \le 2x + 1 \le 3,\ 1 < x \le 2$

$\Rightarrow$ $1 \le 2x \le 2,\ 1 < x \le 2$

$\Rightarrow$ $1/2 \le x \le 1,\ 1 < x \le 2$

$\Rightarrow$ $x = \phi.$

$$g(f(x)) = \begin{cases} (x+1)^2, & -2 \le x < 1 \\ x+3, & x = 1 \end{cases}$$

If we like we can also write $g(f(x)) = (x + 1)^2, -2 \le x \le 1.$

Example – 22

Let $f : R \to R$ defined by $f(x) = x^3 + ax^2 + 3x + 100$. Then find the values of a for which f is a one-one function.

Sol. $f(x) = x^3 + ax^2 + 3x + 100$

$\Rightarrow$ $f'(x) = 3x^2 + 2ax + 3.$

For $f(x)$ to be one-one. $f'(x) \ge 0$ or ≤ 0

But $f(x)$ is a quadratic expression and coefficient of $x^2 > 0$ so that $f'(x) \ge 0$

$\Rightarrow$ $D \le 0$

$\Rightarrow$ $4a^2 - 36 \le 0$

$\Rightarrow$ $a^2 \le 9$

$\Rightarrow$ $-3 \le a \le 3.$

Example – 23

Find whether the given function is even or odd ?

$$f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$$

Sol. We have

$$f(-x)=\frac{-x}{e^{-x}-1}-\frac{x}{2}+1=\frac{-e^{x}.x}{1-e^{x}}-\frac{x}{2}+1$$

$$=\frac{(e^{x}-1+1)x}{(e^{x}-1)}-\frac{x}{2}+1$$

$$=x+\frac{x}{e^{x}-1}-\frac{x}{2}+1=\frac{x}{e^{x}-1}+\frac{x}{2}+1=f(x)$$

Hence $f(x)$ is an even function.

Example – 24

If f is an even function, find the real values of x satisfying the equation $f(x)=f\left(\frac{x+1}{x+2}\right)$.

Sol. Since, $f(x)$ is even, so $f(-x)=f(x)$

Thus, $x=\frac{x+1}{x+2}$ or $-x=\frac{x+1}{x+2}$

$\Rightarrow$ $x^2+2x=x+1$ or $-x^2-2x=x+1$

$\Rightarrow$ $x^2+x-1=0$ or $-x^2-3x-1=0$

$\Rightarrow$ $x=\frac{-1\pm\sqrt{5}}{2}$ or $x=\frac{-3\pm\sqrt{5}}{2}$

Thus, $x\in\left\{\frac{-1+\sqrt{5}}{2},\frac{-1-\sqrt{5}}{2},\frac{-3+\sqrt{5}}{2},\frac{-3-\sqrt{5}}{2}\right\}$

Example – 25

Let $f(x)=\frac{9^x}{9^x+3}$. Show $f(x)+f(1-x)=1$, and hence evaluate

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+f\left(\frac{3}{1996}\right)+...+f\left(\frac{1995}{1996}\right).$$

Sol. $f(x)=\frac{9^x}{9^x+3}$ ...(i)

and $f(1-x)=\frac{9^{1-x}}{9^{1-x}+3}$

$$\Rightarrow f(1-x)=\frac{\frac{9}{9^x}}{\frac{9}{9^x}+3}=\frac{9}{9+3.9^x}$$

$$f(1-x)=\frac{9}{3(3+9^x)} \quad ...(ii)$$

Adding Eqs. (i) and (ii), we get

$$f(x)+f(1-x)=\frac{9^x}{9^x+3}+\frac{9}{3(3+9^x)}$$

$$=\frac{3.9^x+9}{3(9^x+3)}=\frac{3(9^x+3)}{3(9^x+3)}$$

$\therefore$ $f(x)+f(1-x)=1$ ...(iii)

Now, putting $x=\frac{1}{1996},\frac{2}{1996},\frac{3}{1996},....,\frac{998}{1996}$

in (Eq. (iii), we get

$$f\left(\frac{1}{1996}\right)+f\left(\frac{1995}{1996}\right)=1$$

$$\Rightarrow f\left(\frac{2}{1996}\right)+f\left(\frac{1994}{1996}\right)=1$$

$$\Rightarrow f\left(\frac{3}{1996}\right)+f\left(\frac{1993}{1996}\right)=1$$

....

....

$$\Rightarrow f\left(\frac{997}{1996}\right)+f\left(\frac{999}{1996}\right)=1$$

$$\Rightarrow f\left(\frac{998}{1996}\right)+f\left(\frac{998}{1996}\right)=1$$

or $f\left(\frac{998}{1996}\right)=\frac{1}{2}$

Adding all the above expressions, we get

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+...+f\left(\frac{1995}{1996}\right)$$

$$=(1+1+1+....997\text{ times})+\frac{1}{2}$$

$$=997+\frac{1}{2}$$

$$=997.5.$$

INVERSE TRIGONOMETRIC FUNCTIONS

Example – 26

Prove that (i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$,

(ii) $\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$.

Sol. (i) Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$ so that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\left(\text{since range of } \sin^{-1} x \text{ is} \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right)$

$\Rightarrow \quad -\frac{\sqrt{3}}{2} = \sin\theta$

$\Rightarrow \quad \sin\theta = -\sin\frac{\pi}{3} = \sin\left(-\frac{\pi}{3}\right)$ note that $-\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad \theta = -\frac{\pi}{3} \Rightarrow \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

(ii) $\quad \cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{\pi}{3}\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$,

note that $\frac{\pi}{3} \in [0, \pi]$ = range of $\cos^{-1} x$.

Example – 27

Evaluate

(i) $\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$ (ii) $\sin\left(2\sin^{-1}\left(-\frac{4}{5}\right)\right)$

(iii) $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right)$ (iv) $\sin\left(3\sin^{-1}\left(\frac{2}{5}\right)\right)$

Sol. (i) $\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right) = \sin 2\theta$, where $\theta = \sin^{-1}\left(\frac{3}{5}\right)$

$= 2\sin\theta\cos\theta$

$= 2\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

$= 2\times\frac{3}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}$ $\left(\because \cos(\sin^{-1}x) = \sqrt{1-x^2} \text{ for } |x| \le 1\right)$

$= 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

(ii) $\sin\left(2\sin^{-1}\left(-\frac{4}{5}\right)\right) = \sin\left(-2\sin^{-1}\frac{4}{5}\right)$

$(\because \sin^{-1}(-x) = -\sin^{-1}x)$

$= -\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$

$= -\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$

$= -\sin 2\theta$, where $\theta = \sin^{-1}\left(\frac{4}{5}\right) = 2\sin\theta\cos\theta$

$= -2\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

$= -2\left(\frac{4}{5}\right)\sqrt{1-\left(\frac{4}{5}\right)^2}$

$\left(\because \cos(\sin^{-1}x) = \sqrt{1-x^2}\right)$

$= -\frac{8}{5}\times\frac{3}{5} = -\frac{24}{25}$

(iii) $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right) = \sin(2\theta)$, where $\theta = \cos^{-1}\left(-\frac{3}{5}\right)$

$= 2\sin\theta\cos\theta$

$= 2\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

$= 2\sqrt{1-\left(-\frac{3}{5}\right)^2}\left(-\frac{3}{5}\right)$ $\left(\because \sin(\cos^{-1}x) = \sqrt{1-x^2} \text{ for } |x| \le 1\right)$

$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$.

(iv) $\sin\left(3\sin^{-1}\left(\frac{2}{5}\right)\right) = \sin 3\theta$, where $\theta = \sin^{-1}\left(\frac{2}{5}\right)$

$= 3\sin\theta - 4\sin^3\theta$

$= 3\left(\frac{2}{5}\right) - 4\left(\frac{2}{5}\right)^3$

$\left(\because \theta = \sin^{-1}\left(\frac{2}{5}\right), \therefore \sin\theta = \frac{2}{5}\right)$

$= \frac{6}{5} - \frac{32}{125} = \frac{118}{125}$.

Example – 28

Prove that

(i) $\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{84}{85}\right)$

(ii) $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

Sol. (i) Since $\sin^{-1}x$ is an increasing function in [–1, 1] and $\frac{3}{5} > \frac{8}{17}$, therefore,

$\sin^{-1}\left(\frac{3}{5}\right) > \sin^{-1}\left(\frac{8}{17}\right) \Rightarrow \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) > 0.$

$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) \in [0, \pi] = \text{range of } \cos^{-1}x$

$\left(\because 0 \le \sin^{-1}\left(\frac{3}{5}\right) \le \frac{\pi}{2} \text{ and } 0 \le \sin^{-1}\left(\frac{8}{17}\right) \le \frac{\pi}{2}\right)$

Now $\cos\left\{\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right)\right\}$

$= \cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)\cos\left(\sin^{-1}\left(\frac{8}{17}\right)\right) + \sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)\sin\left(\sin^{-1}\left(\frac{8}{17}\right)\right)$

$= \sqrt{1-\frac{9}{25}}\sqrt{1-\frac{64}{289}} + \frac{3}{5}\times\frac{8}{17} = \frac{4}{5}\times\frac{15}{17} + \frac{3}{5}\times\frac{8}{17} = \frac{84}{85}$

$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{84}{85}\right)$.

(ii) We know that

$0 \le \sin^{-1}\frac{3}{5} \le \frac{\pi}{2}$ and $0 \le \sin^{-1}\frac{8}{17} \le \frac{\pi}{2}$, therefore, $0 \le \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) \le \pi$

We compute

$\cos\left(\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right)\right)$

$= \cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)\cos\left(\sin^{-1}\left(\frac{8}{17}\right)\right)$

$- \sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)\sin\left(\sin^{-1}\left(\frac{8}{17}\right)\right)$

$= \sqrt{1-\left(\frac{3}{5}\right)^2}\sqrt{1-\left(\frac{8}{17}\right)^2} - \frac{3}{5}\times\frac{8}{17} = \frac{4}{5}\times\frac{15}{17} - \frac{3}{5}\times\frac{8}{17} = \frac{36}{85}$

$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{36}{85}\right) = \sin^{-1}\left(\sqrt{1-\left(\frac{36}{85}\right)^2}\right)$

$(\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2} \text{ for } 0 \le x \le 1)$

$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$, as desired.

Example – 29

Show that $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$.

Sol. Let $\theta = \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)$

then $0 < \theta < \pi$

$\left(\because 0 < \sin^{-1}\left(\frac{4}{5}\right) < \frac{\pi}{2}, 0 < \sin^{-1}\left(\frac{5}{13}\right) < \frac{\pi}{2}\right)$

Now $\cos\theta = \cos\left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}\right)$

$= \cos\left(\sin^{-1}\frac{4}{5}\right)\cos\left(\sin^{-1}\frac{5}{13}\right) - \sin\left(\sin^{-1}\frac{4}{5}\right)\sin$

$\left(\sin^{-1}\frac{5}{13}\right)$

$(\because \cos(A+B) = \cos A \cos B - \sin A \sin B)$

$= \sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{5}{13}\right)^2} - \frac{4}{5}\cdot\frac{5}{13}$

$(\because \cos(\sin^{-1} x) = \sqrt{1-x^2})$

$= \frac{3}{5}\cdot\frac{12}{13} - \frac{20}{65} = \frac{36-20}{65} = \frac{16}{65} \Rightarrow \theta = \cos^{-1}\left(\frac{16}{65}\right)$

Hence $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$= \left\{\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right\} + \sin^{-1}\left(\frac{16}{65}\right)$

$= \cos^{-1}\left(\frac{16}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$(\because \sin^{-1} t + \cos^{-1} t = \frac{\pi}{2} \text{ for } -1 \le t \le 1)$

$= \frac{\pi}{2}$

Example – 30

Prove that $\cot^{-1}(13) + \cot^{-1}(21) + \cot^{-1}(-8) = \pi$.

Sol. L.H.S. $= \cot^{-1}(13) + \cot^{-1}(21) + \cot^{-1}(-8)$

$= \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \pi - \cot^{-1} 8$

$(\because \cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R)$

$= \tan^{-1}\left(\frac{\frac{1}{13}+\frac{1}{21}}{1-\frac{1}{13}\cdot\frac{1}{21}}\right) + \pi - \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{21+13}{13\times 21-1}\right) + \pi - \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{34}{272}\right)$ $\tan^{-1}\left(\frac{34}{272}\right) = \tan^{-1}\left(\frac{1}{8}\right) + \pi - \tan^{-1}\left(\frac{1}{8}\right)$

$= \pi =$ R.H.S.

Example – 31

Prove that $2\tan^{-1}(-3) = -\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right)$.

Sol. L.H.S. $= 2\tan^{-1}(-3) = -2\tan^{-1} 3 = -2\cot^{-1}\left(\frac{1}{3}\right)$

$= -2\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{3}\right)\right) = -\pi + 2\tan^{-1}\left(\frac{1}{3}\right)$

$= -\pi + \tan^{-1}\left\{\frac{2(1/3)}{1-(1/3)^2}\right\}$

$\left(\because 2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2} \text{ for } |x| < 1\right)$

$= -\pi + \tan^{-1}\frac{3}{4} = -\pi + \cot^{-1}\frac{4}{3} = -\pi + \frac{\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right)$

$= -\frac{\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right) = -\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right) =$ R.H.S.

Example – 32

Find x if $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$.

Sol. Given $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3} \Rightarrow \sin^{-1} 2x = \frac{\pi}{3} - \sin^{-1} x$

$\Rightarrow \quad 2x = \sin\left(\frac{\pi}{3} - \sin^{-1} x\right)$

$\Rightarrow \quad 2x = \sin\frac{\pi}{3}\cos(\sin^{-1} x) - \cos\frac{\pi}{3}\sin(\sin^{-1} x)$

$\Rightarrow \quad 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{1}{2}x \quad \Rightarrow 2x + \frac{x}{2} = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

$\Rightarrow \frac{5x}{2} = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

On squaring both sides, we get

$\frac{25x^2}{4} = \frac{3}{4}(1-x^2)$

$\Rightarrow 28x^2 = 3 \Rightarrow x^2 = \frac{3}{28} \Rightarrow x \pm \sqrt{\frac{3}{28}}$.

$\Rightarrow x = \sqrt{\frac{3}{28}}$

$\left(\because\ x = -\sqrt{\frac{3}{28}}\right.$ does not satisfy the given equation)

Example – 33

If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$,

Prove that $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$.

Sol. Given $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$

$\Rightarrow \quad \cos^{-1}\left(\frac{y}{b}\right) = \alpha - \cos^{-1}\left(\frac{x}{a}\right)$

$\Rightarrow \quad \frac{y}{b} = \cos\left(\alpha - \cos^{-1}\left(\frac{x}{a}\right)\right)$

$\Rightarrow \quad \frac{y}{b} = \cos\alpha\left(\frac{x}{a}\right) + \sin\alpha \sin\left(\cos^{-1}\left(\frac{x}{a}\right)\right)$

$\Rightarrow \quad \frac{y}{b} - \frac{x}{a}\cos\alpha = \sin\alpha\sqrt{1-\left(\frac{x}{a}\right)^2}$

$(\because \sin(\cos^{-1}x) = \sqrt{1-x^2}$ for $|x| \le 1)$

$\Rightarrow \quad \left(\frac{y}{b} - \frac{x}{a}\cos\alpha\right)^2 = \sin^2\alpha\left(1-\frac{x^2}{a^2}\right)$

$\Rightarrow \quad \frac{y^2}{b^2} + \frac{x^2}{a^2}\cos^2\alpha - 2\left(\frac{y}{b}\right)\left(\frac{x}{a}\right)\cos\alpha = \sin^2\alpha\left(1-\frac{x^2}{a^2}\right)$

$\Rightarrow \quad \frac{x^2}{a^2}(\cos^2\alpha + \sin^2\alpha) - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$

Example – 34

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$,

Prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Sol. Let $\cos^{-1}x = A$, $\cos^{-1}y = B$, $\cos^{-1}z = C$

so that $x = \cos A$, $y = \cos B$, $z = \cos C$ and $A + B + C = \pi$.

$\therefore\ x^2 + y^2 + z^2 + 2xyz = \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C$

$= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2} + 2\cos A\cos B\cos C$

$= \frac{3}{2} + \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C) + 2\cos A\cos B\cos C$

$= \frac{3}{2} + \frac{1}{2}(-1 - 4\cos A\cos B\cos C) + 2\cos A\cos B\cos C$

(Using a result from conditional identities)

$= 1$, as required.

Example – 35

$2\sin^{-1}x = \cos^{-1}(1-2x^2)$, $0 \le x \le 1$.

Sol. Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

But $0 \le x \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{2}$

Hence $\cos^{-1}(1-2x^2) = \cos^{-1}(1-2\sin^2\theta)$, $0 \le 2\theta \le \pi$

$= \cos^{-1}(\cos 2\theta)$, $0 \le 2\theta \le \pi = 2\theta$

$\Rightarrow \quad \cos^{-1}(1-2x^2) = 2\theta = 2\sin^{-1}x$.

Example – 36

$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $\frac{1}{2} \le x \le 1$.

Sol. Let $\cos^{-1}x = \theta$ so that

$x = \cos\theta$ and $0 \le \theta \le \pi$

As $\quad x \in \left[\frac{1}{2}, 1\right]$, therefore, $\frac{1}{2} \le x \le 1$

$\Rightarrow \quad \cos\frac{\pi}{3} \le \cos\theta \le \cos 0$

$\Rightarrow \quad \frac{\pi}{3} \ge \theta \ge 0$

Note (that $\cos\theta$ is decreasing in $[0, \pi]$)

$\Rightarrow \quad 0 \le 3\theta \le \pi$, i.e. $3\theta \in [0, \pi]$

$\therefore \quad \cos^{-1}(4x^3 - 3x)$

$= \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$= \cos^{-1}(\cos 3\theta)$

$= 3\theta \quad$ (Note that $0 \le 3\theta \le \pi$)

$= 3\cos^{-1}x$.

Example – 37

Prove that $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)=\sin^{-1}\left(\frac{x}{a}\right)$, $|x|<a$.

Sol. Let $\sin^{-1}\left(\frac{x}{a}\right)=\theta \Rightarrow x=a\sin\theta$, and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$(\because |x|<a)$

Now $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2\cos^2\theta}=a\cos\theta$

$\left(\because -\frac{\pi}{2}<\theta<\frac{\pi}{2}\Rightarrow \cos\theta>0\right)$

Hence $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$

$=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$.

Example – 38

Show that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)=\frac{\pi}{2}-\frac{1}{2}\tan^{-1}x,\ x>0$

Sol. Let $\theta=\tan^{-1}x \Rightarrow x=\tan\theta,\ 0<\theta<\frac{\pi}{2}$

$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right)=\tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}}\right)=\tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$=\tan^{-1}\left(\cot\frac{\theta}{2}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$

$=\frac{\pi}{2}-\frac{\theta}{2}=\frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$.

Example – 39

Prove that $\sin^2\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)=1-x^2$ where $-1\le x<1$.

Sol. L.H.S. $\sin^2\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)=\sin^2(2\theta)$,

where $\theta=\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

ie. $\tan\theta=\sqrt{\frac{1+x}{1-x}}$

Thus L.H.S. $=(\sin 2\theta)^2=\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)^2$

$=\left\{\frac{2\sqrt{1+x}/\sqrt{1-x}}{1+\left(\frac{1+x}{1-x}\right)}\right\}^2$

$=\frac{4(1+x)(1-x)}{(1-x+1+x)^2}=1-x^2=$ R.H.S.

Example – 40

Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$.

Sol. Let $x^2=\cos 2\theta$ so that $0\le 2\theta\le\frac{\pi}{2}$, i.e., $0\le\theta\le\frac{\pi}{4}$.

Now, $\sqrt{1+x^2}=\sqrt{1+\cos 2\theta}=\sqrt{2\cos^2\theta}=\sqrt{2}\cos\theta$

and $\sqrt{1-x^2}=\sqrt{1-\cos 2\theta}=\sqrt{2\sin^2\theta}=\sqrt{2}\sin\theta$

Hence, $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$

$=\tan^{-1}\left(\frac{\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}}{\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}}\right)=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] = \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$

$[\because \cos 2\theta = x^2 \Rightarrow 2\theta = \cos^{-1}(x^2) \Rightarrow \theta = \frac{1}{2}\cos^{-1}(x^2)]$

Example – 41

Solve the equation $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, $|x| < 1$.

Sol. Given equation is $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$

$\left\{\begin{array}{l}\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ for } xy < 1 \\ \text{and for } |x| < 1, \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right) = \frac{1-x^2}{4-x^2} < 1\end{array}\right\}$

$\Rightarrow \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{2x^2-4}{(x^2-4)-(x^2-1)} = 1 \Rightarrow 2x^2-4 = -3$

$\Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$.

Example – 42

Solve the equation $\sin\{2\cos^{-1}(\cot(2\tan^{-1}x))\} = 0$.

Sol. Given equation is $\sin\{2\cos^{-1}(\cot(2\tan^{-1}x))\} = 0$

$\Rightarrow 2\cos^{-1}(\cot(2\tan^{-1}x)) = n\pi, n \in I$

$\Rightarrow \cos^{-1}\{\cot(2\tan^{-1}x)\} = \frac{n\pi}{2}, n \in I$

$\Rightarrow \cos^{-1}\{\cot(2\tan^{-1}x)\} = 0, \frac{\pi}{2}, \pi$

$(\because \cos^{-1}x$ lies in $[0, \pi])$

$\Rightarrow \cot(2\tan^{-1}x) = \cos 0, \cos\frac{\pi}{2}, \cos\pi$

$\Rightarrow \frac{1}{\tan(2\tan^{-1}x)} = -1, 0, 1$ (As $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$)

$\Rightarrow \frac{1-x^2}{2x} = -1, 0, 1$

$\Rightarrow \frac{1-x^2}{2x} = -1, \frac{1-x^2}{2x} = 0$ or $\frac{1-x^2}{2x} = 1$

$\Rightarrow x^2 - 2x - 1 = 0, x^2 = 1$ or $x^2 + 2x - 1 = 0$

$\Rightarrow x = 1 \pm \sqrt{2}, \pm 1$ or $-1 \pm \sqrt{2}$.

Example – 43

Solve : $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Sol. Given that,

$2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\,\text{cosec}\,x$

$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

$\Rightarrow \frac{\cos x}{\sin x} = 1 \quad (\because \sin x \neq 0)$

$\Rightarrow \cot x = 1 \Rightarrow x = \frac{\pi}{4}$ as $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example – 44

Solve for x : $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$.

Sol. We have $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$

$\Rightarrow 1 - x = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$

$\Rightarrow 1 - x = \cos(2\sin^{-1}x)$

$\Rightarrow 1 - x = 1 - 2\sin^2(\sin^{-1}x)$

$\Rightarrow 1-x=1-2\,[\sin(\sin^{-1}x)]^2$

$\Rightarrow 1-x=1-2x^2$

$\Rightarrow 2x^2-x=0$

$\Rightarrow x(2x-1)=0$

$\Rightarrow x=0,\ \frac{1}{2}$

For $x=\frac{1}{2}$, $\sin^{-1}(1-x)-2\sin^{-1}x$

$=\sin^{-1}\left(1-\frac{1}{2}\right)-2\sin^{-1}\frac{1}{2}$

$=\sin^{-1}\frac{1}{2}-2\sin^{-1}\frac{1}{2}$

$=-\sin^{-1}\frac{1}{2}=\frac{-\pi}{6}\neq$ R.H.S.

$x=\frac{1}{2}$ is not a solution of given equation. Hence, $x=0$ is the required solution.

Example – 45

If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, prove that $x+y+z=xyz$.

Sol. Let $\tan^{-1}x=\alpha$, $\tan^{-1}y=\beta$ and $\tan^{-1}z=\gamma$

$\Rightarrow x=\tan\alpha,\ y=\tan\beta$ and $z=\tan\gamma$

Now, given that,

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

$\Rightarrow \alpha+\beta+\gamma=\pi$

$\Rightarrow \alpha+\beta=\pi-\gamma$

$\Rightarrow \tan(\alpha+\beta)=\tan(\pi-\gamma)$

$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\tan\gamma$

Cross multiply, we have

$\Rightarrow \tan\alpha+\tan\beta=-\tan\gamma+\tan\alpha\tan\beta\tan\gamma$

$\Rightarrow \tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$

$\Rightarrow x+y+z=xyz$. Hence, the result.

Example – 46

Express each of the following in the simplest form :

(i) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\},\ -\pi<x<\pi$

(ii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right),\ -\frac{\pi}{2}<x<\frac{\pi}{2}$

Sol. (i) We have,

$$\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}$$

$$=\tan^{-1}\left\{\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right\}=\tan^{-1}\left\{\sqrt{\tan^2\frac{x}{2}}\right\}=\tan^{-1}\left(\left|\tan\frac{x}{2}\right|\right)$$

$$=\begin{cases}\tan^{-1}\left(-\tan\frac{x}{2}\right), & if\ -\pi<x<0\\ \tan^{-1}\left(\tan\frac{x}{2}\right), & if\ 0\le x<\pi\end{cases}$$

$$=\begin{cases}\tan^{-1}\left\{\tan\left(\frac{-x}{2}\right)\right\}=-\frac{x}{2}, & if\ -\pi<x<0\\ \tan^{-1}\left\{\tan\frac{x}{2}\right\}=\frac{x}{2}, & if\ 0<x<\pi\end{cases}$$

(ii) We have,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\tan^{-1}\left\{\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$$

$$=\tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right\}=\tan^{-1}\left\{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right\}$$

$$=\tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$=\frac{\pi}{4}-\frac{x}{2}$$

$$\left[\because -\frac{\pi}{2}<x<\frac{\pi}{2}\Rightarrow -\frac{\pi}{4}<-\frac{x}{2}<\frac{\pi}{4}\Rightarrow 0<\frac{\pi}{4}-\frac{x}{2}<\frac{\pi}{2}\right]$$

ALITER We have,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}=\tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$=\tan^{-1}\left\{\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}\right\}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}=\frac{\pi}{4}-\frac{x}{2}$$

Example – 47

Write the following functions in the simplest form :

(i) $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, -a<x<a$

(ii) $\tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}, -a<x<a$

(iii) $\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

Sol. (i) Putting x = a sin θ, we have

$$\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a\cos\theta}\right\} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}$$

$$\left[\because x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right]$$

$$\left(\text{since } -a<x<a \Rightarrow -1\le\sin\theta\le 1 \Rightarrow -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$$

(ii) Putting x = a cos θ, we have

$$\tan^{-1}\sqrt{\frac{a-x}{a+x}}$$

$$= \tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan^{-1}\left(\left|\tan\frac{\theta}{2}\right|\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\left[\because -a<x<a \Rightarrow 0<\theta<\pi<\frac{\theta}{2}<\frac{\pi}{2}\right]$$

$$\left(\begin{array}{l}\text{since } -1\le\cos\theta\le 1\\ \Rightarrow 0\le\theta\le\pi\\ \Rightarrow 0\le\frac{\theta}{2}\le\frac{\pi}{2}\end{array}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\frac{x}{a}$$

$$\left[\because x = a\cos\theta \Rightarrow \cos\theta = \frac{x}{a} \Rightarrow \theta = \cos^{-1}\frac{x}{a}\right]$$

(iii) Putting x = a tan θ, we have

$$\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$$

$$= \sin^{-1}\left\{\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta+a^2}}\right\}$$

$$= \sin^{-1}\left\{\frac{a\tan\theta}{a\sec\theta}\right\}$$

$$= \sin^{-1}(\sin\theta)$$

$$= \theta = \tan^{-1}\frac{x}{a}$$

$$\left[\because x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\frac{x}{a}\right]$$

Example – 48

Prove that :

$$2\sin^{-1}x = \begin{cases}\sin^{-1}(2x\sqrt{1-x^2}) & ,\text{if } -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}) & ,\text{if } \frac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin(2x\sqrt{1-x^2}) & ,\text{if } -1\le x\le-\frac{1}{\sqrt{2}}\end{cases}$$

Sol. Let $\sin^{-1}x = \theta$. Then,

x = sin θ,

$\Rightarrow$ $\cos\theta = \sqrt{1-x^2}$

[$\because \cos\theta > 0$ for $\theta \in [-\pi/2, \pi/2]$]

$\therefore$ $\sin 2\theta = 2\sin\theta\cos\theta$ ... (i)

$\Rightarrow$ $\sin 2\theta = 2x\sqrt{1-x^2}$

CASE I: When $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

We have,

$$-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

Also, $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$\Rightarrow \quad -1 \le 2x\sqrt{1-x^2} \le 1$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ ... (i)

$\Rightarrow \quad 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

CASE II: When $\frac{1}{\sqrt{2}} \le x \le 1$:

We have,

$$\frac{1}{\sqrt{2}} \le x \le 1$$

$\Rightarrow \quad \frac{1}{\sqrt{2}} \le \sin\theta \le 1$

$\Rightarrow \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} \le 2\theta \le \pi$

$\Rightarrow \quad -\pi \le -2\theta \le -\frac{\pi}{2}$

$\Rightarrow \quad 0 \le \pi - 2\theta \le \frac{\pi}{2}$

Also, $\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow 0 \le 2x\sqrt{1-x^2} < 1$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ from (i)

$\Rightarrow \quad \sin(\pi - 2\theta) = 2x\sqrt{1-x^2}$

(since $\sin(\pi - x) = \sin x$)

$\Rightarrow \quad \pi - 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad \pi - 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1}x = \pi - \sin^{-1}(2x\sqrt{1-x^2})$.

CASE III: When $-1 \le x \le -\frac{1}{\sqrt{2}}$

We have,

$$-1 \le x \le -\frac{1}{\sqrt{2}}$$

$\Rightarrow \quad -1 \le \sin\theta \le -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad -\frac{\pi}{2} \le \theta \le -\frac{\pi}{4}$

$\Rightarrow \quad -\pi \le 2\theta \le -\frac{\pi}{2}$

$\Rightarrow \quad 0 \le \pi + 2\theta \le \frac{\pi}{2}$

Also, $-1 \le x \le -\frac{1}{\sqrt{2}} \Rightarrow -1 \le 2x\sqrt{1-x^2} \le 0$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ [From (i)]

$\Rightarrow \quad -\sin(\pi + 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad \sin(-\pi - 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad -\pi - 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\theta = -\pi - \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1}x = -\pi - \sin^{-1}(2x\sqrt{1-x^2})$

Example – 49

Prove that :

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\} = \frac{\pi}{4}+\frac{x}{2}, 0 < x < \frac{\pi}{2}$$

Sol. We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}$$

$$\left[\because 0<\frac{x}{2}<\frac{\pi}{4} \therefore \cos\frac{x}{2}>0, \sin\frac{x}{2}>0\right]$$

$$= \tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{x}{2}$$

$$\left[\because 0<x<\frac{\pi}{2} \therefore \frac{\pi}{4}<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\right]$$

Example – 50

Prove that :

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\frac{\pi}{4}-\frac{x}{2}, \text{ if } \pi<x<\frac{3\pi}{2}$$

Sol. We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2}\left|\cos\frac{x}{2}\right|+\sqrt{2}\left|\sin\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right|-\sqrt{2}\left|\sin\frac{x}{2}\right|}\right\}$$

$$= \tan^{-1}\left\{\frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}$$

$$\left[\begin{aligned}\because \pi<x<\frac{3\pi}{2} &\Rightarrow -\frac{\pi}{2}\le\frac{\pi}{4}-\frac{x}{2}<-\frac{\pi}{4}\\ &\Rightarrow -\frac{3\pi}{4}<-\frac{x}{2}<-\frac{\pi}{2}\\ &\Rightarrow \frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}\\ &\Rightarrow \pi<x<\frac{3\pi}{2}\end{aligned}\right]$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

$$\left[\because \pi<x<\frac{3\pi}{2} \therefore -\frac{\pi}{2}<\frac{\pi}{4}-\frac{x}{2}<-\frac{\pi}{4}\right]$$

$$\frac{-3\pi}{4}<\frac{-x}{2}<\frac{-\pi}{2}$$

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

Relation and its Types

1. Let R = {(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)} be a relation on the set A = {3, 6, 9, 12}. The relation is

(a) reflexive and symmetric only

(b) an equivalence relation

(c) reflexive only

(d) reflexive and transitive only

2. Let W denotes the words in the English dictionary. Define the relation R by R = {(x, y) ∈ W × W : the words x and y have at least one letter in common}. Then, R is

(a) reflexive, symmetric and not transitive

(b) reflexive, symmetric and transitive

(c) reflexive, not symmetric and transitive

(d) not reflexive, symmetric and transitive

3. Let N be the set of natural numbers and a relation R on N be defined by

$$R=\{(x,y)\in N\times N : x^3-3x^2y-xy^2+3y^3=0\}.$$

Then the relation R is:

(a) reflexive and symmetric, but not transitive

(b) reflexive but neither symmetric nor transitive

(c) an equivalence relation

(d) symmetric but neither reflexive nor transitive

Functions and its classifications

4. Let $f(x)=\frac{\alpha x^2}{x+1}, x\neq -1$. The value of α for which $f(a)=a, (a\neq 0)$ is

(a) $1-\frac{1}{a}$ (b) $\frac{1}{a}$

(c) $1+\frac{1}{a}$ (d) $\frac{1}{a}-1$

5. The domain of $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}$ is

(a) R – {–1, –2} (b) (–2, +∞)

(c) R – {–1, –2, –3} (d) (–3, +∞) – {–1, –2}

6. The range of the function $y=\log_3(5+4x-x^2)$ is

(a) (0, 2] (b) (–∞, 2]

(c) (0, 9] (d) none of these

7. If $e^x+e^{f(x)}=e$, then range of the function of f is

(a) (–∞, 1] (b) (–∞, 1)

(c) (1, ∞) (d) [1, ∞)

8. Range of the function $f(x)=\frac{x^2+x+2}{x^2+x+1}$; x ∈ R is

(a) (1, ∞) (b) $\left(1,\frac{11}{7}\right)$

(c) $\left(1,\frac{7}{3}\right]$ (d) $\left(1,\frac{7}{5}\right)$

9. The equation $2\sin^2\frac{x}{2}\cdot\cos^2 x=x+\frac{1}{x}, 0<x\leq\frac{\pi}{2}$ has

(a) one real solution

(b) no real solution

(c) infinitely many real solutions

(d) none of these

10. Let $f(x)=\frac{x-[x]}{1+x-[x]}$, x ∈ R, then the range of f is :

(a) [0, 1] (b) [0, 1/2]

(c) [0, 1/2) (d) (0, 1)

11. The range of k for which $||x-1|-5|=k$ have four distinct solutions -

(a) [0, 5] (b) (–∞, 5)

(c) [0, ∞) (d) (0, 5)

12. The function $f(x)=\cos\left(\log\left(x+\sqrt{x^2+1}\right)\right)$ is :

(a) even (b) odd

(c) constant (d) None of these

13. Let $f: R\to R$ be a function such that $f(x)=x^3-6x^2+11x-6$. Then

(a) f is one-one and into (b) f is one-one and onto

(c) f is many-one and into (d) f is many-one and onto

14. Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then

(a) f is one-one and into (b) f is many-one and into

(c) f is one-one and onto (d) f is many-one and onto

15. Let f be a function from R to R given by $f(x) = 2x + |\cos x|$. Then f is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

16. Let f be a function from R to R given by $f(x) = \dfrac{x^2-4}{x^2+1}$.

Then $f(x)$ is.

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

17. $f(x) = x + \sqrt{x^2}$ is a function from $R \to R$. Then $f(x)$ is

(a) injective (b) surjective

(c) bijective (d) none of these

18. A function $f : A \to B$, where A $\{x : -1 \le x \le 1\}$ and $B = \{y : 1 \le y \le 2\}$ is defined by the rule $y = f(x) = 1 + x^2$. Which of the following statement is true?

(a) f is injective but not surjective

(b) f is surjective but not injective

(c) f is both injective and surjective

(d) f is neither injective nor surjective

19. $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ and

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$. Then, $f - g$ is

(a) one-one and into (b) neither one-one nor onto

(c) many one and onto (d) one-one and onto

20. If $f : R \to S$, define by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of S is

(a) $[0, 1]$ (b) $[-1, 1]$

(c) $[0, 3]$ (d) $[-1, 3]$

21. If a function $f : [2, \infty) \to B$ defined by $f(x) = x^2 - 4x + 5$ is a bijection, then B is :

(a) R (b) $[1, \infty)$

(c) $[4, \infty)$ (d) $[5, \infty)$

22. Which of the following function has period π ?

(a) $2\cos\left(\dfrac{2\pi x}{3}\right) + 3\sin\left(\dfrac{\pi x}{3}\right)$

(b) $|\tan x| + \cos 2x$

(c) $4\cos\left(2\pi x + \dfrac{\pi}{2}\right) + 2\sin\left(\pi x + \dfrac{\pi}{4}\right)$

(d) none of the above

23. Let $f(x) = \cos 3x + \sin \sqrt{3}x$. Then $f(x)$ is

(a) a periodic function of period 2π.

(b) a periodic function of period $\sqrt{3}\,\pi$.

(c) not a periodic function

(d) none of these

24. The period of $\sin^2 \theta$ is

(a) π^2 (b) π

(c) π^3 (d) $\pi/2$

25. Which one is not periodic

(a) $|\sin 3x| + \sin^2 x$ (b) $\cos\sqrt{x} + \cos^2 x$

(c) $\cos 4x + \tan^2 x$ (d) $\cos^2 x + \sin x$

Composition of a function

26. Let $f(x)$ be a function defined on [0, 1] such that

$$f(x) = \begin{cases} x & x \in Q \\ 1-x, & x \notin Q \end{cases}$$

Then for all $x \in [0, 1]$, $fof(x)$ is

(a) a constant (b) $1 + x$

(c) x (d) none of these

27. If $f(x) = \sqrt{2-x}$ and $g(x) = \sqrt{1-2x}$, then the domain of $f[g(x)]$ is

(a) $(-\infty, 1/2]$ (b) $[1/2, \infty)$

(c) $(-\infty, -3/2]$ (d) none of these

28. Let $f(x) = \sin x$ and $g(x) = \ln |x|$. If the ranges of the composition functions fog and gof are R_1 and R_2 respectively, then

(a) $R_1 = \{u : -1 \le u < 1\}$, $R_2 = \{v : -\infty < v < 0\}$

(b) $R_1 = \{u : -\infty < u < 0\}$, $R_2 = \{v : -1 \le v \le 0\}$

(c) $R_1 = \{u : -1 < u < 1\}$, $R_2 = \{v : -\infty < v < 0\}$

(d) $R_1 = \{u : -1 \le u \le 1\}$, $R_2 = \{v : -\infty < v \le 0\}$

29. If $g\{f(x)\} = |\sin x|$ and $f\{g(x)\} = \left(\sin\sqrt{x}\right)^2$, then

(a) $f(x)=\sin^2 x, g(x)=\sqrt{x}$

(b) $f(x)=\sin x, g(x)=|x|$

(c) $f(x)=x^2, g(x)=\sin\sqrt{x}$

(d) f and g cannot be determined

30. If $f(x) = \sin^2x$, $g(x)=\sqrt{x}$ and $h(x) = \cos^{-1}x$, $0 \le x \le 1$, then -

(a) hogof(x) = gofoh(x)

(b) gofoh(x) = fohog(x)

(c) fohog(x) = hogof(x)

(d) None of these

31. If $f(g(x)) = |\cos x|$, $g(f(x)) = \cos^2\sqrt{x}$, then -

(a) f(x) is a periodic function and g(x) is a non-periodic function.

(b) f(x) is a non-periodic function and g(x) is a periodic function.

(c) Both f(x) and g(x) are periodic functions

(d) Neither f(x) nor g(x) is a periodic function

32. Consider the functions $f(x)=\sqrt{x}$ and $g(x) = 7x + b$. If the function y = fog(x) passes through (4, 6) then the value of b is

(a) 8 (b) – 8

(c) – 25 (d) $4-7\sqrt{6}$

Inverse of a Function

33. The inverse of the function $f(x)=\dfrac{a^x-a^{-x}}{a^x+a^{-x}}$ is

(where codomain of f(x) is (–1, 1))

(a) $\dfrac{1}{2}\log_a\left(\dfrac{1-x}{1+x}\right)$ (b) $\dfrac{1}{2}\log_a\left(\dfrac{1+x}{1-x}\right)$

(c) $\log_a\left(\dfrac{1+x}{1-x}\right)$ (d) none of these

34. Let $f : [-1, \infty) \to R$ be given by $f(x) = (x + 1)^2 - 1$, $x \ge -1$. Then $f^{-1}(x)$, is :

(a) $-1+\sqrt{x+1}$

(b) $-1-\sqrt{x+1}$

(c) does not exist because f is not one-one

(d) does not exist because f is not onto

35. The inverse of the function $y = [1 - (x - 3)^4]^{1/7}$ is

(a) $3 + (1 - x^7)^{1/4}$ (b) $3 - (1 - x^7)^{1/4}$

(c) $3 - (1 + x^7)^{1/4}$ (d) none of these

Functional Equations

36. If $3f(x) + 5f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 3$, $\forall\ x(\ne 0) \in R$, then $f(x)$ is equal to :

(a) $\dfrac{1}{16}\left(\dfrac{3}{x}+5x-6\right)$ (b) $\dfrac{1}{16}\left(-\dfrac{3}{x}+5x-6\right)$

(c) $\dfrac{1}{16}\left(\dfrac{3}{x}-5x-6\right)$ (d) none of these

37. Let $f : R \to R$ be a function given by $f(x + y) = f(x) + f(y)$ for all x, y $\in$ R such that $f(1) = a$. Then, $f(x) =$

(a) a^x (b) ax

(c) x^a (d) a + x

38. Let f be a real valued function satisfying $f(x + y) = f(x) f(y)$ for all x, y $\in$ R such that $f(1) = a$. Then, $f(x) =$

(a) a^x (b) ax

(c) x^a (d) none of these

39. If $a f(x + 1) + bf\left(\dfrac{1}{x+1}\right) = x$, $x \ne -1$, $a \ne -b$, then $f(1)$ is equal to

(a) a + b (b) $a^2 - b^2$

(c) $\dfrac{1}{a+b}$ (d) f(1) = 0

Simplification problems of ITF

40. If $\alpha = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\beta = \tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$, then

(a) $4\alpha = 3\beta$ (b) $3\alpha = 4\beta$

(c) $\alpha - \beta = \frac{7\pi}{12}$ (d) none of these

41. Which one of the following is correct?

(a) $\tan 1 > \tan^{-1} 1$ (b) $\tan 1 < \tan^{-1} 1$

(c) $\tan 1 = \tan^{-1} 1$ (d) None of the above

42. The value of $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ is :

(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

43. $\cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$, then sin x is equal to

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$

(c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

44. If $\tan(\cos^{-1} x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$, then x is equal to :

(a) $\pm\frac{5}{3}$ (b) $\pm\frac{\sqrt{5}}{3}$

(c) $\pm\frac{5}{\sqrt{3}}$ (d) None of these

45. $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$ is equal to :

(a) $\sqrt{\frac{x^2+2}{x^2+3}}$ (b) $\sqrt{\frac{x^2+2}{x^2+1}}$

(c) $\sqrt{\frac{x^2+1}{x^2+2}}$ (d) None of these

Properties of ITF

46. If $\sin^{-1} x = \frac{\pi}{5}$, for some $x \in (-1, 1)$, then the value of $\cos^{-1} x$ is :

(a) $\frac{3\pi}{10}$ (b) $\frac{5\pi}{10}$

(c) $\frac{7\pi}{10}$ (d) $\frac{9\pi}{10}$

47. The value of $\cot^{-1}\frac{3}{4} + \sin^{-1}\frac{5}{13}$ is :

(a) $\sin^{-1}\frac{63}{65}$ (b) $\sin^{-1}\frac{12}{13}$

(c) $\sin^{-1}\frac{65}{68}$ (d) $\sin^{-1}\frac{5}{12}$

48. If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to

(a) $-4\sin^2\alpha$ (b) $4\sin^2\alpha$

(c) 4 (d) $2\sin 2\alpha$

49. If $\tan^{-1}\frac{x-1}{x+2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then x is equal to :

(a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{2}}$

(c) $\sqrt{\frac{5}{2}}$ (d) $\pm\frac{1}{2}$

50. If $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$, then the value of x is :

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) None of these

51. The equation $2\cos^{-1} x + \sin^{-1} x = \frac{11\pi}{6}$ has :

(a) no solution (b) only one solution

(c) two solutions (d) three solutions

52. If $\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)=\frac{\pi}{2}$, then x is equal to :

(a) $\sqrt{ab}$ (b) $\sqrt{2ab}$

(c) 2ab (d) ab

53. If $\cos^{-1}x > \sin^{-1}x$, then :

(a) $x<0$ (b) $-1<x<0$

(c) $0\le x<\frac{1}{\sqrt{2}}$ (d) $-1\le x<\frac{1}{\sqrt{2}}$

54. Set of values of x satisfying $\cos^{-1}\sqrt{x} > \sin^{-1}\sqrt{x}$

(a) $\left(0,\frac{1}{2}\right)$ (b) $\left[0,\frac{1}{2}\right)$

(c) $\left(\frac{1}{2},1\right)$ (d) $\left(\frac{1}{2},1\right]$

55. The value of $\cos(2\cos^{-1}x+\sin^{-1}x)$ at $x=\frac{1}{5}$ is :

(a) 1 (b) 3

(c) 0 (d) $-\frac{2\sqrt{6}}{5}$

56. The value of $\sum_{r=0}^{\infty}\tan^{-1}\left(\frac{1}{1+r+r^2}\right)$ is equal to :

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{4}$

(c) $\frac{\pi}{4}$ (d) None of these

57. If $\tan^{-1}\frac{1}{1+2}+\tan^{-1}\frac{1}{1+(2)(3)}+\tan^{-1}\frac{1}{1+(3)(4)}$

$+.....+\tan^{-1}\frac{1}{1+n(n+1)}=\tan^{-1}\theta$, then $\theta=$

(a) $\frac{n}{n+1}$ (b) $\frac{n+1}{n+2}$

(c) $\frac{n}{n+2}$ (d) $\frac{n-1}{n+2}$

ITF- Domain & Range

58. The domain of $\sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]$ is

(a) [1, 9] (b) [–1, 9]

(c) [–9, 1] (d) [–9, –1]

59. The domain of the function $f(x)=\frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3]

(c) [2, 3) (d) [1, 2)

60. The largest interval lying in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ for which the function $f(x)=4^{-x^2}+\cos^{-1}\left(\frac{x}{2}-1\right)+\log(\cos x)$ is defined, is

(a) $[0,\pi]$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $\left[-\frac{\pi}{4},\frac{\pi}{2}\right)$ (d) $\left[0,\frac{\pi}{2}\right)$

61. If $\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x$, $x\ge 0$, then the smallest interval in which θ lies, is given by :

(a) $\frac{\pi}{2}\le\theta\le\frac{3\pi}{4}$ (b) $-\frac{\pi}{4}\le\theta\le 0$

(c) $0\le\theta\le\frac{\pi}{4}$ (d) $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$

62. Range of $f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$ is

(a) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ (b) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

(c) $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$ (d) None of these

63. Range of $f(x)=\sin^{-1}x+\cot^{-1}x+\tan^{-1}x$ is

(a) $[0,\pi]$ (b) $\left[\frac{\pi}{2},\pi\right]$

(c) $\left[\frac{\pi}{4},\pi\right]$ (d) $[-\pi,\pi]$

Numerical Value Type Questions

64. Let n(A) = 4 and n(B) = 6. Then the number of one-one functions from A to B is

65. The period of the function $f(x) = \sin^4 x + \cos^4 x$ is π/k. Then the value of k is

66. The period of the function $f(x) = |\sin 4x| + |\cos 4x|$ is π/k. Then the value of k is

67. Let [x] denote the greatest integer ≤ x. If $f(x) = [x]$ and $g(x) = |x|$, then the value of $f\left(g\left(\frac{8}{5}\right)\right) - g\left(f\left(-\frac{8}{5}\right)\right)$ is

68. If $f: R \to R$ is given by

$$f(x) = \begin{cases} -1, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases},$$

then $(fof)\left(1-\sqrt{3}\right)$ is equal to

69. The number of real solutions of the equation $e^x = x$ is

70. The number of real solutions of the equation $\log_{0.5} x = |x|$ is

71. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is

72. If f is a real valued function such that $f(x + y) = f(x) + f(y)$ and $f(1) = 5$, then the value of $f(100)$ is

73. If $2f(x+1) + f\left(\frac{1}{x+1}\right) = 2x$ and $x \neq -1$, then $f(2)$ is equal to k/6. Then the value of k is.

74. If $f(x) = ax^2 + bx + c$ satisfies the identity $f(x+1) - f(x) = 8x + 3$ for all $x \in R$. Then, a + b =

75. $\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3)$ is equal to

76. If $\sum_{i=1}^{20} \sin^{-1} x_i = 10\pi$, then $\sum_{i=1}^{20} x_i$ is equal to :

77. If $\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1} x$, then x is equal to

78. If $k \le \sin^{-1} x + \cos^{-1} x + \tan^{-1} x \le K$ and $k + K = m\pi$. Then the value of m is.

79. The value of $\tan^{-1}(1) + \tan^{-1}(0) + \tan^{-1}(2) + \tan^{-1}(3)$ is equal to $k\pi$. Then the value of k is

80. If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the value of $\tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\sin 2x}{5 + 3\cos 2x}\right)$ is kx. Then the value of k is

EXERCISE - 2 : PREVIOUS YEAR JEE MAIN QUESTIONS

1. If $f(x)=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right), x>1$, then f(5) is equal to

(2015/Online Set–1)

(a) $\frac{\pi}{2}$ (b) $\tan^{-1}\left(\frac{65}{156}\right)$

(c) $4\tan^{-1}(5)$ (d) π

2. If $f(x)+2f\left(\frac{1}{x}\right)=3x, x\neq 0$, and **(2016)**

$S=\{x\in R: f(x)=f(-x)\}$; then S :

(a) contains exactly one element

(b) contains exactly two elements.

(c) contains more than two elements.

(d) is an empty set.

3. For $x\in R$, $x\neq 0$, $x\neq 1$, let $f_0(x)=\frac{1}{1-x}$ and $f_{n+1}(x)=f_0(f_n(x))$, n = 0, 1, 2, … Then the value of $f_{100}(3)+\frac{8}{3}f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$ is equal to:

(2016/Online Set–1)

(a) $\frac{8}{3}$ (b) $\frac{5}{3}$

(c) $\frac{4}{3}$ (d) $\frac{1}{3}$

4. The function $f:R\to\left[-\frac{1}{2},\frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$, is: **(2017)**

(a) invertible

(b) injective but not surjective

(c) surjective but not injective

(d) neither injective nor surjective.

5. Let $f(x)=2^{10}.x+1$ and $g(x)=3^{10}.x-1$.

If $(f o g)(x)=x$, then x is equal to : **(2017/Online Set–1)**

(a) $\frac{3^{10}-1}{3^{10}-2^{-10}}$ (b) $\frac{2^{10}-1}{2^{10}-3^{-10}}$

(c) $\frac{1-3^{-10}}{2^{10}-3^{-10}}$ (d) $\frac{1-2^{-10}}{3^{10}-2^{-10}}$

6. The value of $\tan^{-1}\left|\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right|$, $|x|<\frac{1}{2}, x\neq 0$, is equal to : **(2017/Online Set–1)**

(a) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ (b) $\frac{\pi}{4}+\cos^{-1}x^2$

(c) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$ (d) $\frac{\pi}{4}-\cos^{-1}x^2$

7. A value of x satisfying the equation

$\sin[\cot^{-1}(1+x)]=\cos[\tan^{-1}x]$, is :

(2017/Online Set–2)

(a) $-\frac{1}{2}$ (b) -1

(c) 0 (d) $\frac{1}{2}$

8. The function $f: N\to N$ defined by $f(x)=x-5\left[\frac{x}{5}\right]$, where N is the set of natural numbers and [x] denotes the greatest integer less than or equal to x, is :

(2017/Online Set–2)

(a) one-one and onto.

(b) one-one but not onto.

(c) onto but not one-one.

(d) neither one-one nor onto.

9. Let $S = \{(\lambda,\mu) \in R \times R : f(t) = \left[|\lambda| e^{|t|} - \mu\right].$

$\sin(2|t|), t \in R$ is a differentiable function$\}$. Then S is a subset of : **(2018/Online Set–1)**

(a) $R \times [0,\infty)$ (b) $[0,\infty) \times R$

(c) $R \times (-\infty,0)$ (d) $(-\infty,0) \times R$

10. Consider the following two binary relations on the set $A = \{a, b, c\}$:

$R_1 = \{(c, a), (b, b), (a, c), (c, c), (b, c), (a, a)\}$ and

$R_2 = \{(a, b), (b, a), (c, c), (c, a), (a, a), (b, b), (a, c)\}$. Then : **(2018/Online Set–1)**

(a) both R_1 and R_2 are not symmetric.

(b) R_1 is not symmetric but it is transitive.

(c) R_2 is symmetric but it is not transitive.

(d) both R_1 and R_2 are transitive.

11. Let f: $A \to B$ be a function defined as $f(x) = \frac{x-1}{x-2}$, where $A = R - \{2\}$ and $B = R - \{1\}$. Then f is : **(2018/Online Set–2)**

(a) Invertible and $f^{-1}(y) = \frac{3y-1}{y-1}$

(b) Invertible and $f^{-1}(y) = \frac{2y-1}{y-1}$

(c) Invertible and $f^{-1}(y) = \frac{2y+1}{y-1}$

(d) Not invertible

12. If the function f defined as $f(x) = \frac{1}{x} - \frac{k-1}{e^{2x}-1}, x \neq 0$, is continuous at x =0, then the ordered pair (k, f(0)) is equal to : **(2018/Online Set–2)**

(a) (3, 2) (b) (3, 1)

(c) (2, 1) (d) $\left(\frac{1}{3}, 2\right)$

13. If $f(x) = \sin^{-1}\left(\frac{2 \times 3^x}{1+9^x}\right)$, then $f'\left(-\frac{1}{2}\right)$ equals: **(2018/Online Set–2)**

(a) $-\sqrt{3}\log_e\sqrt{3}$ (b) $\sqrt{3}\log_e\sqrt{3}$

(c) $-\sqrt{3}\log_e 3$ (d) $\sqrt{3}\log_e 3$

14. Let N denote the set of all natural numbers. Define two binary relations on N as

$R_1 = \{(x, y) \in N \times N : 2x + y = 10\}$ and

$R_2 = \{(x, y)\ N\ N : x + 2y = 10\}$. Then : **(2018/Online Set–3)**

(a) Range of R_1 is $\{2, 4, 8\}$.

(b) Range of R_2 is $\{1, 2, 3, 4\}$.

(c) Both R_1 and R_2 are symmetric relations.

(d) Both R_1 and R_2 are transitive relations.

15. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to : **(2019-04-08/Shift-1)**

(a) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

(c) $\tan^{-1}\left(\frac{9}{14}\right)$ (d) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

16. If $f(x) = \log_e\left(\frac{1-x}{1+x}\right), |x| < 1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to : **(2019-04-08/Shift-1)**

(a) $2f(x)$ (b) $2f(x^2)$

(c) $(f(x))^2$ (d) $-2f(x)$

17. The sum of the solutions of the equation $|\sqrt{x}-2| + \sqrt{x}(\sqrt{x}-4) + 2 = 0, (x > 0)$ is equal to: **(2019-04-08/Shift-1)**

18. Let $\sum_{k=1}^{10} f(a+k) = 16(2^{10}-1)$, where the function f satisfies $f(x+y) = f(x)f(y)$ for all natural numbers x, y and $f(1) = 2$. Then the natural number 'a' is: **(2019-04-09/Shift-1)**

19. Let f(x) = x^2 , $x \in$ R. For any A $\subseteq$ R, define g(A) = $\{x \in R: f(x) \in A\}$. If S = [0, 4], then which one of the following statements is not true ?

(2019-04-10/Shift-1)

(a) $g(f(S)) \neq S$ (b) $f(g(S)) = S$

(c) $g(f(S)) = g(S)$ (d) $f(g(S)) \neq f(S)$

20. Let $f(x) = \log_e(\sin x), (0 < x < \pi)$

and $g(x) = \sin^{-1}(e^{-x})(x \geq 0)$. If α is a positive real number such that $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then: **(2019-04-10/Shift-2)**

(a) $a\alpha^2 + b\alpha + a = 0$ (b) $a\alpha^2 - b\alpha - a = 1$

(c) $a\alpha^2 - b\alpha - a = 0$ (d) $a\alpha^2 + b\alpha - a = -2a^2$

21. If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, where

$-1 \leq x \leq 1, -3 \leq y \leq 2, x \leq \frac{y}{2}$, then for all

$x, y, 4x^2 - 4xy\cos\alpha + y^2$ is equal to:

(2019-04-10/Shift-2)

(a) $4\sin^2\alpha$ (b) $2\sin^2\alpha$

(c) $4\sin^2\alpha - 2x^2y^2$ (d) $4\cos^2\alpha + 2x^2y^2$

22. For $x \in \left(0, \frac{3}{2}\right)$ let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1-x^2}{1+x^2}$. If $\phi(x) = ((hof)og)(x)$, then $\phi\left(\frac{\pi}{3}\right)$ is equal to **(2019-04-12/Shift-1)**

(a) $\tan\frac{\pi}{12}$ (b) $\tan\frac{11\pi}{12}$

(c) $\tan\frac{7\pi}{12}$ (d) $\tan\frac{5\pi}{12}$

23. The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to

(2019-04-12/Shift-1)

(a) $\pi - \sin^{-1}\left(\frac{63}{65}\right)$ (b) $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$

(c) $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$ (d) $\pi - \cos^{-1}\left(\frac{33}{65}\right)$

24. If [x] denotes the greatest integer $\leq$ x, then the system of linear equations $[\sin\theta]x + [-\cos\theta]y = 0, [\cot\theta]x + y = 0$

(2019-04-12/Shift-2)

(a) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and has a unique solution if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(b) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(c) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and have infinitely many solutions if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(d) have infinitely many solutions if

$\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

25. If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to **(2019-01-09/Shift-1)**

(a) $\frac{\sqrt{145}}{12}$ (b) $\frac{\sqrt{145}}{10}$

(c) $\frac{\sqrt{146}}{12}$ (d) $\frac{\sqrt{145}}{11}$

26. If x = $\sin^{-1}$ (sin 10) and y = $\cos^{-1}$(cos10), then y – x is equal to: **(2019-01-09/Shift-2)**

(a) 0 (b) 10

(c) 7 π (d) π

27. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1 + \sum_{p=1}^{n} 2p\right)\right)$ is:

(2019-01-10/Shift-2)

(a) $\frac{21}{19}$ (b) $\frac{19}{21}$

(c) $\frac{22}{23}$ (d) $\frac{23}{22}$

28. All x satisfying the inequality

$\left(\cot^{-1} x\right)^2 - 7\left(\cot^{-1} x\right) + 10 > 0$, lie in the interval :

(2019-01-11/Shift-2)

(a) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

(b) $(\cot 2, \infty)$

(c) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

(d) $(\cot 5, \cot 4)$

29. Considering only the principal values of inverse functions, the set

$$A = \left\{x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}\right\}$$

(2019-01-12/Shift-1)

(a) contains two elements

(b) contains more than two elements

(c) is a singleton

(d) is an empty set

30. The domain of the function $f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$ is

$(-\infty, -a] \cup [a, \infty)$. Then a is equal to :

(2020-09-02/Shift-1)

(a) $\frac{\sqrt{17}-1}{2}$ (b) $\frac{\sqrt{17}}{2}$

(c) $\frac{1+\sqrt{17}}{2}$ (d) $\frac{\sqrt{17}}{2}+1$

31. Let $f : R \to R$ be a function which satisfies

$f(x+y) = f(x) + f(y)\ \forall x, y \in R$. If $f(1) = 2$ and

$g(n) = \sum_{k=1}^{(n-1)} f(k), n \in N$ then the value of n, for which

$g(n) = 20$, is : **(2020-09-02/Shift-2)**

(a) 9 (b) 5

(c) 4 (d) 20

32. $2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ is equal to :

(2020-09-03/Shift-1)

(a) $\frac{5\pi}{4}$ (b) $\frac{3\pi}{2}$

(c) $\frac{7\pi}{4}$ (d) $\frac{\pi}{2}$

33. Let R_1 and R_2 be two relations defined as follows :

$R_1 = \{(a, b) \in R^2 : a^2 + b^2 \in Q\}$ and

$R_2 = \{(a, b) \in R^2 : a^2 + b^2 \notin Q\}$, where Q is the set of all rational numbers. Then : **(2020-09-03/Shift-2)**

(a) R_1 is transitive but R_2 is not transitive

(b) R_1 and R_2 are both transitive

(c) R_2 is transitive but R_1 is not transitive

(d) Neither R_1 nor R_2 is transitive

34. Let A={a,b,c} and B={1,2,3,4}. Then the number of elements in the set $C = \{f : A \to B \mid 2 \in f(A)$ and f is not one-one$\}$ is_____ **(2020-09-05/Shift-2)**

35. For a suitably chosen real constant a, let a function, f: R-{-a} $\to$ R be defined by $f(x) = \frac{a-x}{a+x}$. Further suppose that for any real number $x \neq -a$ and $f(x) \neq -a$, $(fof)(x) = x$. Then $f\left(-\frac{1}{2}\right)$ is equal to:

(2020-09-06/Shift-2)

(a) -3 (b) 3

(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

36. If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$

then, $f\left(\frac{5}{4}\right)$ is equal to **(2020-01-07/Shift-1)**

(a) $-\frac{3}{2}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

37. The inverse function of $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}, x \in (-1, 1)$, is

(2020-01-08/Shift-1)

(a) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1-x}{1+x}\right)$ (b) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1+x}{1-x}\right)$

(c) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$ (d) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

38. Let $f(x)=\sin^{-1}x$ and $g(x)=\dfrac{x^2-x-2}{2x^2-x-6}$. If $g(2)=\lim\limits_{x\to 2} g(x)$ then the domain of the function fog is **(26-02-2021/Shift-2)**

(a) $(-\infty,-1]\cup[2,\infty)$ (b) $(-\infty,-2]\cup\left[-\frac{4}{3},\infty\right)$

(c) $(-\infty,-2]\cup[-1,\infty)$ (d) $(-\infty,-2]\cup\left[-\frac{3}{2},\infty\right)$

39. Let $A=\{1, 2, 3,\ldots,10\}$ and $f:A\to A$ be defined as

$$f(k)=\begin{cases}k+1 & \text{if } k \text{ is odd}\\ k & \text{if } k \text{ is even}\end{cases}$$

Then the number of possible functions $g:A\to A$ such that $gof=f$ is **(26-02-2021/Shift-2)**

(a) 10^5 (b) ${}^{10}C_5$

(c) $5!$ (d) 5^5

40. Let R = {(P, Q)| P and Q are at the same distance from the origin} be a relation, then the equivalence class of $(1,-1)$ is the set: **(26-02-2021/Shift-1)**

(a) $S=\{(x,y)\mid x^2+y^2=4\}$

(b) $S=\{(x,y)\mid x^2+y^2=1\}$

(c) $S=\{(x,y)\mid x^2+y^2=2\}$

(d) $S=\{(x,y)\mid x^2+y^2=\sqrt{2}\}$

41. Let $f,g:N\to N$ such that $f(n+1)=f(n)+f(1)\ \forall\ n\in N$ and g be any arbitrary function. Which of the following statements is **NOT** true? **(25-02-2021/Shift-1)**

(a) If g is onto, then fog is one-one

(b) f is one-one

(c) If f is onto, then $f(n)=n\ \forall\ n\in N$

(d) If fog is one-one, then g is one-one

42. A possible value of $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is : **(24-02-2021/Shift-2)**

(a) $2\sqrt{2}-1$ (b) $\frac{1}{\sqrt{7}}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\sqrt{7}-1$

43. If $a+\alpha=1, b+\beta=2$ and $af(x)+\alpha f\left(\frac{1}{x}\right)=bx+\frac{\beta}{x}, x\neq 0$, then the value of the expression $\dfrac{f(x)+f\left(\frac{1}{x}\right)}{x+\frac{1}{x}}$ is ______. **(24-02-2021/Shift-2)**

44. Let $f:R\to R$ be defined as $f(x)=2x-1$ and $g:R-\{1\}\to R$ be defined as $g(x)=\dfrac{x-\frac{1}{2}}{x-1}$. Then the composition function $f(g(x))$ is **(24-02-2021/Shift-2)**

(a) One-one but not onto

(b) Both one-one and onto

(c) Neither one-one nor onto

(d) Onto but not one-one

45. Let $f:R-\{3\}\to R-\{1\}$ be defined by $f(x)=\dfrac{x-2}{x-3}$.

Let $g:R\to R$ be given as $g(x)=2x-3$. Then, the sum of all the values of x for which $f^{-1}(x)+g^{-1}(x)=\frac{13}{2}$ is equal to. **(18-03-2021/Shift-2)**

(a) 2 (b) 7

(c) 5 (d) 3

46. Let A = {2, 3, 4, 5, ..., 30} and '' be an equivalence relation on A × A, defined by (a, b) (c, d), if and only if ad = bc. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to **(16-03-2021/Shift-2)**

(a) 6 (b) 5

(c) 8 (d) 7

47. The inverse of $y = 5^{\log x}$ is : **(17-03-2021/Shift-1)**

(a) $x = y^{\log 5}$ (b) $x = y^{\frac{1}{\log 5}}$

(c) $x = 5^{\log y}$ (d) $x = 5^{\frac{1}{\log y}}$

48. If

$\cot^{-1}(\alpha) = \cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + ...$

upto 100 terms, then α is **(17-03-2021/Shift-1)**

(a) 1.03 (b) 1.02

(c) 1.01 (d) 1.00

49. The sum of possible values of x for

$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$ is :

(17-03-2021/Shift-1)

(a) $-\frac{32}{4}$ (b) $-\frac{31}{4}$

(c) $-\frac{30}{4}$ (d) $-\frac{33}{4}$

50. If $f(x) = \sin\left(\cos^{-1}\left(\frac{1-2^{2x}}{1+2^{2x}}\right)\right)$ and its first derivative with respect to x is $-\frac{b}{a}\log_e 2$ when x = 1, where a and b are integers, then the minimum value of $|a^2 - b^2|$ is .... **(17-03-2021/Shift-1)**

51. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is : **(17-03-2021/Shift-2)**

(a) 4 (b) 5

(c) 3 (d) 2

52. Let $[x]$ denote the greatest integer $\le x$, where $x \in R$. If the domain of the real valued function $f(x) = \sqrt{\frac{|[x]|-2}{|[x]|-3}}$ is $(-\infty, a) \cup [b, c) \cup [4, \infty)$, $a < b < c$, then the value of $a + b + c$ is: **(20-07-2021/Shift-1)**

(a) –3 (b) 1

(c) –2 (d) 8

53. Let $f : R - \left\{\frac{\alpha}{6}\right\} \to R$ be defined by $f(x) = \frac{5x+3}{6x-\alpha}$. Then the value of α for which $(fof)(x) = x$, for all $x \in R - \left\{\frac{\alpha}{6}\right\}$, is ? **(20-07-2021/Shift-2)**

(a) No such α exists (b) 5

(c) 6 (d) 8

54. Let $g : N \to N$ be defined as

$g(3n+1) = 3n+2$,

$g(3n+2) = 3n+3$,

$g(3n+3) = 3n+1$, for all $n \ge 0$.

Then which of the following statements is true ? **(25-07-2021/Shift-1)**

(a) gogog = g

(b) There exists an onto function $f : N \to N$ such that fog = f

(c) There exists a one-one function $f : N \to N$ such that fog = f

(d) There exists an function $f : N \to N$ such that gof = f

55. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Then the number of possible functions $f : S \to S$ such that $f(m.n) = f(m).f(n)$ for every $m, n \in S$ and $m.n \in S$ is equal to _____ **(27-07-2021/Shift-1)**

56. Let $f: R \to R$ be defined as

$f(x+y)+f(x-y)=2f(x)f(y), f\left(\frac{1}{2}\right)=-1$. Then, the value of $\sum_{k=1}^{20} \frac{1}{\sin(k)\sin(k+f(k))}$ is equal to:

(27-07-2021/Shift-2)

(a) $\operatorname{cosec}^2(1)\operatorname{cosec}(21)\sin(20)$

(b) $\sec^2(1)\sec(21)\cos(20)$

(c) $\operatorname{cosec}^2(21)\cos(20)\cos(2)$

(d) $\sec^2(21)\sin(20)\sin(2)$

57. Let $[x]$ denote the greatest integer less than or equal to x. Then, the values of $x \in R$ satisfying the equation $[e^x]^2+[e^x+1]-3=0$ lie in the interval:

(22-07-2021/Shift-2)

(a) $[\ln 2, \ln 3)$ (b) $\left[0, \frac{1}{e}\right)$

(c) $[0, \ln 2)$ (d) $[1, e)$

58. Consider function $f: A \to B$ and $g: B \to C (A, B, C \subseteq R)$ such that $(gof)^{-1}$ exists, then:

(25-07-2021/Shift-2)

(a) f and g both are one-one

(b) f is onto and g is one-one

(c) f is one-one and g is onto

(d) f and g both are onto

59. The range of the function

$f(x)=\log_{\sqrt{5}}\left(3+\cos\left(\frac{3\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{3\pi}{4}-x\right)\right)$ **(01-09-2021/Shift-2)**

(a) $\left[\frac{1}{\sqrt{5}}, \sqrt{5}\right]$ (b) $(0, \sqrt{5})$

(c) $[0, 2]$ (d) $[-2, 2]$

60. If be a polynomial of degree 3 such that $f(k)=-\frac{2}{k}$ for $k=2,3,4,5$. Then the value $52-10f(10)$ is equal to ____________

(01-09-2021/Shift-2)

61. The domain of the function $\operatorname{cosec}^{-1}\left(\frac{1+x}{x}\right)$ is :

(26-08-2021/Shift-2)

(a) $\left[-\frac{1}{2}, \infty\right)-\{0\}$ (b) $\left(-\frac{1}{2}, \infty\right)-\{0\}$

(c) $\left[-\frac{1}{2}, 0\right)\cup[1, \infty)$ (d) $\left(-1, -\frac{1}{2}\right]\cup(0, \infty)$

62. Let z be the set of all integers,

$A=\{(x,y)\in Z\times Z; (x-2)^2+y^2 \le 4\}$

$B=\{(x,y)\in Z\times Z; x^2+y^2 \le 4\}$ and

$C=\{(x,y)\in Z\times Z; (x-2)^2+(y-2)^2 \le 4\}$

If the total number of relation from $A \cap B$ to $A \cap C$ is 2^p, then the value of p is : **(27-08-2021/Shift-2)**

(a) 16 (b) 25

(c) 49 (d) 9

63. Let M and m respectively be the maximum and minimum values of the function $f(x)=\tan^{-1}(\sin x+\cos x)$ in $\left[0, \frac{\pi}{2}\right]$. Then the value of $\tan(M-m)$ is equal to:

(27-08-2021/Shift-2)

(a) $3-2\sqrt{2}$ (b) $3+2\sqrt{2}$

(c) $2-\sqrt{3}$ (d) $2+\sqrt{3}$

64. Which of the following is not correct for relation R on the set of real numbers? **(31-08-2021/Shift-1)**

(a) $(x,y)\in R \Leftrightarrow |x-y|\le 1$ is reflexive and symmetric.

(b) $(x,y)\in R \Leftrightarrow 0<|x-y|\le 1$ is symmetric but not transitive.

(c) $(x,y)\in R \Leftrightarrow |x|-|y|\le 1$ is reflexive but not symmetric.

(d) $(x,y)\in R \Leftrightarrow 0<|x|-|y|\le 1$ is neither transitive nor symmetric.

65. Let $f:N\to N$ be a function such that

$f(m+n)=f(m)+f(n)$ for every $m,n\in N$. If

$f(6)=18$, then $f(2)\cdot f(3)$ is equal to:

(31-08-2021/Shift-2)

(a) 36 (b) 6

(c) 18 (d) 54

66. If the domain of the function $f(x)=\dfrac{\cos^{-1}\sqrt{x^2-x+1}}{\sqrt{\sin^{-1}\left(\dfrac{2x-1}{2}\right)}}$ is the interval $(\alpha,\beta]$, then $\alpha+\beta$ is equal to :

(22-07-2021/Shift-2)

(a) 2 (b) $\frac{3}{2}$

(c) $\frac{1}{2}$ (d) 1

67. The number of real roots of the equation

$e^{4x}-e^{3x}-4e^{2x}-e^{x}+1=0$ is equal to ______.

(27-07-2021/Shift-2)

68. If $x^2+9y^2-4x+3=0,\ x,\ y\in R$, then x and y respectively lie in the intervals: **(27-08-2021/Shift--1)**

(a) $[1,3]$ and $\left[-\frac{1}{3},\frac{1}{3}\right]$

(b) $\left[-\frac{1}{3},\frac{1}{3}\right]$ and $[1,3]$

(c) $[1,3]$ and $[1,3]$

(d) $\left[-\frac{1}{3},\frac{1}{3}\right]$ and $\left[-\frac{1}{3},\frac{1}{3}\right]$

69. The number of solutions of the equation

$32^{\tan^2 x}+32^{\sec^2 x}=81, 0\le x\le\frac{\pi}{4}$ is: **(31-08-2021/Shift-2)**

(a) 0 (b) 1

(c) 2 (d) 3

70. If the functions are defined as $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$, then what is the common domain of the following functions $f+g,\ f-g,\ \frac{f}{g},\ \frac{g}{f},\ g-f$ where

$$(f\pm g)(x)=f(x)\pm g(x),\ (f/g)(x)=\frac{f(x)}{g(x)}$$

(18-03-2021/Shift-1)

(a) $0\le x<1$ (b) $0<x\le 1$

(c) $0\le x\le 1$ (d) $0<x<1$

71. The real valued function $f(x)=\dfrac{\operatorname{cosec}^{-1}x}{\sqrt{x-[x]}}$, where $[x]$ denotes the greatest integer less than or equal to x, is defined for all x belonging to : **(18-03-2021/Shift-1)**

(a) all integers except 0, –1, 1

(b) all reals except the interval [–1, 1]

(c) all reals except integers

(d) all non-integers except the interval [–1, 1]

72. Let x denote the total number of one - one functions from a set A with 3 elements to a set B with 5 elements and y denote the total number of one one functions from the set A to the set $A\times B$. Then: **(25-02-2021/Shift-2)**

(a) $y=91x$ (b) $2y=91x$

(c) $y=273x$ (d) $2y=273x$

73. A function $f(x)$ is given by $f(x)=\dfrac{5^x}{5^x+5}$, then the sum of the series $f\left(\frac{1}{20}\right)+f\left(\frac{2}{20}\right)+f\left(\frac{3}{20}\right)+.....+f\left(\frac{39}{20}\right)$

(25-02-2021/Shift-2)

(a) $\frac{49}{2}$ (b) $\frac{29}{2}$

(c) $\frac{39}{2}$ (d) $\frac{19}{2}$

74. Let f be any function defined on R and let it satisfy the condition $|f(x)-f(y)| \le |(x-y)^2|, \forall (x,y) \in R$

If $f(0)=1$, then : **(26-02-2021/Shift-1)**

(a) $f(x) > 0, \forall x \in R$

(b) $f(x) = 0, \forall x \in R$

(c) $f(x) < 0, \forall x \in R$

(d) $f(x)$ can take any value in R

75. $\operatorname{cosec}\left[2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to:

(25-02-2021/Shift-2)

(a) $\frac{65}{56}$ (b) $\frac{65}{33}$

(c) $\frac{75}{56}$ (d) $\frac{56}{33}$

76. If $\frac{\sin^{-1}x}{a}=\frac{\cos^{-1}x}{b}=\frac{\tan^{-1}y}{c}; 0<x<1$, then the value of $\cos\left(\frac{\pi c}{a+b}\right)$ is: **(26-02-2021/Shift-1)**

(a) $\frac{1-y^2}{1+y^2}$ (b) $\frac{1-y^2}{y\sqrt{y}}$

(c) $\frac{1-y^2}{2y}$ (d) $1-y^2$

77. If $0<a, b<1$, and $\tan^{-1}a+\tan^{-1}b=\frac{\pi}{4}$, then the value of

$$(a+b)-\left(\frac{a^2+b^2}{2}\right)+\left(\frac{a^3+b^3}{3}\right)-\left(\frac{a^4+b^4}{4}\right)+.......... \text{ is:}$$

(26-02-2021/Shift-2)

(a) $\log_e\left(\frac{e}{2}\right)$ (b) $\log_e 2$

(c) e^2-1 (d) e

78. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right)+\sin^{-1}\left(\frac{4x}{5}\right)=\sin^{-1}x$ is equal to

(16-03-2021/Shift-2)

(a) 1 (b) 3

(c) 2 (d) 0

79. The number of solutions of the equation

$\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$, for $x \in [-1, 1]$, and $[x]$ denotes the greatest integer less than or equal to x, is:

(17-03-2021/Shift-2)

(a) 4 (b) 0

(c) Infinite (d) 2

80. The number of real roots of the equation $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{4}$ is:

(20-07-2021/Shift-1)

(a) 0 (b) 4

(c) 1 (d) 2

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

Objective Questions I [Only one correct option]

1. The domain of the function $f(x) = \sqrt[4]{\log_3\left(\frac{1}{|\cos x|}\right)}$ is :

(a) $(-\infty, \infty)$

(b) $(-\infty, \infty) - \{n\pi | n \in I\}$

(c) $(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$

(d) none of the above

2. The range of the function $f(x) = \cos[x]$,

for $-\pi/2 < x < \pi/2$ contains.

(a) $\{-1, 1, 0\}$ (b) $\{\cos 1, 1, \cos 2\}$

(c) $\{\cos 1, -\cos 1, 1\}$ (d) $[-1, 1]$

3. Let $f: R \to R$ be a function defined by

$f(x) = \frac{x^2-8}{x^2+2}$, then f is :

(a) one-one but not onto (b) one-one and onto

(c) onto but not one-one (d) neither one-one nor onto

4. Let $f : R - \{n\} \to R$ be a function defined by

$f(x) = \frac{x-m}{x-n}$, where $m \neq n$. This function is-

(a) one-one onto (b) one-one into

(c) many-one onto (d) many one into

5. Let A = $(x_1, x_2, \ldots, x_8)$, B = (y_1, y_2, y_3), the total no. of functions f : A → B that are onto and there are exactly four elements (x) in A such that $f(x) = y_3$, is equal to

(a) $16 \times {}^8C_4$ (b) $14 \times {}^8C_4$

(c) $16 \times {}^4C_4$ (d) None of these

6. If $f(x+y) = f(x).f(y)$ for all real x, y and $f(0) \neq 0$ then the function $g(x) = \frac{f(x)}{1+(f(x))^2}$ is

(a) even function (b) odd function

(c) odd if $f(x) > 0$ (d) neither even nor odd

7. Let $f: (-\infty, 2] \to (-\infty, 4]$ be a function defined by $f(x) = 4x - x^2$. Then $f^{-1}(x)$ is :

(a) $2-\sqrt{4-x}$ (b) $2+\sqrt{4-x}$

(c) $\sqrt{4-x}$ (d) $\sqrt{4+x}$

8. Let $f : R \to R$, $g : R \to R$ be two functions given by $f(x) = 2x - 3$, $g(x) = x^3 + 5$. Then $(fog)^{-1}(x)$ is equal to :

(a) $\left(\frac{x-7}{2}\right)^{1/3}$ (b) $\left(\frac{x+7}{2}\right)^{1/2}$

(c) $\left(x - \frac{7}{2}\right)^{1/3}$ (d) $\left(\frac{x-2}{7}\right)^{1/3}$

9. Let $f : \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right]$ be a function defined as

$f(x) = \sqrt{3}\sin x - \cos x + 2$. The $f^{-1}(x)$ is given by

(a) $\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}$ (b) $\sin^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6}$

(c) $\frac{2\pi}{3} + \cos^{-1}\left(\frac{x-2}{2}\right)$ (d) none of these

10. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\left(\frac{1}{2}\right)\left[1+\sqrt{1+4\log_2 x}\right]$

(c) $\left(\frac{1}{2}\right)\left[1-\sqrt{1+4\log_2 x}\right]$ (d) not defined

11. If the function $f: R \to R$ be such that $f(x) = x - [x]$, where [x] denotes the greatest integer less than or equal to x, then $f^{-1}(x)$ is

(a) $\frac{1}{x-[x]}$ (b) $[x] - x$

(c) not defined (d) none of these

12. The range of the function $f(x) = |x-1| + |x-2|$, $-1 \le x \le 3$, is

(a) $[1, 3]$ (b) $[1, 5]$

(c) $[3, 5]$ (d) none of these

13. The domain of the function $f(x)=\sqrt{\left(\frac{1}{\sin x}-1\right)}$ is :

(a) $\left\{2n\pi,\ 2n\pi+\frac{\pi}{2}\right\},\ \forall\ n\in I$

(b) $(2n\pi, (2n+1)\pi)\ \forall\ n\in I$

(c) $((2n-1)\pi, 2n\pi)\ \forall\ n\in I$

(d) None of the above

14. $f(x)=\frac{\sqrt{|\tan x|+\tan x}}{\sqrt{3x}}$ is defined for :

(a) R^+ (b) $R^+-\left\{\frac{1}{3}\right\}$

(c) $R^+-\left\{n\pi+\frac{\pi}{2}\middle| n\in W\right\}$ (d) none of these

15. The domain of the function $f(x)=x^{\frac{1}{\log x}}$ is :

(a) $(0,\infty)-\{1\}$ (b) $(0,\infty)$

(c) $[0,\infty)$ (d) $[0,\infty)-\{1\}$

16. The minimum value of $2^{(x^2-3)^3+27}$ is :

(a) 1 (b) 2

(c) 2^{27} (d) None of these

17. Let $f(x)=\min\{x, x^2\}$, for every $x\in R$. Then :

(a) $f(x)=\begin{cases}x, & x\geq 1\\ x^2, & 0\leq x<1\\ x, & x<0\end{cases}$ (b) $f(x)=\begin{cases}x^2, & x\geq 1\\ x, & x<1\end{cases}$

(c) $f(x)=\begin{cases}x, & x\geq 1\\ x^2, & x<1\end{cases}$ (d) $f(x)=\begin{cases}x^2, & x\geq 1\\ x, & 0\leq x<1\\ x^2, & x<0\end{cases}$

18. The domain of definition of

$f(x)=\log_2\left(-\log_{1/2}\left(1+\frac{1}{x^{1/4}}\right)-1\right)$ is

(a) (0, 1) (b) (0, 1]

(c) $[1,\infty)$ (d) $(1,\infty)$

19. The domain of the function

$f(x)=\log_3\left[-\log_{1/2}\left(1+\frac{1}{x^{1/5}}\right)-1\right]$ is

(a) $(-\infty, 1)$ (b) (0, 1)

(c) $(1,\infty)$ (d) none of these

20. Let f be a real valued function defined by

$f(x)=\frac{e^x-e^{-|x|}}{e^x+e^{|x|}}$ then range of $f(x)$ is :

(a) R (b) [0, 1]

(c) [0, 1) (d) [0, 1/2)

21. The number of pairs, (x, y), $x, y\in R$, satisfying $4x^2-4x+2=\sin^2 y$ and $x^2+y^2\leq 3$ are

(a) 0 (b) 4

(c) 2 (d) infinite

22. If $f(x)=\frac{x-1}{x+1}$, then $f(2x)$ is :

(a) $\frac{f(x)+1}{f(x)+3}$ (b) $\frac{3f(x)+1}{f(x)+3}$

(c) $\frac{f(x)+3}{f(x)+1}$ (d) $\frac{f(x)+3}{3f(x)+1}$

23. If $f(x)=\begin{cases}|x|, & x\leq 1\\ 2-x, & x>1\end{cases}$, then $f(f(x))$ is equal to

(a) $\begin{cases}2-|x|, & x<-1\\ |x|, & -1\leq x\leq 1\\ |2-x|, & x>1\end{cases}$ (b) $\begin{cases}|x|, & x<-1\\ 2-|x|, & -1\leq x\leq 1\\ |2-x|, & x>1\end{cases}$

(c) $\begin{cases}|2-x|, & x<-1\\ |x|, & -1\leq x\leq 1\\ 2-|x|, & x>1\end{cases}$ (d) none of these

24. If $f\left(x+\frac{y}{8}, x-\frac{y}{8}\right)=xy$, then $f(m,n)+f(n,m)=0$

(a) only when m = n (b) only when m ≠ n

(c) only when m = – n (d) for all m and n

25. If $2 < x^2 < 3$ then the number of positive roots of $\left\{\frac{1}{x}\right\}=\left\{x^2\right\}$, {.} denotes the fractional part of x, is :

(a) 0 (b) 1

(c) 2 (d) 3

26. If [x] denotes the greatest integer ≤ x, then

$\left[\frac{2}{3}\right]+\left[\frac{2}{3}+\frac{1}{99}\right]+\left[\frac{2}{3}+\frac{2}{99}\right]+...+\left[\frac{2}{3}+\frac{98}{99}\right]$ is equal to

(a) 99 (b) 98

(c) 66 (d) 65

27. $f(x)=|x-1|$, $f:R^+\to R$ and $g(x)=e^x$, $g:[-1,\infty)\to R$ If the function fog (x) is defined, then its domain and range respectively are.

(a) $(0,\infty)$ & $[0,\infty)$ (b) $[-1,\infty)$ & $[0,\infty)$

(c) $[-1,\infty)$ & $\left[1-\frac{1}{e},\infty\right)$ (d) $[-1,\infty)$ & $\left[\frac{1}{e}-1,\infty\right)$

28. The number of positive integers satisfying the equation $x+\log_{10}(2^x+1)=x\log_{10}5+\log_{10}6$ is

(a) 0 (b) 1

(c) 2 (d) infinite

29. A certain polynomial P(x), x ∈ R when divided by x – a, x – b, x – c leaves remainder a, b, c respectively. The remainder when P(x) is divided by (x – a) (x – b) (x – c) is (a, b, c and distinct).

(a) 0 (b) x

(c) ax + b – c (d) ax^2+bx+c

30. Complete solution set of the equation $|x^2-1+\cos x|=|x^2-1|+|\cos x|$ belonging to $(-2\pi, \pi)$ is

(a) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\cup(-1,1)$

(b) $\left[-\frac{3\pi}{2},\frac{\pi}{2}\right]\cup[-1,1]\cup\left[\frac{\pi}{2},\pi\right)$

(c) $\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right]\cup\left[\frac{\pi}{2},\pi\right)$

(d) $\left(-2\pi,\frac{-3\pi}{2}\right]\cup\left[-\frac{\pi}{2},-1\right]\cup\left[1,\frac{\pi}{2}\right]$

31. The number points (x, y), where curves $|y| = ln\,|x|$ and $(x-1)^2+y^2-4=0$ cut each other, is

(a) 2 (b) 3

(c) 1 (d) 6

32. Let f be a function satisfying

$$2f(xy)=(f(x))^y+(f(y))^x$$

and $f(1)=k\neq 1$, then $\sum_{r=1}^{n} f(r)$ is equal to :

(a) k^n-1 (b) k^n

(c) k^n+1 (d) None of these

33. If $f(x)+2f(1-x)=x^2+2$, ∀ x ∈ R, then $f(x)$ is given as:

(a) $\frac{(x-1)^2}{3}$ (b) $\frac{(x-2)^2}{3}$

(c) x^2-1 (d) x^2-2

34. The function $f(x)=\sqrt{\cos(\sin x)}+\sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined for :

(a) x ∈ {–1, 1} (b) x ∈ [–1, 1]

(c) x ∈ R (d) x ∈ (–1, 1)

35. If $f(x)=\cos^{-1}\left(\frac{2-|x|}{4}\right)+\left[\log(3-x)\right]^{-1}$, then its domain is :

(a) [–2, 6] (b) $[-6,2)\cup(2,3)$

(c) [–6, 2] (d) $[-2,2)\cup(2,3]$

36. If $1 < x < \sqrt{2}$, then number of solutions of the equation $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$, is/are

(a) 0 (b) 1

(c) 2 (d) 3

37. The complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$ is

(a) $|x-2| < \sqrt{9-2\pi}$ (b) $|x-2| > \sqrt{9-2\pi}$

(c) $|x| < \sqrt{9-2\pi}$ (d) $|x| > \sqrt{9-2\pi}$

38. $\sum_{m=1}^{n}\tan^{-1}\frac{2m}{m^4+m^2+2} =$

(a) $\tan^{-1}(n^2+n+1)$ (b) $\tan^{-1}(n^2-n+1)$

(c) $\tan^{-1}\frac{n^2+n}{n^2+n+2}$ (d) None of these

39. The domain of the function $f(x) = \frac{1}{\sqrt{x^{12}-x^9+x^4-x+1}}$ is

(a) $(-\infty,-1)$ (b) $(1,\infty)$

(c) $(-1,1)$ (d) $(-\infty,\infty)$

40. If $f(x) = \lim_{t\to\infty}\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}$, then range of $f(x)$ is

(a) $\{-1,1\}$ (b) $\{0,1\}$

(c) $\{-1,1\}$ (d) $\{-1,0,1\}$

41. Range of the function f defined by $f(x)=\left[\frac{1}{\sin\{x\}}\right]$ (where [.] and {.} respectively denote the greatest integer and the fractional part functions) is.

(a) I, the set of integers

(b) N, the set of natural numbers.

(c) W, the set of whole numbers

(d) $\{2,3,4,....\}$

42. Let f be a real valued function with domain R satisfying

$$f(x+k)=1+[2-5f(x)+10(f(x))^2-10(f(x))^3+5(f(x))^4-(f(x))^5]^{1/5}$$

for all real x and some positive constant k, then the period of the function $f(x)$ is :

(a) k (b) 2k

(c) non periodic (d) none of these

43. Let f be a real valued function with domain R satisfying $0 < f(x) \le \frac{1}{2}$ and for some fixed a,

$$f(x+a)=\frac{1}{2}-\sqrt{f(x)-(f(x))^2}\ \forall\ x\in R$$

then the peroid of the function $f(x)$ is :

(a) a (b) 2a

(c) non periodic (d) none of these

44. Let $f(x)=\max\{1+\sin x, 1-\cos x, 1\}\ \forall\ x\in[0,2\pi]$ and $g(x)=\max\{1,|x-1|\}\ \forall\ x\in R$. Then gof is :

(a) 2 (b) 1

(c) $\begin{cases}1+\sin x & x\le 0\\ 1-\cos x & x\ge 0\end{cases}$ (d) None of these

45. Find all possible values of x satisfying :

$$\frac{[x]}{[x-2]}-\frac{[x-2]}{[x]}=\frac{8\{x\}+12}{[x-2][x]}$$

(where [ ] denotes the greatest function { } is fractional part).

(a) $\left\{4,\frac{11}{2}\right\}$ (b) $\{4,5\}$

(c) $\left\{5,\frac{11}{2}\right\}$ (d) $\left\{\frac{11}{2},\frac{15}{4}\right\}$

46. Total number of solutions of the equation $\sin\pi x=|l n_e|x||$ are :

(a) 8 (b) 10

(c) 9 (d) 6

47. The number of roots of the question $1+\log_2(1-x)=2^{-x}$ is :

(a) 0 (b) 1

(c) 2 (d) many

48. If $2f(x-1)-f\left(\frac{1-x}{x}\right)=x$, then $f(x)$ is :

(a) $\frac{1}{3}\left[2(1+x)+\frac{1}{1+x}\right]$ (b) $2(x-1)-\frac{1-x}{x}$

(c) $x^2+\frac{1}{x^2}+3$ (d) None of these

49. Let $f_1(n) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$, then

$f_1(1) + f_1(2) + f_1(3) + ... + f_1(n)$ is equal to :

(a) $n f_1(n) - 1$ (b) $(n+1) f_1(n) - n$

(c) $(n+1) f_1(n) + n$ (d) $n f_1(n) + n$

50. If $f(x) = \frac{4^x}{4^x + 2}$, then

$f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + ... + f\left(\frac{1996}{1997}\right)$ is equal to

(a) 1997 (b) 998

(c) 0 (d) none of these

Objective Questions II [One or more than one correct option]

51. If f is an even function defined on the interval [–5, 5], then the real values of x satisfying the equation

$f(x) = f\left(\frac{x+1}{x+2}\right)$, are

(a) $\frac{-1 \pm \sqrt{5}}{2}$ (b) $\frac{-3 \pm \sqrt{5}}{2}$

(c) $\frac{-2 \pm \sqrt{5}}{2}$ (d) none of these

52. Let R = $\{(x, y) : x, y \in R.\ x^2 + y^2 \le 25\}$ and

$R' = \left\{(x, y) : x, y \in R,\ y \ge \frac{4}{9}x^2\right\}$ then is

(a) dom $R \cap R' = [-3, 3]$

(b) Range $R \cap R' \supset [0, 4]$

(c) Range $R \cap R' = [0, 5]$

(d) $R \cap R'$ defines a function

53. Let $f(x)$ and $g(x)$ be two real valued function given by, $f(x) = -l nx$ and $g(x) = e^{-x}$. Let $h(x) = f(x) - x$ and $m(x) = g(x) - x$. Further more let the number of solutions of $h(x) = 0$ and $m(x) = 0$ be a and b, then.

(a) $a \ne b$ (b) $a = b$

(c) $a = 1$ and $b = 1$ (d) None of these

54. Let $f(x)$ be invertible function and let $f^{-1}(x)$ be its inverse. Let equation $f\{f^{-1}(x)\} = f^{-1}(x)$ has two real roots α and β (within domain of $f(x)$), then

(a) $f(x) = x$, also have same two real roots.

(b) $f^{-1}(x) = x$, also have same two real roots.

(c) $f(x) = f^{-1}(x)$, also have same two real roots.

(d) Area formed by (0, 0) $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$ is 1 unit.

55. 2π is fundamental period of the function

(a) $\frac{(1+\sin x)}{\cos x(1+\operatorname{cosec} x)}$ (b) $|\sin x| + |\cos x|$

(c) $\sin 2x + \cos 3x$ (d) $\cos(\sin x) + \cos(\cos x)$

56. Which of the following functions are periodic ?

(a) $f(x) = \operatorname{sgn}(e^{-x})$

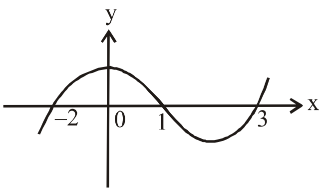
(b) $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ 0 \text{ if } x \text{ is a irrational number} \end{cases}$

(c) $f(x) = \sqrt{\frac{8}{1+\cos x} + \frac{8}{1-\cos x}}$

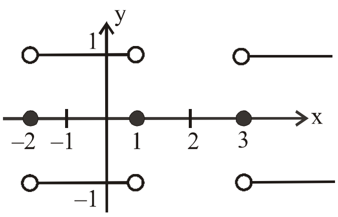
(d) $f(x) = \left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] + 2[-x]$

(where [ ] denotes greatest integer function)

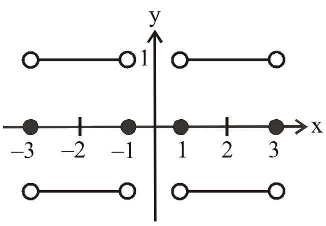
57. The graph of the function $y = f(x)$ is as shown in the figure. Then which one of the following graphs are correct?



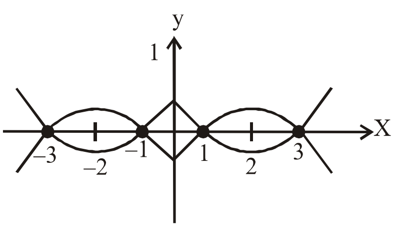
(a) $|y| = \text{sgn}(f(x))$



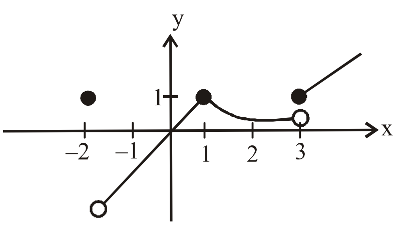
(b) $|y| = \text{sgn}(-f(|x|))$



(c) $|y| = |f|x||$



(d) $y = x^{\text{sgn}(f(x))}$



58. Let $f(x)$ be defined on $[-\pi, \pi]$ and is given by,

$$f(x) = \begin{cases} \sin x & -\pi \le x \le 0 \\ \cos x & 0 < x \le \pi \end{cases}$$

Let $g(x) = f|x| + |f(x)|$, $\forall\ x \in [-\pi, \pi]$, then

(a) $g(x) = 0$, has no real roots

(b) $g(x) = 0$, has infinitely many real roots

(c) $g(x) = 0$

(d) limit does not exist at $x = 0$

Numerical Value Type Questions

59. The number of integer values of m for which $f(x) = x^3 - mx^2 + 3x - 11$ invertible is

60. If $f(1) = 2$ and $f(x + y) = f(x)\ f(y)$ for all natural numbers x, y, the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, is

61. The function $f(x) = \frac{x+5}{\sqrt{x^2+1}}$ takes exactly k integer values, then k must be

62. Let S be the set of points (x, y) given by

$S = \{(x, y)\};\ x^2 + y^2 - 10x + 16 = 0\}$

and $f: S \to R$ be given by $f(x, y) = \frac{y}{x}$

If range of f is $\left[-\frac{3}{k}, \frac{3}{k}\right]$ where $k > 0$. then k must be

Assertion & Reason

(A) **If ASSERTION is true, REASON is true, REASON is a correct explanation for ASSERTION.**

(B) **If ASSERTION is true, REASON is true, REASON is not a correct explanation for ASSERTION.**

(C) **If ASSERTION is true, REASON is false.**

(D) **If ASSERTION is false, REASON is true.**

63. **Assertion :** A function $y = f(x)$ is defined by $x^2 - \text{arc}\cos y = \pi$, then domain of $f(x)$ is R.

Reason : $\cos^{-1} y \in [0, \pi]$.

(a) A
(b) B
(c) C
(d) D

64. **Assertion :**

$\text{cosec}^{-1}\frac{3}{2} + \cos^{-1}\frac{2}{3} - 2\cot^{-1}\frac{1}{7} - \cot^{-1}7 = \cot^{-1}7.$

Reason : $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$,

$\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$, and for $x > 0$, $\cot^{-1}x = \tan^{-1}\frac{1}{x}$

(a) A (b) B

(c) C (d) D

65. **Assertion :** If a is twice the tangent of the arithmetic mean of $\sin^{-1}x$ and $\cos^{-1}x$, be is the geometric mean of tan x and cot x, then $x^2 - ax + b = 0 \Rightarrow x = 1$

Reason : $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right) = 1$

(a) A (b) B

(c) C (d) D

66. **Assertion :** $\sin^{-1}\left\{x - \frac{x^2}{2} + \frac{x^3}{4} - ....\right\}$

$= \frac{\pi}{2} - \cos^{-1}\left\{x^2 - \frac{x^4}{2} + \frac{x^6}{4}...\right\}$ for $0 < |x| < \sqrt{2}$ has a unique solution.

Reason : $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ has no solution for $-\sqrt{2} < x < 0$.

(a) A (b) B

(c) C (d) D

67. **Assertion :** Let $f(x)$ be a function satisfying $f(x-1) + f(x+1) = \sqrt{2}\, f(x)$ for all $x \in R$. Then $f(x)$ is periodic with period 8.

Reason : For every natural number n there exists a periodic function with period n.

(a) A (b) B

(c) C (d) D

68. **Assertion :** $\sin^{-1}\left[\tan\left\{\tan^{-1}x + \tan^{-1}(1-x)\right\}\right] = \frac{\pi}{2}$ has no non-zero integral solution.

Reason : The greatest and least values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ are $\frac{7\pi^3}{8}$ and $\frac{\pi^3}{32}$ respectively.

(a) A (b) B

(c) C (d) D

Match the Following

Each question has two columns. Four options are given representing matching of elements from Column-I and Column-II. Only one of these four options corresponds to a correct matching.For each question, choose the option corresponding to the correct matching.

69. Match the column.

	Column–I	Column–II
(A)	The number of possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{\pi}{2}$, is	**(P)** 1
(B)	Numbers of elements in the domain of $f(x) = \tan^{-1}x + \sin^{-1}x + \sec^{-1}x$ is	**(Q)** 2
(C)	Period of the function $f(x) = \sin\left(\frac{\pi x}{2}\right) . \cos\left(\frac{\pi x}{2}\right)$ is	**(R)** 3
(D)	If the range of the function $f(x) = \cos^{-1}[5x]$ is $\{a, b, c\}$ & $a + b + c = \frac{\lambda\pi}{2}$, then λ is equal to	**(S)** 4

(where [.] denotes greatest integer)

(a) $A\to Q, B\to Q, C\to Q, D\to R$

(b) $A\to P, B\to Q, C\to Q, D\to R$

(c) $A\to Q, B\to P, C\to R, D\to R$

(d) $A\to Q, B\to Q, C\to Q, D\to P$

70. Functions in column I can take values of column II

	Column-I	Column-II
(A)	$x + 2\sqrt{x}$ can be	(P) 38
(B)	$\frac{x-1}{x+1}$ can be	(Q) 0
(C)	$2x^3 - 9x^2 + 12x + 6$ can be	(R) $\frac{3}{5}$
(D)	$\left[[x] - \frac{x}{2}\right]$ can be	(S) – 1

where [.] is G.I.F. Correct matching is

(a) $A \to Q,R; B \to P,Q,R,S; C \to P,Q,R; D \to P,Q,S$

(b) $A \to P,Q,R; B \to P,Q,R,S; C \to P,Q,R,S; D \to P,Q,S$

(c) $A \to P,Q,R; B \to P,Q,R; C \to P,Q,R; D \to P,Q$

(d) $A \to P,Q; B \to P,Q; C \to P,Q,R,S; D \to P,Q$

71.

	Column-I	Column-II
(A)	Let $X = \{a_1, a_2, \ldots, a_6\}$ and $Y = \{b_1, b_2, b_3\}$. The number of functions f from X to Y such that it is onto and there are exactly three elements x in X such that $f(x) = b_1$, is greater than	(P) 2
(B)	The number of real solutions for x, y if $y = \|\sin x\|$ and $y = \sin^{-1}(\sin x)$ where $x \in [-2\pi, 2\pi]$, is	(Q) 5
(C)	If a, b and c are distinct positive real numbers such that $a + b + c = 1$, then $\frac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)}$ can be	(R) 120
(D)	The period of the function $[6x + 7] + \cos \pi x - 6x$, where [] denotes the greatest integer function, is	(S) 80
		(T) 10

The correct matching is

(a) $A \to P,Q,S,T,\ B \to Q, C \to R,S,T,\ D \to P$

(b) $A \to P,Q,\ B \to Q,\ C \to R,S,\ D \to P$

(c) $A \to P,\ B \to Q,\ C \to R,S,\ D \to R,S$

(d) $A \to P,\ B \to T,\ C \to R,S,\ D \to Q$

72. **Match of the column.**

	Column-I	Column-II
(A)	$\log_{10} 5 . \log_{10} 20 + (\log_{10} 2)^2 =$	(P) 4
(B)	Least value of x satisfying $\|2x\| - \|x - 4\| = x + 4$ is	(Q) – 4
(C)	If $\cos^{-1} x = \frac{k}{2} \sin^{-1}\sqrt{\frac{1-x}{2}}$, for all $x \in (-1, 1)$, then k is equal to	(R) 1
(D)	If f: $[0, 2] \to [2, 0]$ is bijective function defined by $f(x) = ax^2 + bx + c$, where a, b, c are non-zero real numbers then f(2) is equal to	(S) 0

The correct matching is

(a) $A \to R, B \to Q\ \ C \to P, D \to S$

(b) $A \to P, B \to Q\ \ C \to R, D \to S$

(c) $A \to Q, B \to R\ \ C \to P, D \to S$

(d) $A \to R, B \to Q\ \ C \to S, D \to P$

73. Column II contain the ranges of the functions given in Column I.

	Column I	Column II
(A)	$y = \frac{e^x}{1+[x]}; x \geq 0$ where [ ] denotes greatest integer function	(P) $\left[\frac{3}{4}, \infty\right)$
(B)	$\cot^{-1}(2x - x^2)$	(Q) $[1, \infty)$
(C)	$4^x - 2^x + 1$	(R) $[0, \infty)$
(D)	$ln(1 + x^2)$	(S) $\left[\frac{\pi}{4}, \pi\right)$

The correct matching is

(a) $A \to (Q); B \to (S); C \to (P); D \to (R)$

(b) $A \to (S); B \to (Q); C \to (P); D \to (R)$

(c) $A \to (Q); B \to (S); C \to (R); D \to (P)$

(d) $A \to (R); B \to (S); C \to (P); D \to (Q)$

Using the following passage, solve Q.74 to Q.76

Passage –1

A function f from a set X to Y is called onto, if for every $y \in Y$ there exist $x \in X$ such that $f(x) = y$. Unless the contrary is specified, a real function is onto if it takes all real values, otherwise it is called into function. Thus, if X and Y are finite sets, then f cannot be onto If Y contains more element than X.

74. The polynomial function

$P(x) = a_0x^n + a_1x^{n-1} + a_1x^{n-2} + ... + a_n$,

where $a_0 \neq 0$, is onto, for

(a) all positive integers n

(b) all even positive integers n

(c) all odd positive integers n

(d) no positive integer

75. Which of the following is not true ?

(a) A one-one function from the set {a, b, c} to {α, β, γ} is onto also.

(b) An onto function from an infinite set to a finite set cannot be one-one.

(c) An onto function is always invertible.

(d) The function tan x and cot x are onto

76. The function $f(x) = \dfrac{x^2+2x+c}{x^2+4x+3c}$ is onto, if

(a) $0 < c < 2$ (b) $0 < c < 4$

(c) $-\dfrac{1}{2} < c < \dfrac{1}{2}$ (d) $0 < c < 1$

Using the following passage, solve Q.77 to Q.79

Passage –2

Let $f : R \to R$ is a function satisfying $f(2-x) = f(2+x)$ and $f(20-x) = f(x)$, $\forall\ x \in R$. For this function f answer the following.

77. If $f(0) = 5$, then minimum possible number of values of x satisfying $f(x) = 5$, for $x \in [0, 170]$ is.

(a) 21 (b) 12

(c) 11 (d) 22

78. Graph of $y = f(x)$ is

(a) symmetrical about $x = 18$

(b) symmetrical about $x = 5$

(c) symmetrical about $x = 8$

(d) symmetrical about $x = 20$

79. If $f(2) \neq f(6)$, then

(a) fundamental period of $f(x)$ is 1

(b) fundamental period of $f(x)$ may be 1

(c) period of $f(x)$ can't be 1

(d) fundamental period of $f(x)$ is 8

Text

80. Find the range of values of t for which

$$2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1},\quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

Objective Questions I [Only one correct option]

1. The domain of definition of the function y (x) is given by the equation $2^x + 2^y = 2$, is **(2000)**

(a) $0 < x \le 1$ (b) $0 \le x \le 1$

(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

2. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x<0 \\ 0, & x=0, \\ 1, & x>0 \end{cases}$ then for all x, $f[g(x)]$ is equal to **(2001)**

(a) x (b) 1

(c) $f(x)$ (d) g (x)

3. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$ then $f^{-1}(x)$ equals. **(2001)**

(a) $\frac{x+\sqrt{x^2-4}}{2}$ (b) $\frac{x}{1+x^2}$

(c) $\frac{x-\sqrt{x^2-4}}{4}$ (d) $1+\sqrt{x^2-4}$

4. The domain of definition of $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}$ is **(2001)**

(a) $R/\{-1,-2\}$ (b) $(-2, \infty)$

(c) $R/\{-1,-2,-3\}$ (d) $(-3, \infty)/\{-1,-2\}$

5. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let m (b) be the minimum value of $f(x)$. As b varies, the range of m (b) is **(2001)**

(a) [0, 1] (b) $\left[0, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1]

6. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$, Then, the number of onto functions from E to F is **(2001)**

(a) 14 (b) 16

(c) 12 (d) 8

7. Let $f(x) = \frac{\alpha x}{x+1}$, $x \ne -1$. Then, for what value of α is $f[f(x)] = x$? **(2001)**

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 1 (d) -1

8. Suppose $f(x) = (x + 1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals **(2002)**

(a) $-\sqrt{x}-1,\ x\ge 0$ (b) $\frac{1}{(x+1)^2}, x > -1$

(c) $\sqrt{x+1},\ x\ge -1$ (d) $\sqrt{x}-1,\ x\ge 0$

9. Let function $f: R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then f is **(2002)**

(a) one-to-one and onto

(b) one-to-one but NOT onto

(c) onto but NOT one-to-one

(d) neither one-to-one nor onto

10. If $f: [0, \infty) \to [0, \infty)$ and $f(x) = \frac{x}{1+x}$, then f is **(2003)**

(a) one-one and onto

(b) one-one but not onto

(c) onto but not one-one

(d) neither one-one nor onto

11. Range of the function $f(x) = \frac{x^2+x+2}{x^2+x+1}$; $x \in R$ is **(2003)**

(a) $(1, \infty)$ (b) $\left(1, \frac{11}{7}\right)$

(c) $\left(1, \frac{7}{3}\right)$ (d) $\left(1, \frac{7}{5}\right)$

12. Domain of definition of the function

$f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ for real valued x, is **(2003)**

(a) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{2},\frac{1}{2}\right]$

(c) $\left(-\frac{1}{2},\frac{1}{9}\right)$ (d) $\left[-\frac{1}{4},\frac{1}{4}\right]$

13. If $f(x)=\sin x+\cos x$, $g(x)=x^2-1$, then $g(f(x))$ is invertible in the domain. **(2004)**

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

(c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (d) $[0,\pi]$

14. $f(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational}\end{cases}$

$g(x)=\begin{cases} 0, \text{ if } x \text{ is rational} \\ x, \text{ if } x \text{ is irrational}\end{cases}$.

Then $f-g$ is. **(2005)**

(a) neither one-one nor onto

(b) one-one and onto

(c) one-one and into

(d) many one and onto

15. Suppose X and Y are two sets and $f: X\to Y$ is a function. For a subset A of X, define f(A) to be the subset $\{f(a): a\in A\}$ of Y. For a subset B of Y, define $f^{-1}(B)$ to be the subset $\{x\in X: f(x)\in B\}$ of X. Then which of the following statements is true ? **(2005)**

(a) $f^{-1}(f(A))=A$ for every $A\subset X$

(b) $f^{-1}(f(A))=A$ for every $A\subset X$ if and only if $f(X)=Y$

(c) $f(f^{-1}(B))=B$ for every $B\subset Y$

(d) $f(f^{-1}(B))=B$ for every $B\subset Y$ if and only if $f(X)=Y$

16. Let $S=\{1,2,3,4\}$. The total number of unordered pairs of disjoint subsets of S is equal to **(2010)**

(a) 25 (b) 34

(c) 42 (d) 41

17. Let $f(x)=x^2$ and $g(x)=\sin x$ for all $x\in R$. Then, the set of all x satisfying $(fogogof)(x)=(gogof)(x)$, where $(fog)(x)=f(g(x))$, is **(2011)**

(a) $\pm\sqrt{n\pi}, n\in\{0,1,2,...\}$

(b) $\pm\sqrt{n\pi}, n\in\{1,2,...\}$

(c) $\frac{\pi}{2}+2n\pi, n\in\{...,-2,-1,0,1,2,....\}$

(d) $2n\pi, n\in\{...,-2,-1,0,1,2,....\}$

18. The function $f:[0,3]\to[1,29]$, defined by $f(x)=2x^3-15x^2+36x+1$, is **(2012)**

(a) one-one and onto

(b) onto but not one-one

(c) one-one but not onto

(d) neither one-one nor onto

19. For any positive integer n, define $f_n:(0,\infty)\to R$ as

$f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$

for all $x\in(0,\infty)$ (Here, the inverse trigonometric function $\tan^{-1}x$ assumes value in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.) Then, which of the following statement(s) is (are) TRUE? **(2018)**

(a) $\sum_{j=1}^{5}\tan^2(f_j(0))=55$

(b) $\sum_{j=1}^{10}\left(1+f_1'(0)\right)\sec^2(f_j(0))=10$

(c) For any fixed positive integer n, $\lim_{x\to\infty}\tan(f_n(x))=\frac{1}{n}$

(d) For any fixed positive integer n, $\lim_{x\to\infty}\sec^2(f_n(x))=1$

20. If the function $f: R\to R$ is defined by $f(x)=|x|(x-\sin x)$, then which of the following statements is **TRUE** ? **(2020)**

(a) f is one-one, but **NOT** onto

(b) f is onto, but **NOT** one-one

(c) f is **BOTH** one-one and onto

(d) f is **NEITHER** one-one **NOR** onto

Objective Questions II [One or more than one correct option]

21. Let $f: (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then, the value(s) of $f\left(\frac{1}{3}\right)$ is/are **(2012)**

(a) $1-\sqrt{\frac{3}{2}}$ (b) $1+\sqrt{\frac{3}{2}}$

(c) $1-\sqrt{\frac{2}{3}}$ (d) $1+\sqrt{\frac{2}{3}}$

22. Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by

$f(x) = (\log(\sec x + \tan x))^3$.

Then **(2014)**

(a) $f(x)$ is an odd function

(b) $f(x)$ is a one-one function

(c) $f(x)$ is an onto function

(d) $f(x)$ is an even function

23. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is (are) **(2015)**

(a) $\cos\beta > 0$ (b) $\sin\beta < 0$

(c) $\cos(\alpha+\beta) > 0$ (d) $\cos\alpha < 0$

24. For any positive integer n, let $S_n : (0, \infty) \to R$ be defined by $S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$,

Where for any $x \in R, \cot^{-1}x \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) TRUE? **(2021)**

(a) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(b) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(c) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(d) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

Numerical Value Type Questions

25. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is _____.

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively) **(2018)**

26. Let $f: [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is **(2014)**

27. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals **(2019)**

28. Let the function $f:[0,1] \to R$ be defined by $f(x) = \frac{4^x}{4^x+2}$. Then the value of

$$f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+f\left(\frac{3}{40}\right)+......+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$$

is….. **(2020)**

Assertion & Reason

For the following questions choose the correct answer from the codes (A), (B), (C) and (D) defined as follows.

(A) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.

(B) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.

(C) Statement I is true, Statement II is false.

(D) Statement I is false, Statement II is true.

29. Let $f(x) = 2 + \cos x$ for all real x.

Statement I : For each real t, there exists a point c in $[t, t+\pi]$ such that $f'(c) = 0$.

Because

Statement II : $f(t) = f(t + 2\pi)$ for each real t. **(2007)**

(a) A (b) B

(c) C (d) D

30. **Statement I :** The curve $y=-\frac{x^2}{2}+x+1$ is symmetric with respect to the line x = 1.

Because

Statement II : A parabola is symmetric about its axis. **(2007)**

(a) A (b) B

(c) C (d) D

Match the Columns

Each question has two columns. Four options are given representing matching of elements from Column-I and Column-II. Only one ofthese four options corresponds to a correct matching.For each question, choose the option corresponding to the correct matching.

31. Let $f(x)=\frac{x^2-6x+5}{x^2-5x+6}$. **(2007)**

Column I	**Column II**
(A) If $-1<x<1$, then $f(x)$ satisfies	**(P)** $0<f(x)<1$
(B) If $1<x<2$, then $f(x)$ satisfies	**(Q)** $f(x)<0$
(C) If $3<x<5$, then $f(x)$ satisfies	**(R)** $f(x)>0$
(D) If $x>5$, then $f(x)$ satisfies	**(S)** $f(x)<1$

The correct matching is

(a) A–P,R,S; B–Q; C–Q,S; D–P,R

(b) A–P,R,S; B–Q,S; C–Q,S; D–P,R,S

(c) A–P; B–Q,S; C–Q; D–P,R,S

(d) A–R,S; B–S; C–Q,S; D–P,R

32. Let $E_1=\left\{x\in\mathbb{R}:x\neq 1 \text{ and } \frac{x}{x-1}>0\right\}$ and

$$E_2=\left\{x\in E_1:\sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)\text{ is a real number}\right\}.$$

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$)

Let f : $E_1\to\mathbb{R}$ be the function defined by

$f(x)=\log e\left(\frac{x}{x-1}\right)$

and g : $E_2\to\mathbb{R}$ be the function defined by

$g(x)=\sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$ **(2018)**

Columns A	**Column B**
(A) The range of f is	**(P)** $\left(-\infty,\frac{1}{1-e}\right]\cup\left[\frac{e}{e-1},\infty\right)$
(B) The range of g contains	**(Q)** (0, 1)
(C) The domain of f contains	**(R)** $\left[-\frac{1}{2},\frac{1}{2}\right]$
(D) The domain of g is	**(S)** $(-\infty,0)\cup(0,\infty)$
	(T) $\left(-\infty,\frac{e}{e-1}\right]$

The correct matching is

(a) A–S; B–Q; C–P, D–P

(b) A–S; B–Q; C–P, D–S

(c) A–Q; B–Q; C–P, D–P

(d) A–S; B–P; C–S, D–P

Using the following passage, solve Q.33 to Q.35

Passage

If a continuous function 'f' defined on the real line R, assumes positive and negative values in R, then the equation $f(x)=0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum values is negative, then the equation $f(x)=0$ has a root in R. Consider $f(x)=ke^x-x$ for all real x where k is real constant. **(2007)**

33. The line $y=x$ meets $y=ke^x$ for $k\le 0$ at

(a) no point (b) one point

(c) two points (d) more than two points

34. The positive value of k for which $ke^x-x=0$ has only one root is

(a) $\frac{1}{e}$ (b) 1

(c) e (d) $\log_e 2$

35. For $k>0$, the set of all values of k for which $ke^x-x=0$ has two distinct roots, is

(a) $\left(0,\frac{1}{e}\right)$ (b) $\left(\frac{1}{e},1\right)$

(c) $\left(\frac{1}{e},\infty\right)$ (d) (0, 1)

Answer Key

CHAPTER -2 | RELATIONS , FUNCTIONS & INVERSE TRIGONOMETRIC FUNCTIONS

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

1. (d)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (d)
6. (b)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (c)
11. (d)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (b)
16. (c)	**17.** (d)	**18.** (b)	**19.** (d)	**20.** (d)
21. (b)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (b)
26. (c)	**27.** (d)	**28.** (d)	**29.** (a)	**30.** (d)
31. (b)	**32.** (a)	**33.** (b)	**34.** (d)	**35.** (a)
36. (b)	**37.** (b)	**38.** (a)	**39.** (d)	**40.** (a)
41. (a)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (c)
46. (a)	**47.** (a)	**48.** (b)	**49.** (c)	**50.** (b)
51. (a)	**52.** (a)	**53.** (d)	**54.** (b)	**55.** (d)
56. (a)	**57.** (c)	**58.** (a)	**59.** (c)	**60.** (d)
61. (d)	**62.** (c)	**63.** (a)	**64.** (360)	**65.** (2)
66. (8)	**67.** (-1)	**68.** (-1)	**69.** (0)	**70.** (1)
71. (0)	**72.** (500)	**73.** (10)	**74.** (3)	**75.** (15)
76. (20)	**77.** (1)	**78.** (1)	**79.** (1)	**80.** (1)

EXERCISE - 2 : PREVIOUS YEAR JEE MAIN QUESTIONS

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (c)
11. (b)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (d)
16. (a)	**17.** 10.00	**18.** 3.00	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (a)
26. (d)	**27.** (a)	**28.** (b)	**29.** (c)	**30.** (c)
31. (b)	**32.** (b)	**33.** (d)	**34.** (19)	**35.** (b)
36. (b)	**37.** (b)	**38.** (b)	**39.** (a)	**40.** (c)
41. (a)	**42.** (b)	**43.** (2.00)	**44.** (a)	**45.** (c)
46. (d)	**47.** (b)	**48.** (c)	**49.** (a)	
50. (481.00)	**51.** (c)	**52.** (c)	**53.** (b)	**54.** (b)
55. (490.00)	**56.** (a)	**57.** (c)	**58.** (c)	**59.** (c)
60. (26.00)	**61.** (a)	**62.** (b)	**63.** (a)	**64.** (b)
65. (d)	**66.** (b)	**67.** (2.00)	**68.** (a)	**69.** (b)
70. (d)	**71.** (d)	**72.** (b)	**73.** (c)	**74.** (a)
75. (a)	**76.** (a)	**77.** (b)	**78.** (b)	**79.** (b)
80. (a)				

ANSWER KEY

CHAPTER -2 | RELATIONS , FUNCTIONS & INVERSE TRIGONOMETRIC FUNCTIONS

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

1. (c)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (b)
6. (a)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (b)
11. (c)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (a)
16. (a)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (d)
21. (c)	**22.** (b)	**23.** (a)	**24.** (d)	**25.** (b)
26. (c)	**27.** (b)	**28.** (b)	**29.** (b)	**30.** (d)
31. (b)	**32.** (d)	**33.** (b)	**34.** (a)	**35.** (b)
36. (a)	**37.** (a)	**38.** (c)	**39.** (d)	**40.** (d)
41. (b)	**42.** (b)	**43.** (b)	**44.** (b)	**45.** (a)
46. (d)	**47.** (c)	**48.** (a)	**49.** (b)	**50.** (b)
51. (a,b)	**52.** (a,b,c)	**53.** (b,c)	**54.** (a,b,c)	**55.** (c)
56. (a,b,c,d)		**57.** (a,b)	**58.** (b,d)	
59. (7)	**60.** (3)	**61.** (6)	**62.** (4)	**63.** (d)
64. (d)	**65.** (a)	**66.** (c)	**67.** (b)	**68.** (d)
69. (a)	**70.** (b)	**71.** (a)	**72.** (a)	**73.** (a)
74. (c)	**75.** (c)	**76.** (d)	**77.** (d)	**78.** (a)

79. (c) **80.** $t \in \left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

1. (d)	**2.** (b)	**3.** (a)	**4.** (d	**5.** (d)
6. (a)	**7.** (d)	**8.** (d)	**9.** (a	**10.** (b)
11. (c)	**12.** ([illegible])	**13.** (b)	**14.** (b)	**15.** (d)
16. (d)	**17.** ([illegible])	**18.** (b)	**19.** (c)	**20.** (c)
21. (a,b)	**22.** (a,b,c)	**23.** (b,c,d)	**24.** (a,b)	**25.** (2)
26. (3)	**27.** (0)	**28.** (19)	**29.** (b)	**30.** (a)
31. (b)	**32.** (a)	**33.** (b)	**34.** (a)	**35.** (a)