LIMIT, CONTINUITY & DIFFERENTIABILITY

1. DEFINITION :

In mathematics, a limit is the value that a function "approaches" as the input "approaches" some value.

In formulas a limit of function is usually written as $\lim_{x \to c} f(x) = L$ and is read as the limit of f(x) as x approahes c is equal to L. The fact that a function f approaches the limit L as x approaches to c is sometimes denoted by a right arrow ($\to$) as in $f(x) \to L$ as $x \to c$

1.1. Left Hand Limit and Right Hand Limit of a Function :

The value to which f(x) approaches, as x tends to 'a' from the left hand side ($x \to a^-$) is called left hand limit of f(x) at x = a. Symbolically, LHL = $\lim_{x \to a^-} f(x) = \lim_{h \to 0^+} f(a - h)$.

The value to which f(x) approaches, as x tends to 'a' from the right hand side ($x \to a^+$) is called right hand limit of f(x) at x = a. Symbolically, RHL = $\lim_{x \to a^+} f(x) = \lim_{h \to 0^+} f(a + h)$.

If a is a point in the interior doman of f(x) then limit of a function f(x) is said to exist as $x \to$ a when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ = Finite quantity.

Example :

Graph of y = f(x)

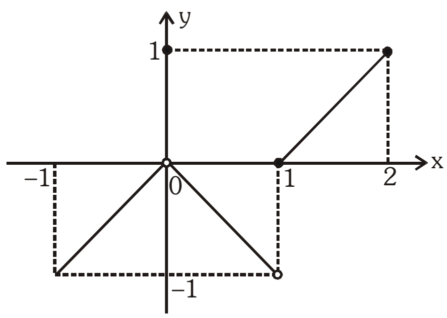


Fig. 1

$\lim_{x \to -1^+} f(x) = f(-1^+) = -1$

$\lim_{x \to 0^-} f(x) = f(0^-) = 0$, $\lim_{x \to 0^+} f(x) = f(0^+) = 0$

$\Rightarrow \lim_{x \to 0} f(x) = 0$

$\lim_{x \to 1^-} f(x) = f(1^-) = -1$, $\lim_{x \to 1^+} f(x) = f(1^+) = 0$

$\Rightarrow \lim_{x \to 1} f(x)$ does not exist.

$\lim_{x \to 2^-} f(x) = f(2^-) = 1$

Important note : Note that we are not interested in knowing about what happens at x =a . Also note that if L.H.L. and R.H.L. are both tending towards '∞' or '$-\infty$', then it is said to be infinite limit.

Remember, '$x \to a$' means that x is approaching to 'a' but not equal to 'a'.

2. Fundamental Theorems on Limits :

Let $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$. If l & m exist finitely then :

(a) Sum rule : $\lim_{x \to a} \{f(x) + g(x)\} = l + m$

(b) Difference rule : $\lim_{x \to a} \{f(x) - g(x)\} = l - m$

(c) Product rule : $\lim_{x \to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$; where k is constant.

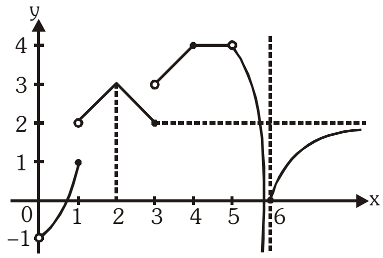
(f) $\lim\limits_{x \to a} f\left[g(x)\right] = f\left(\lim\limits_{x \to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim\limits_{x \to a} \ell n(g(x)) = \ell n[\lim\limits_{x \to a} g(x)]$

$= \ell n\ (m)$; provided ℓnx is continuous at x = m, m = $\lim\limits_{x \to a} g(x)$.

Example # 1 : Consider the adjacent graph of y = f (x)

Find the number of integral point in x ∈ (0, 7) where limit does not exist



Solution : (a) $\lim\limits_{x \to 1^-} f(x) = 1$, $\lim\limits_{x \to 1^+} f(x) = 2 \Rightarrow \lim\limits_{x \to 1} f(x)$ doesnot exist

(b) $\lim\limits_{x \to 2^-} f(x) = 3$, $\lim\limits_{x \to 2^+} f(x) = 3 \Rightarrow \lim\limits_{x \to 6} f(x)$ exist and equal to 3

(c) $\lim\limits_{x \to 3^-} f(x) = 2$, $\lim\limits_{x \to 3^+} f(x) = 3 \Rightarrow \lim\limits_{x \to 3} f(x)$ doesnot exist

(d) $\lim\limits_{x \to 4^-} f(x) = 4$, $\lim\limits_{x \to 4^+} f(x) = 4 \Rightarrow \lim\limits_{x \to 4} f(x)$ exist and equal to 4

(e) $\lim\limits_{x \to 5^-} f(x) = 4$, $\lim\limits_{x \to 5^+} f(x) = 4 \Rightarrow \lim\limits_{x \to 5} f(x)$ exist and equal to 4

(f) $\lim\limits_{x \to 6^-} f(x) = -\infty$, $\lim\limits_{x \to 6^+} f(x) = 0 \Rightarrow \lim\limits_{x \to 6} f(x)$ doesnot exist

Example # 2 : Evaluate the following limits :-

(i) $\ell\text{im}_{x \to 2}$ (x + 2) (ii) $\ell\text{im}_{x \to 0}$ cos (sin x)

Solution : (i) x + 2 being a polynomial in x, its limit as x → 2 is given by $\ell\text{im}_{x \to 2}$ (x + 2) = 2 + 2 = 4

(ii) $\ell\text{im}_{x \to 0}$ cos (sin x) = cos $\left(\ell\text{im}_{x \to 0} \sin x\right)$ = cos 0 = 1

Self practice problems

(1) Which of the following statements about the function y = f(x) graphed here are true, and which are false ?

(a) $\lim\limits_{x \to -1^+} f(x) = 1$ (b) $\lim\limits_{x \to 2} f(x)$ does not exist

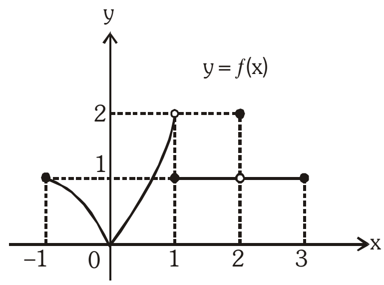
(c) $\lim\limits_{x \to 2} f(x) = 2$ (d) $\lim\limits_{x \to 1^-} f(x) = 2$

(e) $\lim\limits_{x \to 1} f(x)$ does not exist (f) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

(g) $\lim\limits_{x \to c} f(x)$ exists at every c ∈ (–1, 1)

(h) $\lim\limits_{x \to c} f(x)$ exists at every c ∈ (1, 3)

(i) $\lim\limits_{x \to 1^-} f(x) = 0$ (j) $\lim\limits_{x \to 3^+} f(x)$ does not exist.



(2) Evaluate the following limits : -

(i) $\ell\text{im}_{x \to 2}$ x(x – 1) (ii) $\ell\text{im}_{x \to 2} \dfrac{x^2 + 4}{x + 2}$

Answers : **(1)** (a) T (b) F (c) F (d) T (e) T (f) T (g) T (h) T (i) F (j) T

(2) (i) 2 (ii) 2

3. INDETERMINANT FORMS :

If on putting x = a in f(x), any one of $\frac{0}{0}, \frac{\infty}{\infty}$, **$0 \times \infty, \infty - \infty, \infty^0, 0^0, 1^\infty$** form is obtained (here 0,1 are not exact, infact both are approaching to their corresponding values), then the limit has an indeterminate form. All the above forms are interchangeable, i.e. we can change one form to other by suitable substitutions etc. In such cases $\lim_{x \to a} f(x)$ may exist.

Consider f(x) = $\frac{x^2 - 9}{x - 3}$. Here $\lim_{x \to 3} x^2 - 9 = 0$ and $\lim_{x \to 3} x - 3 = 0$

$\therefore$ $\lim_{x \to 3} f(x)$ has an indeterminate form of the type $\frac{0}{0}$.

$\lim_{x \to \infty} \frac{x^2 + 1}{x^2 + 2}$ has an indeterminate form of type $\frac{\infty}{\infty}$.

$\lim_{x \to 0} (1 + x)^{1/x}$ is an indeterminate form of the type 1^∞

NOTE :

(a) $\infty + \infty \to \infty$ (b) $\infty \times \infty \to \infty$ (c) $\infty^\infty \to \infty$ (d) $0^\infty \to 0$

(e) $\lim_{x \to 0} \frac{x^2}{x^2}$ is an indeterminate form whereas $\lim_{x \to 0} \frac{[x^2]}{x^2}$ is not an indeterminate form (where [.] represents greatest integer function)

Students may remember these forms alongwith the prefix 'tending to'

i.e. $\frac{\text{approaching to zero}}{\text{approaching to zero}}$ is an indeterminate form whereas $\frac{\text{exactly zero}}{\text{approaching to zero}}$ is not an indeterminate form, its value is zero.

similarly (approaching to one)$^{\text{approaching to } \infty}$ is indeterminate form whereas (exactly one)$^{\text{approaching to } \infty}$ is not an indeterminate form, its value is one.

4. METHODS TO EVALUATE INDETERMINANT FORMS :

To evaluate a limit, we must always put the value where ' x ' is approaching to in the function. If we get a determinate form, then that value becomes the limit otherwise if an indeterminate form comes, we have to remove the indeterminancy, once the indeterminancy is removed the limit can be evaluated by putting the value of x, where it is approaching.

Basic methods of removing indeterminancy are

(1) Factorisation **(2) Rationalisation** **(3) Double Rationalisation**

(4) Uses Standard limits **(5) Expansion of functions**

(6) L. Hospital rule (Used in $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

(7) Taking log (used in ∞^0, 0^0, 1^∞ inderminate form) **(8) Sandwitch Theorem**

4.1 Factorisation method :-

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

Example # 3 : (i) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$ (ii) $\lim\limits_{x\to 2} \left[\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution : (i) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3} = \lim\limits_{x\to 3} \dfrac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

(ii) $\lim\limits_{x\to 2} \left[\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim\limits_{x\to 2} \left[\dfrac{x^2-5x+6}{x(x-1)(x-2)}\right]$

$= \lim\limits_{x\to 2} \left[\dfrac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim\limits_{x\to 2} \left[\dfrac{x-3}{x(x-1)}\right] = -\dfrac{1}{2}$

4.2 Rationalisation method :-

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Example # 4 : Evaluate :

(i) $\lim\limits_{x\to 1} \dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$ (ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution : (i) $\lim\limits_{x\to 1} \dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}} = \lim\limits_{x\to 1} \dfrac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$

$= \lim\limits_{x\to 1} \dfrac{(15-15x)}{(3-3x)} \times \dfrac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}} = \dfrac{5}{2}$

(ii) The form of the given limit is $\dfrac{0}{0}$ when $x \to 0$. Rationalising the numerator, we get

$\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim\limits_{x\to 0} \left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}\right] = \lim\limits_{x\to 0} \left[\dfrac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \dfrac{2}{2} = 1$

4.3 Double Ralionalisation :-

Example # 5 : Evaluate : $\lim\limits_{x\to 1}\left(\dfrac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Solution : This is of the form $\dfrac{3-3}{2-2}=\dfrac{0}{0}$ if we put x = 1

To eliminate the $\dfrac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$$\therefore \text{ Limit} = \lim_{x\to 1}\left(\sqrt{x^2+8}-\sqrt{10-x^2}\right)\frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\times\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to 1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\times\frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}=\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)\times 1=\frac{2+2}{3+3}=\frac{2}{3}$$

Self practice problems

Evaluate the following limits :-

(3) (i) $\lim\limits_{x\to 1}\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$ (ii) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-(\sin x)^{1/3}}{1-(\sin x)^{2/3}}$

(iii) $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ (iv) $\lim\limits_{x\to a}\dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$

(v) $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}}$

Answers : (3) (i) $\dfrac{-1}{10}$ (ii) $\dfrac{1}{2}$ (iii) $\dfrac{1}{2\sqrt{x}}$ (iv) $\dfrac{1}{4a\sqrt{a-b}}$ (v) 0

4.4 Standard limits :

(a) (i) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

(ii) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}=\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}=1$

(iii) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$; $\lim\limits_{x\to 0}(1+ax)^{\frac{1}{x}}=e^a$

(iv) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$; $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^x=e^a$

(v) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$; $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a=\ln a$,a > 0

(vi) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$

(vii) $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

(b) If $f(x) \to 0$, when $x \to a$, then

(i) $\lim\limits_{x\to a} \dfrac{\sin f(x)}{f(x)} = 1$ (ii) $\lim\limits_{x\to a} \dfrac{\tan f(x)}{f(x)} = 1$

(iii) $\lim\limits_{x\to a} \dfrac{e^{f(x)}-1}{f(x)} = 1$ (iv) $\lim\limits_{x\to a} \dfrac{b^{f(x)}-1}{f(x)} = \ell n\, b,\ (b > 0)$

(v) $\lim\limits_{x\to a} \dfrac{\ell n(1+f(x))}{f(x)} = 1$ (vi) $\lim\limits_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$

(c) $\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} \phi(x) = B$(a finite quantity), then $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = A^B$.

Example # 6 : Evaluate :

(i) $\lim\limits_{x\to 0} \dfrac{(1+x)^n-1}{x}$ (ii) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{x/2}$

(iii) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$ (iv) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

Solution :

(i) $\lim\limits_{x\to 0} \dfrac{(1+x)^n-1}{x} = \lim\limits_{x\to 0} \dfrac{(1+x)^n-1}{(1+x)-1} = n$

(ii) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{x/2} = \lim\limits_{x\to 0}\ 2 \times 3\ \dfrac{e^{3x}-1}{3x} = 6.$

(iii) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{\tan x(1-\cos x)}{x^3} = \lim\limits_{x\to 0} \dfrac{\tan x \,.\, 2\sin^2 \frac{x}{2}}{x^3}$

$= \lim\limits_{x\to 0} \dfrac{1}{2} \,.\, \dfrac{\tan x}{x} \,.\, \left(\dfrac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \dfrac{1}{2}.$

(iv) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0} \left[\dfrac{\sin 2x}{2x} . \dfrac{2x}{3x} . \dfrac{3x}{\sin 3x}\right] = \left[\lim\limits_{2x\to 0} \dfrac{\sin 2x}{2x}\right] . \dfrac{2}{3} . \left[\lim\limits_{3x\to 0} \dfrac{3x}{\sin 3x}\right]$

$= 1 \,.\, \dfrac{2}{3} \times \left[\lim\limits_{3x\to 0} \dfrac{\sin 3x}{3x}\right] = \dfrac{2}{3} \times 1 = \dfrac{2}{3}$

Example # 7 : Evaluate : $\lim\limits_{x\to\infty} \left(1+\dfrac{2}{x}\right)^x$

Solution : $\lim\limits_{x\to\infty} \left(1+\dfrac{2}{x}\right)^x = e^{\lim\limits_{x\to\infty} \frac{2}{x}.x} = e^2.$

Example # 8 : Evaluate : (i) $\lim\limits_{x\to 3} \dfrac{e^x - e^3}{x-3}$ (ii) $\lim\limits_{x\to 0} \dfrac{x(e^x-1)}{1-\cos x}$

Solution : (i) Put $y = x - 3$. So, as $x \to 3 \Rightarrow y \to 0$. Thus

$\lim\limits_{x\to 3} \dfrac{e^x - e^3}{x-3} = \lim\limits_{y\to 0} \dfrac{e^{3+y} - e^3}{y}$

$= \lim\limits_{y\to 0} \dfrac{e^3 \,.\, e^y - e^3}{y} = e^3 \lim\limits_{y\to 0} \dfrac{e^y - 1}{y} = e^3 \,.\, 1 = e^3$

(ii) $\lim\limits_{x\to 0} \dfrac{x(e^x-1)}{1-\cos x} = \lim\limits_{x\to 0} \dfrac{x(e^x-1)}{2\sin^2 \frac{x}{2}} = \dfrac{1}{2} . \lim\limits_{x\to 0} \left[\dfrac{e^x-1}{x} . \dfrac{x^2}{\sin^2 \frac{x}{2}}\right] = 2.$

Self practice problems

(4) Evaluate the following limits : -

(i) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x}$ (ii) $\lim\limits_{x\to 0} \dfrac{8}{x^8}\left(1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)$

(iii) $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}$ (iv) $\lim\limits_{x\to 0} \dfrac{5^x-9^x}{x}$

(v) $\lim\limits_{x\to \infty} (1 + a^2)^x \sin \dfrac{b}{(1+a^2)^x}$, where $a \neq 0$

Answers : (4) (i) 2 (ii) $\dfrac{1}{32}$ (iii) does not exist (iv) $\ell n\, \dfrac{5}{9}$

(v) b

4.4.1 Use of substitution in solving limit problems

Sometimes in solving limit problem we convert $\lim\limits_{x\to a} f(x)$ into $\lim\limits_{h\to 0} f(a + h)$ or $\lim\limits_{h\to 0} f(a - h)$ according as need of the problem. (here h is approaching to zero.)

Example # 9 : Evaluate $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{1-\tan x}{1-\sqrt{2}\sin x}$

Solution : Put $x = \dfrac{\pi}{4} + h$

$\because \quad x \to \dfrac{\pi}{4} \Rightarrow h \to 0$

$$\lim_{h\to 0} \frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)} = \lim_{h\to 0} \frac{1-\frac{1+\tan h}{1-\tan h}}{1-\sin h-\cos h} = \lim_{h\to 0} \frac{\frac{-2\tan h}{1-\tan h}}{2\sin^2\frac{h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$= \lim_{h\to 0} \frac{-2\tan h}{2\sin\frac{h}{2}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tanh)} = \lim_{h\to 0} \frac{-2\frac{\tanh}{h}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tanh)} = \frac{-2}{-1} = 2.$$

4.4.2 Limit when $x \to \infty$

In these types of problems we usually cancel out the greatest power of x common in numerator and denominator both. Also sometime when $x \to \infty$, we use to substitute $y = \dfrac{1}{x}$ and in this case $y \to 0^+$.

Example # 10 : Evaluate

(i) $\lim\limits_{x\to \infty} x \sin \dfrac{1}{x}$ (ii) $\lim\limits_{x\to \infty} \dfrac{x-2}{2x-3}$

(iii) $\lim\limits_{x\to \infty} \dfrac{x^2-4x+5}{3x^2-x^3+2}$ (iv) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Solution : (i) $\lim\limits_{x \to \infty} x \sin \frac{1}{x} = \lim\limits_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$

(ii) $\lim\limits_{x \to \infty} \frac{x-2}{2x-3} = \lim\limits_{x \to \infty} \frac{1-\frac{2}{x}}{2-\frac{3}{x}} = \frac{1}{2}$.

(iii) $\lim\limits_{x \to \infty} \frac{x^2-4x+5}{3x^2-x^3+2} = \lim\limits_{x \to \infty} \frac{\frac{1}{x}-\frac{4}{x^2}+\frac{5}{x^3}}{\frac{3}{x}-1+\frac{2}{x^3}} = 0$

(iv) $\lim\limits_{x \to -\infty} \frac{\sqrt{3x^2+2}}{x-2}$ (Put $x = -\frac{1}{t}$, $x \to -\infty \Rightarrow t \to 0^+$)

$= \lim\limits_{t \to 0^+} \frac{\sqrt{3+2t^2} \cdot \frac{1}{\sqrt{t^2}}}{\frac{(-1-2t)}{t}} = \lim\limits_{t \to 0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)} \frac{t}{|t|} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$.

Some important notes :

(i) $\lim\limits_{x\to\infty} \frac{\ell nx}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \frac{x}{e^x} = 0$ (iii) $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$

(iv) $\lim\limits_{x\to\infty} \frac{(\ell nx)^n}{x} = 0$ (v) $\lim\limits_{x\to 0^+} x(\ell nx)^n = 0$

As $x \to \infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim\limits_{n\to\infty} (1 - h)^n = 0$ and $\lim\limits_{n\to\infty} (1 + h)^n \to \infty$, where $h \to 0^+$.

Example # 11 : Evaluate $\lim\limits_{x\to\infty} \frac{x^{1000}}{e^x}$

Solution : $\lim\limits_{x\to\infty} \frac{x^{1000}}{e^x} = 0$

4.5 Limits using expansion

(a) $a^x = 1 + \frac{x\,\ell n a}{1!} + \frac{x^2 \ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + \ldots\ldots\ldots, a > 0$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(c) $\ell n\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$, for $-1 < x \le 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(h) for $|x| < 1$, $n \in R$; $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + ...\infty$

(i) $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ....\right)$

Example # 12 : Evaluate

(i) $\lim\limits_{x \to 0} \frac{e^x - 1 - x}{x^2}$ (ii) $\lim\limits_{x \to 1} \frac{(7+x)^{\frac{1}{3}} - 2}{x - 1}$

(iii) $\lim\limits_{x \to 0} \frac{\ell n(1+x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$ (iv) $\lim\limits_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{\tan x}$

Solution : (i) $\lim\limits_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim\limits_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + ...\right) - 1 - x}{x^2} = \frac{1}{2}$

(ii) Put $x = 1 + h$

$$\lim_{h \to 0} \frac{(8+h)^{\frac{1}{3}} - 2}{h} = \lim_{h \to 0} \frac{2.\left(1 + \frac{h}{8}\right)^{\frac{1}{3}} - 2}{h}$$

$$= \lim_{h \to 0} \frac{2\left\{1 + \frac{1}{3}.\frac{h}{8} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)\left(\frac{h}{8}\right)^2}{1.2} + ... - 1\right\}}{h} = \lim_{h \to 0} 2 \times \frac{1}{24} = \frac{1}{12}$$

(iii) $$\lim_{x \to 0} \frac{\ell n(1+x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x} = \lim_{x \to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - .....\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - .....\right) + \frac{x^2}{2}}{x^3 . \frac{\tan x}{x} . \frac{\sin x}{x}} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

(iv) $$\lim_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{\tan x} = \lim_{x \to 0} \frac{e - e\left(1 - \frac{x}{2} + ...\right)}{\tan x} = \lim_{x \to 0} \frac{e}{2} \times \frac{x}{\tan x} = \frac{e}{2}$$

Example # 13 : Find the values of a,b and c so that $\lim\limits_{x\to 0}\dfrac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}=2$

Solution : $\lim\limits_{x\to 0}\dfrac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}=2$ .....(1)

at $x \to 0$ numerator must be equal to zero

$\therefore$ a – b + c = 0 $\Rightarrow$ b = a + c .....(2)

From (1) & (2), $\lim\limits_{x\to 0}\dfrac{ae^{x}-(a+c)\cos x+ce^{-x}}{x\sin x}=2$

$$\Rightarrow \lim_{x\to 0}\frac{a\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......\right)-(a+c)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)+c\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+......\right)}{x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....\right)}=2$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{(a-c)}{x}+(a+c)+\frac{x}{3!}(a-c)+.....}{\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)}=2$$

Since R.H.S is finite,

$\therefore$ a – c = 0 $\therefore$ a = c, then $\dfrac{0+2a+0+....}{1}=2$

$\therefore$ a = 1 then c = 1

From (2), b = a + c = 1 + 1 = 2

4.6 L' Hospital's Rule :-

L'Hospital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim\limits_{x\to c}f(x)=\lim\limits_{x\to c}g(x)=0$ or $\pm\infty$, $g'(x)\neq 0$ for all x in I with x ¹ c, and $\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$.

Example # 14 : Evaluate $\lim\limits_{h\to 0}\dfrac{f(a+2h)-f(a-3h)}{h}$, if $f'(a) = 3$

Solution : $\because$ $\lim\limits_{h\to 0}\dfrac{f(a+2h)-f(a-3h)}{h}=\lim\limits_{h\to 0}\dfrac{f'(a+2h)(2)-f'(a-3h)(-3)}{1}$

$= f'(a) \times 5 = 3 \times 5 = 15$

Self practice problems

(5) If $f'(2) = 4$, then find the value of $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2+\sinh)}{h.\sinh.\tanh}$.

Answers : (28) 2/3

4.7 Limits of form 1^∞, 0^0, ∞^0

(A) All these forms can be converted into $\frac{0}{0}$ form in the following ways

(a) If $x \to 1$, $y \to \infty$, then $z = (x)^y$ is of 1^∞ form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{\ell n x}{\frac{1}{y}} \qquad \left(\frac{0}{0}\text{ form}\right)$

As $y \to \infty \Rightarrow \frac{1}{y} \to 0$ and $x \to 1 \Rightarrow \ell n x \to 0$

(b) If $x \to 0$, $y \to 0$, then $z = x^y$ is of (0^0) form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{y}{\frac{1}{\ell n x}} \qquad \left(\frac{0}{0}\text{ form}\right)$

(c) If $x \to \infty$, $y \to 0$, then $z = x^y$ is of $(\infty)^0$ form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{y}{\frac{1}{\ell n x}} \qquad \left(\frac{0}{0}\text{ form}\right)$

(B) $(1)^\infty$ type of problems can be solved by the following method

(a) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(b) $\lim\limits_{x\to a} [f(x)]^{g(x)}$; where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$$= \lim_{x\to a} [1+f(x)-1]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)} = \lim_{x\to a}\left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)g(x)} = e^{\lim\limits_{x\to a}[f(x)-1]\,g(x)}$$

Example # 15 : Evaluate

(i) $\lim\limits_{x\to\infty}\left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$ (ii) $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$ (iii) $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ (iv) $\lim\limits_{x\to 0^+} x^x$

Solution : (i) Since it is in the form of 1^∞

$$\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2} = \exp\left(\lim_{x\to\infty}\left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)\right) = e^{-8}$$

(ii) Since it is in the form of 1^∞ so $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$

$$= e^{2\times\frac{\tan \pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

(iii) $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ put x = a + h

$$= \lim_{h\to 0}\left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)} = \lim_{h\to 0}\left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)} = e^{\lim_{h\to 0}-\cot\frac{\pi h}{2a}\cdot\left(1+\frac{h}{a+h}-1\right)}$$

$$= e^{\lim_{h\to 0}-\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right)\cdot\frac{\frac{2a}{\pi}}{a+h}} = e^{-\frac{2}{\pi}}$$

(iv) Let $y = \lim_{x\to 0^+} x^x$

$$\Rightarrow \quad \ell n\, y = \lim_{x\to 0^+} x\,\ell n\, x = \lim_{x\to 0^+} -\frac{\ell n\frac{1}{x}}{\frac{1}{x}} = 0, \quad \text{as} \quad \frac{1}{x}\to\infty$$

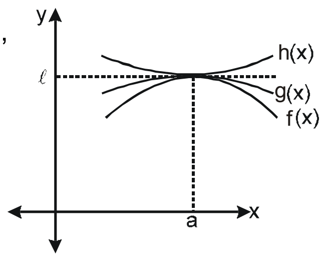
$$\Rightarrow \quad y = 1$$

4.8 Sandwitch theorem or squeeze play theorem:

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that

$\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$,

Then $\lim_{x\to a} g(x) = \ell$.



Example # 16 : Evaluate $\lim_{n\to\infty}\frac{[x]+[2x]+[3x]+....+[nx]}{n^2}$, where [.] denotes greatest integer function.

Solution : We know that, $x - 1 < [x] \le x$

$2x - 1 < [2x] \le 2x$

$3x - 1 < [3x] \le 3x$

. . .

. . .

$nx - 1 < [nx] \le nx$

$\therefore \quad (x + 2x + 3x + ... + nx) - n < [x] + [2x] + ... +[nx] \le (x + 2x + ... + nx)$

$$\Rightarrow \quad \frac{xn(n+1)}{2} - n < \sum_{r=1}^{n}[r\,x] \le \frac{x.n(n+1)}{2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \lim_{n\to\infty}\frac{[x]+[2x]+....+[nx]}{n^2} \le \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)$$

$$\Rightarrow \quad \frac{x}{2} < \lim_{n\to\infty}\frac{[x]+[2x]+....+[nx]}{n^2} \le \frac{x}{2}$$

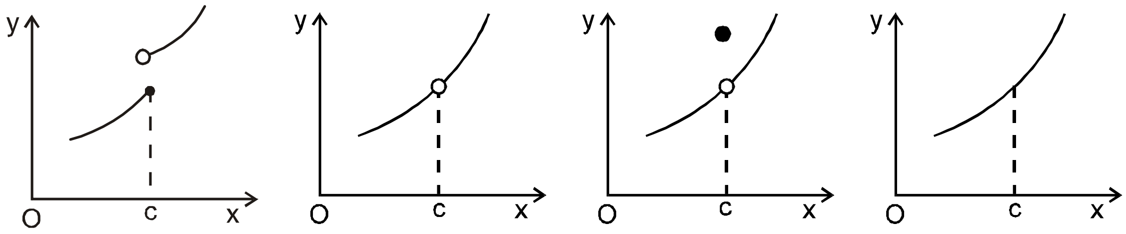
$$\therefore \quad \lim_{n\to\infty}\frac{[x]+[2x]+....+[nx]}{n^2} = \frac{x}{2}$$

5. Continuity :

A function f(x) is said to be continuous at x = c, if $\underset{x\to c}{\text{Limit}}\ f(x) = f(c)$

$\Rightarrow$ If $\underset{h\to 0^+}{\text{Limit}}\ f(c-h) = \underset{h\to 0^+}{\text{Limit}}\ f(c+h) = f(c)$, then f(x) in continous at x = c

If a function f (x) is continuous at x = c, the graph of f (x) at the corresponding point (c, f (c)) will not be broken. But if f (x) is discontinuous at x = c, the graph will be broken when x = c



((i), (ii) and (iii) are discontinuous at x = c) and ((iv) is continuous at x = c)

A function f can be discontinuous due to any of the following two reasons:

(i) $\underset{x\to c}{\text{Limit}}\ f(x)$ does not exist i.e. $\underset{x\to c^-}{\text{Limit}}\ f(x) \neq \underset{x\to c^+}{\text{Limit}}\ f(x)$ (Example figure (i))

This is non removable discontinuity

(iii) $\underset{x\to c}{\text{Limit}}\ f(x) \neq f(c)$ (Example figure (ii), (iii))

This is removable discontinuity
Geometrically, the graph of the function will exhibit a break at x= c.

Example # 17 : If $f(x) = \begin{cases} \sin\dfrac{\pi x}{2} & , \quad x < 1 \\ [x] & , \quad x \geq 1 \end{cases}$, then find whether f(x) is continuous or not at x = 1, where

[.] is greatest integer function.

Solution : $f(x) = \begin{cases} \sin\dfrac{\pi x}{2} & , \quad x < 1 \\ [x] & , \quad x \geq 1 \end{cases}$

For continuity at x = 1, we determine f(1), $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$.

Now, f(1) = [1] = 1

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \sin\dfrac{\pi x}{2} = \sin\dfrac{\pi}{2} = 1$

and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} [x] = 1$

so $f(1) = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 1$ $\therefore$ f(x) is continuous at x = 1

Self practice problems :

(6) If possible find value of λ for which f(x) is continuous at $x = \dfrac{\pi}{2}$

$$f(x) = \begin{cases} \dfrac{1-\sin x}{1+\cos 2x}, & x < \dfrac{\pi}{2} \\ \lambda \quad , & x = \dfrac{\pi}{2} \\ \dfrac{\sqrt{2x-\pi}}{\sqrt{4+\sqrt{2x-\pi}}-2}, & x > \dfrac{\pi}{2} \end{cases}$$

(7) Find the values of a and b such that the function

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x & ; \quad 0\le x<\frac{\pi}{4} \\ 2x\cot x+b & ; \quad \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x & ; \quad \frac{\pi}{2}<x\le\pi \end{cases}$$ is continuous at $x=\frac{\pi}{4}$ and $x=\frac{\pi}{2}$

(8) If $f(x)=\begin{cases} (1+ax)^{1/x} & ; \quad x<0 \\ b & ; \quad x=0 \\ \frac{(x+c)^{1/3}-1}{x} & ; \quad x>0 \end{cases}$, then find the values of a, b, c, for which f(x) is continuous at x = 0

Answers : (6) discontinuous (7) $a=\frac{\pi}{6}$, $b=\frac{-\pi}{12}$

(8) $a=-\ell n\ 3$, $b=\frac{1}{3}$, $c=1$

6. Theorems on continuity :

(i) If f(x) and g(x) are two functions which are continuous at x = c, then the functions defined by: $h_1(x)=f(x)\pm g(x)$; $h_2(x)=K\,f(x)$, K is any real number ; $h_3(x)=f(x).g(x)$ are also continuous at x = c. Further, if g (c) is not zero, then $h_4(x)=\frac{f(x)}{g(x)}$ is also continuous at x = c.

(ii) If f(x) is continuous & g(x) is discontinuous at x = a, then the product function $\phi_1(x)=f(x).\,g(x)$ may or may not be continuous but sum or difference function $\phi_2(x)=f(x)\pm g(x)$ will necessarily be discontinuous at x = a.

e.g. f (x) = x & $g(x)=\begin{cases}\sin\frac{\pi}{x} & x\ne 0\\ 0 & x=0\end{cases}$

(iii) If f (x) and g(x) both are discontinuous at x = a, then the sum, difference, product and division of functions f(x) and g(x) is not necessarily be discontinuous at x = a.

e.g. f (x) = g(x) = $\begin{cases}1 & , \ x\ge 0\\ -1 & , \ x<0\end{cases}$

and atmost one out of f(x) + g(x) and f(x) – g(x) is continuous at x = a.

Example # 18 : If f(x) = [sin(x–1)] – {sin(x–1)}. Comment on continuity of f(x) at $x=\frac{\pi}{2}+1$

(where [.] denotes G.I.F. and { . } denotes fractional part function).

Solution : f(x) = [sin (x – 1)] – {sin (x – 1)}

Let g(x) = [sin (x – 1)] + {sin (x – 1)} = sin (x – 1)

which is continuous at $x=\frac{\pi}{2}+1$

as [sin (x – 1)] and { sin (x – 1)} both are discontinuous at $x=\frac{\pi}{2}+1$

∴ At most one of f(x) or g(x) can be continuous at $x=\frac{\pi}{2}+1$

As g(x) is continuous at $x=\frac{\pi}{2}+1$, therefore, f(x) must be discontinuous

7. Continuity of composite functions :

If f is continuous at x = c and g is continuous at x = f(c), then the composite g(f(x)) is continuous at x = c. eg. $f(x) = \frac{x\sin x}{x^2+2}$ and $g(x) = |x|$ are continuous at x = 0, hence the composite function $(gof)(x) = \left|\frac{x\sin x}{x^2+2}\right|$ will be continuous at x = 0.

Self practice problem :

(9) $f(x) = \begin{cases} 1+x^3 & , \ x<0 \\ x^2-1 & , \ x \ge 0 \end{cases}$ and $g(x) = \begin{cases} (x-1)^{\frac{1}{3}} & , \ x<0 \\ (x+1)^{\frac{1}{2}} & , \ x \ge 0 \end{cases}$

Then define fog (x) and comment on the continuity of gof(x) at x = 1

Answer : [fog(x) = x, $x \in R$ and gof(x) is discontinous at x = 1]

8. Continuity in an Interval :

(a) A function f is said to be continuous in interval (a, b) if f is continuous at each and every point belongs to interval (a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if:

(i) f is continuous in the open interval (a, b),

(ii) f is right continuous at 'a' i.e. $\underset{x\to a^+}{\text{Limit}}$ f(x) = f(a) = a finite quantity and

(iii) f is left continuous at 'b' i.e. $\underset{x\to b^-}{\text{Limit}}$ f(x) = f(b) = a finite quantity.

Note-

(i) All Polynomial functions, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point of their respective domains.

(ii) Continuity of a function should be checked at the points where definition of a function changes.

(iii) Continuity of {f(x)} and [f(x)] should be checked at all points where f(x) becomes integer.

(iv) Continuity of sgn (f(x)) should be checked at the points where f(x) = 0

(v) In case of composite function f(g(x)) continuity should be checked at all possible points of discontinuity of g(x) and at the points where g(x) = c, where x = c is a possible point of discontinuity of f(x).

(vi) Continuity of a function must be disscussed only at pionts which are in the domain of function.

Example # 19: If $f(x) = \begin{cases} [\sin \pi x] & ; \ 0 \le x < 1 \\ \left\{x - \frac{2}{3}\right\} \cdot \text{sgn}\left(x - \frac{5}{4}\right) & ; \ 1 \le x \le 2 \end{cases}$, where { . } represents fractional part function and [.] is greatest integer function, then comment on the continuity of function in the interval [0, 2].

Solution : (i) Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2

(ii) For [sin πx], continuity should be checked at all values of x at which $\sin \pi x \in I$

i.e. $x = 0, \frac{1}{2}$

(iii) For $\left\{x-\frac{2}{3}\right\}\cdot\text{sgn}\left(x-\frac{5}{4}\right)$, continuity should be checked when $x - \frac{5}{4} = 0$

(as sgn (x) is discontinuous at x = 0) i.e. $x = \frac{5}{4}$ and when $x - \frac{2}{3} \in I$

i.e. $x = \frac{5}{3}$ (as {x} is discontinuous when $x \in I$)

$\therefore$ overall discontinuity should be checked at $x = 0, \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$ and 2

check the discontinuity your self.

discontinuous at $x = \frac{1}{2}, 1\ \frac{5}{4}, \frac{5}{3}$

Example # 20 : If $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x).

Solution : $f(x) = \frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain
and f is not defined at x = 1

$\therefore$ f is discontinuous at x = 1

$g(x) = \frac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2

$\therefore$ g is discontinuous at x = 2

Now fog (x) will be discontinuous at

(i) x = 2 (point of discontinuity of g(x))

(ii) g(x) = 1 (when g(x) = point of discontinuity of f(x))

if g(x) = 1 $\Rightarrow$ $\frac{1}{x-2} = 1$ $\Rightarrow$ x = 3

$\therefore$ discontinuity of fog(x) should be checked at x = 2 and x = 3

at x = 2

$$\text{fog}(x) = \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$$

fog (2) is not defined

$$\lim_{x\to 2} \text{fog}(x) = \lim_{x\to 2} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x\to 2} \frac{1+x-2}{1-x+2} = 1$$

$\therefore$ fog (x) is discontinuous at x = 2 and it is removable discontinuity at x = 2

fog (3) = not defined

$$\lim_{x\to 3^+} \text{fog}(x) = \lim_{x\to 3^+} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

$$\lim_{x\to 3^-} \text{fog}(x) = \lim_{x\to 3^-} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

$\therefore$ fog (x) is discontinuous at x = 3 and it is non removable discontinuity of II^{nd} kind.

Self practice problem :

(10) If $f(x) = \begin{cases} [\ell n\, x].\text{sgn}\left(\left\{x - \frac{1}{2}\right\}\right) & ; \quad 1 < x \le 3 \\ \{x^2\} & ; \quad 3 < x \le 3.5 \end{cases}$. Find the pointswhere the continuity of f(x), should be checked, where [.] is greatest integer function and {.} fractional part function.

Answer : $\{ 1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5 \}$

9. Intermediate value theorem :

A function f(x) which is continuous in interval $[a, b]$ possesses the following properties:

(i) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).

(ii) If f(a) and f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

Example # 21 : Given that a > b > c > d, then prove that the equation (x – a) (x – c) + 2(x – b) (x – d) = 0 will have real and distinct roots.

Solution : $(x - a)(x - c) + 2(x - b)(x - d) = 0$

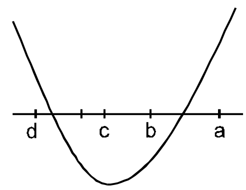
$f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

$f(a) = (a - a)(a - c) + 2(a - b)(a - d) = +ve$

$f(b) = (b - a)(b - c) + 0 = -ve$

$f(c) = 0 + 2(c - b)(c - d) = -ve$

$f(d) = (d - a)(d - c) + 0 = +ve$



hence $(x - a)(x - c) + 2(x - b)(x - d) = 0$

have real and distinct roots

Self practice problem :

(11) If $f(x) = xe^x - 2$, then show that f(x) = 0 has exactly one root in the interval (0, 1).

Example # 22: Let $f(x) = \underset{n\to\infty}{\text{Lim}} \frac{1}{1+n\sin^2 x}$, then find $f\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Solution : Let $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2 x}$

$$f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n\,.\sin^2\frac{\pi}{4}} = \lim_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$$

Now

$$f(0) = \lim_{n\to\infty} \frac{1}{n\,.\sin^2(0)+1} = \frac{1}{1+0} = 1$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty} \frac{1}{1+n\sin^2 x}\right] = 0$$

{here $\sin^2 x$ is very small quantity but not zero and very small quantity when multiplied with ∞ becomes ∞}

∴ f(x) is not continuous at x = 0

Self practice problem :

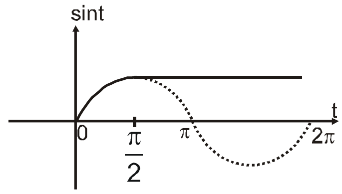
(12) If $f(x) = \lim\limits_{n\to\infty} (1 + x)^n$.

Comment on the continuity of f(x) at x = 0 and explain $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$.

Answer : Discontinous (non-removable)

Example # 23 : f(x) = maximum (sin t, 0 ≤ t ≤ x), 0 ≤ x ≤ 2π discuss the continuity of this function at x= $\frac{\pi}{2}$

Solution : f(x) = maximum (sin t, 0 ≤ t ≤ x), 0 ≤ x ≤ 2π



if $x \in \left[0, \frac{\pi}{2}\right]$, sin t is increasing function

Hence if t ∈ [0, x], sin t will attain its maximum value at t = x.

∴ $f(x) = \sin x$ if $x \in \left[0, \frac{\pi}{2}\right]$

if $x \in \left(\frac{\pi}{2}, 2\pi\right]$ and t ∈ [0, x]

then sin t will attain its maximum value when t = $\frac{\pi}{2}$

∴ $f(x) = \sin\frac{\pi}{2} = 1$ if $x \in \left(\frac{\pi}{2}, 2\pi\right]$ ∴ $f(x) = \begin{cases} \sin x, & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 1, & \text{if } x \in \left(\frac{\pi}{2}, 2\pi\right] \end{cases}$

Now $f\left(\frac{\pi}{2}\right) = 1$

$\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-} \sin x = 1$

$\lim\limits_{x\to\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} 1 = 1$

as f(π/2) = L.H.S. = R.H.S. ∴ f(x) is continuous at x = $\frac{\pi}{2}$

10. CONTINUITY OVER COUNTABLE SET :

There are functions which are continuous over a countable set and else where discontinuous.

Example # 24 : If $f(x) = \begin{bmatrix} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{bmatrix}$, find the points where f(x) is continuous

Solution : Let x = a be the point at which f(x) is continuous.

$\Rightarrow \lim\limits_{\substack{x\to a \\ \text{through rational}}} f(x) = \lim\limits_{\substack{x\to a \\ \text{through irrational}}} f(x)$

⇒ a = –a

⇒ a = 0 ⇒ function is continuous at x = 0.

Self practice problem :

(13) (i) If $g(x) = \begin{bmatrix} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{bmatrix}$, then find the points where function is continuous.

(ii) If $f(x) = \begin{cases} x^2 & ; x \in \mathbb{Q} \\ 1 - x^2 & ; x \notin \mathbb{Q} \end{cases}$, then find the points where function is continuous.

11. MEANING OF DERIVATIVE :

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let 'f' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of 'f' then $\Delta f = f(x + \Delta x) - f(x)$

$$\Rightarrow \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

If $\Delta f/\Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is a function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx},\ \frac{d}{dx}f(x) \text{ or } \frac{df(x)}{dx}$$

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a),\ \left.\frac{df(x)}{dx}\right|_{x=a},\ f'(x)_{x=a}$, etc.

12. Differentiability of a function at a point :

(i) The right hand derivative of f (x) at x = a denoted by $f'(a^+)$ is defined by:

R.H.D. = $f'(a^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

(ii) The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined by:

L.H.D. = $f'(a^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$, provided the limit exists.

A function f(x) is said to be differentiable at x = a if $f'(a^+) = f'(a^-)$ = finite

By definition $f'(a) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$

Example # 25: Comment on the differentiability of $f(x) = \begin{cases} x, & x<1 \\ x^2, & x \ge 1 \end{cases}$ at x = 1.

Solution : R.H.D. = $f'(1^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(1+h)-f(1)}{h}$

$$= \underset{h\to 0^+}{\text{Limit}} \frac{(1+h)^2-1}{h} = \underset{h\to 0^+}{\text{Limit}} \frac{1+h^2+2h-1}{h} = \underset{h\to 0^+}{\text{Limit}} (h+2) = 2$$

L.H.D. = $f'(1^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(1-h)-f(1)}{-h} = \underset{h\to 0^+}{\text{Limit}} \dfrac{1-h-1}{-h} = 1$

As L.H.D. $\neq$ R.H.D. Hence f(x) is not differentiable at x = 1.

Example # 26 : If f(x) = $\begin{cases} A+Bx^2 & , \quad x<1 \\ 3Ax-b+2 & , \quad x \ge 1 \end{cases}$, then find A and B so that f(x) become differentiable at x = 1.

Solution : $f'(1^+) = \lim_{h\to0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+} \frac{3A\,(1+h)-B+2-3A+B-2}{h} = \lim_{h\to0^+} \frac{3Ah}{h} = 3A$

$f'(1^-) = \lim_{h\to0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to0^+} \frac{A+B(1-h)^2-3A+B-2}{-h}$

$= \lim_{h\to0^+} \frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$

hence for this limit to be defined

$-2A + 2B - 2 = 0$

$B = A + 1$

$\therefore$ $f'(1^-) = \lim_{h\to0} -(Bh - 2B) = 2B$

For f(x) to be differentiable at x = 1

$\therefore$ $f'(1^-) = f'(1^+)$

$\Rightarrow$ $3A = 2B = 2(A + 1)$ $\quad \because B = A + 1$

A = 2, B = 3

Example # 27 : If f(x) = $\begin{cases} [\cos \pi x], & x \le 1 \\ 2\{x\}-1, & x > 1 \end{cases}$, then comment on the derivability at x = 1,

where [.] is greatest integer function and {.} is fractional part function.

Solution : $f'(1^-) = \lim_{h\to0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to0^+} \frac{[\cos(\pi-\pi h)]+1}{-h} = \lim_{h\to0^+} \frac{-1+1}{-h} = 0$

$f'(1^+) = \lim_{h\to0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+} \frac{2\{1+h\}+1-1}{h} = \lim_{h\to0^+} \frac{2h}{h} = 2$

$\because$ $f'(1^+) \ne f'(1^-)$

f(x) is not differentiable at x = 1.

Self Practice Problems :

(14) If f(x) = $\begin{cases} [2x]+x & , \quad x<1 \\ \{x\}+1 & , \quad x \ge 1 \end{cases}$, then comment on the continuity and differentiable at x = 1,

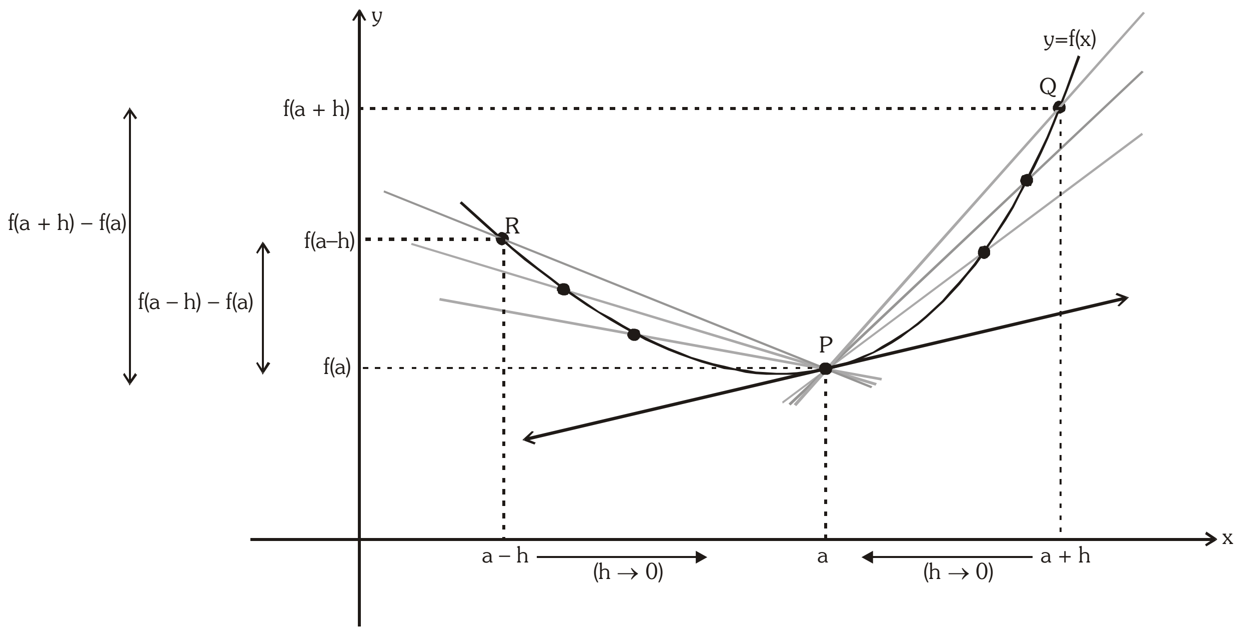
where [.] is greatest integer function and {.} is fractional part function.

(15) If f(x) = $\begin{cases} x\tan^{-1} 1/x, & x \ne 0 \\ 0 \quad , & x = 0 \end{cases}$, then comment on the derivability of f(x) at x = 0.

Answers : (20) Discontinuous and non-differentiable at x = 1

(21) non-differentiable at x = 0

13. Concept of tangent and its association with derivability :



Tangent :- The tangent is defined as the limiting case of a chord or a secant.

slope of the line joining (a,f(a)) and (a + h, f(a + h)) = $\frac{f(a+h)-f(a)}{h}$

Slope of tangent at P = $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

The tangent to the graph of a continuous function f at the point P(a, f(a)) is
(i) the line through P with slope $f'(a)$ if $f'(a)$ exists ;
(ii) the line x = a if L.H.D. and R.H.D. both are either ∞ or $-\infty$.
If neither (i) nor (ii) holds then the graph of f does not have a tangent at the point P.
In case (i) the equation of tangent is $y - f(a) = f'(a)(x - a)$.
In case (ii) it is x = a

Note : (i) A function is said to be derivable at x = a if there exist a tangent of finite slope at that point. $f'(a^+) = f'(a^-)$ = finite value
(ii) It is not necessary that curve is only one side of tangent.
(Example : $y = x^3$ has x-axis as tangent at origin)

Example#28 : Find the equation of tangent to $y = (x)^{1/3}$ at x = 1 and x = 0.

Solution : At x = 1 Here $f(x) = (x)^{1/3}$

L.H.D = $f'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+} \frac{(1-h)^{1/3}-1}{-h} = \frac{1}{3}$

R.H.D. = $f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^{1/3}-1}{h} = \frac{1}{3}$

As R.H.D. = L.H.D. = $\frac{1}{3}$

$\therefore$ slope of tangent = $\frac{1}{3}$ $\therefore$ $y - f(1) = \frac{1}{3}(x - 1)$

$y - 1 = \frac{1}{3}(x - 1)$

$\Rightarrow$ $3y - x = 2$ is tangent to $y = x^{1/3}$ at (1, 1)

At x = 0

$$\text{L.H.D.} = f'(0^-) = \lim_{h \to 0^+} \frac{(0-h)^{1/3} - 0}{-h} = +\infty$$

$$\text{R.H.D.} = f'(0^+) = \lim_{h \to 0^+} \frac{(0+h)^{1/3} - 0}{h} = +\infty$$

As L.H.D. and R.H.D are infinite. y = f(x) will have a vertical tangent at origin.

$\therefore$ x = 0 is the tangent to $y = x^{1/3}$ at origin.

Self Practice Problems :

(16) If possible find the equation of tangent to the following curves at the given points.

(i) $y = x^3 + 3x^2 + 28x + 1$ at x = 0.

(ii) $y = (x - 8)^{2/3}$ at x = 8.

Answers : (i) y = 28x + 1 (ii) x = 8

14. Relation between differentiability & continuity:

(i) If f ′(a) exists, then f(x) is continuous at x = a.

(ii) If f(x) is differentiable at every point of its domain of definition, then it is continuous in that domain.

Note : The converse of the above result is not true i.e. "If 'f' is continuous at x = a, then 'f' may or may not be differentiable at x = a.

e.g. the functions f(x) = | x | is continuous at x = 0 but not differentiable at x = 0.

If f(x) is a function such that R.H.D = $f'(a^+) = \ell$ and L.H.D. = $f'(a^-) = m$.

Case - I

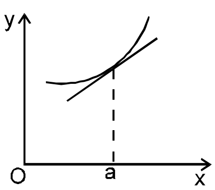
If $\ell = m$ = some finite value, then the function f(x) is differentiable as well as continuous.

Case - II

if $\ell \neq m$ = but both have some finite value, then the function f(x) is non differentiable but it is continuous.

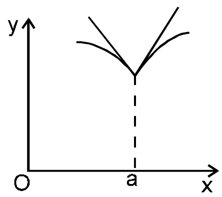
Case - III

If at least one of the ℓ or m is infinite, then the function is non differentiable but we can not say about continuity of f(x).



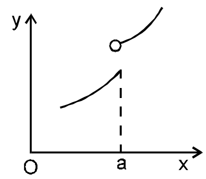
(i)

continuous and differentiable



(ii)

continuous but not differentiable



(iii)

neither continuous nor differentiable

Example # 29 : If f(x) is differentiable at x = a, prove that it will be continuous at x = a.

Solution : $f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \ell$

$\lim_{h \to 0^+} [f(a+h) - f(a)] = h\ell$

as $h \to 0$ and ℓ is finite, then $\lim_{h \to 0^+} f(a+h) - f(a) = 0$

$\Rightarrow$ $\lim_{h \to 0^+} f(a+h) = f(a)$.

Similarly $\lim_{h \to 0^+} [f(a-h) - f(a)] = -h\ell$ $\Rightarrow$ $\lim_{h \to 0^+} f(a-h) = f(a)$

$\therefore$ $\lim_{h \to 0^+} f(a+h) = f(a) = \lim_{h \to 0^+} f(a-h)$

Hence, f(x) is continuous.

Example # 30 : If $f(x) = \begin{cases} x^2 \,\text{sgn}[x] + \{x\}, & 0 \le x < 2 \\ \sin x + |x-3|, & 2 \le x < 4 \end{cases}$, comment on the continuity and differentiability of f(x), where [.] is greatest integer function and {.} is fractional part function, at x = 1, 2.

Solution : Continuity at x = 1

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^2 \,\text{sgn}[x] + \{x\}) = 1 + 0 = 1$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 \,\text{sgn}\,[x] + \{x\})$

$= 1 \,\text{sgn}\,(0) + 1 = 1$

$\because \quad f(1) = 1$

$\therefore$ L.H.L = R.H.L = f(1). Hence f(x) is continuous at x = 1.

Now for differentiability,

$\text{R.H.D.} = f'(1^+) = \lim\limits_{h \to 0^+} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{(1+h)^2 \,\text{sgn}[1+h] + \{1+h\} - 1}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{(1+h)^2 + h - 1}{h} = \lim\limits_{h \to 0^+} \dfrac{1 + h^2 + 2h + h - 1}{h} = \lim\limits_{h \to 0^+} \dfrac{h^2 + 3h}{h} = 3$

and $\text{L.H.D.} = f'(1^-) = \lim\limits_{h \to 0^+} \dfrac{f(1-h) - f(1)}{-h} = \lim\limits_{h \to 0^+} \dfrac{(1-h)^2 \,\text{sgn}[1-h] + 1 - h - 1}{-h} = 1$

$\Rightarrow \quad f'(1^+) \ne f'(1^-)$.

Hence f(x) is non differentiable at x = 1.

Now at x = 2

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 \,\text{sgn}\,[x] + \{x\}) = 4 \,.\, 1 + 1 = 5$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (\sin x + |x - 3|) = 1 + \sin 2$

Hence L.H.L $\ne$ R.H.L

Hence f(x) is discontinuous at x = 2 and then f(x) also be non differentaible at x = 2.

Self Practice Problem :

(17) If $f(x) = \begin{cases} \left(\dfrac{e^{[x]} + |x| - 1}{[x] + \{2x\}}\right) & x \ne 0 \\ 1/2 & x = 0 \end{cases}$, comment on the continuity at x = 0 and differentiability at x = 0, where [.] is greatest integer function and {.} is fractional part function.

Answer : discontinuous hence non-differentiable at x = 0

15. Differentiability of sum, product & composition of functions :

(i) If f(x) & g(x) are differentiable at x = a, then the functions f(x) $\pm$ g(x), f(x). g(x) will also be differentiable at x = a & if g (a) $\ne$ 0, then the function f(x)/g(x) will also be differentiable at x = a.

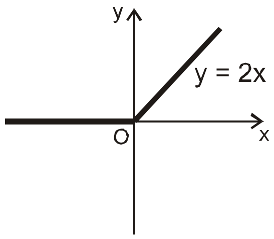
(ii) If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a
e.g. f(x) = $|x|$ and g(x) = x^2.

(iii) If f(x) & g(x) both are not differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$.

(iv) If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

Example#31 : Discuss the differentiability of $f(x) = x + |x|$.

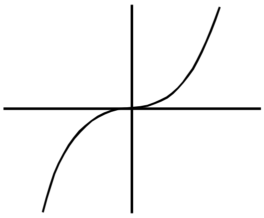
Solution :



Non-differentiable at $x = 0$.

Example#32 : Discuss the differentiability of $f(x) = x|x|$

Solution : $\because$ $f(x) = \begin{cases} x^2 & , \quad x \geq 0 \\ -x^2 & , \quad x < 0 \end{cases}$



Differentiable at $x = 0$

Example#33 : If f(x) is differentiable and g(x) is differentiable, then prove that f(x) . g(x) will be differentiable.

Solution : Given, f(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h} = f'(a)$

g(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{g(a+h) - g(a)}{h} = g'(a)$

let $p(x) = f(x) . g(x)$

Now, $\lim\limits_{h \to 0^+} \dfrac{p(a+h) - p(a)}{h} = \lim\limits_{h \to 0^+} \dfrac{f(a+h).g(a+h) - f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{f(a+h)g(a+h) + f(a+h).g(a) - f(a+h).g(a) - f(a).g(a)}{h}$

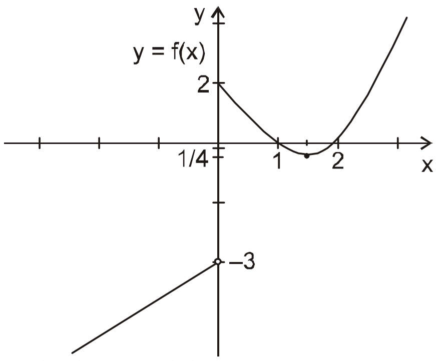
$= \lim\limits_{h \to 0^+} \left[\dfrac{f(a+h)\,(g\,(a+h) - g(a))}{h} + \dfrac{g(a)(f(a+h) - f(a))}{h}\right]$

$= \lim\limits_{h \to 0^+} \left[f(a+h).\dfrac{g(a+h) - g(a)}{h} + g(a).\dfrac{f(a+h) - f(a)}{h}\right] = f(a) . g'(a) + g(a) f'(a) = p'(a)$

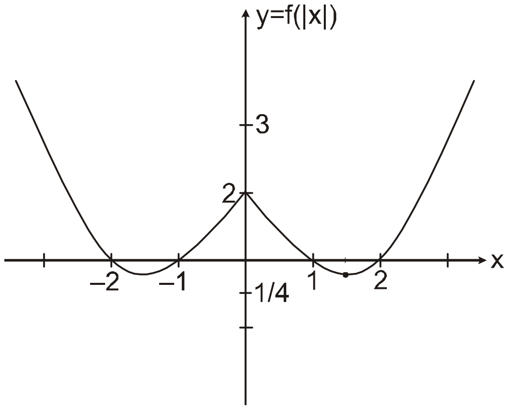
Hence p(x) is differentiable.

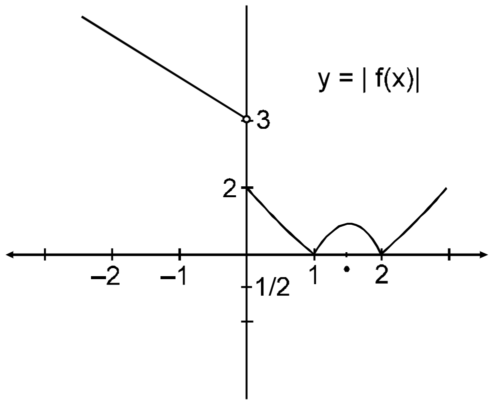
Example # 34 : If $f(x) = \begin{cases} x-3 & , \ x < 0 \\ x^2 - 3x + 2 & , \ x \geq 0 \end{cases}$ and g(x) = f(|x|) + |f(x)|, then comment on the continuity and differentiability of g(x) by drawing the graph of f(|x|) and, |f(x)|.

Solution :



Graph of f(|x|) and |f(x)|





If f(|x|) and |f(x)| are continous, then g(x) is continuous. At x = 0 f(|x|) is continuous, and |f(x)| is discontinuous therefore g(x) is discontineous at x = 0.

$\therefore$ g(x) is non differentiable at x = 0, 1, 2, (find the reason yourself).

16. Differentiability over an Interval :

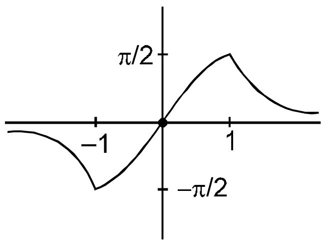
f (x) is said to be differentiable over an open interval if it is differentiable at each point of the interval and f(x) is said to be differentiable over a closed interval [a, b] if :

(i) for the points a and b, $f'(a^+)$ and $f'(b^-)$ exist finitely

(ii) for any point c such that a < c < b, $f'(c^+)$ & $f'(c^-)$ exist finitely and are equal.

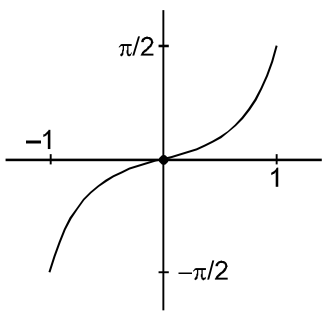
All polynomial, exponential, logarithmic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Graph of $y = \sin^{-1} \frac{2x}{1+x^2}$



Non differentiable at x = 1 & x = – 1

Graph of $y = \sin^{-1} x$.



Non differentiable at x = 1 & x = – 1

Example # 35 : If f(x) = $\begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & , \ 0 \le x < 1 \\ [2x]\,\text{sgn}\left(x-\frac{4}{3}\right) & , \ 1 \le x \le 2 \end{cases}$, find those points at which continuity and differentiability should be checked, where [.] is greatest integer function and {.} is fractional part function. Also check the continuity and differentaibility of f(x) at x = 1.

Solution : f(x) = $\begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & , \ 0 \le x < 1 \\ [2x]\,\text{sgn}\left(x-\frac{4}{3}\right) & , \ 1 \le x \le 2 \end{cases}$ The points, where we should check the continuity and differentiability are x = 0, $\frac{1}{2}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{3}{2}$, 2

At x = 1

L.H.L. = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \left\{x+\frac{1}{3}\right\}[\sin \pi x] = 0$

R.H.L. = $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} [2x]\ \text{sgn} \left\{x-\frac{4}{3}\right\} = 2\,(-1) = -2$

∴ L.H.L ≠ R.H.L. hence f(x) is discontinuous at x = 1 and hence it is non diffferentiable at x = 1.

Self Practice Problems:

(18) If f(x) = [x] + [1 – x] , – 1 ≤ x ≤ 3, then draw its graph and comment on the continuity and differentiability of f(x), where [.] is greatest integer function.

(19) If f(x) = $\begin{cases} |1-4x^2| & , \ 0 \le x < 1 \\ [x^2-2x] & , \ 1 \le x \le 2 \end{cases}$, then draw the graph of f(x) and comment on the differentiability and continuity of f(x), where [.] is greatest integer function.

Answers : (18) f(x) is discontinuous at x = –1, 0, 1, 2, 3 hence non-differentiable.

(19) f(x) is discontinuous at x = 1, 2 & non differentiable at x = $\frac{1}{2}$, 1, 2.

17. Problems of finding functions satisfying given conditions :

Example # 36 : If f(x) is a function satisfies the relation for all x, y ∈ R, f(x + y) = f(x) + f(y) and if f′(0) = 2 and function is differentiable every where, then find f(x).

Solution : $f'(x) = \lim\limits_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0^+} \frac{f(x)+f(h)-f(x)-f(0)}{h}$ (∵ f(0) = 0)

$= \lim\limits_{h\to 0^+} \frac{f(h)-f(0)}{h} = f'(0)$

$f'(x) = 2 \quad \Rightarrow \quad \int f'(x)\,dx = \int 2\,dx$

f(x) = 2x + c

∵ f(0) = 2.0 + c as f(0) = 0

∴ c = 0 ∴ f(x) = 2x

Second Method :

Since f(x + y) = f(x) + f(y) is true for all values of x and y is independent of differentiating both sides w.r.t x (here y is constant with respect to x).

$f'(x + y) = f'(x)$

put x = 0

$f'(y) = f'(0) \Rightarrow \int f'(y)\,dy = \int 2\,dy$

$f(y) = 2y + c \Rightarrow f(0) = 0 + c = 0$

$\therefore$ c = 0

$\therefore$ $f(y) = 2y \Rightarrow f(x) = 2x.$

Example # 37 : $f(x + y) = f(x) \,.\, f(y)\ \forall\ x, y \in R$ and f(x) is a differentiable function and $f'(0) = 1$, $f(x) \neq 0$ for any x. Find f(x)

Solution : f(x) is a differentiable function

$$\therefore \quad f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0^+} \frac{f(x).f(h)-f(x).f(0)}{h} \qquad (\because f(0) = 1)$$

$$= \lim_{h\to 0^+} \frac{f(x).(f(h)-f(0))}{h} = f(x)\,.\,f'(0) = f(x)$$

$\therefore$ $f'(x) = f(x)$

$$\therefore \quad \int \frac{f'(x)}{f(x)}\,dx = \int 1\,dx$$

$\Rightarrow$ $\ell n\, f(x) = x + c$ $\qquad \therefore\ \ell n\, 1 = 0 + c$

$\Rightarrow$ c = 0 $\qquad \therefore\ \ell n\, f(x) = x$

$\Rightarrow$ $f(x) = e^x$

Example # 38 : $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}\ \forall\ x, y \in R$ and f(0) = 1 and $f'(0) = -1$ and function is differentiable for all x, then find f(x).

Solution :

$$f'(x) = \lim_{h\to 0} \frac{f\left(\frac{2x+2h}{2}\right) - f\left(\frac{2x+0}{2}\right)}{h} = \lim_{h\to 0} \frac{\frac{f(2x)+f(2h)}{2} - \frac{f(2x)+f(0)}{2}}{h}$$

$$= \lim_{h\to 0} \frac{f(2h)-f(0)}{2h} = f'(0) = -1$$

$f'(x) = -1$

integrating both sides, we get

$f(x) = -x + c$

$\because$ c = + 1 (as f(0) = 1)

$\therefore$ $f(x) = -x + 1 = 1 - x$

Self Practice Problem:

(20) $f\left(\frac{x}{y}\right) = f(x) - f(y)\ \forall\ x, y \in R^+$ and $f'(1) = 1$, then show that $f(x) = \ell nx$.

Self Practice Problems :

(21) If f(x) and g(x) are differentiable, then prove that f(x) ± g(x) will be differentiable.

17.1 Some known functional equation & its solution

If x, y are independent variables and f(x) is continuous function, then:

(i) $f(xy) = f(x) + f(y)\ \forall x, y \in R^+ \Rightarrow f(x) = k \ln x$ or $f(x) = 0$, where k is a constant.

(ii) $f(xy) = f(x).\ f(y)\ \forall x, y \in R \Rightarrow f(x) = x^n$

(iii) $f(x + y) = f(x).\ f(y)\ \forall x, y \in R \Rightarrow f(x) = a^{kx}$, $f(x) = 0$, where k is a constant.

(iv) $f(x + y) = f(x) + f(y)\ \forall x, y \in R \Rightarrow f(x) = kx$, where k is a constant.

(v) If f(x) is a polynomial function satisfying $f(x) . f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall x \in R - \{0\}$, then

$f(x) = 1 \pm x^n$

Example # 39 : If f(x) is a polynomial function satisfying $f(x) . f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall x \in R - \{0\}$ and $f(2) = 9$, then find f (3)

Solution : $f(x) = 1 \pm x^n$

As $f(2) = 9$ $\therefore$ $f(x) = 1 + x^3$

Hence $f(3) = 1 + 3^3 = 28$

Self practice problems

(22) If f(x) is a polynomial function satisfying $f(x) . f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall x \in R - \{0\}$ and $f(3) = -8$, then find f(4)

(23) If $f(x + y) = f(x) . f(y)$ for all real x, y and $f(0) \neq 0$, then prove that the function, $g(x) = \frac{f(x)}{1+f^2(x)}$ is an even function.

Answer : (22) –15

Exercise-1

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Definition of LHL/RHL and Indeterminate forms

A-1. Evaluate the following limits :

(i) $\lim_{x\to 2} (x^2 + \cos x)$

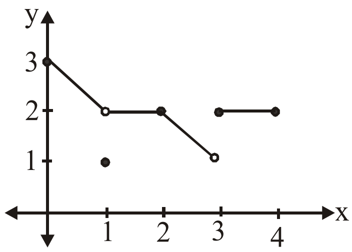
(ii) $\lim_{x\to 3} \tan x + 2^x$

(iii) $\lim_{x\to \frac{3}{4}} x\cos x$

(iv) $\lim_{x\to 5} (\ell n x)^x$

(v) $\lim_{x\to 1} \frac{e^x}{\sin x}$

A-2. Examine the graph of y = f(x) as shown and evaluate the following limits :



(i) $\lim_{x\to 1} f(x)$

(ii) $\lim_{x\to 2} f(x)$

(iii) $\lim_{x\to 3} f(x)$

(iv) $\lim_{x\to 1.99} f(x)$

(v) $\lim_{x\to 3^+} f(x)$

A-3 Evaluate the following limits,
Where [.] represents greatest integer function and {.} represents fractional part function

(i) $\lim_{x\to 0} [\cos x]$

(ii) $\lim_{x\to 3} \left\{\frac{x}{3}\right\}$

(iii) $\lim_{x\to 0}$ sgn[tan x]

(iv) $\lim_{x\to 1} \cos^{-1}(\ell n x)$

A-4. (i) If f(x) = $\begin{cases} 3x+1, & x<1 \\ 4x^2, & x\ge 1 \end{cases}$ evaluate $\lim_{x\to 1} f(x)$

(ii) Let f(x) = $\begin{cases} x+\lambda, & x<1 \\ 3x+2, & x\ge 1 \end{cases}$, if $\lim_{x\to 1} f(x)$ exist, then find value of λ.

A-5 If f(x) = $\begin{cases} x^2+2, & x\ge 2 \\ 8-x, & x<2 \end{cases}$ and g(x) = $\begin{cases} 2x, & x>6 \\ 3-x, & x\le 6 \end{cases}$, evaluate $\lim_{x\to 2} g(f(x))$.

A-6. Which of the followings are indeterminate forms. Also state the type.

(i) $\lim_{x\to 0^+} \frac{[x]}{x}$, where [.] denotes the greatest integer function

(ii) $\lim_{x\to -\infty} \sqrt{x^2+1} - x$

(iii) $\lim_{x\to \frac{\pi}{2}} (\tan x)^{\tan 2x}$

(iv) $\lim_{x\to 1^+} (\{x\})^{\frac{1}{\ell n x}}$, where { . } denotes the fractional part function

Section (B) : Evaluation of limits of form 0/0, ∞/∞, ∞–∞, 0 × ∞, Use of L-Hospital Rule and Expansion

B-1. Evaluate each of the following limits, if exists

(i) $\lim\limits_{x\to 0}\dfrac{x^3-3x+1}{x-1}$ (ii) $\lim\limits_{x\to 1}\dfrac{4x^3-x^2+2x-5}{x^6+5x^3-2x-4}$ (iii) $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a \neq 0$

B-2. Evaluate the following limits, if exists

(i) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\tan^{-1}2x}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin^2 5x}{x^2}$

(iii) $\lim\limits_{x\to 0}\dfrac{\ell n(1+2x)}{2^x-1}$

(iv) $\lim\limits_{x\to 0}\dfrac{e^{bx}-e^{ax}}{x}$, where $0 < a < b$

(v) $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{1-\cos 4x}$

(vi) $\lim\limits_{x\to 0}\dfrac{x\left(e^{2+x}-e^2\right)}{1-\cos x}$

(vii) $\lim\limits_{x\to \frac{\pi}{6}}\dfrac{\sqrt{3}\sin x-\cos x}{x-\dfrac{\pi}{6}}$ (viii) $\lim\limits_{x\to 0}\dfrac{\ell n(2+x)+\ell n\ 0.5}{x}$

(ix) $\lim\limits_{x\to 0}\dfrac{(a+x)^2\sin(a+x)-a^2\sin a}{x}$

(x) $\lim\limits_{x\to 0}\dfrac{\sqrt{\dfrac{1-\cos 2x}{2}}}{x}$

(xi) $\lim\limits_{x\to 0}\left(\dfrac{\tan 2x-3x}{3x-\sin^2 x}\right)$

(xii) Find $n \in N$, if $\lim\limits_{x\to 3}\dfrac{x^n-3^n}{x-3} = 405$

B-3. Evaluate the following limits

(i) $\lim\limits_{x\to \infty}\dfrac{3x^2+2x+9}{x^2+1}$

(ii) $\lim\limits_{x\to \infty}\dfrac{x^3+x+\cos^2 x}{2x^3+\sin x}$

(iii) $\lim\limits_{x\to \infty}\left(\dfrac{1}{x^2}+\dfrac{2}{x^2}+....+\dfrac{x}{x^2}\right)$

(iv) $\lim\limits_{n\to \infty}\dfrac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$, $n\in N$

(v) $\text{Lim}\limits_{x\to -\infty}\dfrac{(3x^4+2x^2)\sin\dfrac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$

B-4 Evaluate following limits :

(i) $\lim\limits_{x\to \infty}\dfrac{x\ell n\left(1+\dfrac{\ell nx}{x}\right)}{2\ell n\ x}$

(ii) $\lim\limits_{x\to \infty}\dfrac{e^x\sin\left(\dfrac{x^n}{e^x}\right)}{x^n}$

B-5. Evaluate the following limits

(i) $\lim\limits_{x\to \infty}\left((x+1)^{\frac{4}{3}}-(x-1)^{\frac{4}{3}}\right)$

(ii) $\lim\limits_{x\to \infty}(\sqrt{x^2+8x}-x)$

(iii) $\lim\limits_{x\to \infty}\cos\left(\sqrt{x+1}\right)-\cos\left(\sqrt{x}\right)$

(iv) $\lim\limits_{x\to \infty}\left(((x+1)(x+2)(x+3)(x+4))^{\frac{1}{4}}-x\right)$

B-6. Evaluate the following limits using expansions :

(i) $\lim\limits_{x\to a}\dfrac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}$ (ii) $\lim\limits_{x\to 2}\dfrac{(x+2)^{\frac{1}{2}}-(15x+2)^{\frac{1}{5}}}{(7x+2)^{\frac{1}{4}}-x}$

(ii) $\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x-\dfrac{\tan^2 x}{2}}{x^3}$ (iv) $\lim\limits_{x\to 1}\dfrac{(\ell n(1+x)-\ell n2)(3.4^{x-1}-3x)}{[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}]\sin(x-1)}$

B-7 If $\lim\limits_{x\to 0}\dfrac{a+b\sin x-\cos x+ce^x}{x^3}$ exists, find the values of a, b, c. Also find the limit

B-8 Find the values of a and b so that :

(i) $\lim\limits_{x\to 0}\dfrac{1-ax\sin x+b\cos x}{x^4}$ may have a finite limit.

(ii) $\lim\limits_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right) = 4$

(iii) $\lim\limits_{x\to 0}\dfrac{axe^x-b\,\ell n(1+x)+cxe^{-x}}{x^2\sin x}=2$

B-9. Find the following limit using expansion : $\lim\limits_{x\to 0}\left(\dfrac{\ell n(1+x)^{(1+x)}}{x^2}-\dfrac{1}{x}\right)$

B-10 Prove that $\lim\limits_{x\to 2}\dfrac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-2}$ = $\cos^2\alpha\,\ell n(\cos\alpha)-\sin^2\alpha\,\ell n(\sin\alpha)$, $\alpha\in\left(0,\dfrac{\pi}{2}\right)$

B-11 Find the value of $\lim\limits_{h\to 0}\dfrac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$

Section (C) : Limit of form 0° ∞°, 1^∞, Sandwich theorem and Miscellaneous problems on limits

C-1 Evaluate the following limits :

(i) $\lim\limits_{x\to 0}(|x|)^{x^2}$

(ii) $\lim\limits_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}$

(iii) $\lim\limits_{x\to 1^-}([x])^{1-x}$, where [.] denotes greatest integer function

(iv) $\lim\limits_{x\to\frac{\pi}{2}} e^{\tan x}$

C-2 Evaluate the following limits :

(i) $\text{Lim}_{x\to 1}\left(\tan\dfrac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$ (ii) $\lim\limits_{x\to\infty}\left(\dfrac{1+3x}{1+4x}\right)^x$

(iii) $\lim\limits_{x\to 1}(1+\pi\,\ell n\,x)^{\sec\frac{\pi x}{2}}$ (iv) $\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{\frac{1}{x}}$

C-3 If $\lim_{x\to 1}(1+ax-bx^2)^{\frac{c}{x-1}} = e^3$, then find conditions on a, b and c.

C-4. Evaluate $\lim_{n\to\infty} \frac{[1.3x]+[2.4x]+.....+[n.(n+2)x]}{n^3}$, where [.] denotes greatest integer function.

C-5. If $f(x) = \lim_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1}$,$n\in N$ find range of f(x).

C-6. If $\ell = \lim_{n\to\infty} \sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1} - r\sin\frac{\pi}{r}\right)$, then find $\{\ell\}$. (where {} denotes the fractional part function)

Section (D) : Continuity at a point, Continuity in an interval, Continuity of composite functions & IMVT

D-1. Determine the values of a, b & c for which the function $f(x) = \begin{cases} \frac{\sin(a-1)x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \frac{(x+bx^2)^{1/2} - x^{1/2}}{b x^{3/2}} & \text{for } x > 0 \end{cases}$

is continuous at x = 0.

D-2. Find the values of a, b & c so that the function, $f(x) = \begin{cases} \frac{1+\sin^3 x}{\cos^2 x} & , x < -\frac{\pi}{2} \\ a & , x = -\frac{\pi}{2} \\ \frac{c(b+\sin x)}{(\pi+2x)^2} & , x > -\frac{\pi}{2} \end{cases}$ is continuous at

$x = -\frac{\pi}{2}$.

D-3. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \frac{f(x)}{x-3} & ,x \neq 3 \\ K & ,x = 3 \end{cases}$, then

(a) find all zeros of f

(b) find the value of K that makes h continuous at x = 3

(c) using the value of K found in (b), determine whether h is an even function.

D-4. If f(x) = {x} & g(x) = [x] (where {.} & [.] denotes the fractional part and the integral part functions respectively), then discuss the continuity of :

(i) h(x) = f(x). g(x) at x = 1 and 2

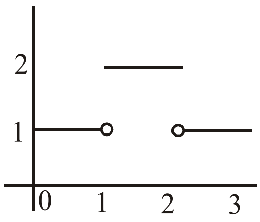
(ii) h(x) = f(x) + g(x) at x = 1

(iii) h(x) = f(x) – g(x) at x = 1

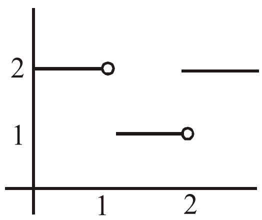
(iv) $h(x) = g(x) + \sqrt{f(x)}$ at x = 1 and 2

D-5. If $f(x) = \frac{\sin x + A\sin 2x + B\sin 3x}{x^5}$ $(x \neq 0)$ is continuous at x = 0. Find A & B. Also find f (0).

D-6. If graph of function y = f(x) is



and graph of function
y = g(x) is



then discuss the continuity of f(x)g(x) at x = 1 and x = 2

D-7. Find interval for which the function given by the following expressions are continuous :

(i) $f(x) = \frac{3x+7}{x^2+2x+3}$ (ii) $f(x) = \frac{1}{x^2-5|x|+6}+\frac{x}{2}$

(iii) $f(x) = \frac{\sqrt{x^2+1}}{1+\cos^2 x}$ (iv) $f(x) = \tan\left(\frac{\pi x}{2}\right)$

D-8. Find the number of points of discontinuity of the function f(x) = [5x] + {3x} in [0, 5] where [y] and {y} denote largest integer less than or equal to y and fractional part of y respectively.

D-9. If $f(x) = \frac{e^x}{x^3-1}$ and g(x) = sec x, then discuss the continuity of fog (x).

D-10. Find the point of discontinuity of y = f(u), where $f(u) = \frac{3}{2u^2+5u-3}$ and $u = \frac{1}{x+2}$.

D-11. Let $f(x) = \begin{cases} 1+x & , 0 \le x \le 2 \\ 3-x & , 2 < x \le 3 \end{cases}$ and $g(x) = \begin{cases} x+8, & 0 \le x < 1 \\ x^2, & 1 \le x \le 3 \end{cases}$. Find the point of discontinuity of f(f(x)) and g(f(x))

D-12. Show that the function $f(x) = \frac{x^3}{4} - \sin \pi x + 3$ takes the value $\frac{7}{3}$ within the interval [–2, 2].

D-13. If $g(x) = (|x-1| + |4x-11|)\,[x^2-2x-2]$, then find the number of point of discontinuity of g(x) in $\left(\frac{1}{2}, \frac{5}{2}\right)$ {where [.] denotes GIF}

Section (E) : Derivability at a point & Derivability in an interval

E-1 Test the continuity and differentiability of the function defined as under at x = 1 and x = 2

$$f(x) = \begin{cases} x^2+1 & ; \quad x<1 \\ 3-x & ; \quad 1 \le x \le 2 \\ -1+3x-x^2 & ; \quad x>2 \end{cases}$$

E-2 A function f is defined as follows: $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+|\sin x| & \text{for } 0 \le x < \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < \infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = $\pi/2$.

E-3. Let f(x) = x cos x, then prove that f(|x|) is not differentiable at x = 0

E-4. Show that the function $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right) & ; \quad x>0 \\ 0 & ; \quad x=0 \end{cases}$ is

(i) differentiable at x = 0, if m >1.
(ii) continuous but not differentiable at x = 0, if $0 < m \le 1$.
(iii) neither continuous nor differentiable, if $m \le 0$

E-5. Examine the differentiability of $f(x) = \sqrt{1-e^{-x^2}}$ at x = 0.

E-6. If $f(x) = \begin{cases} ax^2-b & \text{if } |x|<1 \\ \dfrac{2}{|x|} & \text{if } |x| \ge 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

E-7. Examine the continuity and differentiability of f(x) = | x | + | x – 1 | + | x – 2 |, $x \in \mathbb{R}$. Also draw the graph of f(x).

E-8. Discuss the continuity & the derivability in [0, 2] of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \ge 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x<1 \end{cases}$

where [.] denotes the greatest integer function

E-9. Discuss continuity and differentiability of y = f(x) in [–2, 5] where [.] denotes GIF & {.} denotes FPF

$$f(x) = \begin{cases} [x] & , \quad x \in [-2,0] \\ \{x\} & , \quad x \in (0,2) \\ \dfrac{x^2}{4} & , \quad x \in [2,4] \\ 4+\log_4(x-3) & , \quad x \in (4,5] \end{cases}$$

E-10. Check differentiability of $f(x) = \text{sgn}(x^{2/3}) + \left[\cos\left(\frac{x^2}{1+x^2}\right)\right] + |x-1|^{2/3}$ in [–2, 2] where [.] denotes GIF.

E-11. Discuss the continuity and differentiablility of h(x) = f(x)g(x) in (0, 3) if f(x) = $\frac{e^x - e}{[x]+1}$ {where [.] denotes GIF}

and g(x) = $\begin{cases} \frac{|x-1|+|x-2|}{2}, & x \in (0,1) \\ |x-1|+|x-2|, & x \in [1,2) \\ \frac{3(|x-1|+|x-2|)}{2}, & x \in [2,3) \end{cases}$

Section (F) : Functional equations and Miscellaneous

F-1. If $\lim\limits_{x\to 0} \frac{f(3-\sin x)-f(3+x)}{x} = 8$, then $|f'(3)|$ is

F-2. If f is a differentiable function such that $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)+f(0)}{3}$, ∀ x,y ∈ R and $f'(0) = 2$, find $f(x)$

F-3. Let a function f : R → R be given by f(x + y) = f(x) f(y) for all x, y ∈ R and f(x) ≠ 0 for any x ∈ R. If the function f (x) is differentiable at x = 0, show that f′(x) = f′(0) f(x) for all x ∈ R. Also, determine f(x).

F–4. Let f(x) be a polynomial function satisfying the relation f(x). f $\left(\frac{1}{x}\right)$ = f(x) + f $\left(\frac{1}{x}\right)$ ∀ x ∈ R – {0} and f(4) = 65. Determine f(1) + f′(1).

F-5. Let function f(x) satisfying the relation f(x + y)+ f(x – y) = 2f(x).f(y), then prove that it is even function

F-6. Let f(x) be a bounded function. $L_1 = \lim\limits_{x\to\infty} (f'(x) - \lambda f(x))$ and $L_2 = \lim\limits_{x\to\infty} f(x)$ where λ > 0. If L_1, L_2 both exist then prove that $L_2 = -\frac{L_1}{\lambda}$.

F-7. Let R be the set of real numbers and f : R → R, be a differentiable function such that $|f(x) - f(y)| \le |x - y|^3$ ∀ x,y ∈ R. If $f(10) = 100$, then find the value of $f(20)$.

PART - II : SINGLE OPTION CORRECT TYPE

Section (A) : Definition of LHL/RHL and Indeterminate forms

A-1 $\lim\limits_{x\to 0} \cos^{-1} [\sec x]$ is equal to (where [·] denotes greatest integer function)

(A) $\frac{\pi}{2}$ (B) 1 (C) 0 (D) Does not exist

A-2. Consider the following statements :

S_1 : $\lim\limits_{x\to 0^+} \frac{[x]}{x}$ is an indeterminate form (where [.] denotes greatest integer function).

S_2 : $\lim\limits_{x\to -\infty} \frac{\sin(3^x)}{3^x} = 1$

S_3 : $\lim\limits_{x\to\infty} \sqrt{\frac{x+\sin x}{x-\cos^2 x}}$ does not exist.

S_4 : $\lim\limits_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+3)!}$ (n ∈ N) =0

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) FTFT (B) FTTT (C) FTFF (D) TTFT

A-3 $\ell im_{x \to 2} (1 - x + [x - 2] + [2 - x])$ is equal to (wherre [.] denotes greatest integer function)

(A) 0 (B) 2 (C) –2 (D) does not exist

A-4 $\ell im_{x \to 0} \frac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$ is equal to

(A) 0 (B) 1 (C) –1 (D) does not exist

SECTION (B) : Evaluation of limits of form 0/0, ∞/∞, ∞ – ∞, 0 × ∞, Use of L-Hospital Rule & Expansion

B-1. $\ell im_{x \to 2} \frac{(x^3 + 8)\ \ell n(x - 1)}{(x^3 - 8)}$ is equal to

(A) 16 (B) $\frac{3}{4}$ (C) $\frac{4}{3}$ (D) $\frac{-4}{3}$

B-2. $\ell im_{x \to 0} \frac{(4^x - 1)^3}{\sin x\, \ell n\left(1 + \frac{x^2}{3}\right)}$ is equal to

(A) $9(\ell n\, 4)$ (B) $3(\ell n\, 4)^3$ (C) $12(\ell n\, 4)^3$ (D) $27(\ell n\, 4)^2$

B-3. $\ell im_{x \to 0} \frac{\tan\left(e^x - 1\right)}{\ell n(1 + x)}$ is equal to

(A) 0 (B) – 1 (C) 2 (D) 1

B-4. The value of $\ell im_{x \to 1} \frac{\sin\left(\ell n\ x\right)}{\ell n\ (1 + \sin(x - 1))}$ is equal to

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D*) 1

B-5. $\ell im_{x \to 0} \frac{\sqrt{1 - \cos 3x}}{x}$

(A) exists and it equals $\frac{3}{\sqrt{2}}$

(B) exists and it equals $-\frac{3}{\sqrt{2}}$

(C) does not exist because $x \to 0$

(D) does not exist because left hand limit is not equal to right hand limit.

B-6. the valfue of $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$ is equal to

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{2}}{4}$ (C) $\frac{-\sqrt{2}}{8}$ (D) $\frac{\sqrt{2}}{8}$

B-7. $\ell im_{x \to 0^+} \frac{\cos^{-1}(1 - x)}{\sqrt{x}}$ is equal to

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 1 (D) 0

B-8. $\lim\limits_{x\to 2}\dfrac{\left(\sum\limits_{k=1}^{100}(x-1)^k\right)-100}{(x-2)}$ is equal to

(A) 0 (B) 5050 (C) 4550 (D) –5050

B-9. $\ell im_{x\to\infty}\dfrac{x^2\ \sin\dfrac{1}{x}+x+1}{x^2+x+1}$ is equal to

(A*) 0 (B) $\dfrac{1}{2}$ (C) 1 (D) does not exist

B-10. $\lim\limits_{x\to-\infty}\dfrac{x^3\tan^{-1}\left(\dfrac{1}{x^2}\right)}{\sqrt{4x^2+3x+1}}$ is equal to

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) 0 (D) does not exist

B-11. $\ell im_{n\to\infty}\dfrac{7^n+5^n-4^{n+1}}{7^{n+1}+2^n+3^{n+2}}$, $n\in N$ is equal to

(A) $\dfrac{1}{9}$ (B) $\dfrac{1}{7}$ (C) 1 (D) zero

B-12. $\lim\limits_{n\to\infty} n\cos\left(\dfrac{\pi}{7n}\right)\sin\left(\dfrac{\pi}{3n}\right)$, $n\in N$ is equal to

(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{6}$ (D) does not exist

B-13. $\ell im_{x\to\frac{-\pi}{2}}\left[\dfrac{x+\dfrac{\pi}{2}}{\cos x}\right]$ is equal to (where [.] represents greatest integer function)

(A) 1 (B) 0 (C) – 2 (D) does not exist

B-14 $\ell im_{n\to\infty}\dfrac{-3n+(-1)^n}{4n-(-1)^n}$ is equal to ($n\in N$)

(A) $-\dfrac{3}{4}$ (B) $-\dfrac{3}{4}$ if n is even ; $\dfrac{3}{4}$ if n is odd

(C) not exist if n is even ; $-\dfrac{3}{4}$ if n is odd (D) 1 if n is even ; does not exist if n is odd

B-15. $\lim\limits_{x\to 1}\left(\dfrac{2}{1-x^2}+\dfrac{1}{x-1}\right)$ is equal to

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) – 1 (D) Does not exist

B-16. The value of $\lim\limits_{x\to 0}\dfrac{\sqrt[5]{1-x^2}-\sqrt[3]{1+3x}}{x-x^2}$ is equal to

(A) $\dfrac{1}{2}$ (B) 1 (C) – 1 (D) $-\dfrac{1}{2}$

B-17. $\underset{x\to\infty}{\ell im}\ \sqrt{x}-x\ \ n\left(1+\frac{1}{\sqrt{\ }}\right)$ is equal to :

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $\frac{1}{3}$ (D) 1

B-18. $\underset{x\to 0}{\ell im}\left(\frac{e^{-\frac{x^2}{2}}+\cos x+x^2-2}{x^2\sin^2 x}\right)$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{6}$ (C) $\frac{1}{12}$ (D) $\frac{1}{8}$

B-19. $\underset{x\to 0}{\ell im}\ \frac{\sin(3x^2)}{\ell n\cos(x^2-x)}$ is equal to

(A) 12 (B) – 12 (C) 6 (D) – 6

B-20 $\lim_{h\to 0}\frac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3}$ is equal to

(A) cos a (B) –cosa (C) sin a (D) sina cosa

SECTION (C) : Limit of form 0^0, ∞^0, 1^∞, Sandwich theorem and Miscellaneous problems on limits.

C-1. $\underset{x\to\infty}{\ell im}\left(\frac{x}{x+1}\right)^{2x+3}$ is equal to

(A) e^4 (B) e^{-4} (C) e^{-2} (D) e^2

C-2. $\underset{x\to 0^+}{\ell im}\left(1+\sin^2\sqrt{x}\right)^{\frac{1}{x}}$ is equal to

(A) e^5 (B) e^2 (C) e (D) e^{-1}

C-3. The value of $\underset{x\to\frac{\pi}{2}}{\ell im}\ ([x])^{\tan x}$ is equal to (where [.] denotes the greatest integer function)

(A) 0 (B) 1 (C) e (D) e^{-1}

C-4. $\lim_{x\to\infty}\left(\frac{x^3+x^2-2x+3}{x^3-x^2+5}\right)^x$ is equal to

(A) 1 (B) 2 (C) e^2 (D) e

C-5. The limiting value of $(\sin x)^{\sec x}$ at $x=\frac{\pi}{2}$ is:

(A) 1 (B) e (C) 0 (D) none of these

C-6. $\underset{x\to 3}{\ell im}\left(2-\frac{3}{x}\right)^{\tan\left(\frac{\pi x}{6}\right)}$ is equal to

(A) $e^{-\frac{3}{\pi}}$ (B) $e^{-\frac{6}{\pi}}$ (C) $e^{-\frac{2}{\pi}}$ D) 1

C-7. $\lim_{n\to\infty}\left(\left|\sin\frac{x}{\sqrt{n}}\right|^n+\left|\sin\frac{x}{\sqrt{n}}\right|\right)^{1/n}$, (x ≠ 0) is

(A) e^{-x^2} (B) $e^{\frac{-x^2}{2}}$ (C) 1 (D) e^{-1}

C-8. If [x] denotes greatest integer less than or equal to x, then $\lim_{n\to\infty}\frac{1}{n^3}([1^2x]+[2^2x]+......+[n^2x])$ is equal to

(A) $\frac{x}{2}$ (B) $\frac{x}{3}$ (C) $\frac{x}{6}$ (D) $\frac{x}{4}$

Section (D) : Continuity at a point, Continuity in an interval, Continuity of composite functions & IMVT

D-1. If $f(x)=\frac{x^2-bx+25}{x^2-7x+10}$ for x ≠ 5 and f is continuous at x = 5, then $f(5)$ has the value equal to-

(A) 0 (B) 5 (C) 10 (D) 25

D-2. Let f(x) = $\begin{cases}\frac{\cos(\sin x)-\cos x}{x^2}, & -2\le x<0\\ x-[x], & 0\le x\le 2\end{cases}$. whree [.] represents greatest integer function. Then

(A) f(x) is continuous at x = 2 (B) f(x) is continuous at x = 1
(C) f(x) is discontinuous at x = 0 (D) f(x) is continuous at x = 0

D-3. The function f(x) is defined by f(x) = $\begin{cases}\log_{4x-3}(x^2-2x+5), & \text{if } \frac{3}{4}<x<1 \text{ or } x>1\\ 4, & \text{if } x=1\end{cases}$

(A) is continuous at x = 1
(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists
(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists
(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists

D-4. If f(x) = x sin$\left(\frac{\pi}{2}(x+2[x])\right)$, then f(x) is {where [.] denotes GIF}

(A) Discontinuous at x = 2
(B) Discontinuous at x = 1
(C) Continuous at x = 1
(D) Continuous at x = 3

D-5. If $f(x)=\begin{cases}-4\sin x+\cos x & \text{for } x\le-\frac{\pi}{2}\\ a\sin x+b & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \cos x+2 & \text{for } x\ge\frac{\pi}{2}\end{cases}$ is continuous then :

(A) a = –1, b = 3 (B) a = 1, b = –3 (C) a = 1, b = 3 (D) a = –1, b = –3

D-6. Let f(x) = sgn(x) and g(x) = x (x^2 – 5x + 6). The function f(g(x)) is discontinuous at

(A) infinitely many points (B) exactly one point
(C) exactly three points (D) no point

D-7. If $f(x) = \dfrac{1}{(x-1)(x-2)}$ and g(x) = $\dfrac{1}{x^2}$, then set of points in domain of *f*og(x) at which *f*og(x) is discontinuous.

(A) $\left\{-1, 0, 1, \dfrac{1}{\sqrt{2}}\right\}$

(B) ϕ

(C) {0,1}

(D) $\left\{0, 1, \dfrac{1}{\sqrt{2}}\right\}$

D-8. The equation 2 tan x + 5x – 2 = 0 has

(A) no solution in [0, π/4]

(B) at least one real solution in [0, π/4]

(C) two real solution in [0, π/4]

(D) None of these

Section (E) : Derivability at a point & Derivability in an interval

E-1. If the right hand derivative of f(x) = [x] tan πx at x = 7 is kπ, then k is equal to ([y] denotes greatest integer $\leq$ y)

(A) 6 (B) 7 (C) –7 (D) 49

E-2. If f(x) = $\begin{cases} \dfrac{x(5e^{1/x}-2)}{3+e^{1/x}} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$, then f(x) is

(A) continuous as well differentiable at x = 0

(B) continuous but not differentiable at x = 0

(C) neither differentiable at x = 0 nor continuous at x = 0

(D) none of these

E-3. If f(x) = $\dfrac{x}{\sqrt{x+1}-\sqrt{x}}$ be a real valued function, then

(A) f(x) is continuous, but f′(0) does not exist

(B) f(x) is differentiable at x = 0

(C) f(x) is not continuous at x = 0

(D) f(x) is not differentiable at x = 0

E-4. The function f(x) = $\cos^{-1}$(sin x) is:

(A) continuous as well differentiable at x = 0

(B) continuous but not differentiable at x = 0

(C) neither differentiable at x = 0 nor continuous at x = 0

(D) none of these

E-5. If f(x) = $\begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, where { . } denotes the fractional part function, then:

(A) f is continuous & differentiable at x = 0

(B) f is continuous but not differentiable at x = 0

(C) f is continuous & differentiable at x = 2

(D) none of these.

E-6. The function f(x) = $(x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at :

(A) –1 (B) 0 (C) 1 (D) 2

E-7. If f (x) = $\begin{cases} \dfrac{x^2-1}{x^2+1} & , \quad 0 < x \le 2 \\ \frac{1}{4}\left(x^3 - x^2\right) & , \quad 2 < x \le 3 \\ \frac{9}{4}\left(|x-4| + |2-x|\right) & , \quad 3 < x < 4 \end{cases}$, then:

(A) f (x) is differentiable at x = 2 & x = 3

(B) f (x) is non-differentiable at x = 2 & x = 3

(C) f (x) is differentiable at x = 3 but not at x = 2

(D) f (x) is differentiable at x = 2 but not at x = 3.

E-8. The set of all points where the function f(x) = $\sqrt[3]{x^2\,|x|}$ is differentiable is:

(A) $(-\infty, \infty)$ (B) $[\,0, \infty)$ (C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$

E-9. If f (x) is differentiable everywhere, then:

(A) $|f|$ is differentiable everywhere

(B) $|f|^2$ is differentiable everywhere

(C) f $|f|$ is not differentiable at some point

(D) f + $|f|$ is differentiable everywhere

E-10. Let f(x) be defined in [–1, 1] by f(x) = $\begin{cases} \max(\sqrt{1-x}, \sqrt{1+x}) & , \quad -1 \le x \le 0 \\ \min(\sqrt{1-x}, \sqrt{1+x}) & , \quad 0 < x \le 1 \end{cases}$, then f(x) :

(A) is continuous at all points (B) is not continuous at more than one point

(C) is not differentiable only at one point (D) is not differentiable at more than one point

E-11. If $f(x) = (x^5 + 1)\,|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$, then $f(x)$ is not differentiable at -

(A) 2 points (B) 3 points (C) 4 points (D) zero points

E-12. Let f(x) = x – x² and g(x) = $\begin{cases} \max f(t), 0 \le t \le x, 0 \le x \le 1 \\ \sin \pi x \quad , x > 1 \end{cases}$, then in the interval $[0, \infty)$

(A) g(x) is everywhere continuous except at two points

(B) g(x) is everywhere differentiable except at two points

(C) g(x) is everywhere differentiable except at x = 1

(D) none of these

E-13. Consider the following statements :

S_1 : Number of points where f(x) = | x sgn (1 – x²) | is non-differentiable is 3.

S_2 : Number of points where f(x) = $\dfrac{1}{\sin^{-1}(\sin x)}$ is non-differentiable in the interval $(0, 3\pi)$ is 3.

S_3 : Let f(x) = |[x] x| for $-1 \le x \le 2$, where [.] is greatest integer function, then f is not differentiable at x = 2.

S_4 : If f(x) takes only rational values for all real x and is continuous, then f′(10) = 10.

(A) FTTT (B) TTTF (C) TFTF (D) FFTF

E-14. for what triplets of real number (a, b, c) with a ≠ 0 the function

$f(x) = \begin{cases} x & , \quad x \le 1 \\ ax^2 + bx + c & , \quad \text{otherwise} \end{cases}$ is differentiable for all real x ?

(A) {(a, 1 – 2a, a) | a ∈ R, a ≠ 0}
(B) {(a, 1 – 2a, c) | a, c ∈ R, a ≠ 0}
(C) {(a, b, c) | a, b, c ∈ R, a + b + c = 1}
(D) {(a, 1 – 2a, 0) | a ∈ R, a ≠ 0}

Section (F) : Functional Equations and Miscellaneous

F-1. Let f be differentiable at x = 0 and $f'(0) = 1$. Then $\lim_{h \to 0} \frac{f(h) - f(-2h)}{h} =$

(A) 3 (B) 2 (C) 1 (D) –1

F-2. If f(x + y) = f(x) . f(y), ∀ x, y ∈ R and f(1) = $\frac{1}{3}$, then the value of $\sum_{n=1}^{\infty} f(n)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) 3

F-3. If f(1) = 1 and f(n + 1) = 2f(n) + 1 if n ≥ 1, then f(n) is equal to

(A) $2^n + 1$ (B) 2^n (C) $2^n - 1$ (D) $2^{n-1} - 1$

F-4. If y = f(x) satisfies the condition $f\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = x + \frac{1}{x}$, (x > 0), then f(x) is equal to

(A) $-x^2 + 2$ (B) $-x^2 - 2$
(C) $x^2 - 2$, x ∈ (0, ∞) (D) $x^2 - 2$, x ∈ [2, ∞)

F-5. A function f : R → R satisfies the condition $xf(x) + f(1 - x) = x - x^2$. Then f(x) is:

(A) $\frac{x(1-x^2)}{x^2 - x + 1}$ (B) $\frac{x^2(1-x)}{x^2 + x + 1}$

(C) $\frac{x^2(1-x)}{x^2 - x + 1}$ (D) $\frac{x^2(x-1)}{x^2 - x + 1}$

F-6. If f: R → R be a differentiable function, such that f(x + 2y) = f(x) + f(2y) + 4xy ∀ x, y ∈ R. then

(A) f′(1) = f′(0) + 1 (B) f′(1) = f′(0) – 1 (C) f′(0) = f′(1) + 2 (D) f′(0) = f′(1) – 2

PART - III : MATCH THE COLUMN

1. Let [.] denotes the greatest integer function.

Column – I	**Column – II**
(A) If P(x) = [2 cos x], x ∈ [–π, π], then P(x)	(p) is discontinuous at exactly 7 points
(B) If Q(x) = [2 sin x], x ∈ [–π, π], then Q(x)	(q) is discontinuous at exactly 4 points
(C) If R(x) = [2 tan x/2], x ∈ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then R(x)	(r) is non differentiable at some points
(D) If S(x) = $\left[3\,\text{cosec}\,\frac{x}{3}\right]$, x ∈ $\left[\frac{\pi}{2}, 2\pi\right]$, then S(x)	(s) is continuous at infinitely many values

2. **Column - I** | **Column - II**

Column - I	Column - II
(A) If f(x) is derivable at x = 3 & f'(3) = 2, then $\underset{h\to 0}{\text{Limit}}\frac{f(3+h^2)-f(3-h^2)}{2h^2}$ equals	(p) 0
(B) Let f(x) be a function satisfying the condition f(–x) = f(x) for all real x. If f'(0) exists, then its value is equal to	(q) 1
(C) For the function f(x) = $\begin{cases} \frac{x}{1+e^{1/x}}, x \neq 0 \\ 0, \ x = 0 \end{cases}$ the left hand derivative of f(x) at x = 0 is	(r) 2
(D) The number of points at which the function f(x) = max. {a – x, a + x, b}, –∞ < x < ∞, 0 < a < b cannot be differentiable is	(s) 3

Exercise-2

PART - I : SINGLE OPTION CORRECT TYPE

1. The value of $\lim\limits_{x\to 2}\dfrac{\sec^x\theta-\tan^x\theta-1}{x-2}$ is equal to

(A) $\sec^2\theta.\ \ell n\sec\theta+\tan^2\theta.\ \ell n\tan\theta$
(B) $\sec^2\theta.\ \ell n\tan\theta+\tan^2\theta.\ \ell n\sec\theta$
(C) $\sec^2\theta.\ \ell n\tan\theta-\tan^2\theta.\ \ell n\sec\theta$
(D) $\sec^2\theta.\ \ell n\sec\theta-\tan^2\theta.\ \ell n\tan\theta$

2. $\ell im_{x\to a^-}\left(\dfrac{|x|^3}{a}-\left[\dfrac{x}{a}\right]^3\right)$ (a < 0), where [x] denotes the greatest integer less than or equal to x, is equal to

(A) $-a^2-1$ (B) $-a^2+1$ (C) a^2-1 (D) $-a^2$

3. $\ell im_{n\to\infty}\cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\cos\dfrac{x}{2^3}\cos\dfrac{x}{2^4}.......\cos\dfrac{x}{2^n}$ is equal to

(A) 1 (B) -1 (C) $\dfrac{\sin x}{x}$ (D) $\dfrac{x}{\sin x}$

4. $\ell im_{\theta\to 0}\left(\left[\dfrac{m\sin\theta}{\theta}\right]+\left[\dfrac{m\tan\theta}{\theta}\right]\right)$, where [.] represents greatest integer function and $m\in Z$, is equal to

(A) 2m (B) 2m – 1 (C) 2m +1 (D) does not exist

5. $\ell im_{x\to-\infty}\dfrac{x^2\sin\left(\frac{1}{x}\right)}{\sqrt{4x^2+x+1}}$ is equal to

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) 0 (D) does not exist

6. If α and β be the roots of equation $ax^2+bx+c=0$, then $\ell im_{x\to\alpha}\left(1+ax^2+bx+c\right)^{\frac{1}{x-\alpha}}$ is equal to

(A) $a(\alpha-\beta)$ (B) $\ell n|a(\alpha-\beta)|$ (C) $e^{a(\alpha-\beta)}$ (D) $e^{a|\alpha-\beta|}$

7. $\ell im_{x\to\infty}\dfrac{e^x\left(\left(2^{x^n}\right)^{\frac{1}{e^x}}-\left(3^{x^n}\right)^{\frac{1}{e^x}}\right)}{x^n}$, $n\in N$, is equal to

(A) 0 (B) $\ell n2-\ell n3$ (C) $\ell n3-\ell n2$ (D) none of these

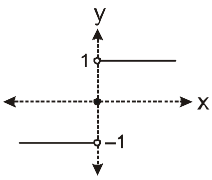
8. $\ell im_{y\to 0}\left(\ell im_{x\to\infty}\dfrac{\exp\left(x\,\ell n\left(1+\frac{11y}{x}\right)\right)-\exp\left(x\,\ell n\left(1+\frac{5y}{x}\right)\right)}{y}\right)$ is equal to

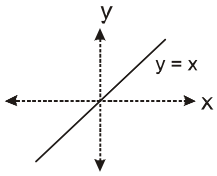
(A) 16 (B) –16 (C) 6 (D) –6

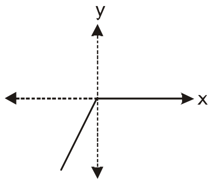
9. The value of $\ell im_{x\to\frac{\pi}{2}}\dfrac{1-\sin x+\ell n\sin x}{\sin x-(\sin x)^{\sin x}}$ is

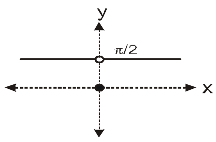
(A) 1 (B) 1/2 (C) 3 (D) $\pi/2$

10. The graph of the function $f(x) = \lim_{t\to 0}\left(\frac{2x}{\pi}\cot^{-1}\frac{x}{t^2}\right)$ is

(A) 

(B) 

(C*) 

(D) 

11. Let [x] denote the integral part of $x \in R$ and $g(x) = x - [x]$. Let f(x) be any continuous function with $f(0) = f(1)$, then the function $h(x) = f(g(x))$:

(A) has finitely many discontinuities (B) is continuous on R
(C) is discontinuous at some x = c (D) is a constant function.

12. The function f(x) is defined by $f(x) = \begin{cases} \log_{(4x-3)}(x^2-2x+5) & , \text{ if } \frac{3}{4}<x<1 \text{ or } x>1 \\ 4 & , \text{ if } x=1 \end{cases}$

(A) is continuous at x = 1
(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists
(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists
(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists.

13. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 4 & \text{if } x \text{ is rational} \end{cases}$, then:

(A) f(x) is discontinuous for all x
(B) discontinuous for all x except at x = 0
(C) discontinuous for all x except at x = 2 or – 2
(D) none of these

14. A point (x, y), where function f(x) = [sin [x]] in (0, 2π) is not continuous, is ([.] denotes greatest integer ≤ x).

(A) (3, 0) (B) (2, 0) (C) (1, 0) (D) (4, –1)

15. The function f defined by $f(x) = \lim_{t\to\infty} \cdot \left\{\frac{(1+\sin\pi x)^t - 1}{(1+\sin\pi x)^t + 1}\right\}$ is

(A) everywhere continuous (B) discontinuous at all integer values of x
(C) continuous at x = 0 (D) none of these

16. If $f(x) = \begin{cases} \sqrt{x}\left(1+x\sin\frac{1}{x}\right) & , \quad x>0 \\ -\sqrt{-x}\left(1+x\sin\frac{1}{x}\right), & \quad x<0 \\ 0 & , \quad x=0 \end{cases}$, then f(x) is

(A) continuous as well as diff. at x = 0
(B) continuous at x = 0, but not diff. at = 0
(C) neither continuous at x = 0 nor diff. at x = 0
(D) none of these

17. The functions defined by $f(x) = \max\{x^2, (x-1)^2, 2x(1-x)\}$, $0 \le x \le 1$

(A) is differentiable for all x
(B) is differentiable for all x except at one point
(C) is differentiable for all x except at two points
(D) is not differentiable at more than two points.

18. For what triplets of real numbers (a, b, c) with a $\neq$ 0 the function

$f(x)=\begin{cases} x & , \quad x\le 1 \\ ax^2+bx+c & , \quad \text{otherwise} \end{cases}$ is differentiable for all real x?

(A) $\{(a, 1-2a, a) \mid a \in R, a \neq 0\}$

(B) $\{(a, 1-2a, c) \mid a, c \in R, a \neq 0\}$

(C) $\{(a, b, c) \mid a, b, c \in R, a + b + c = 1\}$

(D) $\{(a, 1-2a, 0) \mid a \in R, a \neq 0\}$

19. [x] denotes the greatest integer less than or equal to x. If f(x) = [x] [sin πx] in (–1,1), then f(x) is:

(A) continuous at x = 0

(B) continuous in (–1, 0)

(C) differentiable in (–1,1)

(D) none

20. Let f(x) = [n + 11 sin x], x $\in$ (0, π), n $\in$ N and [x] is greatest integer less than or equal to x. The number of points at which f(x) is not differentiable is

(A) 11 (B) 10 (C) 23 (D) 21

21. Let f: R $\to$ R be any function and $g(x)=\frac{1}{f(x)}$. Then g is

(A) onto if f is onto

(B) one-one if f is one-one

(C) continuous if f is continuous

(D) differentiable if f is differentiable

22. Let f(x) = $x^3 - x^2 + x + 1$ and $g(x)=\begin{cases} \max\{f(t) \text{ for } 0\le t\le x\} & \text{for } 0\le x\le 1 \\ 3-x+x^2 & \text{for } 1<x\le 2 \end{cases}$, then:

(A) g(x) is continuous & derivable at x = 1

(B) g(x) is continuous but not derivable at x = 1

(C) g(x) is neither continuous nor derivable at x = 1

(D) g(x) is derivable but not continuous at x = 1

23. Let f : R $\to$ R be a function such that $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{3}$, f(0) = 0 and f′(0) = 3, then

(A) $\frac{f(x)}{x}$ is differentiable in R

(B) f(x) is continuous but not differentiable in R

(C) f(x) is continuous in R

(D) f(x) is bounded in R

24. If a differentiable function f satisfies $f\left(\frac{x+y}{3}\right)=\frac{4-2(f(x)+f(y))}{3}$ $\forall$ x, y $\in$ R, then f(x) is equal to

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{8}{7}$ (D) $\frac{4}{7}$

PART - II : NUMERICAL TYPE

1. Let $f(x) = \dfrac{\sin^{-1}(1-\{x\}) \,.\, \cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\ (1-\{x\})}$ and $\left(\dfrac{\lim\limits_{x\to0^+} f(x)}{\lim\limits_{x\to0^-} f(x)}\right) = P$, then P is equal to

(where {.} denotes the fractional part function)

2. Let f (x) = $x \sin\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x^2}\right), x > 0$, then $\lim\limits_{x\to\infty}$ f(x) is equal to

3. $\lim\limits_{x\to0}\left(\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}\right)$ is equal to

4. If $\lim\limits_{x\to\infty}$ f(x) exists and is finite and nonzero and $\lim\limits_{x\to\infty}\left(2f(x) + \dfrac{6f(x)-1}{4f^2(x)}\right)$ = 3, then the value of $\lim\limits_{x\to\infty}$ f(x) is equal to

5. If $f(x) = \begin{cases} x-1, & x \ge 1 \\ 2x^2-2, & x < 1 \end{cases}$, $g(x) = \begin{cases} x+1, & x > 0 \\ -x^2+1, & x \le 0 \end{cases}$ and h(x) = |x|,

then $\lim\limits_{x\to0}$ f(g(h(x))) is equal to

6. If $f(x) = \begin{cases} \sin x & , \ x \ne n\pi,\ n = 0, \pm1, \pm2, ... \\ 2 & , \ \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} x^2+1 & , x \ne 0, 2 \\ 4 & , x = 0 \\ 5 & , x = 2 \end{cases}$, then $\lim\limits_{x\to0}$ g (f(x)) is equal to

7. $\lim\limits_{n\to\infty}\left(\sum\limits_{r=0}^{2n}\dfrac{1}{\sqrt{n^2+r}}\right)$ is equal to

8. The value of $\lim\limits_{x\to0} x^4\left[\dfrac{1}{x^4}\right]$ where [.] denotes G.I.F., is

9. $\lim\limits_{x\to0}\left(\dfrac{\sin^{-1}x - \tan^{-1}x}{x^3}\right)$ is equal to

10. The value of $\left[\lim\limits_{x\to0}\dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}\right]$ where [.] denotes GIF is

11. If $\lim\limits_{x\to0}\dfrac{x^3}{\sqrt{a+x}\,(bx-\sin x)}$ = 1, then the value of $\dfrac{a}{10\,b}$ where a > 0, is

12. If $f(x) = \sum\limits_{\lambda=1}^{n}\left(x-\dfrac{5}{\lambda}\right)\left(x-\dfrac{4}{\lambda+1}\right)$, then $\lim\limits_{n\to\infty}$ f(0) is equal to

13. Let $f(x) = \begin{cases} (-1)^{[x^2]} & \text{if} \quad x < 0 \\ \ell im_{n\to\infty} \dfrac{1}{1+x^n} & \text{if} \quad x \geq 0 \end{cases}$. Then $\ell im_{x\to 0^-} f(x) + \ell im_{x\to 0^+} f(x)$ equals (where [.] represents greatest integer function)

14. If $\ell im_{x\to 0} \dfrac{e^{-nx} + e^{nx} - 2\cos\frac{nx}{2} - kx^2}{(\sin x - \tan x)}$ exists and finite (n, k $\in$ N), then the least value of $\dfrac{n^2}{k}$ is :

15. If $\ell im_{n\to\infty} \dfrac{1^2\, n + 2^2\,(n-1) + 3^2\,(n-2) + ... + n^2 . 1}{1^3 + 2^3 + 3^3 + ... + n^3}$ = P, then P is equal to

16. If $\ell im_{n\to\infty} \dfrac{n^{98}}{n^x - (n-1)^x} = \dfrac{1}{99}$, then the value of x equals

17. The number of points of discontinuity of $f(x) = \begin{cases} |4x-5|[x] & \text{for} x > 1 \\ [\cos \pi x] & \text{for} x \leq 1 \end{cases}$

(where [x] is the greatest integer not greater than x) in [0, 2] is

18. If $f(x) = \begin{cases} 4x^2 + 24x + 32, & -4 \leq x \leq -2 \\ 2 - |x| & , \quad -2 < x \leq 1 \\ 4x - x^2 - 2 & , \quad 1 < x \leq 13 \end{cases}$, then the maximum length of interval for which f(|x|) is continuous is

19. Let $f(x) = \dfrac{1-\sin x}{(\pi - 2x)^2} . \dfrac{\ell n\,(\sin x)}{\ell n\,(1 + \pi^2 - 4\pi x + 4x^2)}$, $x \neq \dfrac{\pi}{2}$. The value of $f\left(\dfrac{\pi}{2}\right)$ so that the function is continuous at x = $\dfrac{\pi}{2}$ is λ and $|\lambda|\alpha^\beta = 1$ where $\alpha, \beta \in$ N then least value of $\dfrac{\alpha}{\beta}$ is

20. If the function f(x) defined as $f(x) = \begin{cases} (\sin x + \cos x)^{\text{cosec}x} & , \quad -\dfrac{\pi}{2} < x < 0 \\ a & , \quad x = 0 \\ \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}}}{ae^{-2+\frac{1}{x}} + be^{-1+\frac{3}{x}}} & , \quad 0 < x < \dfrac{\pi}{2} \end{cases}$

is continuous at x = 0 , then the value of $\log_{e^4} a + 9b$ is :

21. The number of points of non differentiability of the function f (x) = |sin x| + sin |x| in [–4π, 4π] is

22. If $f(x) = \begin{cases} \dfrac{\sin[x^2]\pi}{x^2 - 3x + 8} + ax^3 + b & , 0 \leq x \leq 1 \\ 2\cos \pi x + \tan^{-1} x & , 1 < x \leq 2 \end{cases}$ is differentiable in [0 , 2] , then the value of $a - b + \dfrac{\pi}{4}$ is

23. If f(x) = $\begin{cases} x^2 e^{2(x-1)} & \text{for } 0 \le x \le 1 \\ a\,\text{sgn}(x+1)\cos(2x-2) + bx^2 & \text{for } 1 < x \le 2 \end{cases}$ is differentiable at x = 1 then $\frac{a^2+6}{b^2}$ =

24. Find number of points of non-differentiability of f(x) = $\lim_{n\to\infty} \frac{\{e^x\}^n - 1}{\{e^x\}^n + 1}$ in interval [0, 1] where {.} represents fractional part function

25. Let [x] denote the greatest integer less than or equal to x. The number of integral points in [–1, 1] where f(x) = [x sin πx] is differentiable are

26. Let f″(x) be continuous at x = 0 and f″(0) = 4 then value of $\lim_{x\to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is

27. Let f : R → R is a function satisfying f(10 – x) = f(x) and f(2 – x) = f(2 +x) ∀ x ∈ R. If f(0) = 101, then the minimum possible number of values of x satisfying f(x) = 101 for x ∈ [0,30] is

28. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 2048(2^n - 1)$, where the function 'f' satisfies the relation f(x + y) = f(x) . f(y) for all natural numbers x & y and further f(1) = 2

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. Let f(x) = $\frac{x^2 - 9x + 20}{x - [x]}$ (where [x] denotes greatest integer less than or equal to x), then

(A) $\lim_{x\to 5^-} f(x) = 0$

(B) $\lim_{x\to 5^+} f(x) = 1$

(C) $\lim_{x\to 5} f(x)$ does not exist

(D) none of these

2. If $\ell = \lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = \lim_{x\to 0} \frac{1+a\cos x}{x^2} - \lim_{x\to 0} \frac{b\sin x}{x^3}$, where ℓ ∈ R, then

(A) (a, b) = (–1, 0)

(B) a & b are any real numbers

(C) ℓ = 0

(D) $\ell = \frac{1}{2}$

3. If $\ell = \lim_{x\to a} \frac{\sqrt{3x^2+a^2} - \sqrt{x^2+3a^2}}{(x-a)}$ then -

(A) ℓ = 1 ∀ a ∈ ℝ (B) ℓ = 1 ∀ a > 0 (C) ℓ = –1 ∀ a < 0 (D) ℓ = D.N.E. if a = 0

4. Let f(x) = $\frac{|x+\pi|}{\sin x}$, then

(A) $f(-\pi^+) = -1$

(B) $f(-\pi^-) = 1$

(C) $\lim_{x\to -\pi} f(x)$ does not exist

(D) $\lim_{x\to \pi} f(x)$ does not exist

5. Let α, β be the roots of equation $ax^2 + bx + c = 0$, where 1 < α < β and $\lim_{x\to x_0} \frac{|ax^2+bx+c|}{ax^2+bx+c} = 1$, then which of the following statements is correct

(A) a > 0 and $x_0 < 1$

(B) a > 0 and $x_0 > \beta$

(C) a < 0 and $\alpha < x_0 < \beta$

(D) a < 0 and $x_0 < 1$

6. $\lim\limits_{x\to 0}\left[\left(1-e^{x}\right)\frac{\sin x}{|x|}\right]$, where [·] represents greatest integer function, is equal to

(A) – 1 (B) 1 (C) $\log_{\sqrt{2}+1}(\sqrt{2}-1)$ (D) does not exist

7. Given a real valued function f such that

$$f(x)=\begin{cases}\dfrac{\tan^2[x]}{(x^2-[x]^2)}, & x>0\\ 1, & x=0\\ \sqrt{\{x\}\cot\{x\}}, & x<0\end{cases}$$

where [.] represents greatest integer function and {.} represents fractional part function, then

(A) $\lim\limits_{x\to 0} f(x) = 1$

(B) $\lim\limits_{x\to 0^-} f(x) = \sqrt{\cot 1}$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$

(D) $\lim\limits_{x\to 0^+} f(x) = 0$

8. If $\lim\limits_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$ = p (finite), then

(A) a = – 2 (B) a = – 1 (C) p = – 2 (D) p = – 1

9. $\lim\limits_{x\to\infty}\frac{(ax+1)^n}{x^n+A}$ is equal to

(A) a^n if $n \in N$

(B) ∞ if $n \in Z^-$ & a = A = 0

(C) $\frac{1}{1+A}$ if n = 0

(D) a^n if $n \in Z^-$, A = 0 & $a \neq 0$

10. If $\ell = \lim\limits_{x\to\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})$ and $m = \lim\limits_{x\to -\infty}[\sin\sqrt{x+1}-\sin\sqrt{x}]$, where [.] denotes the greatest integer function, then :

(A) ℓ = 0

(B) m = 0

(C) m is undefined

(D) ℓ is undefined

11. Let $\lim\limits_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4} = \ell$ and $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}$ = m, then

(A) $\ell = \frac{-1}{6}$ (B) $m = \frac{1}{6}$ (C) ℓ + m = 0 (D) ℓ = m

12. If f(x) = $|x|^{\sin x}$, then

(A) $\lim\limits_{x\to 0^-} f(x) = 1$

(B) $\lim\limits_{x\to 0^+} f(x) = 1$

(C) $\lim\limits_{x\to 0} f(x) = 1$

(D) limit does not exist at x = 0

13. If $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{\frac{1}{x}} = e^2$, then the possible values of 'a' & 'b' are :

(A) a = 1 , b = 2 (B) a = 2 , b = 1 (C) a = 3, b = 2/3 (D) a = 2/3 , b = 3

14. $\lim\limits_{x\to 0^+}\log_{\sin\frac{x}{2}}\sin x$ is equal to

(A) 1 (B) 0 (C) $\lim\limits_{x\to 0} x^{\sin x}$ (D) $\lim\limits_{x\to 0^+}(\tan x)^{\sin x}$

15. $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$, $n \in$ integer number, is true for

(A) no value of n
(B) all values of n
(C) negative values of n
(D) positive values of n

16. If $f(x) = \underset{n\to\infty}{\text{Limit}} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n}+1}$ $(n \in N)$, then

(A) $\lim_{x\to 1^+} f(x) = -\sin 1$
(B) $\lim_{x\to 1^-} f(x) = \log 3$
(C) $\lim_{x\to 1} f(x) = \sin 1$
(D) $f(1) = \frac{\log 3 - \sin 1}{2}$

17. Which of the following function(s) defined below has/have single point continuity.

(A) $f(x) = \begin{bmatrix} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$
(B) $g(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{bmatrix}$
(C) $h(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$
(D) $k(x) = \begin{bmatrix} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{bmatrix}$

18. The function $f(x) = \begin{cases} |x-3| & ,x \geq 1 \\ \left(\frac{x^2}{4}\right) - \left(\frac{3x}{2}\right) + \left(\frac{13}{4}\right) & ,x < 1 \end{cases}$ is:

(A) continuous at x = 1
(B) differentiable at x = 1
(C) continuous at x = 3
(D) differentiable at x = 3

19. If $f(x) = \frac{1}{2}x - 1$, then on the interval $[0, \pi]$

(A) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both continuous

(B) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both discontinuous

(C) $\tan(f(x))$ and $f^{-1}(x)$ are both continuous

(D) $\tan(f(x))$ is continuous but $\frac{1}{f(x)}$ is not.

20. Let f(x) and g(x) be defined by f(x) = [x] and $g(x) = \begin{cases} 0 & , x \in I \\ x^2 & , x \in R - I \end{cases}$ (where [.] denotes the greatest integer function), then

(A) $\lim_{x\to 1} g(x)$ exists, but g is not continuous at x = 1

(B) $\lim_{x\to 1} f(x)$ does not exist and f is not continuous at x = 1

(C) gof is continuous for all x
(D) fog is continuous for all x

21. Let $f(x) = [x] + \sqrt{x - [x]}$, where [.] denotes the greatest integer function. Then

(A) f(x) is continuous on R^+
(B) f(x) is continuous on R
(C) f(x) is continuous on R – I
(D) discontinuous at x = 1

22. Which of the following function(s) is/are discontinuous at x = 0 ?

(A) $f(x) = \sin\frac{\pi}{2x}$, $x \neq 0$ and $f(0) = 1$

(B) $g(x) = x\ \sin\left(\frac{\pi}{x}\right)$, $x \neq 0$ and $g(0) = \pi$

(C) $h(x) = \frac{|x|}{x}$, $x \neq 0$ and $h(0) = 1$

(D) $k(x) = \frac{1}{1+e^{\cot x}}, x \neq 0$ and $k(0) = 0$.

23. A function $f(x)$ is defined as $f(x) = \frac{A\sin x + \sin 2x}{x^3}, (x \neq 0)$. If the function is continuous at x = 0, then -

(A) A = –2 (B) $f(0) = -1$ (C) A = 1 (D) $f(0) = 1$

24. In which of the following cases the given equations has atleast one root in the indicated interval ?

(A) x – cos x = 0 in (0, π/2)

(B) x + sinx = 1 in (0, π/6)

(C) $\frac{a}{x-1} + \frac{b}{x-3} = 0$, a, b > 0 in (1, 3)

(D) f(x) – g(x) = 0 in (a, b) where f and g are continuous on [a, b] and f(a) > g(a) and f(b) < g(b).

25. The points at which the function, $f(x) = |x - 0.5| + |x - 1| + \tan x$ does not have a derivative in the interval (0, 2) are:

(A) 1 (B) π/2 (C) π/4 (D) 1/2

26. $f(x) = (\sin^{-1}x)^2 . \cos(1/x)$ if $x \neq 0$; $f(0) = 0$, f(x) is:

(A) continuous no where in $-1 \leq x \leq 1$

(B) continuous everywhere in $-1 \leq x \leq 1$

(C) differentiable no where in $-1 \leq x \leq 1$

(D) differentiable everywhere in $-1 < x < 1$

27. Which of the following statements is/are correct ?

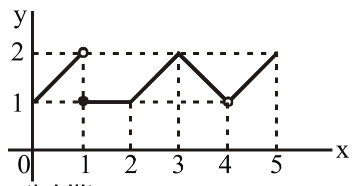
(A) There exist a function $f : [0,1] \to \mathbb{R}$ which is discontinuous at every point in [0,1] & $|f(x)|$ is continuous at every point in [0,1]

(B) Let $F(x) = f(x). g(x)$. If $f(x)$ is differentiable at x = a , $f(a) = 0$ and g(x) is continuous at x = a then F(x) is always differentiable at x = a.

(C) If $Rf'(a)$ = 2 & $Lf'(a)$ = 3, then $f(x)$ is non-differentiable at x = a but will be always continuous at x = a

(D) If $f(a)$ and $f(b)$ possess opposite signs then there must exist at least one solution of the equation $f(x) = 0$ in (a,b) provided f is continuous on [a,b]

28. Graph of $f(x)$ is shown in adjacent figure, then in [0, 5]



(A) $f(x)$ has non removable discontinuity at two points

(B) $f(x)$ is non differentiable at three points in its domain

(C) $\lim\limits_{x\to 1} f(f(x)) = 1$

(D) Number of points of discontinuity = number of points of non-differentiability

29. If $f(x) = a_0 + \sum\limits_{k=1}^{n} a_k |x|^k$, where a_i's are real constants, then f(x) is

(A) continuous at x = 0 for all a_i

(B) differentiable at x = 0 for all $a_i \in R$

(C) differentiable at x = 0 for all $a_{2k-1} = 0$

(D) none of these

30. Let $f : R \to R$ be a function such that f(0) = 1 and for any x, y $\in$ R, f(xy + 1) = f(x) f(y) – f(y) – x + 2. Then f is

(A) one-one (B) onto (C) many one (D) into

31. Suppose that f is a differentiable function with the property that f(x + y) = f(x) + f(y) + xy and

$\lim\limits_{h\to 0} \frac{1}{h} f(h) = 3$

where [.] represents greatest integer function, then

(A) f is a linear function

(B) $2f(1) = \left[\lim\limits_{x\to 0}(1+2x)^{1/x}\right]$

(C) $f(x) = 3x + \frac{x^2}{2}$

(D) f '(1) = 4

PART - IV : COMPREHENSION

Comprehension # 1

Consider $f(x) = \frac{\sin x + ae^x + be^{-x} + c\,\ell n(1+x)}{x^3}$, where a, b, c are real numbers.

1. If $\lim\limits_{x\to 0^+} f(x)$ is finite, then the value of a + b + c is

(A) 0 (B) 1 (C) 2 (D) – 2

2. If $\lim\limits_{x\to 0^+} f(x) = \ell$ (finite), then the value of ℓ is

(A) – 2 (B) $-\frac{1}{2}$ (C) – 1 (D) $-\frac{1}{3}$

3. Using the values of a, b, c as found in Q.No. 1 or Q. No.2 above, the value of $\lim\limits_{x\to 0^+} x f(x)$ is

(A) 0 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 2

Comprehension # 2

If both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist finitely and are equal , then the function f is said to have removable discontinuity at x = c

If both the limits i.e. $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist finitely and are not equal, then the function f is said to have non-removable discontinuity at x = c and in this case $\left|\lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x)\right|$ is called jump of the discontinuity.

4. Which of the following function has non-removable discontinuity at the origin ?

(A) $f(x) = \dfrac{1}{\ell n|x|}$

(B) $f(x) = x\sin\dfrac{\pi}{x}$

(C) $f(x) = \dfrac{1}{1+2^{\cot x}}$

(D) $f(x) = \cos\left(\dfrac{|\sin x|}{x}\right)$

5. Which of the following function not defined at x = 0 has removable discontinuity at the origin ?

(A) $f(x) = \dfrac{1}{1+2^{\frac{1}{x}}}$

(B) $f(x) = \tan^{-1}\dfrac{1}{x}$

(C) $f(x) = \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

(D) $f(x) = \dfrac{1}{\ell n|x|}$

6. If $f(x) = \begin{bmatrix} \tan^{-1}(\tan x); & x \le \dfrac{\pi}{4} \\ \pi[x]+1 \quad ; & x > \dfrac{\pi}{4} \end{bmatrix}$, then jump of discontinuity is

(where [.] denotes greatest integer function)

(A) $\dfrac{\pi}{4} - 1$

(B) $\dfrac{\pi}{4} + 1$

(C) $1 - \dfrac{\pi}{4}$

(D) $-1 - \dfrac{\pi}{4}$

Comprehension # 3

Let $f(x) = \begin{cases} x\,g(x) & , \quad x \le 0 \\ x + ax^2 - x^3 & , \quad x > 0 \end{cases}$, where $g(t) = \lim_{x\to 0} (1 + a\tan x)^{t/x}$, a is positive constant, then

7. If a is even prime number, then g(2) =

(A) e^2 (B) e^3 (C) e^4 (D) none of these

8. Set of all values of a for which function f(x) is continuous at x = 0

(A) (–1, 10) (B) $(-\infty, \infty)$ (C) $(0, \infty)$ (D) none of these

9. If f(x) is differentiable at x = 0, then a $\in$

(A) (–5, –1) (B) (–10, 3) (C) $(0, \infty)$ (D) none of these

Comprehension # 4

Let $f : R \to R$ be a function defined as,

$$f(x) = \begin{cases} 1-|x| & , \ |x| \le 1 \\ 0 & , \ |x| > 1 \end{cases} \text{ and } \ g(x) = f(x-1) + f(x+1), \ \forall \ x \in R. \text{ Then}$$

10. The value of g(x) is :

(A) $g(x) = \begin{cases} 0 & , & x \le -3 \\ 2+x & , & -3 \le x \le -1 \\ -x & , & -1 < x \le 0 \\ x & , & 0 < x \le 1 \\ 2-x & , & 1 < x \le 3 \\ 0 & , & x > 3 \end{cases}$

(B) $g(x) = \begin{cases} 0 & , & x \le -2 \\ 2+x & , & -2 \le x \le -1 \\ -x & , & -1 < x \le 0 \\ x & , & 0 < x \le 1 \\ 2-x & , & 1 < x \le 2 \\ 0 & , & x > 2 \end{cases}$

(C) $g(x) = \begin{cases} 0 & , & x \le 0 \\ 2+x & , & 0 < x < 1 \\ -x & , & 1 \le x \le 2 \\ x & , & 2 < x < 3 \\ 2-x & , & 3 \le x < 4 \\ 0 & , & 4 \le x \end{cases}$

(D) none of these

11. The function g(x) is continuous for, $x \in$

(A) $R - \{0, 1, 2, 3, 4\}$

(B) $R - \{-2, -1, 0, 1, 2\}$

(C) R

(D) none of these

12. The function g(x) is differentiable for, $x \in$

(A) R

(B) $R - \{-2, -1, 0, 1, 2\}$

(C) $R - \{0, 1, 2, 3, 4\}$

(D) none of these

Exercise-3

PART - I : PREVIOUS YEARS JEE ADVANCE PROBLEMS

*** Marked Questions may have more than one correct option.**

1. If $\lim_{x\to 0}\left[1+x\ln(1+b^2)\right]^{\frac{1}{x}} = 2b\sin^2\theta$, b > 0 and $\theta \in (-\pi, \pi]$, then the value of θ is

[IIT-JEE 2011, Paper-2, (3, –1), 80]

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

2*. Let f : **R** → **R** be a function such that f(x + y) = f(x) + f(y), ∀ x, y ∈ **R**. If f(x) is differentiable at x = 0, then

(A) f(x) is differentiable only in a finite interval containing zero

(B) f(x) is continuous ∀ x ∈ **R**

(C) f′(x) is constant ∀x ∈ **R**

(D) f(x) is differentiable except at finitely many points

3*. If $f(x) = \begin{cases} -x-\frac{\pi}{2} & , \quad x \le -\frac{\pi}{2} \\ -\cos x & , \quad -\frac{\pi}{2} < x \le 0 \\ x-1 & , \quad 0 < x \le 1 \\ \ell n\, x & , \quad x > 1 \end{cases}$, then **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

(A) f(x) is continuous at $x = -\frac{\pi}{2}$

(B) f(x) is not differentiable at x = 0

(C) f(x) is differentiable at x = 1

(D) f(x) is differentiable at $x = -\frac{3}{2}$

4. Let f : (0, 1) → **R** be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that 0 < b < 1. Then

(A) f is not invertible on (0, 1)

(B) $f \ne f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0, 1)

5. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$, then **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) a = 1, b = 4

(B) a = 1, b = –4

(C) a = 2, b = –3

(D) a = 2, b = 3

6. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2 + \left(\sqrt{1+a}-1\right)x + \left(\sqrt[6]{1+a}-1\right) = 0$ where a > –1.

Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and –1 (C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

7. Let $f(x) = \begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $x \in \mathbb{R}$, then f is

(A) differentiable both at x = 0 and at x = 2 **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(B) differentiable at x = 0 but not differentiable at x = 2

(C) not differentiable at x = 0 but differentiable at x = 2

(D) differentiable neither at x = 0 nor at x = 2

8*. For every integer n, let a_n and b_n be real numbers. Let function $f : \mathbb{R} \to \mathbb{R}$ be given by

$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}$, for all integers n.

If f is continuous, then which of the following hold(s) for all n ? **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$ (C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

9*. For every pair of continuous functions $f, g : [0, 1] \to R$ such that

max {f(x) : x ∈ [0,1]} = max {g(x) : x ∈ [0, 1]}, **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

the correct statement(s) is (are) :

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some c ∈ [0, 1]

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some c ∈ [0, 1]

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some c ∈ [0, 1]

(D) $(f(c))^2 = (g(c))^2$ for some c ∈ [0, 1]

10. The largest value of the non-negative integer a for which $\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$ is

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

11. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$

by $h(x) = \begin{cases} \max \{f(x), g(x)\} & \text{if } x \le 0, \\ \min \{f(x), g(x)\} & \text{if } x > 0. \end{cases}$ The number of points at which h(x) is not differentiable is

12. Let $f_1 : R \to R$, $f_2 : [0, \infty) \to R$, $f_3 : R \to R$ and $f_4 : R \to [0, \infty)$ be defined by

$f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \ge 0; \end{cases}$ **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

$f_2(x) = x^2$;

$f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \ge 0 \end{cases}$

and

$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \ge 0 \end{cases}$

List I		List II	
P.	f_4 is	**1.**	onto but not one-one
Q.	f_3 is	**2.**	neither continuous nor one-one
R.	$f_2 o f_1$ is	**3.**	differentiable but not one-one
S.	f_2 is	**4.**	continuous and one-one

13*. Let g: R $\to$ R be a differentiable function with g(0) = 0, g'(0) = 0 and g'(1) $\neq$ 0. Let $f(x) = \begin{cases} \frac{x}{|x|}g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

and h(x) = $e^{|x|}$ for all x $\in$ R. Let (foh)(x) denote f(h(x)) and (hof)(x) denote h(f(x)). Then which of the following is(are) true?

(A) f is differentiable at x = 0
(B) h is differentiable at x = 0
(C) foh is differentiable at x = 0
(D) hof is differentiable at x = 0

14*. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all x $\in$ R and $g(x) = \frac{\pi}{2}\sin x$ for all x $\in$ R. Let (fog) (x) dentoe f(g(x)) and (gof) (x) denote g(f(x)). Then which of the following is(are)true?

(A) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(B) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(C) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)} = \frac{\pi}{6}$
(D) There is an x $\in$ R such that (gof)(x) = 1

15. Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$, then the value of $\frac{m}{n}$ is

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

16. Let α, β $\in \mathbb{R}$ be such that $\lim\limits_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then 6($\alpha$ + β) equals

[JEE(Advanced)-2016, 3]

17*. Let $f:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ and $g:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ be function defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|\,f(x) + |4x-7|\,f(x)$, where [y] denotes the greatest integer less than or equal to y for y $\in \mathbb{R}$. Then **[JEE(Advanced)-2016, 4(–2)]**

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

18*. Let a, b $\in \mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is -

(A) differentiable at x = 0 if a = 0 and b = 1
(B) differentiable at x = 1 if a = 1 and b = 0
(C) **NOT** differentiable at x = 0 if a = 1 and b = 0
(D) **NOT** differentiable at x = 1 if a = 1 and b = 1 **[JEE(Advanced)-2016, 4(–2)]**

19*. Let $f(x) = \frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for x $\neq$ 1. Then **[JEE(Advanced)-2017, 4]**

(A) $\lim_{x\to 1^+} f(x)$ does not exist
(B) $\lim_{x\to 1^-} f(x)$ does not exist
(C) $\lim_{x\to 1^-} f(x) = 0$
(D) $\lim_{x\to 1^+} f(x) = 0$

20*. Let [x] be the greatest integer less than or equal to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous ? **[JEE(Advanced)-2017, 4]**

(A) x = –1 (B) x = 0 (C) x = 2 (D) x = 1

21 For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4]**

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} (1+f_j'(0))\sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

22. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1-e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in \mathbb{R}$, [t] denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ **[JEE(Advanced)-2018, 3(–1)]**

	List-I		List-II
P.	The function f_1 is	**1.**	**NOT** continuous at x = 0
Q.	The function f_2 is	**2.**	continuous at x = 0 and **NOT** differentiable at x = 0
R.	The function f_3 is	**3.**	differentiable at x = 0 and its derivative is **NOT** continuous at x = 0
S.	The function f_4 is	**4.**	differentiable at x = 0 and its derivative is continuous at x = 0

The correct option is :

(A) P → 2; Q → 3, R → 1; S → 4 (B) P → 4; Q → 1; R → 2; S → 3

(C) P → 4; Q → 2, R → 1; S → 3 (D) P → 2; Q → 1; R → 4; S → 3

23. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

24*. Let $F : \mathbb{R} \to \mathbb{R}$ be a function. We say that f has

PROPERTY 1 if $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(A) f(x) = x|x| has PROPERTY 2 (B) $f(x) = x^{2/3}$ has PROPERTY 1

(C) f(x) = sin x has PROPERTY 2 (D) f(x) = |x| has PROPERTY 1

PART - II : PREVIOUS YEARS AIEEE & JEE MAINS PROBLEMS

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a positive increasing function with $\lim\limits_{x\to\infty} \dfrac{f(3x)}{f(x)} = 1$. Then $\lim\limits_{x\to\infty} \dfrac{f(2x)}{f(x)}$.

[AIEEE– 2010, (8, –2), 144]

(1) $\dfrac{2}{3}$ (2) $\dfrac{3}{2}$ (3) 3 (4) 1

2. $\lim\limits_{x\to 2} \left(\dfrac{\sqrt{1-\cos\{2(x-2)\}}}{x-2} \right)$ **[AIEEE– 2011, I, (4, –1), 120]**

(1) does not exist (2) equals $\sqrt{2}$ (3) equals $-\sqrt{2}$ (4) equals $\dfrac{1}{\sqrt{2}}$

3. Let $f : R \to [0,\infty)$ be such that $\lim\limits_{x\to 5} f(x)$ exists and $\lim\limits_{x\to 5} \dfrac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$ **[AIEEE– 2011, II,(4, –1), 120]**

Then $\lim\limits_{x\to 5} f(x)$ equals :

(1) 0 (2) 1 (3) 2 (4) 3

4. The value of p and q for which the function $f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x} & , \quad x < 0 \\ q & , \quad x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & , \quad x > 0 \end{cases}$ is continuous for all x in R, are:

[AIEEE 2011, I,(4, –1), 120]

(1) $p = \dfrac{1}{2}, q = -\dfrac{3}{2}$ (2) $p = \dfrac{5}{2}, q = \dfrac{1}{2}$

(3) $p = -\dfrac{3}{2}, q = \dfrac{1}{2}$ (4) $p = \dfrac{1}{2}, q = \dfrac{3}{2}$

5. Define F(x) as the product of two real functions $f_1(x) = x$, $x \in R$, and $f_2(x) = \begin{cases} \sin\frac{1}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$

as follows :

$F(x) = \begin{cases} f_1(x).f_2(x), & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$ **[AIEEE 2011, II,(4, –1), 120]**

Statement - 1 : F(x) is continuous on R.

Statement - 2 : $f_1(x)$ and $f_2(x)$ are continuous on R.

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(2) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1

(3) Statement-1 is true, Statement-2 is false

(4) Statement-1 is false, Statement-2 is true

6. If function f(x) is differentiable at x = a, then $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$ is : **[AIEEE 2011, II,(4, –1), 120]**

(1) $-a^2 f'(a)$ (2) $af(a) - a^2 f'(a)$ (3) $2af(a) - a^2 f'(a)$ (4) $2af(a) + a^2 f'(a)$

7. If $f : R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where [x] denotes the greatest integer function, then f is : **[AIEEE- 2012, (4, –1), 120]**

(1) continuous for every real x.

(2) discontinuous only at x = 0.

(3) discontinuous only at non-zero integral values of x.

(4) continuous only at x = 0.

8. Consider the function, $f(x) = |x - 2| + |x - 5|$, $x \in R$. **[AIEEE- 2012, (4, –1), 120]**

Statement-1 : $f'(4) = 0$

Statement-2 : f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5).

(1) Statement-1 is false, Statement-2 is true.

(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.

(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.

(4) Statement-1 is true, statement-2 is false.

9. $\lim_{x \to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to **[AIEEE - 2013, (4, –¼),360]**

(1) $-\frac{1}{4}$ (2) $\frac{1}{2}$ (3) 1 (4) 2

10. $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to : **[JEE(Main) 2014, (4, – ¼), 120]**

(1) $-\pi$ (2) π (3) $\pi/2$ (4) 1

11. $\lim_{x \to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to **[JEE(Main) 2015, (4, – ¼), 120]**

(1) 4 (2) 3 (3) 2 (4) $\frac{1}{2}$

12. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of k+ m is;

[JEE(Main) 2015, (4, – ¼), 120]

(1) 2 (2) $\frac{16}{5}$ (3) $\frac{10}{3}$ (4) 4

13. Let $p=\lim_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to - **[JEE(Main)-2016]**

(1) $\frac{1}{4}$ (2) 2 (3) 1 (4) $\frac{1}{2}$

14. For x ∈ R, f(x) = |log2 – sinx| and g(x) = f(f(x)), then : **[JEE(Main)-2016]**

(1) g is differentiable at x = 0 and g'(0) = –sin(log2)

(2) g is not differentiable at x = 0

(3) g'(0) = cos(log2)

(4) g'(0) = –cos(log2)

15. $\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$ equals :- **[JEE(Main)-2017]**

(1) $\frac{1}{4}$ (2) $\frac{1}{24}$ (3) $\frac{1}{16}$ (4) $\frac{1}{8}$

16. For each t ∈ R, let [t] be the greatest integer less than or equal to t. Then

$\lim_{x\to 0+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+....+\left[\frac{15}{x}\right]\right)$ **[JEE(Main)-2018]**

(1) is equal to 15. (2) is equal to 120.

(3) does not exist (in R) (4) is equal to 0.

17. Let S = {t ∈ R : f(x) = |x – π|·($e^{|x|}$ – 1) sin|x| is not differentiable at t}. Then the set S is equal to: **[JEE(Main)-2018]**

(1) {0} (2) {π} (3) {0, π} (4) ϕ (an empty set)

18. For each t ∈ R, let [t] be the greatest integer less than or equal to t.

Then, $\lim_{x\to 1+}\frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$ **[JEE(Main)-Jan 19]**

(1) equals –1 (2) equals 1 (3) does not exist (4) equals 0

19. Let f be a differentiable function from R to R such that $\left|f(x)-f(y)\right|\le 2\left|x-y\right|^{\frac{3}{2}}$, for all x, y ∈ R.

If $f(0)$ = 1 then $\int_0^1 f^2(x)dx$ is equal to **[JEE(Main)-Jan 19]**

(1) 0 (2) $\frac{1}{2}$ (3) 2 (4) 1

20. Let $f(x)=\begin{cases}\max\{|x|,x^2\}, & |x|\le 2\\ 8-2|x|, & 2<|x|\le 4\end{cases}$. Let S be the set of points in the interval (–4,4) at which f is not differentiable. Then S : **[JEE(Main)-Jan 19]**

(1) is an empty set (2) equals {–2, –1, 1, 2}

(3) equals {–2, –1, 0, 1, 2} (4) equals {–2, 2}

21. Let f : R $\rightarrow$ R be a differentiable function satisfying $f'(3) + f'(2) = 0$.

Then $\lim_{x\to 0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}}$ is equal to **[JEE(Main)-Apr 19]**

(1) e^2 (2) e (3) e^{-1} (4) 1

22. If $\lim_{x\to 1}\frac{x^2-ax+b}{x-1}=5$, then a + b is equal to :- **[JEE(Main)-Apr 19]**

(1) –7 (2) – 4 (3) 5 (4) 1

23. If $f(x) = \begin{cases} \frac{\sin(p+1)+\sin x}{x} & , \ x<0 \\ q & , \ x=0 \\ \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}} & , \ x>0 \end{cases}$ is continuous at x = 0, then the ordered pair (p,q) is equal to :

[JEE(Main)-Apr 19]

(1) $\left(\frac{5}{2},\frac{1}{2}\right)$ (2) $\left(-\frac{3}{2},-\frac{1}{2}\right)$ (3) $\left(-\frac{1}{2},\frac{3}{2}\right)$ (4) $\left(-\frac{3}{2},\frac{1}{2}\right)$

24. Let $f(x) = 15 - |x - 10|$; x $\in$ R. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is : **[JEE(Main)-Apr 19]**

(1) {5,10,15,20} (2) {10,15} (3) {5,10,15} (4) {10}

25. If $f(x)=\begin{cases} \frac{\sin(a+2)x+\sin x}{x} & ; \ x<0 \\ b & ; \ x=0 \\ \frac{(x+3x^2)^{\frac{1}{3}}-x^{-\frac{1}{3}}}{x^{\frac{4}{3}}} & ; \ x>0 \end{cases}$ **[JEE(Main)-Jan 20]**

is continuous at x = 0, then a + 2b is equal to :

(1) –1 (2) 1 (3) –2 (4) 0

26. Let [t] denote the greatest integer $\leq$ t and $\lim_{x\to 0} x\left[\frac{4}{x}\right]=A$. Then the function, $f(x) = [x^2]\sin(\pi x)$ is discontinuous, when x is equal to : **[JEE(Main)-Jan 20]**

(1) $\sqrt{A+5}$ (2) $\sqrt{A+1}$

(3) $\sqrt{A}$ (4) $\sqrt{A+21}$

ANSWER KEY

Exercise-1

PART - I : SUBJECTIVE QUESTIONS

SECTION - (A)

A-1. (i) $4 + \cos 2$ (ii) $\tan 3 + 8$ (iii) $\frac{3}{4}\cos\frac{3}{4}$ (iv) $(\ell n 5)^5$

(v) $\frac{e}{\sin 1}$

A-2. (i) 2 (ii) 2 (iii) Limit does not exist (iv) 2
(v) 2

A-3 (i) 0 (ii) Limit does not exist

(iii) Limit does not exist (iv) $\frac{\pi}{2}$

A-4. (i) 4 (ii) $\lambda = 4$

A-5 12

A-6. (i) No (ii) No (iii) Yes, ∞^0 form (iv) No

SECTION - (B)

B-1. (i) –1 (ii) $\frac{12}{19}$ (iii) $\frac{2}{3\sqrt{3}}$

B-2. (i) $\frac{3}{2}$ (ii) 25 (iii) $\frac{2}{\ell n 2}$ (iv) $(b - a)$

(v) $\frac{25}{16}$ (vi) $2e^2$ (vii) 2 (viii) $\frac{1}{2}$

(ix) $2a \sin a + a^2 \cos a$ (x) limitdoes not exist. (xi) $-\frac{1}{3}$ (xii) 5

B-3. (i) 3 (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$ (iv) 1

(v) –2

B-4 (i) $\frac{1}{2}$ (ii) 1

B-5. (i) limit does not exist (ii) 4 (iii) 0 (iv) 5/2

B-6. (i) $\frac{3}{2}(a+2)^{1/2}$ (ii) $-\frac{2}{25}$ (iii) $\frac{1}{3}$ (iv) $-\frac{9}{4}\ell n\frac{4}{e}$

B-7 $a = 2, b = 1, c = -1$ and limit $= -\frac{1}{3}$

B-8 (i) $a = \frac{1}{2}, b = -1$ (ii) $a = 2, b \in R, c = 5, d \in R$ (iii) $a = 3, b = 12, c = 9$

B-9. $\frac{1}{2}$

B-11. $2(\sec^2 a)\tan a$

SECTION- (C)

C-1 (i) 1 (ii) 1 (iii) 0 (iv) Limit does not exist

C-2 (i) e^{-1} (ii) 0 (iii) e^{-2} (iv) e^{2}

C-3 a = b and bc = –3

C-4. $\frac{x}{3}$

C-5. {–1, 0, 1}

C-6. $\pi - 3$

SECTION- (D)

D-1. $a = \frac{1}{2}$, $b \neq 0$, $c = \frac{1}{2}$

D-2. $a = \frac{3}{2}$, b = 1 and c = 12

D-3. (a) –2, 2, 3 (b) K = 5 (c) even

D-4. (i) continuous at x = 1 (ii) continuous (iii) discontinuous (iv) continuous at x = 1, 2

D-5. $A = \frac{-4}{5}$, $B = \frac{1}{5}$, $f(0) = \frac{1}{5}$

D-6. continuous at x = 1 but discontinuous at x = 2

D-7. (i) $x \in R$ (ii) $x \in R - \{-3, -2, 2, 3\}$ (iii) $x \in R$ (iv) $x \in R - \{(2n + 1), n \in I\}$

D-8. 30

D-9. discontinuous at $2m\pi$, $(2n + 1)\frac{\pi}{2}$, m, $n \in I$

D-10. $-\frac{7}{3}$, – 2, 0

D-11. f(f(x)) is discontinuous at x = 1 & x = 2, g(f(x)) is continuous $\forall\, x \in R$

D-12. f(x) is continuous and $\frac{7}{3} \in [f(-2), f(2)]$, by intermediate value theorem (IVT), there exists a point

$c \in (-2, 2)$ such that $f(c) = \frac{7}{3}$

D-13. 2

SECTION - (E)

E-1 Continuous at both points but differentiable only at x = 2

E-2 continuous but not differentiable at x = 0; differentiable & continuous at $x = \pi/2$

E-5. not differentiable at x = 0

E-6. a = –1, b = –3

E-7. Continuous $\forall\, x \in R$ but not differentiable at x = 0, 1 & 2

E-8. f is continuous at x = 1, 3/2 & discontinuous at x = 2, f is not differentiable at x = 1,3/2, 2

E-9. discontinuous and non-differentiable at –1, 0, 1,continuous but non-differentiable at x = 4

E-10. non-differentiable at x = 1

E-11. Continuos everywhere in (0, 3) but non differentiable at x = 2

SECTION- (F)

F-1. 4

F-2. $f(x) = 2x + c$

F-3. $f(x) = e^{xf'(0)}\ \forall\ x \in R$

F–4. 5

F-7. 100

PART - II : SINGLE OPTION CORRECT TYPE

SECTION - (A)

A-1. (C) **A-2** (A) **A-3** (C) **A-4** (D)

SECTION - (A)

B-1. (C) **B-2.** (B) **B-3.** (D) **B-4.** (D) **B-5.** (D) **B-6.** (D)

B-7. (B) **B-8.** (B) **B-9.** (A) **B-10.** (B) **B-11.** (B) **B-12.** (A)

B-13. (A) **B-14** (A) **B-15.** (A) **B-16.** (C) **B-17.** (A) **B-18.** (B)

B-19. (D) **B-20** (B)

SECTION - (C)

C-1. (C) **C-2.** (D) **C-3.** (B) **C-4.** (C) **C-5.** (A) **C-6.** (C)

C-7. (C) **C-8.** (B)

SECTION - (D)

D-1. (A) **D-2.** (D) **D-3.** (D) **D-4.** (B) **D-5.** (A) **D-6.** (C)

D-7. (B) **D-8.** (B)

SECTION - (E)

E-1. (B) **E-2.** (B) **E-3.** (B) **E-4.** (A) **E-5.** (D) **E-6.** (D)

E-7. (B) **E-8.** (C) **E-9.** (B) **E-10.** (A) **E-11.** (A) **E-12.** (C)

E-13. (C) **E-14.** (A)

SECTION - (F)

F-1. (A) **F-2.** (B) **F-3.** (C) **F-4.** (D) **F-5.** (C) **F-6.** (D)

PART - III : MATCH THE COLUMN

1. (A) $\rightarrow$ (p, r, s), (B) $\rightarrow$ (p, r, s), (C) $\rightarrow$ (q, r, s), (D) $\rightarrow$ (r, s)

2. (A) $\rightarrow$ (r), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)

Exercise-2

PART - I : SINGLE OPTION CORRECT TYPE

1.	(D)	**2.**	(A)	**3.**	(C)	**4.**	(B)	**5.**	(B)	**6.**	(C)
7.	(B)	**8.**	(C)	**9.**	(B)	**10.**	(C)	**11.**	(B)	**12.**	(D)
13.	(C)	**14.**	(D)	**15.**	(B)	**16.**	(B)	**17.**	(C)	**18.**	(A)
19.	(B)	**20.**	(D)	**21.**	(B)	**22.**	(C)	**23.**	(C)	**24.**	(D)

PART - II : NUMERICAL TYPE

1.	01.41	**2.**	1	**3.**	1.5	**4.**	0.5	**5.**	0	**6.**	1
7.	2	**8.**	1	**9.**	0.5	**10.**	1	**11.**	3.6	**12.**	20
13.	2	**14.**	0.8	**15.**	0.33	**16.**	99	**17.**	4	**18.**	26
19.	0.33	**20.**	9.25	**21.**	7	**22.**	2.33	**23.**	1.75	**24.**	0
25.	3	**26.**	12	**27.**	11	**28.**	10				

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1.	(A,B,C)	**2.**	(A,D)	**3.**	(B,C,D)	**4.**	(A,B,C,D)	**5.**	(A,B,C)	**6.**	(A,C)
7.	(B,C,D)	**8.**	(A,D)	**9.**	(A,B,C,D)	**10.**	(A,C)	**11.**	(B,D)	**12.**	(A,B,C)
13.	(A,B,C,D)	**14.**	(A,D)	**15.**	(B,C,D)	**16.**	(A,B,D)	**17.**	(B,C,D)	**18.**	(A,B,C)
19.	(C,D)	**20.**	(A,B,C)	**21.**	(A,B,C)	**22.**	(A,B)	**23.**	(A,B)	**24.**	(A,B,C,D)
25.	(A,B,D)	**26.**	(B,D)	**27.**	(A,B,C,D)	**28.**	(B,C)	**29.**	(A,C)	**30.**	(A,B)
31.	(B,C,D)										

PART - IV : COMPREHENSION

1.	(A)	**2.**	(D)	**3.**	(A)	**4.**	(C)	**5.**	(D)	**6.**	(C)
7.	(C)	**8.**	(C)	**9.**	(C)	**10.**	(B)	**11.**	(C)	**12.**	(B)

Exercise-3

PART - I : PREVIOUS YEARS JEE ADVANCE PROBLEMS

1.	(D)	**2.**	(B,C,D) or (B,C)			**3.**	(A,B,C,D)	**4.**	(A)	**5.**	(B)
6.	(B)	**7.**	(B)	**8.**	(B,D)	**9.**	(AD)	**10.**	(0)	**11.**	(3)
12.	(D)	**13.**	(AD)	**14.**	(A,B,C)	**15.**	(2)	**16.**	(7)	**17.**	(B,C)
18.	(A,B)	**19.**	(A,C)	**20.**	(A,C,D)	**21.**	(D)	**22.**	(D)	**23.**	(2)
24.	(BD)										

PART - II : PREVIOUS YEARS AIEEE & JEE MAINS PROBLEMS

1.	(4)	**2.**	(1)	**3.**	(4)	**4.**	(3)	**5.**	(3)	**6.**	(3)
7.	(1)	**8.**	(3)	**9.**	(4)	**10.**	(2)	**11.**	(3)	**12.**	(1)
13.	(4)	**14.**	(3)	**15.**	(3)	**16.**	(2)	**17.**	(4)	**18.**	(4)
19.	(4)	**20.**	(3)	**21.**	(4)	**22.**	(1)	**23.**	(4)	**24.**	(3)
25.	(4)	**26.**	(2)								

Advanced Level Problems

SUBJECTIVE QUESTIONS

1. If $\lim\limits_{x\to 0} \dfrac{1-\cos(x).\cos(2x).\cos(3x).......\cos(nx)}{x^2} = \dfrac{385}{2}$, then find n.

2. Find the value of $\lim\limits_{x\to \pi} \dfrac{1}{(x-\pi)}\left(\sqrt{\dfrac{4\cos^2 x}{2+\cos x}}-2\right)$

3. $f_1(x) = \dfrac{x}{3}+10$

$f_n(x) = f_1(f_{n-1}(x))$ $\quad\quad n \ge 2$

then evaluate $\lim\limits_{n\to\infty} f_n(x)$

4. Let $f : R \to R$ be a real function. The function f is derivable and there exists $n \in N$ and $p \in R$ such that

$\lim\limits_{x\to\infty} x^n f(x) = p$, then evaluate $\lim\limits_{x\to\infty} (x^{n+1}.f'(x))$.

5. Let $\langle x_n \rangle$ denotes a sequence, $x_1 = 1$, $x_{n+1} = \sqrt{x_n^2+1}$, then evaluate $\lim\limits_{n\to\infty}\left(\dfrac{x_{n+1}}{x_n}\right)^{2n}$

6. Let $P_n = \dfrac{2^3-1}{2^3+1}.\dfrac{3^3-1}{3^3+1}.\dfrac{4^3-1}{4^3+1}..........\dfrac{n^3-1}{n^3+1}$. Prove that $\lim\limits_{n\to\infty} P_n = \dfrac{2}{3}$.

7. Evaluate $\lim\limits_{n\to\infty} 2\left(\dfrac{1}{n^2+1}+\dfrac{2}{n^2+2}+\dfrac{3}{n^2+3}+.......+\dfrac{n}{n^2+n}\right)$

8. Evaluate $\lim\limits_{x\to 1}\left(\dfrac{10}{1-x^{10}}-\dfrac{3}{1-x^3}\right)$

9. Evaluate : $\lim\limits_{x\to\infty} x^3\left\{\sqrt{2x^2+\sqrt{4+4x^4}}-2x\right\}$

10. Evaluate $\lim\limits_{x\to\infty}\dfrac{\log_e(\log_e x)}{e^{\sqrt{x}}}$

11. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\log_e(\sin 5x)}{\log_e(\sin 25x)}$

21. $f(x) = \begin{cases} \dfrac{\cos^{-1}(2x\sqrt{1-x^2})}{x-\dfrac{1}{\sqrt{2}}} & x \neq \dfrac{1}{\sqrt{2}} \\ k & x = \dfrac{1}{\sqrt{2}} \end{cases}$

Then find 'k' if possible for which function is continuous at $x = \dfrac{1}{\sqrt{2}}$

22. Discuss the continuity of the function

$f(x) = \lim\limits_{n\to\infty} \dfrac{(1+\sin x)^n + \log x}{2+(1+\sin x)^n}$

23. Let f be a continuous function on R such that $f\left(\dfrac{1}{4x}\right) = \left(\sin e^x\right) e^{-x^2} + \dfrac{x^2}{x^2+1}$, then find the value of f(0).

24. Examine the continuity at x = 0 of the sum function of the infinite series:

$\dfrac{x}{x+1} + \dfrac{x}{(x+1)(2x+1)} + \dfrac{x}{(2x+1)(3x+1)} + \ldots\ldots\ldots\ldots\infty$.

25. If f(x) is continuous in [a, b] such that f(a) = b and f(b) = a, then prove that there exists at least one $c \in (a, b)$ such that f(c) = c.

26. If f(x . y) = f(x). f(y) for all x, y and f(x) is continuous at x = 1. Prove that f(x) is continuous for all x except possibly at x = 0. Given $f(1) \neq 0$.

27. $g(x) = \lim\limits_{n\to\infty} \dfrac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$, $x \neq 1$

$g(1) = \lim\limits_{x\to 1} \dfrac{\sin^2 \pi \cdot 2^x}{\log_e \sec(\pi \cdot 2^x)}$ be a continous function at x = 1, then find the value of $\dfrac{h(1)-g(1)}{f(1)}$, assume that f(x) and h(x) are continuous at x = 1

$g(x) = \lim\limits_{n\to\infty} \dfrac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$, $x \neq 1$

28. Let f(x) is defined only for $x \in (0, 5)$ and defined as $f^2(x) = 1 \ \forall\ x \in (0, 5)$. Function f(x) is continuous for all $x \in (0, 5) - \{1, 2, 3, 4\}$ (at x = 1, 2, 3, 4 f(x) may or may not be continuous). Find the number of possible function f(x) if it is discontinuous at

(i) One integral points in (0, 5)

(ii) Two integral points in (0, 5)

(iii) Three integral points in (0, 5)

(iv) Four integral points in (0, 5)

ANSWER KEY

SUBJECTIVE QUESTIONS

1. (10) **2.** (0) **3.** (15) **4.** –np **5.** e **7.** (1)

8. (3.5) **9.** (0.25) **10.** (0) **11.** (0.04) **12.** k = 0 **13.** (0)

15. Domain = R – $\left\{2k\pi-\frac{\pi}{2}, k\in Z\right\}$ = R – $\left\{2k\pi-\frac{\pi}{2}, k\in Z\right\}$

Range = {0} $\cup$ $\left\{k\pi+\frac{\pi}{4}, k\in Z\right\}$ = {0} $\cup$ $\left\{k\pi+\frac{\pi}{4}, k\in Z\right\}$

16. (0.5) **17.** (1) **18.** (7.5) **19.** no value of f(0)

20. $\frac{1}{\sqrt{2}}, \frac{1}{8}(\ell n\ 2)^2$ **21.** $k \in \phi$

22. f(x) is discontinuous at integral multiples of π

23. (1)

24. Discontinuous

27. (2.25)

28. (i) (24) (ii) 108 (iii) 216 (iv) 162

29. discontinuous and non-differentiable

30. f is continuous & derivable at x = – 1 but f is neither continuous nor derivable at x = 1

31. continuous in $0 \le x \le 1$ & not differentiable at x = 0

32. f(x) is continuous and non-differentiable for integral points

33. discontinuous at x = 0 and not differentiable at x = 0, 2

34. $\frac{x^2}{2}+c$ **35.** 0.5 **36.** f(10) = 10 **37.** $\frac{x+1}{x-1}$ **38.** 26

39. 0 **40.** f(x) = log(x)

42. (i) 4 (ii) 8 **43.** x = 0

44. $h(x)=\begin{cases}\alpha, & x\in(-\infty,\beta)\\ \beta, & x=\beta\\ \gamma, & x\in(\beta,\infty)\end{cases}$