QUADRATIC EQUATION

1. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $\mathbf{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then ;

(i) $\alpha + \beta = -b/a$ **(ii)** $\alpha\beta = c/a$ **(iii)** $|\alpha - \beta| = \sqrt{D} / |a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. $x^2 -$ (sum of roots) x + product of roots = 0.

(e) If α, β are roots of equation $ax^2 + bx + c = 0$, we have identity in x as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

2. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. $(p, q \in R$ & $i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

3. COMMON ROOTS OF TWO QUADRATIC EQUATIONS

(a) Atleast one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\dfrac{\alpha^2}{bc'-b'c} = \dfrac{\alpha}{a'c-ac'} = \dfrac{1}{ab'-a'b}$

Therefore, $\alpha = \dfrac{ca'-c'a}{ab'-a'b} = \dfrac{bc'-b'c}{a'c-ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(b) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

4. ROOTS UNDER PARTICULAR CASES

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(b) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(c) If $a = c \Rightarrow$ roots are reciprocal to each other

(d) If $ac < 0 \Rightarrow$ roots are of opposite signs

(e) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(f) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c

$\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a

$\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

5. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSION :

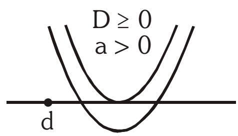
Maximum or Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

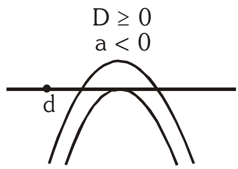
$$y \in \left[\frac{-D}{4a}, \infty\right) \text{ if } a > 0 \qquad \& \qquad y \in \left(-\infty, \frac{-D}{4a}\right] \text{ if } a < 0.$$

6. LOCATION OF ROOTS :

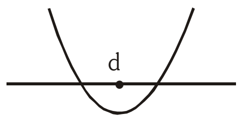
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

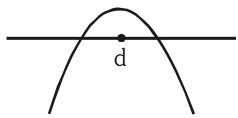
(a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $\mathbf{D \geq 0}$ **and** $\mathbf{a.f(d) > 0}$ **&** $\mathbf{(-b/2a) > d}$**.**



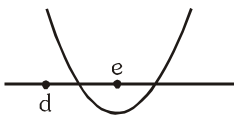


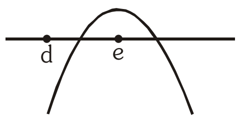
(b) Condition for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $\mathbf{a.f(d) < 0}$.





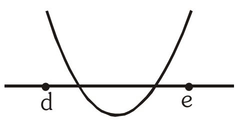
(c) Condition for exactly one root of $f(x) = 0$ to lie in the interval (d,e) i.e. $d < x < e$ is $\mathbf{f(d).\ f(e) < 0}$

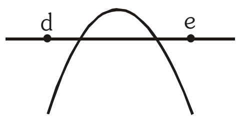




(d) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$\mathbf{D \geq 0}$ **and** $\mathbf{a.f(d) > 0}$ **&** $\mathbf{af(e) > 0}$ **and** $\mathbf{d < (-b/2a) < e}$





7. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

8. THEORY OF EQUATIONS :

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ are the roots of the equation ;

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n = 0$

where $a_0, a_1, \ldots\ldots a_n$ are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \ \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}, \ \sum \alpha_1\alpha_2\alpha_3$$

$$= -\frac{a_3}{a_0}, \ldots\ldots\ldots, \alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note :

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its constant term, when coefficient of highest degree term is (+)ve {If not then make it (+) ve}.
Ex. $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose constant term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

(iv) Rational root theorem : If a rational number $\frac{p}{q}$ $(p, q \in Z_0)$ is a root of polynomial equation with integral coefficient $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_0 = 0$, then p divides a_0 and q divides a_n.