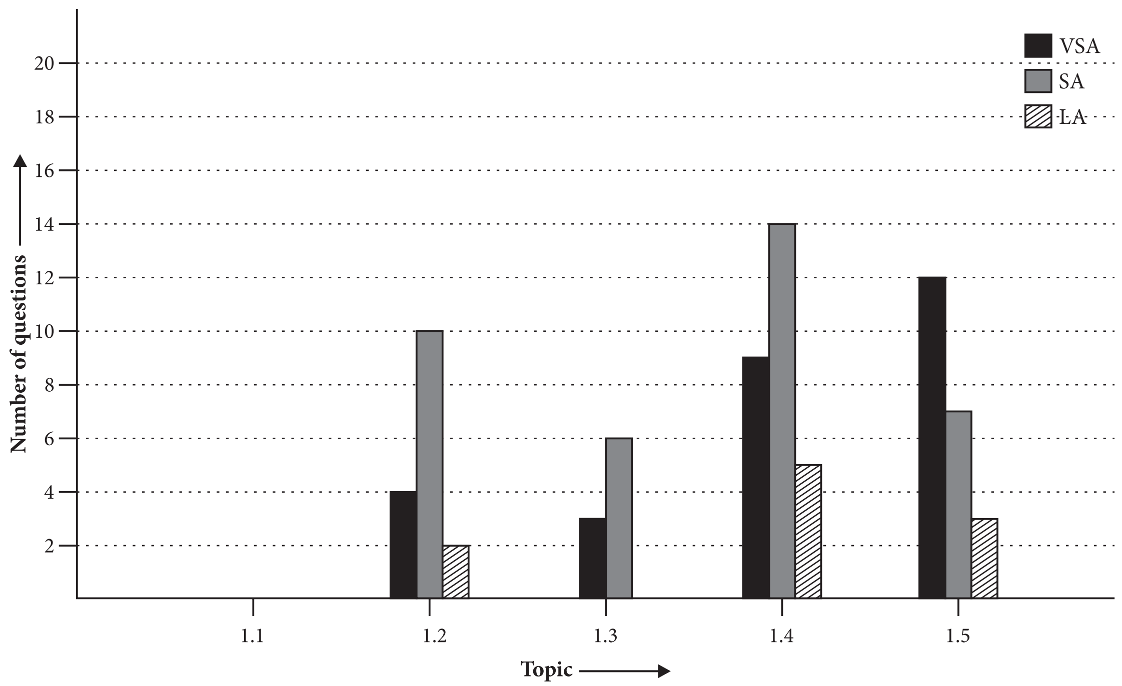
01 Relations and Functions

Topicwise Analysis of Last 10 Years' CBSE Board Questions



- Maximum weightage is of *Composition of Functions and Invertible Function*
- Maximum VSA type Questions were asked from *Binary Operations*
- Maximum SA & LA type questions were asked from *Composition of Functions and Invertible Functions*
- No VBQ type questions were asked till now

QUICK RECAP

RELATION

- A relation R from a set A to a set B is a subset of $A \times B$. So, we say $R \subseteq A \times B$. A relation from a set A to itself is called a relation in A.

Empty Relation

- If no element of A is related to any element of A. Then relation R in A is called an empty relation *i.e.*, $R = \phi \subset A \times A$.

Universal Relation

- If each element of A is related to every element of A, then relation R in A is called universal relation *i.e.*, $R = A \times A$.

A relation R in a set A is called
(i) reflexive, if $(a, a) \in R$, for all $a \in A$
(ii) symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$
(iii) transitive, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, for all $a, b, c \in A$

- A relation R in a set A is called an **equivalence relation**, if it is reflexive, symmetric and transitive.
- In a relation R in a set A, the set of all elements related to any element $a \in A$ is denoted by $[a]$ *i.e.*, $[a] = \{x \in A : (x, a) \in R\}$
  Here, $[a]$ is called an equivalence class of $a \in A$.

FUNCTION

A relation f from a set A to a set B is called a function if
(i) for each $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$ *i.e.*, $f(a) = b$
(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

- A function $f: A \to B$ is called
  (i) **one-one or injective function**, if distinct elements of A have distinct images *i.e.*, for $a, b \in A, f(a) = f(b) \Rightarrow a = b$
  (ii) **onto or surjective function**, if for every element $b \in B$, there exists some $a \in A$ such that $f(a) = b$.
- A function $f: A \to B$ is called **bijective function**, if it is both one-one and onto function.

Composition of Functions

- Let $f: A \to B$ and $g: B \to C$ be any two functions, then the function $gof: A \to C$ defined as $gof(x) = g(f(x))$, for all $x \in A$, is called the composition of f and g.

Invertible Functions

- A function $f: A \to B$ is said to be invertible, if there exists a function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$. Here, g is called the inverse of f.
- Also, f is an invertible function iff it is a bijective function.

BINARY OPERATIONS

A function $*$ on a set A *i.e.*, $* : A \times A \to A$ is called a binary operation *i.e.*, $a, b \in A \Rightarrow a * b \in A$

- A binary operation $*$ on a set A is
  (i) commutative, if $a * b = b * a$, for all $a, b \in A$
  (ii) associative, if $(a * b) * c = a * (b * c)$, for all $a, b, c \in A$
  (iii) distributive over another binary operation 'o', if $a * (b \; o \; c) = (a * b) \; o \; (a * c)$, for all $a, b, c \in A$.

Identity element

- An element $e \in A$ is the identity element for binary operation $* : A \times A \to A$, if $a * e = a = e * a$, for all $a \in A$.

Invertible element

- An element $a \in A$ is the invertible element for binary operation $* : A \times A \to A$, if there exists a unique element $b \in A$ such that $a * b = e = b * a$. Here, b is called the inverse of a.

Previous Years' CBSE Board Questions

1.2 Types of Relations

VSA (1 mark)

1. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, write the range of R. *(AI 2014)*

2. Let $R = \{(a, a^3) : a$ is a prime number less than $5\}$ be a relation. Find the range of R. *(Foreign 2014)*

3. Let R be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2$ divides $(a - b)\}$. Write the equivalence class $[0]$. *(Delhi 2014 C)*

4. State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ not to be transitive. *(Delhi 2011)*

SA (4 marks)

5. Let $A = \{1, 2, 3, ..., 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for (a, b), (c, d) in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$. *(Delhi 2014)*

6. Let R be a relation defined on the set of natural numbers N as follow :
$R = \{(x, y) \mid x \in N, y \in N$ and $2x + y = 24\}$
Find the domain and range of the relation R. Also, find if R is an equivalence relation or not. *(Delhi 2014 C)*

7. Show that the relation S in the set R of real numbers defined as $S = \{(a, b) : a, b \in$ R and $a \le b^3\}$ is neither reflexive, nor symmetric, nor transitive. *(Delhi 2010)*

8. Let Z be the set of all integers and R be the relation on Z defined as $R = \{(a, b) : a, b \in Z$ and $(a - b)$ is divisible by $5\}$. Prove that R is an equivalence relation. *(Delhi 2010)*

9. Show that the relation S in the set $A = \{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in Z, |a - b|$ is divisible by $4\}$ is an equivalence relation. Find the set of all elements related to 1. *(AI 2010)*

10. Show that the relation R defined by $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation. *(AI 2010, 2008)*

11. Let $f : X \to Y$ be a function, define a relation R on X given by $R = \{(a, b) : f(a) = f(b)\}$. Show that R is an equivalence relation on X. *(AI 2010 C)*

12. Prove that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b|$ is even$\}$, is an equivalence relation. *(Delhi 2009)*

13. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive. *(AI 2007)*

LA (6 marks)

14. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b)\ R(c, d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation. *(Delhi 2015)*

15. Show that the relation R in the set $A = \{1,2,3,4,5\}$ given by $R\ \{(a, b) : |a - b|$ is divisible by $2\}$ is an equivalence relation. Write all the equivalence classes of R. *(AI 2015 C)*

1.3 Types of Functions

VSA (1 mark)

16. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B, state whether f is one-one or not. *(AI 2011)*

17. What is the range of the function $f(x) = \dfrac{|x-1|}{(x-1)}$? *(Delhi 2010)*

18. State whether the function $f : N \to N$ given by $f(x) = 5x$ is injective, surjective or both. *(AI 2008 C)*

SA (4 marks)

19. Show that $f : N \to N$, given by
$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$
is both one-one and onto. *(AI 2012)*

20. Let $f: N \to N$ be defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, \text{if } n \text{ is odd} \\ \dfrac{n}{2}, \quad \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in N$$

Find whether the function f is bijective. *(Delhi 2012 C, AI 2009)*

21. If $f: R \to R$ be the function defined by $f(x) = 4x^3 + 7$, show that f is a bijection. *(Delhi 2011 C)*

22. Show that the function $f: W \to W$ defined by

$$f(n) = \begin{cases} n+1, \text{ if } n \text{ is even} \\ n-1, \text{ if } n \text{ is odd} \end{cases}$$

is a bijective function. *(AI 2011 C)*

23. Show that the function $f: R \to R$ given by $f(x) = ax + b$, where $a, b \in R$, $a \neq 0$ is a bijective function. *(Delhi 2010 C)*

1.4 Composition of Functions and Invertible Function

VSA (1 mark)

24. Let $f: \{1, 3, 4\} \to \{1, 2, 5\}$ and $g: \{1, 2, 5\} \to \{1, 3\}$ given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof. *(AI 2014 C)*

25. If $f: R \to R$ defined as $f(x) = \dfrac{2x-7}{4}$ is an invertible function, write $f^{-1}(x)$. *(Delhi 2012 C, 2008 C)*

26. If $f: R \to R$ is defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$. *(AI 2010)*

27. If $f: R \to R$ is defined by $f(x) = 3x + 2$, find $f(f(x))$. *(Delhi 2010 C)*

28. If the function $f: R \to R$, defined by $f(x) = 3x - 4$, is invertible, find f^{-1}. *(AI 2010C)*

29. If $f: R \to R$ defined by $f(x) = \dfrac{3x+5}{2}$ is an invertible function, find f^{-1}. *(AI 2009 C)*

30. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, find $(fog)(7)$. *(Delhi 2008)*

31. If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$. *(AI 2007)*

SA (4 marks)

32. If the function $f: R \to R$ be given by $f(x) = x^2 + 2$ and $g: R \to R$ be given by $g(x) = \dfrac{x}{x-1}, x \neq 1$, find fog and gof and hence find $fog\ (2)$ and $gof\ (-3)$. *(AI 2014)*

33. Let $f: W \to W$, be defined as $f(x) = x - 1$, if x is odd and $f(x) = x + 1$, if x is even. Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers. *(Foreign 2014)*

34. Let $A = R - \{3\}$, $B = R - \{1\}$. Let $f: A \to B$ be defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$, for all $x \in A$. Then show that f is bijective. Hence find $f^{-1}(x)$. *(Delhi 2014 C, 2012)*

35. Let $f, g: R \to R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, for all $x \in R$. Then find fog and gof. *(AI 2014 C)*

36. Show that the function f in $A = R - \left\{\dfrac{2}{3}\right\}$ defined as $f(x) = \dfrac{4x+3}{6x-4}$ is one-one and onto. Hence find f^{-1}. *(Delhi 2013)*

37. Consider $f: R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers. *(AI 2013)*

38. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f: A \to B$ is a function defined by $f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Hence find f^{-1}. *(Delhi 2013 C)*

39. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f: A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto ? Justify your answer. *(AI 2012C)*

40. Let $f: R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g: R \to R$ such that $gof = fog = I_R$. *(AI 2011)*

41. If the function $f: R \to R$ is given by $f(x) = \dfrac{x+3}{2}$ and $g : R \to R$ is given by $g(x) = 2x - 3$. Find (i) *fog* and (ii) *gof*. Is $f^{-1} = g$? *(Delhi 2009 C)*

42. If $f : R \to R$ and $g : R \to R$ are defined respectively as $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$. Find (a) *fog* (b) *gof*. *(AI 2009 C, 2008)*

43. If f be a greatest integer function and g be an absolute value function, find the value of
$(fog)\left(\dfrac{-3}{2}\right) + (gof)\left(\dfrac{4}{3}\right)$. *(Delhi 2007)*

LA (6 marks)

44. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f: N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$. *(Delhi 2016)*

45. If $f, g : R \to R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, $\forall x \in R$. Then find *fog* and *gof*. Hence find *fog* (–3), *fog* (5), and *gof* (–2). *(Foreign 2016)*

46. Consider $f : R_+ \to [-9, \infty[$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \left(\dfrac{\sqrt{54+5y}-3}{5}\right)$. *(AI 2015)*

47. Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f: N \to S$, where S is the range of f, is invertible. Also find the inverse of f. *(Foreign 2015, AI 2013 C)*

1.5 Binary Operations

VSA (1 mark)

48. Let $*$ be a binary operation on the set of all non-zero real numbers, given by $a * b = \dfrac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$. *(Delhi 2014)*

49. Let $* : R \times R \to R$, given by $(a, b) \to a + 4b^2$ is a binary operation. Compute $(-5) * (2 * 0)$. *(AI 2014 C)*

50. Let $*$ be a binary operation on N given by $a * b$ = L.C.M. (a, b) for all $a, b \in N$. Find $5 * 7$. *(Delhi 2012)*

51. The binary operation $* : R \times R \to R$, is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$. *(AI 2012)*

52. If the binary operation $*$ on the set of integers Z, is defined by $a * b = a + 3b^2$, then find the value of $8 * 3$. *(AI 2012 C)*

53. Let $*$ be a binary operation defined on the set of integers by $a * b = 2a + b - 3$. Find $3 * 4$. *(Delhi 2011 C, AI 2008)*

54. Let $*$ be a binary operation defined by $a * b = 3a + 4b - 2$. Find $4 * 5$. *(AI 2011 C, Foreign 2008)*

55. If the binary operation $*$ on the set of integers Z is defined by $a * b = a + 3b^2$, then find the value $2 * 4$. *(Delhi 2009)*

56. Let $*$ be a binary operation on N given by $a * b$ = H.C.F. (a, b); $a, b \in N$. Write the value of $22 * 4$. *(AI 2009)*

57. Let $*$ be a binary operation on set Q of rational numbers defined as $a * b = \dfrac{ab}{5}$, write the identity for $*$, if any. *(Delhi 2009 C)*

SA (4 marks)

58. Let S be the set of all rational numbers except 1 and $*$ be defined on S by $a * b = a + b - ab$, for all $a, b \in S$.
Prove that
(i) $*$ is a binary operation on S.
(ii) $*$ is commutative as well as associative. *(Delhi 2014 C)*

59. Consider the binary operations $* : R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $a\ o\ b = a$ for all, $a, b \in R$. Show that '$*$' is commutative but not associative, 'o' is associative but not commutative. *(AI 2012)*

60. Consider the binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ defined by $a * b = \min(a, b)$. Write the operation table of the operation $*$. *(Delhi 2011)*

61. A binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ is defined as :

$$a * b = \begin{cases} a+b \quad , & \text{if } a+b<6 \\ a+b-6 \, , & \text{if } a+b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a. *(AI 2011)*

62. Let $*$ be a binary operation on Q defined by $a * b = \frac{3ab}{5}$. Show that $*$ is commutative as well as associative. Also, find its identity element, if it exists. *(Delhi 2010)*

63. Let $*$ be a binary operation on the set of rational numbers given as $a * b = (2a - b)^2$, $a, b \in Q$. Find $3 * 5$ and $5 * 3$. Is $3 * 5 = 5 * 3$? *(Delhi 2008 C)*

64. Let $*$ be the binary operation on N given by $a * b$ = L.C.M. of a and b. Find the value of $20 * 16$. Is $*$ (i) commutative, (ii) associative ? *(AI 2008 C)*

LA (6 marks)

65. Show that the binary operation $*$ on $A = R - \{-1\}$ defined as $a * b = a + b + ab$ for all $a, b \in A$ is commutative and associative on A. Also find the identity element of $*$ in A and prove that every element of A is invertible. *(AI 2016, 2015)*

66. Let $A = R \times R$ and $*$ be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A. *(Delhi 2015 C)*

Detailed Solutions

1. Here, $R = \{(x, y) : x + 2y = 8\}$, where $x, y \in N$.
For $x = 1, 3, 5, ...$
$x + 2y = 8$ has no solution in N.
For $x = 2$, we have $2 + 2y = 8 \Rightarrow y = 3$
For $x = 4$, we have $4 + 2y = 8 \Rightarrow y = 2$
For $x = 6$, we have $6 + 2y = 8 \Rightarrow y = 1$
For $x = 8, 10, ...$
$x + 2y = 8$ has no solution in N.
$\therefore$ Range of $R = \{y : (x, y) \in R\} = \{ 1, 2, 3\}$

2. Given relation is
$R = \{(a, a^3) : a$ is a prime number less than $5\}$.
$\therefore$ $R = \{(2, 8), (3, 27)\}$
So, the range of R is $\{8, 27\}$.

3. Here, $R = \{(a, b) \in A \times A : 2 \text{ divides } (a - b)\}$
This is the given equivalence relation, where
$A = \{0, 1, 2, 3, 4, 5\}$
$\therefore$ $[0] = \{0, 2, 4\}$.

4. For transitivity of a relation,
If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
We have, $R = \{(1, 2), (2, 1)\}$
$(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$
$\therefore$ R is not transitive.

5. Given $A = \{1, 2, 3, 4,...,9\} \subset N$, the set of natural numbers.
To show : R is an equivalence relation.
(i) Reflexivity : Let (a, b) be an arbitrary element of $A \times A$. Then, we have $(a, b) \in A \times A \Rightarrow a, b \in A$
$\Rightarrow$ $a + b = b + a$
(by commutativity of addition on $A \subset N$)
$\Rightarrow$ $(a, b)\ R\ (a, b)$
Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in A \times A$
So, R is reflexive.
(ii) Symmetry: Let $(a, b), (c, d) \in A \times A$ such that $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow b + c = a + d$
$\Rightarrow$ $c + b = d + a$
(by commutativity of addition on $A \subset N$)
$\Rightarrow$ $(c, d)\ R\ (a, b)$.
Thus, $(a, b)\ R\ (c, d)$
$\Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in A \times A$.
So, R is symmetric.
(iii) Transitivity: Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$
Now, $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ ...(i)
and $(c, d)\ R\ (e, f) \Rightarrow c + f = d + e$ ...(ii)
Adding (i) and (ii), we get
$(a + d) + (c + f) = (b + c) + (d + e)$
$\Rightarrow$ $a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$
Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$.
So, R is transitive.
$\therefore$ R is an equivalence relation.
Equivalence class for $[(2, 5)]$ is $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8) (6, 9)\}$.

6. Here, $R = \{(x, y) \mid x \in N, y \in N \text{ and } 2x + y = 24\}$
Domain of $R = \{1, 2, 3, 4, ..., 11\}$
Range of $R = \{2, 4, 6, 8, 10, 12 ,..., 22\}$
R is not reflexive as if $(2, 2) \in R$
$\Rightarrow$ $2 \times 2 + 2 = 6 \neq 24$
In fact R is neither symmetric nor transitive.
$\Rightarrow$ R is not an equivalence relation.

7. We have $S = \{(a, b) : a \leq b^3\}$ where $a, b \in$ R.
(i) Reflexive : We observe that, $\frac{1}{2} \leq \left(\frac{1}{2}\right)^3$ is not true.
$\therefore$ $\left(\frac{1}{2}, \frac{1}{2}\right) \notin S$. So, S is not reflexive.
(ii) Symmetric : We observe that $1 \leq 3^3$ but $3 \not\leq 1^3$
i.e., $(1, 3) \in S$ but $(3, 1) \notin S$. So, S is not symmetric.
(iii) Transitive : We observe that, $10 \leq 3^3$ and $3 \leq 2^3$ but $10 \not\leq 2^3$
i.e., $(10, 3) \in S$ and $(3, 2) \in S$ but $(10, 2) \notin S$
So, S is not transitive.
$\therefore$ S is neither reflexive, nor symmetric, nor transitive.

8. We have $R = \{(a, b) : (a - b)$ is divisible by $5\}$
(i) Reflexive : For any $a \in Z$,
$a - a = 0$, which is a multiple of 5.
$\Rightarrow$ $(a, a) \in R$
Hence, R is reflexive.
(ii) Symmetric : For any $a, b \in Z$, let $(a, b) \in R$
$\Rightarrow$ $(a - b)$ is a multiple of 5.
$\Rightarrow$ $(a - b) = 5m, m \in Z \Rightarrow (b - a) = -5m$
$\therefore$ $(a, b) \in R \Rightarrow (b, a) \in R$
Hence, R is symmetric.
(iii) Transitive : For any $a, b, c, \in Z$, let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $(a - b) = 5m$ and $(b - c) = 5n; m, n \in Z$
$\Rightarrow$ $a - b + b - c = 5m + 5n; m, n \in Z$
$\Rightarrow$ $a - c = 5(m + n); m, n \in Z$
$\therefore$ $a - c$ is a multiple of 5.
i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
Hence, R is transitive.
$\therefore$ R is an equivalence relation.

9. We have, $A = \{x \in Z : 0 \le x \le 12\}$

$\therefore$ $A = \{0, 1, 2, 3, ..., 12\}$

and $S = \{(a, b) : |a - b|$ is a multiple of 4$\}$

(i) Reflexive : For any $a \in A$,

$|a - a| = 0$ is a multiple of 4.

Thus, $(a, a) \in S$

$\therefore$ S is reflexive.

(ii) Symmetric : For any $a, b \in A$,

Let $(a, b) \in S$

$\Rightarrow$ $|a - b|$ is a multiple of 4

$\Rightarrow$ $|b - a|$ is a multiple of 4 $\Rightarrow$ $(b, a) \in S$

i.e., $(a, b) \in S \Rightarrow (b, a) \in S$

$\therefore$ S is symmetric.

(iii) Transitive : For any $a, b, c \in A$,

Let $(a, b) \in S$ and $(b, c) \in S$

$\Rightarrow$ $|a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4

$\Rightarrow$ $a - b = \pm 4k_1$ and $b - c = \pm 4k_2$; $k_1, k_2 \in N$

$\Rightarrow$ $(a - b) + (b - c) = \pm 4(k_1 + k_2)$; $k_1, k_2 \in N$

$\Rightarrow$ $a - c = \pm 4(k_1 + k_2)$; $k_1, k_2 \in N$

$\Rightarrow$ $|a - c|$ is a multiple of 4 $\Rightarrow (a, c) \in S$

$\therefore$ S is transitive.

Hence, S is an equivalence relation.

The set of elements related to 1 is $\{5, 9\}$.

10. *Refer to answer 5.*

11. We have, $f : X \to Y$ is a function

$R = \{(a, b) : f(a) = f(b)\}$

(i) Reflexivity : For any $a \in X$, we have

$f(a) = f(a) \Rightarrow (a, a) \in R \Rightarrow R$ is reflexive.

(ii) Symmetric : For any $a, b \in X$,

Let $(a, b) \in R \Rightarrow f(a) = f(b)$

$\Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$

So, R is symmetric.

(iii) Transitive : For any $a, b, c \in X$,

Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $f(a) = f(b)$ and $f(b) = f(c)$

$\Rightarrow$ $f(a) = f(c) \Rightarrow (a, c) \in R$

So, R is transitive.

Hence, R is an equivalence relation on X.

12. We have $A = \{1, 2, 3, 4, 5\}$

$R = \{(a, b) : |a - b|$ is even$\}$; $a, b \in A$

(i) Reflexive : For any $a \in A$,

We have $|a - a| = 0$, which is even.

$\Rightarrow$ $(a, a) \in R\ \forall\ a \in A$

So, R is reflexive.

(ii) Symmetric : For any $a, b \in A$,

Let $(a, b) \in R \Rightarrow |a - b|$ is even $\Rightarrow$ $|b - a|$ is even

$\Rightarrow$ $(b, a) \in R$.

So, R is symmetric.

(iii) Transitive : For any $a, b, c \in A$. Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ $a - b = \pm 2k_1$ and $b - c = \pm 2k_2$, for $k_1, k_2 \in N$

$\Rightarrow$ $(a - b) + (b - c) = \pm (2k_1 + 2k_2)$; $k_1, k_2 \in N$

$\Rightarrow$ $a - c = \pm 2(k_1 + k_2)$; $k_1, k_2 \in N$

$\Rightarrow$ $|a - c|$ is even $\Rightarrow (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

So, R is transitive.

Hence, R is an equivalence relation.

13. Here $R = \{(a, b) : b = a + 1\}$

$= \{(a, a + 1) : a, a + 1 \in \{1, 2, 3, 4, 5, 6\}\}$

$= \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

(i) R is not reflexive as $(a, a) \notin R\ \forall\ a$.

(ii) R is not symmetric as $(1, 2) \in R$ but $(2, 1) \notin R$.

(iii) R is not transitive as $(1, 2) \in R$, $(2, 3) \in R$ but $(1, 3) \notin R$.

14. (i) Reflexivity : Let (a, b) be an arbitrary element of $N \times N$. Then, $(a, b) \in N \times N$

$\Rightarrow$ $ab(b + a) = ba(a + b)$

[by commutativity of addition and multiplication on N]

$\Rightarrow$ $(a, b)\ R\ (a, b)$

So, R is reflexive on $N \times N$.

(ii) Symmetry : Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$.

$\Rightarrow$ $ad(b + c) = bc(a + d) \Rightarrow cb(d + a) = da(c + b)$

[by commutativity of addition and multiplication on N]

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$.

So, R is symmetric on $N \times N$.

(iii) Transitivity : Let $(a, b), (c, d), (e, f) \in N \times N$ be such that

$(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$(a, b)\ R\ (c, d) \Rightarrow ad(b + c) = bc(a + d)$

$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad} \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \quad \text{...(i)}$$

and $(c, d)\ R\ (e, f) \Rightarrow cf(d + e) = de(c + f)$

$$\Rightarrow \frac{d+e}{de} = \frac{c+f}{cf} \Rightarrow \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$$\Rightarrow \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \Rightarrow \frac{b+e}{be} = \frac{a+f}{af}$$

$\Rightarrow$ $af(b + e) = be(a + f) \Rightarrow (a, b)\ R\ (e, f)$

So, R is transitive on $N \times N$.

Hence, R is an equivalence relation.

15. *Refer to answer 9.*
Further R has only two equivalence classes, namely $[1] = [3] = [5] = \{1, 3, 5\}$ and $[2] = [4] = \{2, 4\}$.

16. $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and
$f = \{(1, 4), (2, 5), (3, 6)\}$
We have, $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$. Distinct elements of A have distinct images in B. Hence, f is a one-one function.

17. We have, $|x-1| = \begin{cases} x-1, & x \ge 1 \\ 1-x, & x < 1 \end{cases}$

$\therefore \quad f(x) = \dfrac{|x-1|}{(x-1)} = \begin{cases} 1, & x \ge 1 \\ -1, & x < 1 \end{cases}$

$\therefore$ Range $(f) = \{-1, 1\}$

18. We have, $f(x) = 5x$
For $x_1, x_2 \in N$.
Let $f(x_1) = f(x_2) \Rightarrow 5x_1 = 5x_2 \Rightarrow x_1 = x_2$
$\therefore$ The function is one-one.
Now, $f(x)$ is not onto. Since, for $2 \in N$ (co-domain), there does not exist any $x \in N$ (domain) such that $f(x) = 5x = 2$
$\therefore$ $f(x)$ is injective but not surjective.

19. Here, $f : N \to N$ s.t.

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

Let $x, y \in N$ s.t. $f(x) = f(y)$
We shall show that $x = y$
(i) If x and y both are even
$f(x) = f(y) \Rightarrow x - 1 = y - 1 \Rightarrow x = y$
(ii) If x and y both are odd
$f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$
(iii) If x is odd and y is even
$f(x) = f(y) \Rightarrow x + 1 = y - 1$
$\Rightarrow$ $y - x = 2$ ...(1)
R.H.S. is even but L.H.S. is odd.
$\Rightarrow$ Equation (1) in N is not possible.
$\Rightarrow$ (iii) does not arise.
(iv) If x is even and y is odd, does not arise.
In any case, $f(x) = f(y) \Rightarrow x = y$
$\Rightarrow$ f is one-one
For any $y \in N$ (co-domain), y can be even or odd
When y is odd, $y + 1$ is even, so
$f(y + 1) = (y + 1) - 1 = y$
When y is even, $y - 1$ is odd, so
$f(y - 1) = (y - 1) + 1 = y$
$\Rightarrow$ $f : N \to N$ is onto.
Hence, f is both one-one and onto.

20. (i) Injectivity: Here, $f(1) = \dfrac{1+1}{2} = 1, f(2) = \dfrac{2}{2} = 1$,

$f(3) = \dfrac{3+1}{2} = 2, f(4) = \dfrac{4}{2} = 2$

Thus, $f(2k-1) = \dfrac{(2k-1)+1}{2} = k$ and $f(2k) = \dfrac{2k}{2} = k$
$\Rightarrow$ $f(2k - 1) = f(2k)$, where $k \in N$
But, $2k - 1 \ne 2k$, $\Rightarrow f$ is not one-one.
Hence, f is not bijective.

21. Here, $f(x) = 4x^3 + 7$
Let $x_1, x_2 \in R$ s.t.
$f(x_1) = f(x_2)$
$\Rightarrow$ $4x_1^3 + 7 = 4x_2^3 + 7$
$\Rightarrow$ $4x_1^3 = 4x_2^3 \Rightarrow x_1^3 = x_2^3$
$\Rightarrow$ $x_1^3 - x_2^3 = 0$
$\Rightarrow$ $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$
$\Rightarrow$ $x_1 - x_2 = 0$
[$\because$ $x_1^2 + x_1x_2 + x_2^2 = 0$ has no real roots]
$\Rightarrow$ $x_1 = x_2$
$\therefore$ $f : R \to R$ is one-one.
Again $\forall$ $y \in R$ (co-domain), we find $x \in R$ (domain) s.t. $f(x) = y$
$\Leftrightarrow$ $4x^3 + 7 = y \Leftrightarrow 4x^3 = y - 7$
$\Leftrightarrow$ $x^3 = \dfrac{y-7}{4} \Leftrightarrow x = \sqrt[3]{\dfrac{y-7}{4}} \in R$
($\because$ $x^3 = \alpha$, $\alpha \in R$ has always one real root)
Hence, f is onto
So, $f : R \to R$ is a bijection.

22. *Refer to answer 19.*

23. We have $f(x) = ax + b$ where $a, b \in R$ and $a \ne 0$
(i) Injectivity : Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
$\Rightarrow$ $ax_1 + b = ax_2 + b \Rightarrow x_1 = x_2$
$\therefore$ $f(x)$ is one-one.
(ii) Surjectivity : Let $y \in R$ (co-domain) such that $f(x) = y$
$\Rightarrow$ $y = ax + b \Rightarrow x = \dfrac{y-b}{a} \in R$ ($\because$ $a \ne 0$)

$\therefore$ $f\left(\dfrac{y-b}{a}\right) = a\left(\dfrac{y-b}{a}\right) + b = y$

$\therefore$ $f(x)$ is onto.
Hence, f is injective and surjective. So, $f(x)$ is bijective.

24. Here, $f = \{(1,2), (3, 5), (4, 1)\}$
$\Leftrightarrow$ $f(1) = 2; f(3) = 5; f(4) = 1$ ...(1)
$g = \{(1, 3), (2, 3), (5, 1)\}$
$\Leftrightarrow$ $g(1) = 3; g(2) = 3, g(5) = 1$ ...(2)

Now, $gof : \{1, 3, 4\} \to \{1, 3\}$
Using (1) and (2), we get
$(gof)(1) = g(f(1)) = g(2) = 3$
$(gof)(3) = g(f(3)) = g(5) = 1$
$(gof)(4) = g(f(4)) = g(1) = 3$
$\therefore \quad gof = \{(1, 3), (3, 1), (4, 3)\}$.

25. Let $y = f(x) = \dfrac{2x-7}{4}$

$\Rightarrow \quad 4y = 2x - 7 \;\Rightarrow\; x = \dfrac{4y+7}{2}$

As $y = f(x)$ is an invertible function, so

$x = f^{-1}(y)$ *i.e.*, $f^{-1}(y) = \dfrac{4y+7}{2}$

$\Rightarrow \quad f^{-1} : R \to R$ s.t. $f^{-1}(x) = \dfrac{4x+7}{2}, \forall x \in R$.

26. $f : R \to R$ and $f(x) = (3 - x^3)^{1/3}$
$\therefore \quad fof(x) = f(f(x)) = f[3 - x^3)^{1/3}]$
$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3}$
$= [3 - (3 - x^3)]^{1/3} = (3 - 3 + x^3)^{1/3} = x$

27. We have, $f(x) = 3x + 2$
$\therefore \quad f(f(x)) = f(3x + 2) = 3(3x + 2) + 2 = 9x + 8$

28. We have, $f(x) = 3x - 4$
Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\therefore \quad y = 3x - 4 \Rightarrow x = \dfrac{y+4}{3}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{y+4}{3} \Rightarrow f^{-1}(x) = \dfrac{x+4}{3}$

29. We have, $f(x) = \dfrac{3x+5}{2}$
Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\therefore \quad y = \dfrac{3x+5}{2} \Rightarrow x = \dfrac{2y-5}{3}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{2y-5}{3}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{2x-5}{3}$

30. $f(x) = x + 7$ and $g(x) = x - 7$
So, $fog(x) = f(g(x)) = f(x - 7) = x - 7 + 7 = x$
$\Rightarrow \quad fog(x) = x$
$\therefore \quad fog(7) = 7$

31. $f(f(x)) = f(x^2 - 3x + 2)$
$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$
$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$
$= x^4 - 6x^3 + 10x^2 - 3x$.

32. Here $f : R \to R$ s.t. $f(x) = x^2 + 2$

and $g : R \to R$ s.t. $g(x) = \dfrac{x}{x-1}, x \neq 1$

Now $fog : R \to R$ s.t.

$$(fog)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2, x \neq 1$$

$$\Rightarrow \quad (fog)(2) = \left(\frac{2}{2-1}\right)^2 + 2 = 6$$

Also, $gof : R \to R$ s.t.

$$(gof)(x) = g(f(x)) = g(x^2+2) = \frac{x^2+2}{x^2+1}$$

$$\therefore \quad (gof)(-3) = \frac{(-3)^2+2}{(-3)^2+1} = \frac{11}{10}.$$

33. *Refer to answer 19,*
since f is bijective function. So, it is invertible.
Now, let $x, y \in W$ such that $f(x) = y$
$\Rightarrow \quad x + 1 = y$, if x is even and $x - 1 = y$, if x is odd.

$$\Rightarrow \quad x = \begin{cases} y-1, \text{ if } y \text{ is odd} \\ y+1, \text{ if } y \text{ is even} \end{cases}$$

$$\Rightarrow \quad f^{-1}(y) = \begin{cases} y-1, \text{ if } y \text{ is odd} \\ y+1, \text{ if } y \text{ is even} \end{cases}$$

$$\text{Hence, } f^{-1}(x) = \begin{cases} x-1, \text{ if } x \text{ is odd} \\ x+1, \text{ if } x \text{ is even} \end{cases}$$

34. $A = R - \{3\}$; $B = R - \{1\}$
and $f : A \to B$ defined as

$$f(x) = \frac{x-2}{x-3} \;\forall\; x \in A.$$

Here, f is defined $\forall\; x \in A$, as $3 \notin A$.
Also, $f(x) \neq 1$

$[\because f(x) = 1 \Leftrightarrow \dfrac{x-2}{x-3} = 1 \Leftrightarrow x - 2 = x - 3$
$\Leftrightarrow -2 = -3$, which is absurd]

Let $x_1, x_2 \in A$ be such that $f(x_1) = f(x_2)$

$\Rightarrow \quad \dfrac{x_1-2}{x_1-3} = \dfrac{x_2-2}{x_2-3}$

$\Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$
$\Rightarrow \quad x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$
$\Rightarrow \quad -3x_1 - 2x_2 = -2x_1 - 3x_2 \Rightarrow x_1 = x_2$
$\therefore \quad f$ is a one-one function.
Let $y \in B = R - \{1\} \Rightarrow y \neq 1$.
We want to solve $y = f(x)$ for some $x \in A$

$\Leftrightarrow \quad y = \dfrac{x-2}{x-3} \Leftrightarrow xy - 3y = x - 2$

$\Leftrightarrow \quad x(y-1) = -2 + 3y$

$\Leftrightarrow \quad x = \dfrac{3y-2}{y-1} \in A$ (as $y \neq 1$)

$\therefore \quad f$ is onto also.

$\Rightarrow \quad f: A \to B$ is a bijective function.

Now, $f^{-1}(y) = x = \dfrac{3y-2}{y-1} \Leftrightarrow f^{-1}(x) = \dfrac{3x-2}{x-1}$

35. Here, $f, g: R \to R$ s.t.

$$f(x) = |x| + x = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $g(x) = |x| - x = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

$\therefore \quad (fog)(x) = f(g(x)) = \begin{cases} f(0) & \text{if } x \geq 0 \\ f(-2x) & \text{if } x < 0 \end{cases}$

$= \begin{cases} 0 & \text{if } x \geq 0 \\ 2(-2x) & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ -4x & \text{if } x < 0 \end{cases}$

and $(gof)(x) = g(f(x))$

$= \begin{cases} g(2x) & \text{if } x \geq 0 \\ g(0) & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = 0 \ \forall x \in R.$

36. Here, $f(x) = \dfrac{4x+3}{6x-4}$ where $x \in A = R - \left\{\dfrac{2}{3}\right\}$.

(i) Let $f(x_1) = f(x_2)$ ($\forall x_1, x_2 \in A$)

$\Rightarrow \quad \dfrac{4x_1+3}{6x_1-4} = \dfrac{4x_2+3}{6x_2-4}$

$\Rightarrow \quad (4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$

$\Rightarrow \quad 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

$\Rightarrow \quad -34x_1 = -34x_2 \Rightarrow x_1 = x_2$

$\therefore \quad f$ is one-one.

(ii) For $y \in A = R - \left\{\dfrac{2}{3}\right\}$.

Let $f(x) = y$

$\Leftrightarrow \quad \dfrac{4x+3}{6x-4} = y \Leftrightarrow (6x-4)y = 4x+3$

$\Leftrightarrow \quad x = \dfrac{4y+3}{6y-4} \in A \left(\text{as } y \neq \dfrac{2}{3}\right)$

$\Leftrightarrow \quad f$ is onto and $f(x) = y \Leftrightarrow x = f^{-1}(y)$

$\Leftrightarrow \quad f^{-1}(y) = \dfrac{4y+3}{6y-4} \forall y \in A \Leftrightarrow f^{-1}(x) = \dfrac{4x+3}{6x-4} \forall x \in A$

This gives the inverse function of f.

37. $f: R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$

(i) Let $x_1, x_2 \in R_+$ s.t. $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^2 + 4 = x_2^2 + 4 \Rightarrow x_1^2 = x_2^2$

$\Rightarrow \quad x_1 = x_2 \qquad (\because x_1, x_2 \in R_+)$

$\Rightarrow \quad f$ is one-one

(ii) $y = f(x) \ \forall y \in [4, \infty), y \geq 4$

$\Rightarrow \quad x^2 + 4 = y \Rightarrow x = \sqrt{y-4}$

Now, x is defined if

$y - 4 \geq 0 \Rightarrow \sqrt{y-4} \in R_+$ and

$f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$

$\Rightarrow \quad f$ is onto.

$\therefore \quad f$ is one–one and onto.

$\Rightarrow \quad f$ is invertible and f^{-1} exists.

$f^{-1}(y) = \sqrt{y-4}.$

38. Here, $f: A \to B$ is given by $f(x) = \dfrac{x-1}{x-2}$,

where $A = R - \{2\}$ and $B = R - \{1\}$

Let $f(x_1) = f(x_2)$, where $x_1, x_2, \in A$ (*i.e.*, $x_1 \neq 2, x_2 \neq 2$)

$\Rightarrow \quad \dfrac{x_1-1}{x_1-2} = \dfrac{x_2-1}{x_2-2}$

$\Rightarrow \quad (x_1 - 1)(x_2 - 2) = (x_1 - 2)(x_2 - 1)$

$\Rightarrow \quad x_1x_2 - 2x_1 - x_2 + 2 = x_1x_2 - x_1 - 2x_2 + 2$

$\Rightarrow \quad -2x_1 - x_2 = -x_1 - 2x_2$

$\Rightarrow \quad x_1 = x_2 \Rightarrow f$ is one-one.

Next, let $y \in B = R - \{1\}$ *i.e.*, $y \in R$ and $y \neq 1$

Now, $\dfrac{x-1}{x-2} = y \Leftrightarrow (x-2)y = x-1$

$\Leftrightarrow \quad xy - 2y = x - 1 \Leftrightarrow x(y-1) = 2y - 1$

$\Leftrightarrow \quad x = \dfrac{2y-1}{y-1} \qquad \text{...(i)}$

$\therefore \quad f(x) = y$ when $x = \dfrac{2y-1}{y-1} \in A$ (as $y \neq 1$)

Hence, f is onto.

Thus, f is one–one and onto.

From (i), $f^{-1}: B \to A$ is given by $x = f^{-1}(y)$

i.e., $f^{-1}(y) = \dfrac{2y-1}{y-1}.$

39. *Refer to answer 34.*

40. Let $y \in R$ (co-domain) be arbitrary.

By definition, $y = 10x + 7$ for $x \in R$

$\Rightarrow \quad x = \dfrac{y-7}{10}$

So, we define, $g: R \to R$ by $g(y) = \dfrac{y-7}{10}$

Now, $(gof)(x) = g(f(x)) = g(10x+7) = \dfrac{(10x+7)-7}{10} = x$ and

$(fog)(y) = f(g(y)) = f\left(\dfrac{y-7}{10}\right) = 10\left(\dfrac{y-7}{10}\right) + 7 = y$

Thus, $gof = fog = I_R$.

41. (i) $fog(x) = f(g(x)) = \frac{2x-3+3}{2} = \frac{2x}{2} = x$

(ii) $gof\ (x) = g(f(x)) = 2\left(\frac{x+3}{2}\right) - 3$

$= x + 3 - 3 = x.$

So, $fog = gof = I_R$. Hence, $f^{-1} = g$.

42. (a) $fog\ (x) = f(g(x)) = (2x-3)^2 + 3(2x-3) + 1$
$= 4x^2 - 12x + 9 + 6x - 9 + 1 = 4x^2 - 6x + 1$

(b) $gof\ (x) = g(f(x)) = 2(x^2 + 3x + 1) - 3$
$= 2x^2 + 6x + 2 - 3 = 2x^2 + 6x - 1$

43. Here $f(x) = [x]$ and $g(x) = |x|$

$fog\ (x) = f(g(x)) = f(|x|) = [|x|]$

$\therefore\ (fog)\left(\frac{-3}{2}\right) = \left[\left|\frac{-3}{2}\right|\right] = \left[\frac{3}{2}\right] = 1$

$gof\ (x) = g\ (f(x)) = g\ ([x]) = |[x]|$

$\therefore\ (gof)\left(\frac{4}{3}\right) = \left|\left[\frac{4}{3}\right]\right| = |1| = 1$

$\therefore\ (fog)\left(\frac{-3}{2}\right) + (gof)\left(\frac{4}{3}\right) = 1 + 1 = 2.$

44. Let $f : N \to S$, $f(x) = 9x^2 + 6x - 5$

Consider, $f(x_1) = f(x_2)$

$\Rightarrow\ 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow\ 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow\ (x_1 - x_2)[9x_1 + 9x_2 + 6] = 0$

$\Rightarrow\ x_1 = x_2$ $\quad [\because\ x_1, x_2 \in N]$

$\Rightarrow$ f is one-one.

Since, S is the range of f.

$\therefore$ f is onto.

Since, f is one-one and onto.

So, f is invertible.

Let $y \in S$ be arbitrary number.

Consider, $y = f(x) \Rightarrow x = f^{-1}(y)$

$\Rightarrow\ y = 9x^2 + 6x - 5 \Rightarrow y = (3x+1)^2 + 6$

$\Rightarrow\ \sqrt{y+6} = 3x + 1 \Rightarrow x = \frac{\sqrt{y+6}-1}{3}$

Also, $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$ or $f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$

Now, $f^{-1}(43) = \frac{\sqrt{49}-1}{3} = \frac{7-1}{3} = 2$

and $f^{-1}(163) = \frac{\sqrt{169}-1}{3} = \frac{13-1}{3} = 4$

45. *Refer to answer 34,*

We get $fog\ (x) = \begin{cases} 0, & x \geq 0 \\ -4x, & x < 0 \end{cases}$

and $gof\ (x) = \forall\ x \in R$

Now, $fog\ (-3) = -4\ (-3) = 12$

and $fog\ (5) = 0$, $gof\ (-2) = 0$

46. Here $f : R_+ \to [-9, \infty[$ as

$f(x) = 5x^2 + 6x - 9$

First we shall show that f is one-one.

Let $f(x) = f(y)$, for $x, y \in R_+$

$\Rightarrow\ 5x^2 + 6x - 9 = 5y^2 + 6y - 9$

$\Rightarrow\ 5\ (x^2 - y^2) + 6\ (x - y) = 0$

$\Rightarrow\ (x - y)\ [5\ (x + y) + 6] = 0$

$\Rightarrow\ x = y$ $\quad [\because$ for $x, y \in R_+, 5\ (x + y) + 6 \neq 0]$

$\Rightarrow$ f is one-one.

Let $y \in [-9, \infty[$ be such that $f(x) = y$

$\Leftrightarrow\ 5x^2 + 6x - 9 = y \Rightarrow 5x^2 + 6x - (9 + y) = 0$

$\Rightarrow\ x = \frac{-6 \pm \sqrt{6^2 + 4 \cdot 5(9+y)}}{2 \cdot 5} = \frac{-6 \pm \sqrt{216 + 20y}}{10}$

$= \frac{-3 \pm \sqrt{54 + 5y}}{5}$

Taking only +ve sign (as for –ve sign, $x \notin R_+$)

We get $x = \frac{-3 + \sqrt{54 + 5y}}{5} \in R_+$ for which

$f(x) = y$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is both one-one and onto.

$\Rightarrow$ f is invertible and f^{-1} is given by

$f^{-1}(y) = x = \frac{-3 + \sqrt{54 + 5y}}{5}$

47. Let $f : N \to S$, $f(x) = 4x^2 + 12x + 15$

Consider, $f(x_1) = f(x_2)$

$\Rightarrow\ 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow\ 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$

$\Rightarrow\ (x_1 - x_2)\ [4x_1 + 4x_2 + 12] = 0$

$\Rightarrow\ x_1 = x_2$ $\quad [\because\ x_1, x_2 \in N]$

$\therefore$ f is one-one

Since, S is the range of f.

$\therefore$ f is onto.

Since, f is one-one and onto.

Therefore, f is invertible.

Let $y \in S$ be arbitrary number such that $f(x) = y$

$\Rightarrow\ y = 4x^2 + 12x + 15$

$\Rightarrow\ y = (2x + 3)^2 + 6$

$\Rightarrow\ \sqrt{y-6} = 2x + 3 \Rightarrow x = \frac{\sqrt{y-6}-3}{2}$

Also, $f(x) = y \Rightarrow x = f^{-1}(y)$

$\Rightarrow\ f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}$ or $f^{-1}(x) = \frac{\sqrt{x-6}-3}{2}$

48. We have, $2 * (x * 5) = 10 \Rightarrow 2 * \left(\frac{5x}{5}\right) = 10$

$\Rightarrow \quad 2 * x = 10 \Rightarrow \frac{2x}{5} = 10 \Rightarrow x = 25$

49. Here $* : R \times R \to R$ is given by
$a * b = a + 4b^2$.
$(-5) * (2 * 0) = (-5) * (2 + 4 \cdot 0^2) = -5 * (2)$
$= -5 + 4 \cdot 2^2 = -5 + 16 = 11$

50. $5 * 7 =$ L.C.M. $(5, 7) = 35$.

51. $* : R \times R \to R$ given by
$a * b = 2a + b$
$\therefore \quad (2 * 3) * 4 = (2 \times 2 + 3) * 4 = 2 \times 7 + 4 = 18$.

52. Here $a * b = a + 3b^2 \ \forall a, b \in Z$
$\Rightarrow \quad 8 * 3 = 8 + 3 \cdot 3^2 = 8 + 27 = 35$.

53. Here $a * b = 2a + b - 3$
$\therefore \quad 3 * 4 = 2(3) + 4 - 3 = 7$

54. Here $a * b = 3a + 4b - 2$
$\therefore\ 4 * 5 = 3(4) + 4(5) - 2 = 12 + 20 - 2 = 30$

55. Here $a * b = a + 3b^2$
$\therefore \quad 2 * 4 = 2 + 3(4)^2 = 2 + 3 \times 16 = 50$

56. Here $a * b =$ H.C.F. (a, b)
$\therefore \quad 22 * 4 =$ H.C.F. $(22, 4) = 2$

57. Here $a * b = \frac{ab}{5}$.

For identity, $a * e = a = e * a$

$\Rightarrow \quad \frac{ae}{5} = a = \frac{ea}{5} \Rightarrow e = 5$

$\therefore$ Identity element for $*$ is 5.

58. We have, $S = Q - \{1\}$
$a * b = a + b - ab \ \forall a, b \in S$
(i) As $a, b \in S \Rightarrow a, b \in Q$ and $a \neq 1, b \neq 1$, ...(1)
$\therefore \quad a + b - ab \in Q$
We check : $a + b - ab \neq 1$
Suppose $a + b - ab = 1$
$\Rightarrow \quad a + b - ab - 1 = 0$
$\Rightarrow \quad a - 1 + b(1 - a) = 0$
$\Rightarrow \quad -(1 - a) + b(1 - a) = 0$
$\Rightarrow \quad (1 - a)(-1 + b) = 0$
$\Rightarrow$ Either $1 - a = 0$ or $-1 + b = 0$
$\Rightarrow \quad a = 1$ or $b = 1$
This contradicts (1).
$\therefore \quad a + b - ab \neq 1$.
$\Rightarrow \quad a + b - ab \in Q - \{1\} = S$
$\Rightarrow \quad *$ is binary operation on S.

(ii) Let $a, b \in S$
$a * b = a + b - a \cdot b = b + a - b \cdot a = b * a$ $*$ is commutative in S
Let $a, b, c \in S$
Then $a * (b * c) = a * (b + c - bc)$
$= a + b + c - bc - a(b + c - bc)$
$= a + b + c - ab - bc - ca + abc$
$= a + b - ab + c - (a + b - ab)\, c$
$= (a * b) * c$
$\therefore\ *$ is associative

59. $b * a = |b - a| = |a - b| \quad [\because |-x| = |x| \ \forall\ x \in R]$
$= a * b \ \forall a, b \in R$
$\Rightarrow \quad *$ is commutative on R.
Also, for $a = 2, b = 4, c = 5$
$(a * b) * c = (2 * 4) * 5 = |2 - 4| * 5$
$= 2 * 5 = |2 - 5| = 3$
and $a * (b * c) = 2 * (4 * 5) = 2 * |4 - 5|$
$= 2 * 1 = |2 - 1| = 1$.
$\therefore \quad (a * b) * c \neq a * (b * c)$
$\Rightarrow \quad *$ is not associative on R.
Also, $(a \ o \ b) \ o \ c = a \ o \ c = a$
and $a \ o \ (b \ o \ c) = a \ o \ b = a$
$\Rightarrow \quad (a \ o \ b) \ o \ c = a \ o \ (b \ o \ c) \ \ \forall a, b, c \in R$
$\Rightarrow \quad o$ is associative on R.
Also, for $a = 3, b = 2$
$a \ o \ b = 3 \ o \ 2 = 3$
$b \ o \ a = 2 \ o \ 3 = 2$
$\Rightarrow \quad a \ o \ b \neq b \ o \ a$
$\Rightarrow \quad o$ is not commutative on R.

60. Let $A = \{1, 2, 3, 4, 5\}$
$a * b =$ minimum of a and b

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

61.

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Identity : Let e be the identity element, then
$a * e = a = e * a$

Now, $a * 0 = a + 0 = a$ and $0 * a = 0 + a = a$
Thus, $a * 0 = a = 0 * a$. Hence, 0 is the identity element of the operation.
Inverse : Since, each row or column contains the identity element *i.e.*, 0.
So, each element is invertible.
Now, $a * (6 - a) = a + (6 - a) - 6 = 0$
and $(6 - a) * a = (6 - a) + a - 6 = 0$.
Hence, each element a of the set is invertible with inverse $6 - a$.

62. Commutativity : $a * b = \frac{3ab}{5} = \frac{3ba}{5} = b * a$
$\therefore$ $*$ is commutative.
Associativity : $(a * b) * c = \left(\frac{3ab}{5}\right) * c = \frac{9abc}{25}$
and $a * (b * c) = a * \left(\frac{3bc}{5}\right) = \frac{9abc}{25}$
$\Rightarrow$ $(a * b) * c = a * (b * c)$
$\therefore$ $*$ is associative.
Identity : $a * e = a = e * a$, where e is the identity element.
$\Rightarrow$ $\frac{3ae}{5} = a = \frac{3ea}{5} \Rightarrow e = \frac{5}{3} \in Q$
$\therefore$ $\frac{5}{3}$ is the identity element.

63. We have, $a * b = (2a - b)^2$
$\therefore 3 * 5 = (2 \times 3 - 5)^2 = (6 - 5)^2 = 1$
$5 * 3 = (2 \times 5 - 3)^2 = (10 - 3)^2 = 49$
Thus, $3 * 5 \neq 5 * 3$

64. We have, $a * b$ = L.C.M. of a and b
$\therefore$ $20 * 16$ = L.C.M. of 20 and 16 = 80
(i) Commutativity : $a * b$ = L.C.M. of a and b
= L.C.M. of b and $a = b * a$
$\Rightarrow$ $a * b = b * a$
So, $*$ is commutative.
(ii) Associativity : $(a * b) * c$
= [L.C.M. (a, b)] $* c$
= L.C.M. [L.C.M. (a, b), c]
= L.C.M. (a, b, c)
and $a * (b * c) = a *$ [L.C.M. (b, c)]
= L.C.M. [a, L.C.M. (b, c)]
= L.C.M. (a, b, c)
$\Rightarrow$ $a * (b * c) = (a * b) * c$
So, $*$ is associative.

65. We have $a * b = a + b + ab \ \forall \ a, b \in A$, where $A = R - \{-1\}$
Commutativity : Let $a, b \in R - \{-1\}$
We have, $a * b = a + b + ab = b + a + ba = b * a$
Hence, $*$ is commutative.
Associativity : Let $a, b, c \in R - \{-1\}$
We have, $a * (b * c) = a * (b + c + bc)$
$= a + (b + c + bc) + a(b + c + bc)$
$= a + b + c + bc + ab + ac + abc$
$= a + b + ab + c + (a + b + ab)c$
$= (a + b + ab) * c = (a * b) * c$
Hence, $*$ is associative.
Identity : Let $e \in A$ be the identity element. Then,
$a * e = a = e * a$
$a * e = a + e + ae = a$ and $e * a = e + a + ea = a$
$\Rightarrow$ $e(1 + a) = 0 \Rightarrow e = 0$ [$\because$ $a \neq -1$]
Hence, the identity element for $*$ is $e = 0$.
Existence of inverse : Let $a \in R - \{-1\}$ and b be the inverse of a.
Then, $a * b = e = b * a$
$\Rightarrow$ $a + b + ab = 0 = b + a + ba$
$\Rightarrow$ $b = -\frac{a}{a+1}$
Since, $a \in R - \{-1\}$
$\therefore$ $a \neq -1 \Rightarrow a + 1 \neq 0 \Rightarrow b = \frac{-a}{a+1} \in R$
Also, if $-\frac{a}{a+1} = -1$
$\Rightarrow$ $-a = -a - 1 \Rightarrow -1 = 0$, which is not possible.
Hence, $\frac{-a}{a+1} \in R - \{-1\}$
So, every element of $R - \{-1\}$ is invertible and the inverse of an element a is $\frac{-a}{a+1}$.

66. Here $A = R \times R$ and $*$ on A is defined as
$(a, b) * (c, d) = (a + c, b + d) \ \forall (a, b), (c, d), \in R$
Now $(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$
$= (a, b) * (c, d) \ \forall (a, b), (c, d) \in A$
$\Rightarrow$ $*$ is commutative on A.
Again $[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f)$
$= (a + c + e, b + d + f) = (a + (c + e), b + (d + f))$
$= (a, b) * (c + e, d + f)$
$= (a, b) * [(c, d) * (e, f)] \ \forall (a, b), (c, d), (e, f) \in A$
$\Rightarrow$ $*$ is associative on A.
Also $0 \in R$ and $(0, 0) \in A$.
$\therefore$ $\forall (a, b) \in A, (a, b) * (0, 0) = (a + 0, b + 0) = (a, b)$
and $(0, 0) * (a, b) = (0 + a, 0 + b) = (a, b)$
$\Rightarrow$ $(0, 0)$ acts as an identity element in A w.r.t. $*$.