Vectors & Three Dimensional Geometry

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JEE (Advanced) Syllabus

Vectors : Addition of vectors, scalar multiplication, dot and cross products, scalar triple products and their geometrical interpretations.
Three Dimensional Geometry : Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

JEE (Main) Syllabus

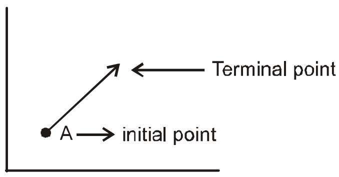
Vectors : Vector and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.
Three Dimensional Geometry : Coordinates of a point in space, distance between two points, section formula, direction ratios and direction cosines, angle between two intersecting lines. skew lines, the shortest distance between them and its equation. Equations of a line and a plane in different forms, intersection of a on line and a plane, coplanar lines.

VECTORS & THREE DIMENSIONAL GEOMETRY

1. INTRODUCTION :

Vector is a quantity which possess both magnitude and direction it is represented by an arrow the direction of which indicates the direction of the quantity and length of which is the magnitude. There is also an additional requirement that such quantity obey the vector law of addition e.g. velocity, force etc. A quantity such as mass, length, time that does not involve the concept of direction is called scalar quantity.

2. MATHEMATICAL DESCRIPTION :



A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

The line of unlimited length of which a directed line segment is a part is called its line of support. Any real number is a scalar.

3. TYPES OF VECTORS :

Zero vector : A vector of zero magnitude i.e. which has the same initial and terminal point, is called a **zero vector.** It is denoted by **O**. **The direction of zero vector is indeterminate.**

Unit vector : A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$, symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Equal vectors : Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.

Collinear vectors : Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called parallel vectors. If they have the same direction they are named as like vectors otherwise unlike vectors.

Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Note that in mathematical frame, vectors are treated as free vector so they can move parallel to themselves.

Coplanar vectors : A given number of vectors are called coplanar if their directed line segments are all parallel to the same plane. Note that two vectors are always coplanar.

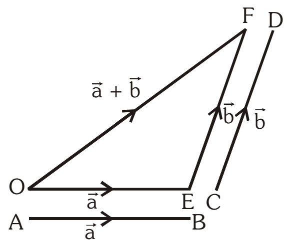
Co-initial vectors : Vectors having same initial point are called Co-initial Vectors.

4. MULTIPLICATION OF A VECTOR BY A SCALAR :

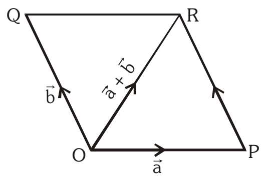
If $\vec{a}$ is a vector and m is a scalar, then m$\vec{a}$ is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called scalar multiplication.

5. ADDITION OF VECTORS :

Triangle law of addition of vectors : If two vectors can be represented in magnitude and direction by the two sides of a triangle, taken in order, then their sum will be represented by the third side in reverse order.



Parallelogram law of addition of vectors : If two vectors be represented in magnitude and direction by the two adjacent sides of a parallelogram then their sum will be represented by the diagonal through the co-initial point.



Subtraction of Vectors : Vector $-\vec{b}$ has length equals to vector $\vec{b}$ but its direction is opposite. Subtraction of vector $\vec{a}$ and $\vec{b}$ is defined as addition of $\vec{a}$ and ($-\vec{b}$). It is written as follows :

$$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$$

Properties of vector addition :

(i) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(ii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(iii) $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$

(iv) $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

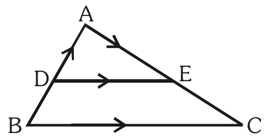
SOLVED EXAMPLE

Example 1 : Prove that the line joining the middle points of two sides of a triangle is parallel to the third side and is of half its length.

Solution : Let the middle points of side AB and AC of a ΔABC be D and E respectively.

$\overrightarrow{BA} = 2\overrightarrow{DA}$ and $\overrightarrow{AC} = 2\overrightarrow{AE}$

Now in ΔABC, by triangle law of addition



$\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$

$2\overrightarrow{DA} + 2\overrightarrow{AE} = \overrightarrow{BC} \Rightarrow \overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}\overrightarrow{BC} \Rightarrow \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

Hence, line DE is parallel to third side BC of triangle and half of it.

Example 2 : If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then find the value of x for which vectors :

$\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3+2x)\vec{a} - 2\vec{b}$ are collinear.

Solution : Since the vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear.

$\therefore$ there exist scalar λ such that $\vec{\alpha} = \lambda\vec{\beta}$

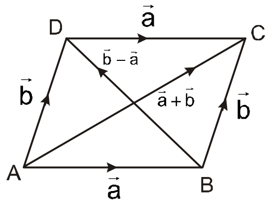
$\Rightarrow$ $(x-2)\vec{a} + \vec{b} = \lambda\{(3+2x)\vec{a} - 2\vec{b}\}$ $\Rightarrow$ $(x-2-\lambda(3+2x))\vec{a} + (1+2\lambda)\vec{b} = \vec{0}$

$\Rightarrow$ $x - 2 - \lambda(3+2x) = 0$ and $1 + 2\lambda = 0$ $\Rightarrow$ $x - 2 - \lambda(3+2x) = 0$ and $\lambda = -\frac{1}{2}$

$\Rightarrow$ $x - 2 + \frac{1}{2}(3+2x) = 0$ $\Rightarrow$ $4x - 1 = 0$ $\Rightarrow$ $x = \frac{1}{4}$. **Ans.**

Example 3 : If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Solution : Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.



Then, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\Rightarrow$ $\overrightarrow{AC} = \vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

$|\overrightarrow{AC}| = \sqrt{9+36+4} = 7$ $\Rightarrow$ $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

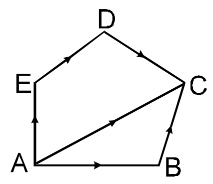
$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \vec{b} - \vec{a} = \hat{i} + 2\hat{j} - 8\hat{k}$ $\Rightarrow$ $|\overrightarrow{BD}| = \sqrt{1+4+64} = \sqrt{69}$

$\therefore$ Unit vector along $\overrightarrow{AC} = \frac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \frac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$ and Unit vector along $\overrightarrow{BD} = \frac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = \frac{1}{\sqrt{69}}$

$\left(\hat{i} + 2\hat{j} - 8\hat{k}\right)$

Example 4 : ABCDE is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Solution : Let $\vec{R}$ be the resultant force



$\therefore$ $\vec{R} = \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$\Rightarrow$ $\vec{R} = (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$

$\Rightarrow$ $\vec{R} = \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$

$\Rightarrow$ $\vec{R} = 3\overrightarrow{AC}$. Hence proved.

Problems for Self Practice-01

(1) ABCD is a parallelogram whose diagonals meet at P. If O is a fixed point, then prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD} = 4\overrightarrow{OP}$

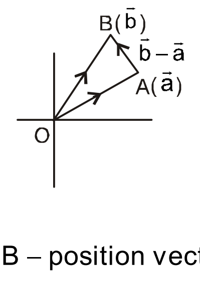
(2) A, B, P, Q, R are five points in any plane. If forces $\overrightarrow{AP},\overrightarrow{AQ},\overrightarrow{AR}$ acts on point A and force $\overrightarrow{PB},\overrightarrow{QB},\overrightarrow{RB}$ acts on point B then find their resultant

(3) Find the unit vector in the direction of $3\hat{i} - 6\hat{j} + 2\hat{k}$.

Answers : (2) $3\overrightarrow{AB}$ (3) $\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}$

6. POSITION VECTOR OF A POINT:

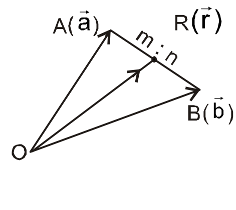
Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then



$\overrightarrow{AB} = \vec{b}-\vec{a}$ = position vector **(p.v.)** of B – position vector **(p.v.)** of A.

7. SECTION FORMULA :

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the p.v. of a point which divides AB in the ratio m: n is given by $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.



SOLVED EXAMPLE

Example 5 : If $A \equiv (2\hat{i} + 3\hat{j})$, $B \equiv (p\hat{i} + 9\hat{j})$ and $C \equiv (\hat{i} - \hat{j})$ are collinear, then find the value of p

Solution : $\overrightarrow{AB} = (p-2)\hat{i} + 6\hat{j}$, $\overrightarrow{AC} = -\hat{i} - 4\hat{j}$

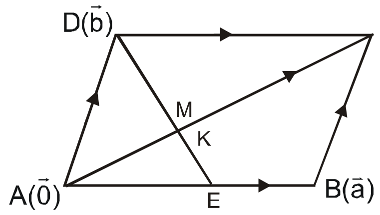
Now A, B, C are collinear $\Leftrightarrow \overrightarrow{AB} \,||\, \overrightarrow{AC} \Leftrightarrow \frac{p-2}{-1} = \frac{6}{-4} \Leftrightarrow p = 7/2$

Example 6 : If ABCD is a parallelogram and E is the mid point of AB. Show by vector method that DE trisect AC and is trisected by AC.

Solution : Let $\overrightarrow{AB} = \vec{a}$ and AD = $\vec{b}$

Then $\overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

Also let K be a point on AC, such that AK : AC = 1 : 3



$\Rightarrow$ AK = $\frac{1}{3}$ AC

$\Rightarrow$ $\overrightarrow{AK} = \frac{1}{3}(\vec{a} + \vec{b})$ ....(i)

Again E being the mid point of AB, we have $\overrightarrow{AE} = \frac{1}{2}\vec{a}$

Let M be the point on DE such that DM : ME = 2 : 1

$\therefore$ $\overrightarrow{AM} = \frac{\overrightarrow{AD} + 2\overrightarrow{AE}}{1+2} = \frac{\vec{b} + \vec{a}}{3}$ ....(ii)

From (i) and (ii) we find that

$\overrightarrow{AK} = \frac{1}{3}(\vec{a} + \vec{b}) = \overrightarrow{AM}$, and so we conclude that K and M coincide. i.e. DE trisect AC and is trisected by AC. Hence proved.

Problems for Self Practice-02

(1) Prove that medians of a triangle are concurrent.

(2) ABCD is a parallelogram. If L, M be the middle point of BC and CD then show that $\overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$.

(3) The position vectors of the points A, B, C, D are $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$, $\hat{i} - 6\hat{j} - \hat{k}$ respectively. Show that the lines AB and CD are parallel and find the ratio of their lengths.

(4) The vertices P, Q and S of a ΔPQS have position vectors $\vec{p}, \vec{q}$ and $\vec{s}$ respectively.

(i) If M is the mid point of PQ, then find position vector of M in terms of $\vec{p}$ and $\vec{q}$

(ii) Find $\vec{t}$, the position vector of T on SM such that ST : TM = 2 : 1, in terms of $\vec{p}, \vec{q}$ and $\vec{s}$.

(iii) If the parallelogram PQRS is now completed. Express $\vec{r}$, the position vector of the point R in terms of $\vec{p}, \vec{q}$ and $\vec{s}$

(5) If $\vec{a}, \vec{b}$ are position vectors of the points(1,–1),(–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

(6) The median AD of a ΔABC is bisected at E and BE is produced to meet the side AC in F. Show that AF = $\frac{1}{3}$ AC and EF = $\frac{1}{4}$BF.

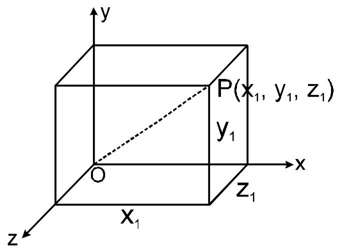
(7) Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio 1 : 3.

Answers : (3) 1 : 2 (4) $\vec{m} = \frac{1}{2}(\vec{p}+\vec{q})$, $\vec{t} = \frac{1}{2}(\vec{p}+\vec{q}+\vec{s})$, $\vec{r} = \frac{1}{2}(\vec{q}+\vec{p}-\vec{s})$ (5) m = 2

8. COORDINATE OF A POINT IN SPACE :

Three mutually perpendicular lines OX, OY, OZ are considered as the three axes. The plane formed with the help of x & y axes is called x-y plane, similarly y & z axes form y-z plane and z & x axes form z - x plane.

Consider any point P in the space, Drop a perpendicular from that point to x -z plane, then the algebraic length of this perpendicular is considered as y-coordinate and from foot of the perpendicular drop perpendiculars to x and z axes. These algebraic lengths of perpendiculars are considered as z and x coordinates respectively.



If coordinate of a point P in space is (x, y, z), then the position vector of the point P with respect to the same origin is $x\hat{i} + y\hat{j} + z\hat{k}$.

Distance formula

Distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

We know that if position vector of two points A and B are given as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ then

AB = $|\overrightarrow{OB} - \overrightarrow{OA}|$ $\Rightarrow$ AB = $|(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})|$

$\Rightarrow$ AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

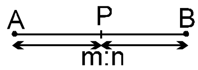
Distance of a point P from coordinate axes

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then

PA = $\sqrt{y^2+z^2}$, PB = $\sqrt{z^2+x^2}$, PC = $\sqrt{x^2+y^2}$

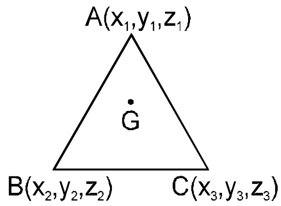
Section formula

If point P divides the distance between the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio of m : n, internally then coordinates of P are given as $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$



Centroid of a triangle

$$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$



Incentre of triangle ABC

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right)$$

Where AB = c, BC = a, CA = b

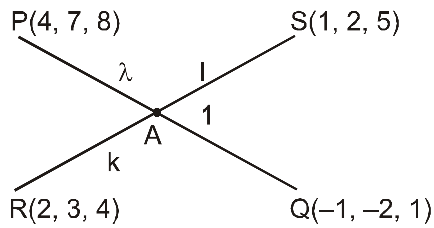
Centroid of a tetrahedron

A (x_1, y_1, z_1) B (x_2, y_2, z_2) C (x_3, y_3, z_3) and D (x_4, y_4, z_4) are the vertices of a tetrahedron, then coordinate of its centroid (G) is given as $\left(\frac{\sum_{i=1}^{4} x_i}{4}, \frac{\sum_{i=1}^{4} y_i}{4}, \frac{\sum_{i=1}^{4} z_i}{4}\right)$

SOLVED EXAMPLE

Example 7 : If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1,2,5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Solution : Let the lines PQ and RS intersect at point A.



Let A divide PQ in the ratio λ : 1, then

$$A \equiv \left(\frac{-\lambda+4}{\lambda+1}, \frac{-2\lambda+7}{\lambda+1}, \frac{\lambda+8}{\lambda+1}\right). \quad \text{.... (1)}$$

Let A divide RS in the ratio k : 1, then

$$A \equiv \left(\frac{k+2}{k+1}, \frac{2k+3}{k+1}, \frac{5k+4}{k+1}\right) \quad \text{..... (2)}$$

From (1) and (2), we have,

$$\frac{-\lambda+4}{\lambda+1}=\frac{k+2}{k+1} \qquad ..... (3)$$

$$\frac{-2\lambda+7}{\lambda+1}=\frac{2k+3}{k+1} \qquad ..... (4)$$

$$\frac{\lambda+8}{\lambda+1}=\frac{5k+4}{k+1} \qquad ..... (5)$$

From (3), $-\lambda k - \lambda + 4k + 4 = \lambda k + 2\lambda + k + 2$

or $\quad 2\lambda k + 3\lambda - 3k - 2 = 0$ ..... (6)

From (4), $-2\lambda k - 2\lambda + 7k + 7 = 2\lambda k + 3\lambda + 2k + 3$

or $\quad 4\lambda k + 5\lambda - 5k - 4 = 0$ ..... (7)

Multiplying equation (6) by 2, and subtracting from equation (7), we get

$-\lambda + k = 0$ or , $\lambda = k$

Putting $\lambda = k$ in equation (6), we get $2\lambda^2 + 3\lambda - 3\lambda - 2 = 0$

or, $\lambda = \pm 1$.

But $\lambda \neq -1$, as the co-ordinates of P would be undefined and in this case PQ || RS, which is not true.

$\therefore \quad \lambda = 1 = k$.

Clearly $\lambda = k = 1$ satisfies eqn. (5).

Hence our assumption is correct

$$\therefore \quad A \equiv \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{1+8}{2}\right) \quad \text{or,} \quad A \equiv \left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right).$$

Example 8 : Find the ratio in which the XY plane divides the line joining the points P(–2, 4, 7) and Q(3, –5, 8).

Solution : Let line PQ cuts XY plane at point R which divides PQ is ratio λ:1 internally

$\because R\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right)$ lies on xy plane so $8\lambda + 7 = 0$

$\Rightarrow \lambda = -7/8$

$\Rightarrow$ xy plane divides line PQ in ratio 7 : 8 externally

Example 9 : Prove by using distance formula that the points P (1, 2, 3), Q (–1, –1, –1) and R(3, 5, 7) are collinear.

Solution : We have PQ = $\sqrt{(-1-1)^2+(-1-2)^2+(-1-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

QR $= \sqrt{(3+1)^2+(5+1)^2+(7+1)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$

and PR = $\sqrt{(3-1)^2+(5-2)^2+(7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since QR = PQ + PR. Therefore the given points are collinear.

Example 10 : Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Solution : Given A ≡ (2, 3, 4), B ≡ (–1, 2, –3), C ≡ (– 4, 1, –10).

Let C divide AB internally in the ratio k : 1, then

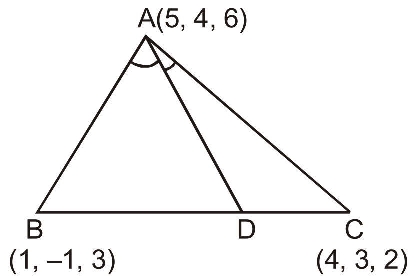
$$C \equiv \left(\frac{-k+2}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+4}{k+1}\right)$$

$$\therefore \quad \frac{-k+2}{k+1} = -4 \quad \Rightarrow \quad 3k = -6 \Rightarrow \quad k = -2$$

For this value of k, $\frac{2k+3}{k+1} = 1$, and $\frac{-3k+4}{k+1} = -10$

Since k < 0, therefore C divides AB externally in the ratio 2 :1 and points A, B, C are collinear.

Example 11 : The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of $\angle$BAC meets BC in D. Find AD.



Solution : AB = $\sqrt{4^2+5^2+3^2} = 5\sqrt{2}$

AC = $\sqrt{1^2+1^2+4^2} = 3\sqrt{2}$

Since AD is the internal bisector of $\angle$BAC

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3}$$

$\therefore$ D divides BC internally in the ratio 5 : 3

$$\therefore \quad D \equiv \left(\frac{5\times4+3\times1}{5+3}, \frac{5\times3+3(-1)}{5+3}, \frac{5\times2+3\times3}{5+3}\right) \qquad \text{or,} \qquad D = \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$$

$$\therefore \quad AD = \sqrt{\left(5-\frac{23}{8}\right)^2 + \left(4-\frac{12}{8}\right)^2 + \left(6-\frac{19}{8}\right)^2}$$

$$= \frac{\sqrt{1530}}{8} \text{ unit}$$

<u>Problems for Self Practice</u>-03

(1) Find the locus of a point which moves such that the sum of its distances from points A(0, 0, – α) and B(0, 0, α) is constant.

(2) Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

(3) Find the locus of point P if $AP^2 - BP^2 = 18$, where A ≡ (1, 2, – 3) and B ≡ (3, – 2, 1).

(4) Show that the points (0, 7, 10), (– 1, 6, 6) and (– 4, 9, 6) form a right angled isosceles triangle.

(5) Find the co-ordinates of the foot of perpendicular drawn from the point A(1, 2, 1) to the line joining the point B(1, 4, 6) and C(5, 4, 4).

(6) Two vertices of a triangle are (4, –6, 3) and (2, –2, 1) and its centroid is $\left(\frac{8}{3}, -1, 2\right)$. Find the third vertex.

(7) If centroid of the tetrahedron OABC, where co-ordinates of A, B, C are (a, 2, 3), (1, b, 2) and (2, 1, c) respectively be (1, 2, 3), then find the distance of point (a, b, c) from the origin, where O is the origin.

(8) Show that $\left(-\frac{1}{2}, 2, 0\right)$ is the circumcentre of the triangle whose vertices are A (1, 1, 0), B (1, 2, 1) and C (– 2, 2, –1) and hence find its orthocentre.

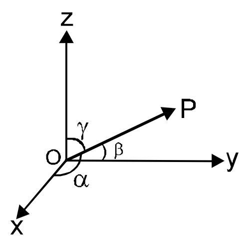
Answers : (1) $\frac{x^2}{a^2-\alpha^2} + \frac{y^2}{a^2-\alpha^2} + \frac{z^2}{a^2} = 1$ (3) $2x - 4y + 4z - 9 = 0$

(5) (3, 4, 5) (6) (2, 5, 2) (7) $\sqrt{75}$ (8) (1, 1, 0)

9. DIRECTION COSINES AND DIRECTION RATIOS :

(i) **Direction cosines :** Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by ℓ, m, n.

Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.



(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) **Direction ratios :** Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L, then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L, then $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\left(\ell = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

or $$\ell = \frac{-a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{-b}{\sqrt{a^2+b^2+c^2}},\ n = \frac{-c}{\sqrt{a^2+b^2+c^2}}$$

(v) If OP = r, when O is the origin and the direction cosines of OP are ℓ, m, n then the coordinates of P are (ℓr, mr, nr).

If direction cosines of the line AB are ℓ, m,n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1) then the coordinates of B is given as $(x_1 + r\ell, y_1 + rm, z_1 + rn)$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

(vii) **Direction cosines of axes :** Since the positive x–axis makes angles 0º, 90º, 90º with axes of x, y and z respectively. Therefore

Direction cosines of x–axis are (1, 0, 0)

Direction cosines of y–axis are (0, 1, 0)

Direction cosines of z–axis are (0, 0, 1)

SOLVED EXAMPLE

Example 12 : If a line makes angles α, β, γ with the co-ordinate axes, prove that $\sin^2\alpha + \sin^2\beta + \sin\gamma^2 = 2$.

Solution : Since a line makes angles α, β, γ with the co-ordinate axes,

hence $\cos\alpha, \cos\beta, \cos\gamma$ are its direction cosines

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$

Example 13: Find the direction cosines ℓ, m, n of a line which are connected by the relations $\ell + m + n = 0$, $2mn + 2m\ell - n\ell = 0$

Solution : Given, $\ell + m + n = 0$ ..... (1)

$2mn + 2m\ell - n\ell = 0$ ..... (2)

From (1), $n = -(\ell + m)$.

Putting $n = -(\ell + m)$ in equation (2), we get,

$-2m(\ell + m) + 2m\ell + (\ell + m)\ell = 0$

or, $\quad -2m\ell - 2m^2 + 2m\ell + \ell^2 + m\ell = 0$

or, $\quad \ell^2 + m\ell - 2m^2 = 0$

or, $\quad \left(\frac{\ell}{m}\right)^2 + \left(\frac{\ell}{m}\right) - 2 = 0 \quad$ [dividing by m^2]

or $\quad \frac{\ell}{m} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$

Case I. when $\frac{\ell}{m} = 1$: In this case $m = \ell$

From (1), $2\ell + n = 0 \quad \Rightarrow \quad n = -2\ell$

$\therefore \quad \ell : m : n = 1 : 1 : -2$

$\therefore$ Direction ratios of the line are 1, 1, – 2

$\therefore$ Direction cosines are

$$\pm \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \pm \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \pm \frac{-2}{\sqrt{1^2 + 1^2 + (-2)^2}}$$

$\Rightarrow \quad \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ or $-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

Case II. When $\frac{\ell}{m} = -2$: In this case $\ell = -2m$

From (1), $-2m + m + n = 0 \quad \Rightarrow \quad n = m$

$\therefore \quad \ell : m : n = -2m : m : m$

$= -2 : 1 : 1$

$\therefore$ Direction ratios of the line are – 2, 1, 1.

$\therefore$ Direction cosines are

$$\pm\frac{-2}{\sqrt{(-2)^2+1^2+1^2}}, \pm\frac{1}{\sqrt{(-2)^2+1^2+1^2}}, \pm\frac{1}{\sqrt{(-2)^2+1^2+1^2}}$$

$$\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \quad \text{or} \quad \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$$

Problems for Self Practice-04

(1) Find the direction cosines of a line lying in the xy plane and making angle 30° with x-axis.

(2) A line makes an angle of 60° with each of x and y axes, find the angle which this line makes with z-axis.

(3) A plane intersects the co-ordinates axes at point A(a, 0, 0), B(0, b, 0), C(0, 0, c) ; O is origin. Find the direction ratio of the line joining the vertex B to the centroid of face AOC.

(4) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

Answers : (1) $\ell = \frac{\sqrt{3}}{2}$, $m = \pm\frac{1}{2}$, $n = 0$ (2) 45° (3) $\frac{a}{3}, -b, \frac{c}{3}$

10. ANGLE BETWEEN TWO VECTORS :

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. Note that $0^\circ \le \theta \le 180^\circ$.

11. SCALAR PRODUCT (DOT PRODUCT) OF TWO VECTORS :

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta, (0 \le \theta \le \pi)$

Properties of Dot product :

(i) If θ is acute, then $\vec{a}\,.\,\vec{b} > 0$ and if θ is obtuse, then $\vec{a}.\vec{b} < 0$.

$\vec{a}\,.\,\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}\ (\vec{a} \neq 0\,,\,\vec{b} \neq 0)$

(ii) $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ **(commutative)**

(iii) $\vec{a}\,.\,(\vec{b}+\vec{c}) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$ **(distributive)**

(iv) $\hat{i}\,.\,\hat{i} = \hat{j}\,.\,\hat{j} = \hat{k}\,.\,\hat{k} = 1;\quad \hat{i}\,.\,\hat{j} = \hat{j}\,.\,\hat{k} = \hat{k}\,.\,\hat{i} = 0$

(v) $\vec{a}\,.\,\vec{a} = |\vec{a}|^2 = \vec{a}^2$

(vi) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\,.\,\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

(vii) $|\vec{a} \pm \vec{b}|^2 = a^2 + b^2 \pm 2\vec{a}.\vec{b}$

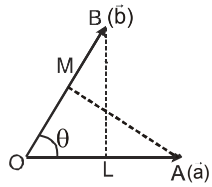
$|\vec{a}+\vec{b}+\vec{c}|^2 = a^2 + b^2 + c^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

12. GEOMETRICAL INTERPRETATION OF SCALAR PRODUCT :

Let $\vec{a}$ and $\vec{b}$ be vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw BL $\perp$ OA and AM $\perp$ OB.

From ΔOBL and ΔOAM, we have OL = OB cos θ and OM = OA cos θ.

Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.



Now, $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = |\vec{a}|(|\vec{b}|\cos\theta)$

$= |\vec{a}|\,(\text{OB} \cos\theta) = |\vec{a}|\,(\text{OL})$

= (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ....(i)

Again $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = |\vec{b}|\,(|\vec{a}|\cos\theta) = |\vec{b}|\,(\text{OA}\cos\theta) = |\vec{b}|\,(\text{OM})$

= (magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) ....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Projection of a line segment on a line

If the coordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

Note that $\ell\,|\vec{r}|$, m $|\vec{r}|$ & n $|\vec{r}|$ are the projection of $\vec{r}$ in OX, OY & OZ axes.

SOLVED EXAMPLE

Example 14 : Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Solution : (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow (3\hat{i} + 2\hat{j} + 9\hat{k}) . (\hat{i} + p\hat{j} + 3\hat{k}) = 0$

$\Rightarrow 3 + 2p + 27 = 0 \Rightarrow p = -15$

(ii) vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \quad \Rightarrow 3 = \frac{2}{p} \quad \Rightarrow p = \frac{2}{3}$$

Example 15 : If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution : We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c} \quad \Rightarrow (\vec{a} + \vec{b}) . (\vec{a} + \vec{b}) = (-\vec{c}) . (-\vec{c})$

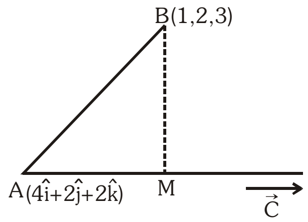
$\Rightarrow \left|\vec{a} + \vec{b}\right|^2 = |\vec{c}|^2 \quad \Rightarrow \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}.\vec{b} = \left|\vec{c}\right|^2$

$\Rightarrow \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left|\vec{a}\right|\left|\vec{b}\right| \cos\theta = \left|\vec{c}\right|^2$

$\Rightarrow 9 + 25 + 2(3)(5)\cos\theta = 49 \quad \Rightarrow \cos\theta = \frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{3}$.

Example 16 : Find the distance of the point B($\hat{i} + 2\hat{j} + 3\hat{k}$) from the line which is passing through A($4\hat{i} + 2\hat{j} + 2\hat{k}$) and which is parallel to the vector $\vec{C} = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

Solution : AB = $\sqrt{3^2 + 1^2} = \sqrt{10}$



$$AM = \overline{AB}.\hat{c} = (-3\hat{i} + \hat{k}).\frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7}$$

= – 6 + 6 = 0

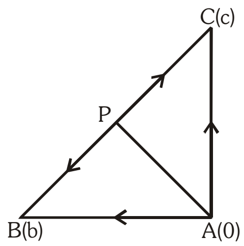
$BM^2 = AB^2 - AM^2$

So, BM = AB = $\sqrt{10}$ **Ans.**

Example 17 : Prove that the medians to the base of an isosceles triangle is perpendicular to the base.

Solution : The triangle being isosceles, we have

AB = AC .... (i)



Now $\overline{AP} = \frac{\vec{b} + \vec{c}}{2}$ where P is mid-point of BC.

Also $\overline{BC} = \vec{c} - \vec{b}$

$\therefore \quad \overrightarrow{AP}.\overrightarrow{BC} = \frac{\vec{b}+\vec{c}}{2}.(\vec{c}-\vec{b}) = \frac{1}{2}(c^2 - b^2)$

$= \frac{1}{2}(AC^2 - AB^2) = 0$ {by (i)}

$\therefore$ Median AP is perpendicular to base BC.

Example 18: If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$. (ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$.

Solution : (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

Here $\vec{a} \,.\, \vec{b} = 2 - 1 + 3 = 4$

$|\vec{a}|^2 = 3$

Hence $\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{4}{3}\vec{a} = \frac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$ is $\vec{b} - \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$.

$= \frac{1}{3}\left(2\hat{i} - 7\hat{j} + 5\hat{k}\right)$

Example 19 : Find the length of projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Solution : The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

$\therefore \quad \ell = \frac{6}{7},\ m = \frac{2}{7}, n = \frac{3}{7}$

The required length of projection is given by

$= |\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)| = \left|\frac{6}{7}[2-(-1)] + \frac{2}{7}(5-0) + \frac{3}{7}(1-3)\right|$

$= \left|\frac{6}{7}\times 3 + \frac{2}{7}\times 5 + \frac{3}{7}\times -2\right| = \left|\frac{18}{7} + \frac{10}{7} - \frac{6}{7}\right| = \left|\frac{18+10-6}{7}\right| = \frac{22}{7}$

Problems for Self Practice-05

(1) Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\,\hat{i} + 4x\,\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\,\hat{k}$ is obtuse.

(2) Find the values of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

(3) Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse.

(4) D is the mid point of the side BC of a ΔABC, show that $AB^2 + AC^2 = 2\,(AD^2 + BD^2)$

(5) ABCD is a tetrahedron and G is the centroid of the base BCD. Prove that
$AB^2 + AC^2 + AD^2 = GB^2 + GC^2 + GD^2 + 3GA^2$

(6) A (6, 3, 2), B (5, 1, 1,), C(3, –1, 3) D (0, 2, 5). Find the projection of line segment AB on CD line.

(7) The projections of a directed line segment on co-ordinate axes are – 2, 3, – 6. Find its length and direction cosines.

(8) Find the projection of the line segment joining (2, – 1, 3) and (4, 2, 5) on a line which makes equal acute angles with co-ordinate axes.

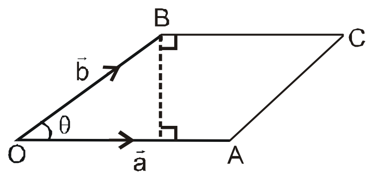
Answers : (1) $x \in (0, 1/2)$ (2) $x = -\dfrac{31}{12}$, $y = \dfrac{41}{12}$ (3) $(-3, -2) \cup (2, 3)$

(6) $\dfrac{5}{\sqrt{22}}$ (7) 7, $\dfrac{-2}{7}, \dfrac{3}{7}, \dfrac{-6}{7}$ (8) $\dfrac{7}{\sqrt{3}}$

13. VECTOR PRODUCT (CROSS PRODUCT) OF TWO VECTORS:

(i) If $\vec{a}, \vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(iv) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ **(distributive)**

(v) $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0\ ,\ \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(vi) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$; $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

(vii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(viii) Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

(ix) If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

(x) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of $\Delta ABC = \dfrac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$. The points A, B and C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

(xi) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

(xii) **Lagrange's Identity :** For any two vectors $\vec{a}$ and $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

SOLVED EXAMPLE

Example 20 : Find the vectors of magnitude 5 which are perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution : Unit vectors perpendicular to $\vec{a}$ & $\vec{b} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$\because$ $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$\therefore$ $$\text{Unit vectors} = \pm\frac{(-5\hat{i} - 5\hat{j} - 5\hat{k})}{5\sqrt{3}}$$

Hence the required vectors are $\pm\dfrac{5\sqrt{3}}{3}(\hat{i} + \hat{j} + \hat{k})$ **Ans.**

Example 21 : For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = \vec{0}$.

Solution : We have, $\vec{a} \times (\vec{b}+\vec{c}) + \vec{b} \times (\vec{c}+\vec{a}) + \vec{c} \times (\vec{a}+\vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$ [Using distributive law]

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} - \vec{a}\times\vec{c} - \vec{b}\times\vec{c} = \vec{0}$ [$\because$ $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$ etc]

Example 22 : For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2|\vec{a}|^2$

Solution : Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$\vec{a}\times\hat{i} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) \times \hat{i} = a_1(\hat{i}\times\hat{i}) + a_2(\hat{j}\times\hat{i}) + a_3(\hat{k}\times\hat{i}) = -a_2\hat{k}+a_3\hat{j}$

$\Rightarrow \quad |\vec{a}\times\hat{i}|^2 = a_2^2 + a_3^2$

$\vec{a}\times\hat{j} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) \times \hat{j} = a_1\hat{k}-a_3\hat{i}$

$\Rightarrow \quad |\vec{a}\times\hat{j}|^2 = a^2{}_1 + a_3^2$

$\vec{a}\times\hat{k} = (a_i\hat{i}+a_2\hat{j}+a_3\hat{k}) \times \vec{k} = -a_i\hat{j}+a_2\hat{i}$

$\Rightarrow \quad |\vec{a}\times\hat{k}|^2 = a_1^2 + a_2^2$

$\therefore \quad |\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = a_2^2 + a_3^3 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$= 2(a_1^2 + a_2^2 + a_3^2) = 2|\vec{a}|^2$

Example 23 : Let $\vec{a}$ & $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a}.\vec{b})\vec{b}$ & $\vec{v} = (\vec{a}\times\vec{b})$, then prove that

$|\vec{v}| = |\vec{u}| + |\vec{u}.\vec{b}|$

Solution : $\vec{u}.\vec{a} = \vec{a}.\vec{a} - (\vec{a}.\vec{b})(\vec{a}.\vec{b})$

$= 1 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$= 1 - \cos^2\theta = \sin^2\theta$

$= |\vec{v}| = |\vec{a}\times\vec{b}| = \sin\theta$

$|\vec{u}| = \sqrt{\vec{u}.\vec{u}} = \sqrt{\vec{a}.\vec{a} - 2(\vec{a}.\vec{b})^2 + (\vec{a}.\vec{b})^2|\vec{b}|^2} = \sqrt{1-(\vec{a}.\vec{b})^2} = \sin\theta$

$\therefore \quad |\vec{v}| = |\vec{u}|$ also $\vec{u}.\vec{b} = 0$

Hence, $|\vec{v}| = |\vec{u}| = |\vec{u}| + |\vec{u}.\vec{b}|$

<u>Problems for Self Practice</u>-06

(1) If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30º. Find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

(2) Prove that the normal to the plane containing the three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lies in the direction $\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}$

(3) Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = 10\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that p = 6q.

Answers : (1) 3/4 sq. units

14. EQUATION OF A STRAIGHT LINE

(i) Equation of a straight line passing through a fixed point with position vector $\vec{p}$ and parallel to a given vector $\vec{q}$ is $\vec{r} = \vec{p} + \lambda\vec{q}$ where λ is a scalar.

Let $\vec{p} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{q} = a\hat{i} + b\hat{j} + c\hat{k}$ then we get $x = x_1 + \lambda a$, $y = y_1 + \lambda b$, $z = z_1 + \lambda c$.

The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda$. This form is called symmetric form. A general point on the line is given by $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c r)$.

(ii) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

(iii) Equation of x-axis is y = 0 and z = 0

(iv) Equation of a line parallel to x–axis is y = p, z = q

Angle between two line segments : If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we can consider two vectors parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

(i) The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Two parallel lines have same direction cosines i.e. $\ell_1 = \ell_2$, $m_1 = m_2$, $n_1 = n_2$

15. ANGLE BISECTORS :

A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$.

Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a} + \hat{b}\right)$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a} - \hat{b}\right)$, $\lambda \in R^+$.

Note that the equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are : $\vec{r} = \vec{a} + t\left(\hat{b}+\hat{c}\right)$ and $\vec{r} = \vec{a} + p\left(\hat{c}-\hat{b}\right)$.

SOLVED EXAMPLE

Example 24 : Find the equation of the line through the points (3, 4, –7) and (1, – 1, 6) in vector form as well as in cartesian form.

Solution : Let $A \equiv (3, 4, -7)$, $B \equiv (1, -1, 6)$

Now $\vec{a} = \overrightarrow{OA} = 3\hat{i} + 4\hat{j} - 7\hat{k}$,

$\vec{b} = \overrightarrow{OB} = \hat{i} - \hat{j} + 6\hat{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + t(-2\hat{i} - 5\hat{j} + 13\hat{k})$ ..... (1)

Equation in cartesian form :

Equation of AB is $\frac{x-3}{3-1} = \frac{y-4}{4+1} = \frac{z+7}{-7-6}$ or, $\frac{x-3}{2} = \frac{y-4}{5} = \frac{z+7}{-13}$.

Example 25: Find the co-ordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which is at a distance of 3 units from point (1, – 2, 3).

Solution : Given line is $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ ..... (1)

Let $P \equiv (1, -2, 3)$

Direction ratios of line (1) are 2, 3, 6

$\therefore$ Direction cosines of line (1) are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

Equation of line (1) may be written as $\frac{x-1}{\frac{2}{7}} = \frac{y+2}{\frac{3}{7}} = \frac{z-3}{\frac{6}{7}}$ ..... (2)

Co-ordinates of any point on line (2) may be taken as $\left(\frac{2}{7}r+1,\ \frac{3}{7}r-2\ ,\ \frac{6}{7}r+3\right)$

Let $Q \equiv \left(\frac{2}{7}r+1,\ \frac{3}{7}r-2,\ \frac{6}{7}r+3\right)$

Distance of Q from P = | r |

According to question | r | = 3

$\therefore$ r = ± 3

Putting the value of r, we have

$Q \equiv \left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right)$ or $Q \equiv \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$

Example 26 : Find the equation of the line drawn through point (1, 0, 2) to meet at right angles the line

$$\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$$

Solution : Given line is $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ ..... (1)

Let $P \equiv (1, 0, 2)$

Co-ordinates of any point on line (1) may be taken as

$Q \equiv (3r - 1, -2r + 2, -r - 1)$

Direction ratios of PQ are $3r - 2, -2r + 2, -r - 3$

Direction ratios of line AB are $3, -2, -1$

Since $PQ \perp AB$

$\therefore$ $3(3r - 2) - 2(-2r + 2) - 1(-r - 3) = 0$

$\Rightarrow$ $9r - 6 + 4r - 4 + r + 3 = 0$ $\Rightarrow 14r = 7$ $\Rightarrow r = \frac{1}{2}$

Therefore, direction ratios of PQ are $-\frac{1}{2}, 1, -\frac{7}{2}$ or, $-1, 2, -7$

Equation of line PQ is $\frac{x-1}{-1}=\frac{y-0}{2}=\frac{z-2}{-7}$ or, $\frac{x-1}{1}=\frac{y}{-2}=\frac{z-2}{7}$

Example 27 : Show that the two lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}$ = z intersect. Find also the point of intersection of these lines.

Solution : Given lines are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ ..... (1)

and $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}$ ..... (2)

Any point on line (1) is $P(2r + 1, 3r + 2, 4r + 3)$

and any point on line (2) is $Q(5\lambda + 4, 2\lambda + 1, \lambda)$

Lines (1) and (2) will intersect if P and Q coincide for some value of λ and r.

$\therefore$ $2r + 1 = 5\lambda + 4$ $\Rightarrow$ $2r - 5\lambda = 3$ ..... (3)

$3r + 2 = 2\lambda + 1$ $\Rightarrow$ $3r - 2\lambda = -1$ ..... (4)

$4r + 3 = \lambda$ $\Rightarrow$ $4r - \lambda = -3$ ..... (5)

Solving (3) and (4), we get $r = -1, \lambda = -1$

Clearly these values of r and λ satisfy eqn. (5)

Now $P \equiv (-1, -1, -1)$

Hence lines (1) and (2) intersect at $(-1, -1, -1)$.

Example 28 : What is the angle between the lines whose direction cosines are

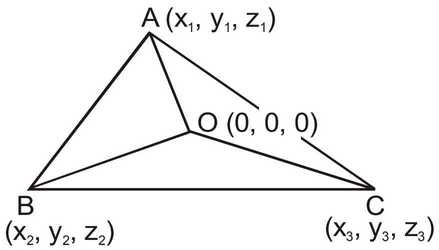
$$-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2} \text{ and } -\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}.$$

Solution : Let θ be the required angle, then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right)+\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)+\left(-\frac{\sqrt{3}}{2}\right).\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}+\frac{1}{16}-\frac{3}{4}=-\frac{1}{2} \quad \Rightarrow \quad \theta = 120°,$$

Example 29 : If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular.

Solution : Let OABC be the tetrahedron, where O is the origin and co-ordinates of A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, x_3) respectively.



Let $OA \perp BC$ and $OB \perp CA$.

We have to prove that $OC \perp BA$.

Now, direction ratios of OA are $x_1 - 0, y_1 - 0, z_1 - 0$ or, x_1, y_1, z_1

direction ratios of BC are $(x_3 - x_2), (y_3 - y_2), (z_3 - z_2)$.

$\because$ $OA \perp BC$.

$\therefore$ $x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$ ..... (1)

Similarly,

$\because$ $OB \perp CA$

$\therefore$ $x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$ ..... (2)

Adding equations (1) and (2), we get

$x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ ($\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$)

Problems for Self Practice-07

(1) Find the equation of the line passing through point (1, 0, 2) having direction ratio 3, – 1, 5. Prove that this line passes through (4, – 1, 7).

(2) Find the equation of the line parallel to line $\frac{x-2}{3}=\frac{y+1}{1}=\frac{z-7}{9}$ and passing through the point (3, 0, 5).

(3) Find the coordinates of the point when the line through (3, 4, 1) and (5, 1, 6) crosses the xy plane.

(4) Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$

(5) Let P (6, 3, 2), Q (5, 1, 4), R (3, 3, 5) are vertices of a Δ find $\angle Q$.

(6) Show that the direction cosines of a line which is perpendicular to the lines having directions cosines $\ell_1\, m_1\, n_1$ and $\ell_2\, m_2\, n_2$ respectively are proportional to $m_1n_2 - m_2n_1,\ n_1\ell_2 - n_2\ell_1,\ \ell_1m_2 - \ell_2m_1$

Answers : (1) $\frac{x-1}{3} = \frac{y}{-1} = \frac{z-2}{5}$ (2) $\frac{x-3}{3} = \frac{y}{1} = \frac{z-5}{9}$

(3) $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ (4) 60° (5) 90°

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16. FOOT AND LENGTH OF PERPENDICULAR FROM A POINT TO A LINE :

(i) Cartesian form : Let equation of the line be $\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n} = r$ (say) ....(i)

and A (α, β, γ) be the point.

Any point on line (i) is P $(\ell r + a, mr + b, nr + c)$ .... (ii)

If it is the foot of the perpendicular from A on the line, then AP is perpendicular to the line. So $\ell(\ell r + a - \alpha) + m(mr + b - \beta) + n(nr + c - \gamma) = 0$ i.e. $r = (\alpha - a)\ell + (\beta - b)m + (\gamma - c)n$ since $\ell^2 + m^2 + n^2 = 1$. Putting this value of r in (ii), we get the foot of perpendicular from point A on the given line. Since foot of perpendicular P is known, then the length of perpendicular is given by

$$AP = \sqrt{(\ell r + a - \alpha)^2 + (mr + b - \beta)^2 + (nr + c - \gamma)^2}$$

(ii) Vector Form : Equation of a line passing through a point having position vector $\vec{\alpha}$ and perpendicular to the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is parallel to $\vec{b}_1 \times \vec{b}_2$. So the vector equation of such a line is $\vec{r} = \vec{\alpha} + \lambda(\vec{b}_1 \times \vec{b}_2)$. Position vector $\vec{\beta}$ of the image of a point $\vec{\alpha}$ in a straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ is given by $\vec{\beta} = 2\vec{a} - \left[\frac{2(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right]\vec{b} - \vec{\alpha}$. Position vector of the foot of the perpendicular on line is $\vec{f} = \vec{a} - \left[\frac{(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right]\vec{b}$. The equation of the perpendicular is $\vec{r} = \vec{\alpha} + \mu\left[(\vec{a}-\vec{\alpha}) - \left(\frac{(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right)\vec{b}\right]$.

SOLVED EXAMPLE

Example 30 : Find the length of the perpendicular from P (2, – 3, 1) to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$.

Solution : Given line is $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$ ..... (1)

$P \equiv (2, -3, 1)$

Co-ordinates of any point on line (1) may be taken as $Q \equiv (2r-1, 3r+3, -r-2)$

Direction ratios of PQ are $2r-3, 3r+6, -r-3$

Direction ratios of AB are 2, 3, – 1

Since $PQ \perp AB$

$\therefore$ $2(2r-3)+3(3r+6)-1(-r-3)=0$

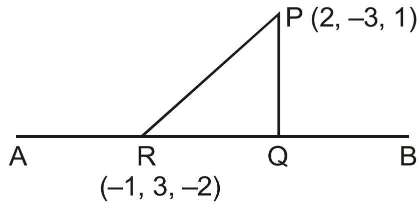
or, $14r+15=0$ $\therefore$ $r=\frac{-15}{14}$

$\therefore$ $Q \equiv \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right)$ $\therefore$ $PQ=\sqrt{\frac{531}{14}}$ units.

Second method : Given line is $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$

$P \equiv (2, -3, 1)$

Direction ratios of line (1) are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}$



RQ = length of projection of RP on AB

$=\left|\frac{2}{\sqrt{14}}(2+1)+\frac{3}{\sqrt{14}}(-3-3)-\frac{1}{\sqrt{14}}(1+2)\right|=\frac{15}{\sqrt{14}}$

$PR^2 = 3^2 + 6^2 + 3^2 = 54$

$\therefore$ $PQ=\sqrt{PR^2-RQ^2}=\sqrt{54-\frac{225}{14}}=\sqrt{\frac{531}{14}}$ units.

Problems for Self Practice-08

(1) Find the length and foot of perpendicular drawn from point (2, –1, 5) to the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$. Also find the image of the point in the line.

(2) Find the image of the point (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

(3) Find the foot and hence the length of perpendicular from (5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. Find also the equation of the perpendicular.

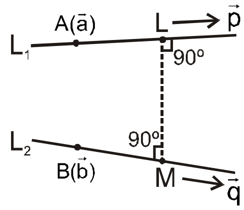
Answers : (1) $\sqrt{14}$, $N \equiv (1, 2, 3)$, $I \equiv (0, 5, 1)$

(2) (1, 0 , 7)

(3) (9, 13, 15) ; 14 ; $\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}$

17. SHORTEST DISTANCE BETWEEN TWO LINES :

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect and are also not parallel are called **skew line**. For Skew lines the direction of the shortest distance would be perpendicular to both the lines.



Let $\overrightarrow{LM}$ be the shortest distance vector between the lines L_1 and L_2. Then $\overrightarrow{LM}$ is perpendicular to both $\vec{p}$ and $\vec{q}$ i.e. $\overrightarrow{LM}$ is parallel to $\vec{p} \times \vec{q}$. Therefore the magnitude of the shortest distance vector (i.e. $|\overrightarrow{LM}|$) would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance.

$$\therefore \quad |\overrightarrow{LM}| = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}\right| = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \vec{p} \times \vec{q}\right|$$

$$= \left|\frac{\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \left|\frac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(i) If the two lines directed along $\vec{p}$ and $\vec{q}$ then $\left[\vec{b}-\vec{a}\ \ \vec{p}\ \ \vec{q}\right] = 0$. Lines are skew if $\left[\vec{b}-\vec{a}\ \ \vec{p}\ \ \vec{q}\right] \neq 0$

(ii) If two parallel lines are given by $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, then distance (d) between them is given by $d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

(iii) The straight lines which are not parallel and non–coplanar i.e. non–intersecting are called skew lines. If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$,

then lines $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ & $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$ are skew lines.

SOLVED EXAMPLE

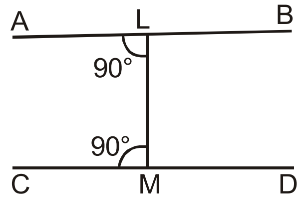
Example 31 : Find the shortest distance and the vector equation of the line of shortest distance between the lines given by

$$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} - \hat{j} + \hat{k}\right) \quad \text{and} \quad \vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu\left(-3\hat{i} + 2\hat{j} + 4\hat{k}\right)$$

Solution : Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} - \hat{j} + \hat{k}\right)$ ..... (1)

and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu\left(-3\hat{i} + 2\hat{j} + 4\hat{k}\right)$ ..... (2)

Equation of lines (1) and (2) in cartesian form is



AB : $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$

and CD : $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$

Let $L \equiv (3\lambda + 3, -\lambda + 8, \lambda + 3)$

and $M \equiv (-3\mu - 3, 2\mu - 7, 4\mu + 6)$

Direction ratios of LM are

$3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$.

Since LM $\perp$ AB

$\therefore$ $3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

or, $11\lambda + 7\mu = 0$ ..... (5)

Again LM $\perp$ CD

$\therefore$ $-3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

or, $-7\lambda - 29\mu = 0$ ..... (6)

Solving (5) and (6), we get $\lambda = 0, \mu = 0$

$\therefore$ $L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t\left(6\hat{i} + 15\hat{j} - 3\hat{k}\right)$

Note : Cartesian equation of LM is $\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$.

Example 32 : Prove that the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0$, $x + z = 0$, $x + y = 0$, $x + y + z = \sqrt{3}a$ is $\sqrt{2}a$.

Solution : Given planes are $y + z = 0$ ..... (i)

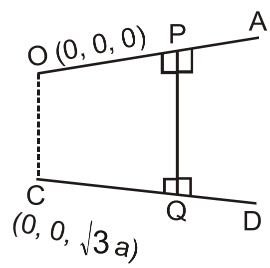
$x + z = 0$ ..... (ii)

$x + y = 0$ ..... (iii)

$x + y + z = \sqrt{3}a$ ..... (iv)

Clearly planes (i), (ii) and (iii) meet at O(0, 0, 0)

Let the tetrahedron be OABC



Let the equation to one of the pair of opposite edges OA and BC be

$y + z = 0, x + z = 0$ ..... (1)

$x + y = 0, x + y + z = \sqrt{3}a$ ..... (2)

equation (1) and (2) can be expressed in symmetrical form as

$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{-1}$ ..... (3)

and, $\frac{x-0}{1}=\frac{y-0}{-1}=\frac{z-\sqrt{3}\,a}{0}$ ..... (4)

d. r. of OA and BC are respectively (1, 1, – 1) and (1, – 1, 0).

Let PQ be the shortest distance between OA and BC having direction cosines (ℓ, m, n)

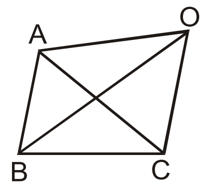
$\therefore$ PQ is perpendicular to both OA and BC.

$\therefore$ $\ell + m - n = 0$

and $\ell - m = 0$

Solving (5) and (6), we get, $\frac{\ell}{1}=\frac{m}{1}=\frac{n}{2}$ = k (say)

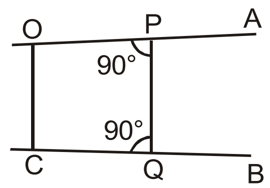
also, $\ell^2 + m^2 + n^2 = 1$



$\therefore$ $k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{6}}$

$\therefore$ $\ell = \pm\frac{1}{\sqrt{6}}$, $m = \pm\frac{1}{\sqrt{6}}$, $n = \pm\frac{2}{\sqrt{6}}$

Shortest distance between OA and BC



i.e. PQ = The length of projection of OC on PQ

$= |(x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n|$

$= \left|0\cdot\frac{1}{\sqrt{6}} + 0\cdot\frac{1}{\sqrt{6}} + \sqrt{3}\,a\cdot\frac{2}{\sqrt{6}}\right| = \sqrt{2}\,a.$

Problems for Self Practice-09

(1) Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$. Find also its equation.

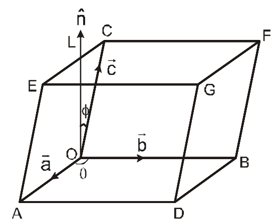
(2) Prove that the shortest distance between the diagonals of a rectangular parallelopiped whose coterminous sides are a, b, c and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

Answers : (1) $\frac{1}{\sqrt{6}}$, $6x - y = 10 - 3y = 6z - 25$

18. SCALAR TRIPLE PRODUCT (BOX PRODUCT) (S.T.P.) :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}|\,|\vec{b}|\,|\vec{c}| \sin\theta \cdot \cos\phi$, where θ is the angle between $\vec{a}, \vec{b}$ **(i.e. $\vec{a}\,\hat{}\,\vec{b} = \theta$)** and ϕ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ **(i.e. $(\vec{a} \times \vec{b})\,\hat{}\,\vec{c} = \phi$)**. It is (i.e. $\vec{a} \times \vec{b} \cdot \vec{c}$) also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $V = \left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$



(iii) In a scalar triple product the position of dot and cross can be interchanged i.e.

$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c} \quad \Rightarrow \quad [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$

(iv) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]$, where $\vec{\ell}, \vec{m}$ and $\vec{n}$ are non-coplanar vectors.

(vi) If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vii) Scalar product of three vectors, two of which are equal or parallel is 0

(viii) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(ix) $[\hat{i}\ \hat{j}\ \hat{k}] = 1$ (x) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ (xi) $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(xii) $\left[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}\right] = 0$ and $\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(xiii) $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Tetrahedron and its properties :

(a) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}, \vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]\right|$

(b) If the position vectors of the vertices of tetrahedron are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

SOLVED EXAMPLE

Example 33 : Find the volume of a parallelopiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$

Solution : Let $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b}=-5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$.

We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a},\vec{b},\vec{c}$ is $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$.

Now $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3\,(-21-15) - 7\,(15+21) + 5\,(25-49)$

$= 108 - 252 - 120 = -264$

So required volume of the parallelopiped = $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right| = |-264| = 264$ cubic units

Example 34 : Simplify $[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}]$

Solution : We have :

$[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}] = \{(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})\}\,.\,(\vec{c}-\vec{a})$ [By definition]

$= (\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{b}+\vec{b}\times\vec{c})\,.\,(\vec{c}-\vec{a})$ [By distribution law]

$= (\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{b}\times\vec{c})\,.\,(\vec{c}-\vec{a})$ [$\because\ \vec{b}\times\vec{b}=\vec{0}$]

$= (\vec{a}\times\vec{b})\cdot\vec{c} - (\vec{a}\times\vec{b})\cdot\vec{a} + (\vec{c}\times\vec{a})\cdot\vec{c} - (\vec{c}\times\vec{a})\cdot\vec{a} + (\vec{b}\times\vec{c})\cdot\vec{c} - (\vec{b}\times\vec{c})\cdot\vec{a}$ [By distribution law]

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{c}\ \vec{a}\ \vec{c}] - [\vec{c}\ \vec{a}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$ [$\because$ When any two vectors are equal, scalar triple product is zero]

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}] = 0$ [$\because\ [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$]

Example 35 : Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Solution : Let four vertices be A, B, C, D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

$\therefore\quad \overrightarrow{DA} = (\vec{a}-\vec{d})$

$\overrightarrow{DB} = (\vec{b}-\vec{d})$

$\overrightarrow{DC} = (\vec{c}-\vec{d})$

Hence volume V = $\frac{1}{6}\,[\vec{a}-\vec{d}\quad \vec{b}-\vec{d}\quad \vec{c}-\vec{d}]$

$= \frac{1}{6}(\vec{a} - \vec{d}) . [(\vec{b} - \vec{d}) \times (\vec{c} - \vec{d})]$

$= \frac{1}{6}(\vec{a} - \vec{d}) . [\vec{b} \times \vec{c} - \vec{b} \times \vec{d} + \vec{c} \times \vec{d}]$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{d}\ \vec{b}\ \vec{c}]\}$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{b}\ \vec{c}\ \vec{d}]\}$

Example 36: Show that the vectors $\vec{a} = -2\hat{i} + 4\hat{j} - 2\hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{c} = -2\hat{i} - 2\hat{j} + 4\hat{k}$ are coplanar.

Solution : $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -2 & 4 & -2 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{vmatrix} = -2(-8-4) - 4(16-4) - 2(-8-4)$

$= 24 - 48 + 24 = 0$

So vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar

<u>Problems for Self Practice</u>-10

(1) Show that $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{c} - \vec{b})\} . \vec{a} = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

(2) Show that $\vec{a} . (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) = 0$

(3) One vertex of a parallelopiped is at the point A (1, –1, –2) in the rectangular cartesian co-ordinate. If three adjacent vertices are at B(–1, 0, 2), C(2, –2, 3) and D(4, 2, 1), then find the volume of the parallelopiped.

(4) Find the value of m such that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + m\hat{j} + 5\hat{k}$ are coplanar.

(5) Show that the vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$, $\vec{a} + \vec{b}$ are coplanar.

Answers : (3) 72 (4) – 4

19. VECTOR TRIPLE PRODUCT :

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product.

Geometrical interpretation of $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ and $(\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ and $(\vec{b} \times \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which lies in the plane of $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b}$ and $\vec{c}$ i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$, where x , y are scalars.

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$
- In general $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

SOLVED EXAMPLE

Example 37: For any vector $\vec{a}$, prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$

Solution : Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$.

Then $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\hat{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$= \{(\hat{i}\,.\,\hat{i})\vec{a}-(\hat{i}\,.\,\vec{a})\hat{i}\} + \{(\hat{j}\,.\,\hat{j})\vec{a}-(\hat{j}\,.\,\vec{a})\hat{j}\} + \{(\hat{k}\,.\,\hat{k})\vec{a}-(\hat{k}\,.\,\vec{a})\hat{k}\}$

$= \{(\vec{a}-(\hat{i}\,.\,\vec{a})\,\hat{i}\}+\{\vec{a}-(\hat{j}\,.\,\vec{a})\hat{j}\} + \{\vec{a}-(\hat{k}\,.\,\vec{a})\hat{k}\}$

$= 3\vec{a}-\{(\hat{i}\,.\,\vec{a})\hat{i}+(\hat{j}\,.\,\vec{a})\hat{j}+(\hat{k}\,.\,\vec{a})\hat{k} = 3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = 3\vec{a}-\vec{a}=2\vec{a}$

Example 38: Prove that $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = (\vec{b}\,.\,\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}.\vec{c})\ (\vec{a}\times\vec{d})$

Solution : We have, $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = \vec{a}\times\{(\vec{b}\,.\,\vec{d})\,\vec{c}-(\vec{b}\,.\,\vec{c})\,\vec{d}\}$

$= \vec{a}\times\{(\vec{b}\,.\,\vec{d})\,\vec{c}\}-\vec{a}\times\{(\vec{b}\,.\,\vec{c})\,\vec{d}\}$ [by dist. law]

$= (\vec{b}\,.\,\vec{d})\,(\vec{a}\times\vec{c})-(\vec{b}\,.\,\vec{c})\,(\vec{a}\times\vec{d})$.

Example 39 : Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value(s) of α, if any, such that $\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\} \times \left(\vec{c}\times\vec{a}\right) = \vec{0}$.

Solution : $\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\} \times \left(\vec{c}\times\vec{a}\right) = \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b} \times \left(\vec{c}\times\vec{a}\right)$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{\left(\vec{a}.\vec{b}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}\right\}$

which vanishes if (i) $\left(\vec{a}.\vec{b}\right)\vec{c} = \left(\vec{b}.\vec{c}\right)\vec{a}$ (ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

(i) $\left(\vec{a}.\vec{b}\right)\vec{c} = \left(\vec{b}.\vec{c}\right)\vec{a}$ leads to the equation $2\alpha^3 + 10\alpha + 12 = 0$, $\alpha^2 + 6\alpha = 0$ and $6\alpha - 6 = 0$, which do not have a common solution.

(ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0 \Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix} = 0 \Rightarrow 3\alpha = 2 \Rightarrow \alpha = \dfrac{2}{3}$

Example 40 : If $\vec{A}+\vec{B} = \vec{a}$, $\vec{A}\,.\,\vec{a} = 1$ and $\vec{A}\times\vec{B} = \vec{b}$, then prove that $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$.

Solution : Given $\vec{A}+\vec{B}=\vec{a}$ .....(i)

$\Rightarrow \quad \vec{a}\,.\left(\vec{A}+\vec{B}\right) = \vec{a}\,.\,\vec{a}$

$\Rightarrow \quad \vec{a}\,.\,\vec{A}+\vec{a}\,.\,\vec{B} = \vec{a}\,.\,\vec{a} \quad \Rightarrow \quad 1 + \vec{a}\,.\,\vec{B} = |\vec{a}|^2$

$\Rightarrow \quad \vec{a}\,.\,\vec{B} = |\vec{a}|^2 - 1$ .....(ii)

Given $\vec{A}\times\vec{B}=\vec{b} \quad \Rightarrow \quad \vec{a}\times\left(\vec{A}\times\vec{B}\right) = \vec{a}\times\vec{b}$

$\Rightarrow \quad \left(\vec{a}\,.\,\vec{B}\right)\vec{A} - \left(\vec{a}\,.\,\vec{A}\right)\vec{B} = \vec{a}\times\vec{b}$

$\Rightarrow \quad \left(|\vec{a}|^2-1\right)\vec{A}-\vec{B} = \vec{a}\times\vec{b}$ .....(iii) [Using equation (ii)]

solving equation (i) and (iii) simultaneously, we get

$\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$

Example 41 : Solve for $\vec{r}$ satisfying the simultaneous equations $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$, $\vec{r}\,.\,\vec{a}=0$ provided $\vec{a}$ is not perpendicular to $\vec{b}$.

Solution : $(\vec{r}-\vec{c})\times\vec{b} = \vec{0} \quad \Rightarrow \quad \vec{r}-\vec{c}$ and $\vec{b}$ are collinear

$\therefore \quad \vec{r}-\vec{c}=k\vec{b} \quad \Rightarrow \quad \vec{r} = \vec{c}+k\vec{b}$ ....(i)

$\because \quad \vec{r}\,.\,\vec{a} = 0 \quad \Rightarrow \quad (\vec{c}+k\vec{b})\,.\,\vec{a} = 0$

$\Rightarrow \quad k = -\dfrac{\vec{a}\,.\,\vec{c}}{\vec{a}\,.\,\vec{b}}$ putting in (i) we get $\vec{r}=\vec{c}-\dfrac{\vec{a}\,.\,\vec{c}}{\vec{a}\,.\,\vec{b}}\,\vec{b}$

Example 42 : If $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$, where k is a scalar and $\vec{a},\vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a},\vec{b}$ and k.

Solution : $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$ ....(i)

Premultiply the given equation vectorially by $\vec{a}$

$\vec{a}\times(\vec{x}\times\vec{a}) + k\,(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$

$\Rightarrow \quad (\vec{a}\,.\,\vec{a})\,\vec{x}-(\vec{a}\,.\,\vec{x})\,\vec{a}+k(\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$ ....(ii)

Premultiply (i) scalarly by $\vec{a}$

$[\vec{a}\ \vec{x}\ \vec{a}] + k\,(\vec{a}\,.\,\vec{x}) = \vec{a}\,.\,\vec{b}$

$k(\vec{a}\,.\,\vec{x})=\vec{a}\,.\,\vec{b}$ ....(iii)

Substituting $\vec{x}\times\vec{a}$ from (i) and $\vec{a}\,.\,\vec{x}$ from (iii) in (ii) we get

$$\vec{x} = \frac{1}{a^2+k^2}\left[k\vec{b}+(\vec{a}\times\vec{b})+\frac{(\vec{a}\,.\,\vec{b})}{k}\,\vec{a}\right]$$

Problems for Self Practice-11

(1) Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$.

(2) Find the unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$.

(3) Prove that $\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\}=(\vec{a}\,.\,\vec{a})\,(\vec{b}\times\vec{a})$.

(4) Given that $\vec{x}+\frac{1}{\vec{p}^2}\,(\vec{p}\,.\,\vec{x})\;\vec{p}=\vec{q}$, show that $\vec{p}\,.\,\vec{x}=\frac{1}{2}\vec{p}\,.\,\vec{q}$ and find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

(5) If $\vec{x}\,.\,\vec{a}=0$, $\vec{x}\,.\,\vec{b}=0$ and $\vec{x}\,.\,\vec{c}=0$ for some non-zero vector $\vec{x}$, then show that $[\vec{a}\;\vec{b}\;\vec{c}]=0$

(6) Prove that $\vec{r}=\frac{(\vec{r}\,.\,\vec{a})\,(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}+\frac{(\vec{r}\,.\,\vec{b})\,(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]}+\frac{(\vec{r}\,.\,\vec{c})\,(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

where $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors

Answers : (2) $\pm\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$ and $\vec{x}=\vec{q}-\left(\frac{\vec{p}.\vec{q}}{2|\vec{p}|^2}\right)\vec{p}$

20. LINEAR COMBINATIONS :

Given a finite set of vectors $\vec{a},\vec{b},\vec{c},......$, then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+........$ is called a linear combination of $\vec{a},\vec{b},\vec{c},......$ for any x, y, z..... ∈ R. We have the following results:

(i) If $\vec{a},\vec{b}$ are non zero, non–collinear vectors, then $x\vec{a}+y\vec{b}=x'\vec{a}+y'\vec{b}\Rightarrow x=x',\;\; y=y'$

(ii) Let $\vec{a},\vec{b}$ be non zero, non collinear vectors, then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$

i.e. there exist some unique x, y ∈ R such that $x\vec{a}+y\vec{b}=\vec{r}$.

(iii) If $\vec{a},\vec{b},\vec{c}$ are non–zero, non–coplanar vectors, then

$x\vec{a}+y\vec{b}+z\vec{c}=x'\,\vec{a}+y'\,\vec{b}+z'\,\vec{c}\Rightarrow x=x'\;,\;y=y'\;,\;z=z'$

(iv) Let $\vec{a},\vec{b},\vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i.e. there exist some unique x,y, z ∈ R such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{r}$.

(v) If $\vec{x}_1,\vec{x}_2,......,\;\vec{x}_n$ are n non zero vectors and $k_1,k_2,.....,k_n$ are n scalars and if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+.......+k_n\vec{x}_n=\vec{0}\Rightarrow k_1=0,\,k_2=0,.....,k_n=0$, then we say that vectors $\vec{x}_1,\,\vec{x}_2,\,......,\,\vec{x}_n$ are linearly independent vectors.

(vi) If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are not linearly independent then they are said to be linearly dependent vectors. i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 ...... + k_r\vec{x}_r + ... + k_n\vec{x}_n = \vec{0}$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$ are said to be linearly dependent vectors.

(vii) Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = \vec{0} \Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq \vec{0}$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

(viii) If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. Conversely if $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ then the vectors are linearly independent.

(ix) Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where x + y + z = 0.

(x) Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$, where x + y + z + w = 0.

SOLVED EXAMPLE

Example 43 : Given that position vectors of points A, B, C are respectively $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$, $-7\vec{b} + 10\vec{c}$ then prove that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Solution : Let A, B, C be the given points and O be the point of reference then

$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\overrightarrow{OB} = 2\vec{a} + 3\vec{b} - 4\vec{c}$ and $\overrightarrow{OC} = -7\vec{b} + 10\vec{c}$

Now $\overrightarrow{AB}$ = p.v. of B – p.v. of A

$= \overrightarrow{OB} - \overrightarrow{OA} = (\vec{a} + 5\vec{b} - 7\vec{c})$ and $\overrightarrow{AC}$ = p.v. of C – p.v of A

$= \overrightarrow{OC} - \overrightarrow{OA} = -(\vec{a} + 5\vec{b} - 7\vec{c}) = -\overrightarrow{AB}$

$\therefore \overrightarrow{AC} = \lambda \overrightarrow{AB}$ where $\lambda = -1$.

Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Example 44 : Prove that the vectors $5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$ are linearly dependent, where $\vec{a}, \vec{b}, \vec{c}$ being linearly independent vectors.

Solution : We know that if these vectors are linearly dependent , then we can express one of them as a linear combination of the other two.

Now let us assume that the given vector are coplanar, then we can write

$5\vec{a} + 6\vec{b} + 7\vec{c} = \ell(7\vec{a} - 8\vec{b} + 9\vec{c}) + m(3\vec{a} + 20\vec{b} + 5\vec{c})$

where ℓ, m are scalars

Comparing the coefficients of $\vec{a}, \vec{b}$ and $\vec{c}$ on both sides of the equation

$5 = 7\ell + 3m$ ....(i)

$6 = -8\ell + 20m$ ....(ii)

$7 = 9\ell + 5m$ ....(iii)

From (i) and (iii) we get

$4 = 8\ell$

$\Rightarrow \quad \ell = \frac{1}{2} = m$ which evidently satisfies (ii) equation too.

Hence the given vectors are linearly dependent .

Example 45 : Show that the vectors $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$ are non-coplanar vectors.

Solution : Let, the given vectors be coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $2\vec{a}-\vec{b}+3\vec{c} = x\left(\vec{a}+\vec{b}-2\vec{c}\right) + y\left(\vec{a}+\vec{b}-3\vec{c}\right)$, for some scalars x and y.

$\Rightarrow$ $2\vec{a}-\vec{b}+3\vec{c} = (x+y)\,\vec{a} + (x+y)\,\vec{b} + (-2x-3y)\,\vec{c}$

$\Rightarrow$ $2 = x + y$, $-1 = x + y$ and $3 = -2x - 3y$.

Solving first and third of these equations, we get $x = 9$ and $y = -7$.

Clearly these values do not satisfy the second equation. Hence the given vectors are not coplanar.

Example 46 : Prove that four points $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.

Solution : Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two.

So let $\overrightarrow{PQ} = x\,\overrightarrow{PR} + y\,\overrightarrow{PS}$

$\Rightarrow$ $-\vec{a}-5\vec{b}+4\vec{c} = x\left(\vec{a}+\vec{b}-\vec{c}\right) + y\left(-\vec{a}-9\vec{b}+7\vec{c}\right)$

$\Rightarrow$ $-\vec{a}-5\vec{b}+4\vec{c} = (x-y)\,\vec{a} + (x-9y)\,\vec{b} + (-x+7y)\,\vec{c}$

$\Rightarrow$ $x - y = -1$, $x - 9y = -5$, $-x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two equations of these three equations, we get $x = -\frac{1}{2}$, $y = \frac{1}{2}$.

These values also satisfy the third equation. Hence the given four points are coplanar.

Problems for Self Practice-12

(1) Does there exist scalars u, v, w such that $u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 = \hat{i}$ where $\vec{e}_1 = \hat{k}$, $\vec{e}_2 = \hat{j}+\hat{k}$, $\vec{e}_3 = -\hat{j}+2\hat{k}$?

(2) Consider a base $\vec{a}, \vec{b}, \vec{c}$ and a vector $-2\vec{a}+3\vec{b}-\vec{c}$. Compute the co-ordinates of this vector relatively to the base $\vec{p}, \vec{q}, \vec{r}$ where $\vec{p} = 2\vec{a}-3\vec{b}$, $\vec{q} = \vec{a}-2\vec{b}+\vec{c}$, $\vec{r} = -3\vec{a}+\vec{b}+2\vec{c}$.

(3) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A} = (x + 4y)\vec{a} + (2x + y + 1)\vec{b}$ and $\vec{B} = (y - 2x + 2)\vec{a} + (2x - 3y - 1)\vec{b}$, find x and y such that $3\vec{A} = 2\vec{B}$.

(4) If vectors $\vec{a}, \vec{b}, \vec{c}$ be linearly independent, then show that

(i) $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-4\vec{c}$, $-\vec{b}+2\vec{c}$ are linearly dependent

(ii) $\vec{a}-3\vec{b}+2\vec{c}$, $-2\vec{a}-4\vec{b}-\vec{c}$, $3a+2\vec{b}-\vec{c}$ are linearly independent.

(5) Given that $\hat{i}-\hat{j}$, $\hat{i}-2\hat{j}$ are two vectors. Find a unit vector coplanar with these vectors and perpendicular to the first vector $\hat{i}-\hat{j}$. Find also the unit vector which is perpendicular to the plane of the two given vectors.

(6) If with reference to a right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$, $\vec{\alpha} = 3\hat{i}-\hat{j}$, $\vec{\beta} = 2\hat{i}+\hat{j}-3\hat{k}$. Express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$ where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

(7) Prove that a vector $\vec{r}$ in space can be expressed linearly in terms of three non-coplanar, non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ in the form $\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}]\ \vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\ \vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\ \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

(8) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that the terminal A, B, C, D of these vectors are coplanar. Find the point (P) at which AC and BD meet. Also find the ratio in which P divides AC and BD.

(9) Show that the vector $\vec{a} - \vec{b} + \vec{c}$, $\vec{b} - \vec{c} - \vec{a}$ and $2\vec{a} - 3\vec{b} - 4\vec{c}$ are non-coplanar, where $\vec{a}, \vec{b}, \vec{c}$ are any non-coplanar vectors.

(10) Find the value of λ for which the four points with position vectors $-\hat{j} - \hat{k}$, $4\hat{i} + 5\hat{j} + \lambda\hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

Answers :

(1) No

(2) (0, – 7/5, 1/5)

(3) x = 2, y = –1

(5) $\pm \frac{1}{\sqrt{2}} (\hat{i} + \hat{j})$; $\hat{k}$

(6) $\vec{\beta}_1 = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$, $\vec{\beta}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

(8) $\vec{p} = \frac{3\vec{a} + \vec{c}}{4}$ P divides AC in 1 : 3 and BD in 1 : 1 ratio

(10) $\lambda = 1$

21. EQUATION OF A PLANE

If line joining any two points on a surface lies completely on it then the surface is a plane.

The line joining any two points on a surface is perpendicular to some fixed straight line. This fixed line is called the normal to the plane.

(i) The equation $(\vec{r} - \vec{r}_0) . \vec{n} = 0$ represents a plane containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane.

The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a (x – x_1) + b(y – y_1) + c (z – z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(ii) The above equation can also be written as $\vec{r} . \vec{n} = d$, where $d = \vec{r}_0 . \vec{n}$.

In cartesian form it is written as ax + by + cz + d = 0, where a, b, c are the direction ratios of the normal to the plane. It is called the general form.

(iii) Normal form of the equation of a plane is $\ell x + my + nz = p$, where, ℓ,m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(iv) Intercept Form: The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Note:

(i) Equation of yz–plane is x = 0, Equation of xz–plane is y = 0, Equation of xy–plane is z = 0.

If a = 0, the plane is parallel to x–axis i.e. equation of the plane parallel to the x–axis is by + cz + d = 0.

(ii) The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$. Angle between these planes is the angle between their normals.

Consider two planes ax + by + cz + d = 0 and a′x + b′y + c′z + d′ = 0. Since direction ratios of their normals are (a, b, c) and (a′, b′, c′) respectively, hence θ, the angle between them, is given by $\cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$

Planes are perpendicular if aa′ + bb′ + cc′ = 0 and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(iii) Any plane parallel to the given plane ax + by + cz + d = 0 is ax + by + cz + λ = 0. Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

(iv) The equation of a plane passing through a point having position vector $\vec{a}$ and parallel to

$\vec{b}$ & $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$ (non parametric form)

(v) Coplanarity of four points

The points $A(x_1 y_1 z_1)$, $B(x_2 y_2 z_2)$ $C(x_3 y_3 z_3)$ and $D(x_4 y_4 z_4)$ are coplaner then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

very similar in vector method the points A $(\vec{r}_1)$, B$(\vec{r}_2)$, C$(\vec{r}_3)$ and D$(\vec{r}_4)$ are coplanar if

$[\vec{r}_4 - \vec{r}_1, \vec{r}_4 - \vec{r}_2, \vec{r}_4 - \vec{r}_3] = 0$

22. ANGLE BETWEEN A PLANE AND A LINE:

If θ is the angle between line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane ax + by + cz + d = 0, then sin θ

$=\left[\frac{a\ell+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right]$.

If θ is the angle between a line $\vec{r}=(\vec{a}+\lambda\vec{b})$ and $\vec{r}\cdot\vec{n}=d$ then $\sin\theta=\left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

23. COPLANAR LINES :

If the given lines are $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$, then condition for intersection/

coplanarity is $\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0$ & equation of plane containing the above two lines is

$\begin{vmatrix}x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0$ or $\begin{vmatrix}x-\alpha' & y-\beta' & z-\gamma' \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0$

SOLVED EXAMPLE

Example 47 : Find the equation of the plane through the points A(2, 2, –1), B(3, 4, 2) and C(7, 0, 6).

Solution : The general equation of a plane passing through (2, 2, –1) is

a (x – 2) + b (y – 2) + c (z + 1) = 0 ....(i)

It will pass through B (3, 4, 2) and C (7, 0, 6) if

a (3 – 2) + b (4 – 2) + c (2 + 1) = 0 or a + 2b + 3c = 0 ....(ii)

and a (7 – 2) + b (0 – 2) + c (6 + 1) = 0 or 5a – 2b + 7c = 0 ....(iii)

Solving (ii) and (iii) by cross-multiplication, we have

$\frac{a}{14+6}=\frac{b}{15-7}=\frac{c}{-2-10}$ or $\frac{a}{5}=\frac{b}{2}=\frac{c}{-3}=\lambda$ (say)

$\Rightarrow$ a = 5λ, b = 2λ and c = -3λ

Substituting the values of a, b and c in (i), we get

$5\lambda(x-2)+2\lambda(y-2)-3\lambda(z+1)=0$

or $5(x-2)+2(y-2)-3(z+1)=0$

$\Rightarrow$ 5x + 2y – 3z = 17, which is the required equation of the plane

Example 48 : A plane meets the co-ordinates axes in A,B,C such that the centroid of the Δ ABC is the point (p,q,r) show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}$ = 3

Solution : Let the required equation of plane be :

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ......(i)$$

Then, the co-ordinates of A, B and C are A(a, 0, 0), B(0, b, 0), C(0, 0, c) respectively

So the centroid of the triangle ABC is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

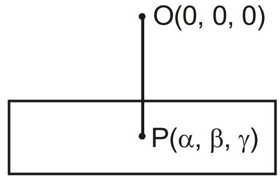
But the co-ordinate of the centroid are (p,q,r)

$\frac{a}{3}$ = p, $\frac{b}{3}$ = q, $\frac{c}{3}$ = r

Putting the values of a, b and c in (i), we get the required plane as $\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}$ = 1

$$\Rightarrow \quad \frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 3$$

Example 49 : A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$



Solution : Let the equation of the variable plane be

$\ell x + my + nz + d = 0$ ..... (1)

Plane passes through the fixed point (a, b, c)

$\therefore \quad \ell a + mb + nc + d = 0$ ..... (2)

Let P (α, β, γ) be the foot of perpendicular from origin to plane (1).

Direction ratios of OP are

$\alpha - 0, \beta - 0, \gamma - 0$

i.e. α, β, γ

From equation (1), it is clear that the direction ratios of normal to the plane i.e. OP are ℓ, m, n; α, β, γ and ℓ, m, n are the direction ratios of the same line OP

$$\therefore \quad \frac{\alpha}{\ell} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{1}{k} \text{ (say)}$$

$\therefore \quad \ell = k\alpha, m = k\beta, n = k\gamma$ ..... (3)

Putting the values of ℓ, m, n in equation (2), we get

$k a\alpha + k b\beta + k c\gamma + d = 0$ ..... (4)

Since α, β, γ lies in plane (1)

$\therefore \quad \ell\alpha + m\beta + n\gamma + d = 0$ ..... (5)

Putting the values of ℓ, m, n from (3) in (5), we get

$k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ..... (6)

or $k\alpha^2 + k\beta^2 + k\gamma^2 - ka\alpha - kb\beta - kc\gamma = 0$

[putting the value of d from (4) in (6)]

or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of perpendicular P (α, β, γ) is

$x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (7)

Example 50 : Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane $3x + 4y + z + 5 = 0$.

Solution : The given line is $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ .... (i)

and the given plane is $3x + 4y + z + 5 = 0$ .... (ii)

If the line (i) makes angle θ with the plane (ii), then the line (i) will make angle $(90° - \theta)$ with the normal to the plane (i). Now direction-ratios of line (i) are 3, –1, –2 and direction-ratios of normal to plane (ii) are 3, 4, 1

$$\therefore \cos(90° - \theta) = \frac{(3)(3) + (-1)(4) + (-2)(1)}{\sqrt{9+1+4}\sqrt{9+16+1}} \Rightarrow \sin\theta = \frac{9-4-2}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{14}\sqrt{26}}$$

Hence $\theta = \sin^{-1}\left(\frac{3}{\sqrt{14}\sqrt{26}}\right)$

Example 51 : Find the equation of the plane passing through (1, 2, 0) which contains the line

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

Solution : Equation of any plane passing through (1, 2, 0) may be taken as

$a(x - 1) + b(y - 2) + c(z - 0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point (– 3, 1, 2) on line (2) lies in plane (1)

$\therefore$ $a(-3 - 1) + b(1 - 2) + c(2 - 0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13}$ = k (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

$6(x - 1) + 2(y - 2) + 13(z - 0) = 0$.

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Problems for Self Practice-13

(1) Find the equation of the plane passing through the points (2, 3, 1), (3, 0, 2) and (–1, 2, 3).

(2) Find the angle between the planes $3x + 4y + z + 7 = 0$ and $-x + y - 2z = 5$

(3) Find the equation of plane passing through (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $x + 3y + 3z = 8$.

(4) Find the equation of the plane parallel to $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}$ and passing through (1, 1, 2).

(5) Find the equation of the plane passing through the point (1, 1, – 1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

(6) Find the plane containing the line $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{5}$ and parallel to the line $\frac{x+1}{1} = \frac{y-1}{-2} = \frac{-z+1}{1}$

(7) Show that the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ & $\frac{x-4}{5} = \frac{y-1}{2} = z$ are intersecting each other.

Find their intersection point and the plane containing the line.

Answers :

(1) $x + y + 2z = 7$

(2) $\theta = \cos^{-1}\left(\frac{1}{\sqrt{156}}\right)$

(3) $3x + 4y - 5z = 9$

(4) $x + y - 2z + 2 = 0$

(5) $17x + 2y - 7z = 26$

(6) $13x + 3y - 7z - 7 = 0$

(7) $(-1, -1, -1)$ & $5x - 18y + 11z - 2 = 0$

24. PLANE & POINT

(i) Two points A $(x_1\ y_1\ z_1)$ and B $(x_2\ y_2\ z_2)$ are on the same side of the plane $ax + by + cz + d = 0$ if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

(ii) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\left|\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}\right|$.

(iii) The coordinates of the foot of perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(iv) Let P (x_1, y_1, z_1) is a given point and $ax + by + cz + d = 0$ is given plane. Let (x', y', z') is the image of the point, then $\frac{x'-x_1}{a} = \frac{y'-y_1}{b} = \frac{z'-z_1}{c} = -2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(v) The distance between two parallel planes $ax + by + cx + d = 0$ and $ax + by + cx + d' = 0$ is $\frac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$

25. ANGLE BISECTORS

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad ....(1)$$

(ii) Bisector of acute/obtuse angle :

If $a_1a_2 + b_1b_2 + c_1c_2 > 0$ then equation (1) with '+' sign gives obtuse angle bisector and equation (1) with '–' sign gives acute angle bisector

If $a_1a_2 + b_1b_2 + c_1c_2 < 0$ then equation (1) with '+' sign gives acute angle bisector and equation (1) with '–' sign gives obtuse angle bisector

(iii) Equation of bisector of the angle containing point P(α, β, γ) :

If $(a_1\alpha + b_1\beta + c_1\gamma + d_1)(a_2\alpha + b_2\beta + c_2\gamma + d_1) > 0$ then

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$ gives the bisector of the angle which contains point P(α, β, γ) .

If $(a_1\alpha + b_1\beta + c_1\gamma + d_1)(a_2\alpha + b_2\beta + c_2\gamma + d_1) < 0$ then

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$ gives the bisector of the angle which contains point P(α, β, γ) .

SOLVED EXAMPLE

Example 52 : Find the perpendicular distance of the point (2, 1, 0) from the plane 2x + y +2z + 5 = 0

Solution : We know that the perpendicular distance of the point (x_1, y_1, z_1) from the plane

ax + by + cz + d = 0 is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

so required distance = $\frac{|2\times 2+1\times 1+2\times 0+5|}{\sqrt{2^2+1^2+2^2}} = \frac{10}{3}$ **Ans.**

Example 53 : Find the distance between the parallel planes 2x – y + 2z + 3 = 0 and 4x – 2y + 4z + 5 = 0.

Solution : Let $P(x_1, y_1, z_1)$ be any point on 2x – y + 2z + 3 = 0, then $2x_1 - y_1 + 2z_1 + 3 = 0$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to 4x – 2y + 4z + 5 = 0 is

$$\left|\frac{4x_1-2y_1+4z_1+5}{\sqrt{4^2+(-2)^2+4^2}}\right| = \left|\frac{2(2x_1-y_1+2z_1)+5}{\sqrt{36}}\right| = \frac{|2(-3)+5|}{6} = \frac{1}{6} \text{ [using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$ **Ans.**

Example 54 : Find the equation of the bisector planes of the angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.

Solution : The two given planes are $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ where $d_1, d_2 > 0$

and $a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 > 0$

$\therefore \quad \dfrac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\dfrac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$ (acute angle bisector)

and $\dfrac{a_1x + b_1y + c_1z + d}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \dfrac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$ (obtuse angle bisector)

i.e., $\dfrac{2x - y + 2z + 3}{\sqrt{4 + 1 + 4}} = \pm\dfrac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}}$

$\Rightarrow \quad (14x - 7y + 14z + 21) = \pm (9x - 6y + 18z + 24)$

Taking positive sign on the right hand side,

we get $5x - y - 4z - 3 = 0$ (obtuse angle bisector)

and taking negative sign on the right hand side,

we get $23x - 13y + 32z + 45 = 0$ (acute angle bisector)

<u>Problems for Self Practice</u>-14

(1) Find the perpendicular distance of the point P(1, 2, 3) from the plane $2x + y + z + 1 = 0$.

(2) Find the position of the point P(2, –2, 1), Q(3, 0, 1) and R(–12, 1, 8) w.r.t. the plane $2x - 3y + 4z - 7 = 0$.

(3) Two given planes are $-2x + y - 2z + 5 = 0$ and $6x - 2y + 3z - 7 = 0$. Find equation of a plane parallel to the plane bisecting the angle between both the two planes and passing through the point (3, 2, 0).

(4) Show that the origin lies in the acute angle between the planes

$x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$

(5) Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line

$\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{-6}$

Answers : (1) $\dfrac{8}{\sqrt{6}}$

(2) P, Q same side & R opposite side

(3) $4x + y - 5z - 14 = 0$ & $32x - 13y + 23z - 70 = 0$

(5) 1

26. NON-SYMMETRICAL FORM OF LINE

A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non–symmetrical form.

Let equation of the line in non–symmetrical form be $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$.
To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes. So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by cross–multiplication as $\dfrac{\ell}{b_1c_2 - b_2c_1} = \dfrac{m}{c_1a_2 - c_2a_1} = \dfrac{n}{a_1b_2 - a_2b_1}$

Let the line is not parallel to xy plane. Let it intersect xy–plane in $(x_1, y_1, 0)$. Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$. Solving these, we get a point on the line.

Note : If lines $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ &
$\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$

are coplanar then $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

27. FAMILY OF PLANES

Any plane passing through the line of intersection of non–parallel planes
$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is
$a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$, where $\lambda \in R$

The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}.\vec{n}_2 = d_2$ is
$\vec{r}.(n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

SOLVED EXAMPLE

Example 55 : Find the angle between the lines $x - 3y - 4 = 0$, $4y - z + 5 = 0$ and $x + 3y - 11 = 0$, $2y - z + 6 = 0$.

Solution : Given lines are $\left.\begin{array}{l} x - 3y - 4 = 0 \\ 4y - z + 5 = 0 \end{array}\right\}$ ..... (1)

and $\left.\begin{array}{l} x + 3y - 11 = 0 \\ 2y - z + 6 = 0 \end{array}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes

x – 3y – 4 = 0 and 4y – z + 5 = 0

$\therefore$ $\ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3)

and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0}$

or, $\frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4}$ = k (let).

Since line (2) is perpendicular to the normals of each of the planes

x + 3y – 11 = 0 and 2y – z + 6 = 0,

$\therefore$ $\ell_2 + 3m_2 = 0$ ..... (5)

and $2m_2 - n_2 = 0$ ..... (6)

$\therefore$ $\ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$ and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

$\therefore$ $\frac{\ell_2}{-3} = \frac{m_2}{1} = \frac{n_2}{2}$ = t (let).

If θ be the angle between lines (1) and (2), then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

= (3k) (– 3t) + (k) (t) + (4k) (2t) = – 9kt + kt + 8kt = 0

$\therefore$ $\theta = 90°$.

Example 56 : Find the equation of plane containing the line of intersection of the plane x + y + z – 6 = 0 and 2x + 3y + 4z + 5 = 0 and passing through (1,1,1).

Solution : The equation of the plane through the line of intersection of the given planes is,

$(x + y + z - 6) + \lambda (2x + 3y + 4z + 5) = 0$ ......(i)

If it is passes through (1,1,1)

$\Rightarrow$ $(1 + 1 + 1 - 6) + \lambda (2 + 3 + 4 + 5) = 0 \Rightarrow \lambda = \frac{3}{14}$

Putting λ =3/14 in (i); we get $(x + y + z - 6) + \frac{3}{14} (2x + 3y + 4z + 5) = 0$

$\Rightarrow$ 20x + 23y + 26z – 69 = 0

Example 57 : The plane x – y – z = 4 is rotated through 90° about its line of intersection with the plane x + y + 2z = 4. Find its equation in the new position.

Solution : Given planes are x – y – z = 4 ..... (1)

and x + y + 2z = 4 ..... (2)

Since the required plane passes through the line of intersection of planes (1) and (2)

$\therefore$ its equation may be taken as

x + y + 2z – 4 + k (x – y – z – 4) = 0

or (1 + k)x + (1 – k)y + (2 – k)z – 4 – 4k = 0 ..... (3)

Since planes (1) and (3) are mutually perpendicular,

$\therefore$ (1 + k) – (1 – k) – (2 – k) = 0 or, $1 + k - 1 + k - 2 + k = 0$

or $k = \frac{2}{3}$

Putting $k = \frac{2}{3}$ in equation (3), we get $5x + y + 4z = 20$

This is the equation of the required plane.

Example 58 : Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equation of the line of greatest slope on the plane $2x+y-5z=0$, passing through the point (2,1,1).

Solution : The required line passing through the point P (2,1,1) in the plane $2x + y - 5z = 0$ and is having greatest slope, so it must be perpendicular to the line of intersection of the planes

$2x + y - 5z = 0$ ......(i)

and $4x - 3y + 7z = 0$ ......(ii)

Let the d.r'.s of the line of intersection of (i) and (ii) be a, b, c

$\Rightarrow$ $2a + b - 5c = 0$ and $4a - 3b + 7c = 0$

{as dr'.s of straight line (a, b,c) is perpendicular to d.r'.s of normal to both the planes}

$$\Rightarrow \quad \frac{a}{4} = \frac{b}{17} = \frac{c}{5}$$

Now let the direction ratio of required line be proportional to ℓ, m, n then its equation be

$$\frac{x-2}{\ell} = \frac{y-1}{m} = \frac{z-1}{n}$$

where $2\ell + m - 5n = 0$ and $4\ell + 17m + 5n = 0$

so, $$\frac{\ell}{3} = \frac{m}{-1} = \frac{n}{1}$$

Thus the required line is $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$

<u>Problems for Self Practice</u>-15

(1) Find the symmetrical form of the line $x - y + 2z = 5$, $3x + y + z = 6$.

(2) Prove that the three planes $2x + y - 4z - 17 = 0$, $3x + 2y - 2z - 25 = 0$, $2x - 4y + 3z + 25 = 0$ intersect at a point and find its co-ordinates.

(3) Find the locus of a point, the sum of squares of whose distances from the planes : $x - z = 0$, $x - 2y + z = 0$ and $x + y + z = 0$ is 36

(4) Prove that the planes $12x - 15y + 16z - 28 = 0$, $6x + 6y - 7z - 8 = 0$ and $2x + 35y - 39z + 12 = 0$ have a common line of intersection.

Answers : (1) $\frac{x-11/4}{-3} = \frac{y+9/4}{5} = \frac{z-0}{4}$

(2) (3, 7, – 1)

(3) $x^2 + y^2 + z^2 = 36$

Exercise # 1

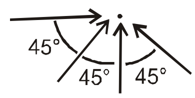
PART-I : SUBJECTIVE QUESTIONS

Section (A) : Position vector, Direction Ratios & Direction cosines

A-1. If $\vec{a} = 2\hat{i} + \mu\hat{j} - 7\hat{k}$ and $\vec{b} = \lambda\hat{i} + \sqrt{3}\,\hat{j} - 7\hat{k}$ are two equal vectors, then find $\lambda^2 + \mu^2$.

A-2. Find the value of λ when $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = 8\hat{i} + \lambda\hat{j} + 4\hat{k}$ are parallel

A-3 Four coplanar forces are applied at a point O. Each of them is equal to k and the angle between two consecutive forces equals 45º as shown in the figure. Then find the magnitude of the resultant :



A-4. If the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$ & $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ constitute the sides of a Δ ABC, then find the length of the median bisecting the vector $\vec{c}$

A-5. If ABCD is a quadrilateral, E and F are the mid-points of AC and BD respectively, then prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

A-6 In a ΔOAB, E is the mid-point of OB and D is a point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.

A-7. What are the direction cosines of a line that passes through the points P(6, –7, –1) and Q(2, –3, 1).

A-8 Find the direction cosines l, m, n of line which are connected by the relations $\ell + m + n = 0$, $2mn + 2m\ell - n\ell = 0$.

Section (B) : Dot Product, Projection and Cross Product

B-1. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ & $\vec{b}$

B-2. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}| = 5$, $|\vec{b}| = 12$ and $|\vec{c}| = 13$, and $\vec{a} + \vec{b} + \vec{c} = \vec{o}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$ ·

B-3. Using vectors, prove that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.

B-4. If $\vec{e}_1$ and $\vec{e}_2$ are two unit vectors such that $\vec{e}_1 - \vec{e}_2$ is also a unit vector, then find the angle θ between $\vec{e}_1$ and $\vec{e}_2$.

B-5 Let position vector of point A be $\hat{i} + \hat{j} + \hat{k}$ and that of point B be $-\hat{i} + \hat{k}$, then find the position vector of point $R(\vec{r})$ such that AR is perpendicular to BR and $\vec{r}$ is not perpendicular to $\left(\vec{r} - \left(\hat{j} + 2\hat{k}\right)\right)$.

B-6. If the three successive vertices of a parallelogram have the position vectors as, A (– 3, – 2, 0); B (3, – 3, 1) and C (5, 0, 2). Then find

(a) position vector of the fourth vertex D

(b) a vector having the same direction as that of $\overrightarrow{AB}$ but magnitude equal to $\overrightarrow{AC}$

(c) the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

B-7. Prove that $\left(\frac{\vec{a}}{\vec{a}^2} - \frac{\vec{b}}{\vec{b}^2}\right)^2 = \left(\frac{\vec{a}-\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)^2$

B-8. The edges of a rectangular parallelopiped are a, b, c; show that the angles between the four diagonals are given by $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$.

B-9. Find the angle between the lines whose direction cosines are given by the equations :
$3\ell + m + 5n = 0$ and $6\,mn - 2n\ell + 5\,\ell m = 0$

B-10. Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where $\vec{a} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}+\hat{k}$.

B-11. P and Q are the points (–1, 2, 1) and (4, 3, 5) respectively. Find the projection of PQ on a line which makes angles of 120º and 135º with y and z axes respectively and an acute angle with x-axis.

B-12 If $\vec{a}\times\vec{b} = \vec{c}$, $\vec{b}\times\vec{c} = \vec{a}$, then find value of $|3\vec{a}+4\vec{b}+12\vec{c}|$ if $\vec{a}, \vec{b}, \vec{c}$ are vectors of same magnitude.

B-13. For any two vectors $\vec{u}$ & $\vec{v}$, prove that

$$(1+|\vec{u}|^2)(1+|\vec{v}|^2) = (1-\vec{u}.\vec{v})^2 + |\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$$

B-14. A vector $\vec{c}$ is perpendicular to the vectors $2\hat{i}+3\hat{j}-\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$ and satisfies the condition $\vec{c}.(2\hat{i}-\hat{j}+\hat{k}) + 6 = 0$. Find the vector $\vec{c}$

B-15. (a) Show that the perpendicular distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is

$$\frac{\left|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}\right|}{\left|\vec{b}-\vec{a}\right|}.$$

(b) Given a parallelogram ABCD with area 12 sq. units. A straight line is drawn through the mid point M of the side BC and the vertex A which cuts the diagonal BD at a point ' O '. Use vectors to determine the area of the quadrilateral OMCD.

B-16 Position vectors of A, B, C are given by $\vec{a}, \vec{b}, \vec{c}$ where $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = 0$. If $\overrightarrow{AC} = 2\hat{i}-3\hat{j}+6\hat{k}$ then find $\overrightarrow{BC}$ if BC = 14.

Section (C) : Line

C-1 Line L_1 is parallel to vector $\vec{\alpha} = -3\hat{i}+2\hat{j}+4\hat{k}$ and passes through a point A(7, 6, 2) and line L_2 is parallel to a vector $\vec{\beta} = 2\hat{i}+\hat{j}+3\hat{k}$ and passes through a point B(5, 3, 4). Now a line L_3 parallel to a vector $\vec{r} = 2\hat{i}-2\hat{j}-\hat{k}$ intersects the lines L_1 and L_2 at points C and D respectively, then $|4\overrightarrow{CD}|$ is equal to :

C-2. Find the distance between points of intersection of

Lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ & $\frac{x-4}{5} = \frac{y-1}{2} = z$ and

Lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ & $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

C-3. Find the equation of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle of $\pi/3$.

C-4. If $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\mu(\hat{i}+\hat{j}-\hat{k})$ are two lines, then find the equation of acute angle bisector of two lines.

C-5. Find the image of the point P with position vector $7\hat{i} - \hat{j} + 2\hat{k}$ in the line whose vector equation is, $\vec{r} = 9\hat{i}+5\hat{j}+5\hat{k}+\lambda(\hat{i}+3\hat{j}+5\hat{k})$

C-6. Show that the foot of the perpendicular from the origin to the join of A(–9, 4, 5) and B (11, 0, –1) is the mid point of AB. Also find distance of point (2, 4, 4) from the line AB.

C-7. Find the shortest distance between the lines :

$$\vec{r} = (4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k}) \text{ and } \vec{r} = (\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$$

C-8. Let L_1 and L_2 be the lines whose equation are $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ respectively. A and B are two points on L_1 and L_2 respectively such that AB is perpendicular both the lines L_1 and L_2.Find points A, B and hence find shortest distance between lines L_1 and L_2

Section (D) : STP, Tetrahedron, VTP, LI/LD

D-1. Given unit vectors $\hat{m}, \hat{n}$ and $\hat{p}$ such that $\left(\hat{m}\,\widehat{\ }\,\hat{n}\right) = \hat{p}\,\widehat{\ }\,(\hat{m} \times \hat{n}) = \alpha$, then find value of $[\hat{n}\ \hat{p}\ \hat{m}]$ in terms of α.

D-2. If three coterminous edges of a tetrahedron are $\vec{a}, \vec{b}, \vec{c}$ such that $|\vec{a}| = 2$, $|\vec{b}| = 3$, $|\vec{c}| = 4$, angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$, $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{4}$ and $\vec{c}$ and $\vec{a}$ is $\frac{\pi}{6}$. The area of the base is 2 sq. units, then find the height of the tetrahedron.

D-3. Examine for coplanarity of the following sets of points

(a) $4\hat{i} + 8\hat{j} + 12\hat{k}$, $2\hat{i} + 4\hat{j} + 6\hat{k}$, $3\hat{i} + 5\hat{j} + 4\hat{k}$, $5\hat{i} + 8\hat{j} + 5\hat{k}$.

(b) $3\vec{a} + 2\vec{b} - 5\vec{c}$, $3\vec{a} + 8\vec{b} + 5\vec{c}$, $-3\vec{a} + 2\vec{b} + \vec{c}$, $\vec{a} + 4\vec{b} - 3\vec{c}$. Where $\vec{a}, \vec{b}, \vec{c}$ are noncoplanar

D-4 If $a_m\hat{i} + b_m\hat{j} + c_m\hat{k}, m = 1,2,3$, are pairwise perpendicular unit vectors, then find the value of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

D-5 Find the value of $\vec{a}.\left\{\left(\vec{b}+\vec{c}\right)\times\left(\vec{a}+\vec{b}+\vec{c}\right)\right\}$

D-6. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors and $\vec{v}\,.\,\vec{a} = \vec{v}\,.\,\vec{b} = \vec{v}\,.\,\vec{c} = 0$, then find value of $\vec{v}$.

D-7. Find the volume of tetrahedron whose vertices are P(2, 3, 2), Q(1, 1, 1), R(3, –2, 1) and S (7, 1, 4).

D-8✍ Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{d} = 3\hat{i} - \hat{j} - 2\hat{k}$, then

find the value of $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}).\vec{d}$

D-9✍ If $\vec{a}$ = – i + j + k and $\vec{b}$ = 2i + k, then find the vector $\vec{c}$ satisfying the conditions.

(i) that it is coplanar with $\vec{a}$ and $\vec{b}$

(ii) that its projection on $\vec{b}$ is 0

D-10. Are the following set of vectors linearly independent?

(i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 6\hat{j} + 9\hat{k}$

(ii) $\vec{a} = -2\hat{i} - 4\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$, $\vec{c} = \hat{i} - 4\hat{j} + 3\hat{k}$

D-11✍ Find value of $x \in R$ for which the vectors $\vec{a}$ = (1, –2, 3), $\vec{b}$ = (–2, 3, – 4), $\vec{c}$ = (1, – 1, x) form a linearly dependent system.

D-12. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the unit vectors such that $\vec{b}$ is not parallel to $\vec{c}$ and $\vec{a} \times \left(2\vec{b} \times \vec{c}\right) = \vec{b}$, then find the angle that $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are respectively:

D-13✍ Given that $\vec{x} + \frac{1}{\vec{p}^2}(\vec{p}\,.\,\vec{x})\,\vec{p} = \vec{q}$, then show that $\vec{p}\,.\,\vec{x} = \frac{1}{2}(\vec{p}\,.\,\vec{q})$ and hence find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

Section (E) : Plane

E-1. Find equation of plane

(i) Which passes through (0, 1, 0), (0, 0, 1), (1, 2, 3)

(ii) Which passes through (0, 1, 0) and contains two vectors $\hat{i} + \hat{j} - \hat{k}$ & $2\hat{i} - \hat{j}$.

(iii) Whose normal is $\hat{i} + \hat{j} + \hat{k}$ & which passes through (1, 2, 1).

(iv) Which makes equal intercepts on co-ordinate axis and passes through (1, 2, 3)

E-2. Find the ratio in which the line joining the points (3, 5,–7) and (–2, 1, 8) is divided by the yz-plane. Find also the point of intersection of the plane and the line

E-3. Find the angle between the plane passing through points (1, 1, 1), (1, – 1, 1), (–7, –3, –5) & x–z plane.

E-4✍ Let P (1, 3, 5) and Q(–2, 1, 4) be two points from which perpendiculars PM and QN are drawn to the x-z plane. Find the angle that the line MN makes with the plane x + y + z = 5.

E-5✍ Find the equation of the plane containing parallel lines $(x - 4) = \frac{3-y}{4} = \frac{z-2}{5}$ and

$(x - 3) = \lambda\,(y + 2) = \mu z$

E-6. If the acute angle that the vector, $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ makes with the plane of the two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $\hat{i}-\hat{j}+2\hat{k}$ is $\cot^{-1}\sqrt{2}$ then find the value of $\alpha(\beta+\gamma)-\beta\gamma$

E-7. Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-2}{5}$.

E-8. Find the equation of the planes passing through points (1, 0, 0) and (0, 1, 0) and making an angle of 0.25 π radians with plane $x + y - 3 = 0$

E-9. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (1, 4, 2) and (2, 3, 5). Also find the coordinates of the foot of the perpendicular and the perpendicular distance of the point (4, 0, 3) from the above found plane.

E-10. Find the locus of the point whose sum of the square of distances from the planes $x + y + z = 0$, $x - z = 0$ and $x - 2y + z = 0$ is 9

E-11. Find the equation of image of the line $\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3}$ in the plane $3x - 3y + 10z = 26$.

E-12. Find the distance of the point (2, 3, 4) from the plane $3x + 2y + 2z + 5 = 0$, measured parallel to the line $\frac{x+3}{3}=\frac{y-2}{6}=\frac{z}{2}$

E-13. Find the equation of plane which passes through the point of intersection of lines $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ and $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and at greatest distance from the point (0, 0, 0)

E-14. If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$ is :

E-15. (i) If $\hat{n}$ is the unit vector normal to a plane and p be the length of the perpendicular from the origin to the plane, find the vector equation of the plane.

(ii) Find the equation of the plane which contains the origin and the line of intersection of the planes $\vec{r}\cdot\vec{a} = p$ and $\vec{r}\cdot\vec{b} = q$

PART-II : OBJECTIVE QUESTIONS

Section (A) : Position vector, Direction Ratios & Direction cosines

A-1. A vector $\vec{a}$ has components 2p and 1 with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counterclockwise sense. If with respect to the new system, $\vec{a}$ has components p + 1 and 1, then integral value of p can be

(A) $p = -\frac{1}{3}$ (B) $p = 1$ (C) $p = -1$ (D) $p = \frac{1}{3}$

A-2. The vectors $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+\hat{j}+4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are :

(A) Not coplaner (B) Coplaner but cannot form a triangle

(C) Coplaner but can form a triangle (D) Coplaner & can form a right angled triangle

A-3. OABCDE is a regular hexagon of side 2 units in the XY–plane in the I[st] quadrant . O being the origin and OA taken along the X–axis. A point P is taken on a line parallel to Z–axis through the centre of the hexagon at a distance of 3 units from O in the positive Z direction. Then vector $\overrightarrow{AP}$ is:

(A) $-\hat{i}+3\hat{j}+\sqrt{5}\hat{k}$ (B) $\hat{i}-\sqrt{3}\hat{j}+5\hat{k}$

(C) $-\hat{i}+\sqrt{3}\hat{j}+\sqrt{5}\hat{k}$ (D) $\hat{i}+\sqrt{3}\hat{j}+\sqrt{5}\hat{k}$

A-4. The ratio in which the line joining the points (1, 2, 3) and (-3, 4, -5) is divided by the xy plane is

(A) $\frac{5}{3}$ (B) $\frac{3}{5}$ (C) $\frac{2}{5}$ (D) none of these

A-5. Points X and Y are taken on the sides QR and RS, respectively of a parallelogram PQRS, so that QX = 4XR and RY = 4YS. The line XY cuts the line PR at Z. Find the ratio PZ : ZR.

(A) 4 : 21 (B) 3 : 4 (C) 21 : 4 (D) 4 : 3

A-6. A line makes angles α, β, γ with the coordinate axes. If $\alpha + \beta = 90°$, then γ =

(A) 0 (B) 90° (C) 180° (D) 45°

A-7. A line in xy-plane makes angle 30° with x-axis, the direction ratio of the line are:

(A) 1 : 1 : 0 (B) $\sqrt{3}$: 1 : 0 (C) 1 : $\sqrt{3}$: 0 (D) 1 : 1 : 1

A-8. If α, β, γ are angles which a line makes with the axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ =

(A) 1 (B) 2 (C) 3 (D) 4

A-9. Direction cosines of the line joining the points (0, 0, 0) and (a, a, a) are

(A) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (B) 1, 1, 1 (C) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (D) none of these

A-10. ABC is a triangle where A = (2, 3, 5), B = (–1, 2, 2) and C(λ, 5, μ), if the median through A is equally inclined to the positive axes, then $\lambda + \mu$ is

(A) 7 (B) 6 (C) 15 (D) 9

Section (B) : Dot Product, Projection and Cross Product

B-1. If a vector $\vec{a}$ of magnitude 50 is collinear with vector $\vec{b} = 6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k}$ and makes an acute angle with positive z-axis then :

(A) $\vec{a} = 4\vec{b}$ (B) $\vec{a} = -4\vec{b}$ (C) $\vec{b} = 4\vec{a}$ (D) none

B-2. The set of values of c for which the angle between the vectors $cx\hat{i} - 6\hat{j} + 3\hat{k}$ & $x\hat{i} - 2\hat{j} + 2cx\hat{k}$ is acute for every x $\in$ R is

(A) (0, 4/3) (B) [0, 4/3] (C) (11/9, 4/3) (D) [0, 4/3)

B-3. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ to $\vec{c}+\vec{a}$ and $\vec{c}$ to $\vec{a}+\vec{b}$. Then $|\vec{a}+\vec{b}+\vec{c}|$ is equal to :

(A) $2\sqrt{5}$ (B) $2\sqrt{2}$ (C) $10\sqrt{5}$ (D) $5\sqrt{2}$

B-4 If the unit vectors $\vec{e}_1$ and $\vec{e}_2$ are inclined at an angle 2θ and $|\vec{e}_1-\vec{e}_2| < 1$, then for $\theta \in [0, \pi]$, θ may lie in the interval :

(A) $\left[0, \frac{\pi}{6}\right)$ (B) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (C) $\left(\frac{5\pi}{6}, \pi\right]$ (D) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

B-5. A, B, C & D are four points in a plane with position vectors $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ respectively such that

$(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = (\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0$. Then for the triangle ABC, D is its:

(A) incentre (B) circumcentre (C) orthocentre (D) centroid

B-6. Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 = n^2$.

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{2}$ (D) 0

B-7. The projections of a vector on the three coordinate axes are 6, –3, 2 respectively. The direction cosines of the vector are.

(A) 6, –3, 2 (B) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$ (C) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ (D) $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

B-8. If the vector $6\hat{i} - 3\hat{j} - 6\hat{k}$ is decomposed into vectors parallel and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ then the vectors are :

(A) $-(\hat{i}+\hat{j}+\hat{k})$ & $7\hat{i} - 2\hat{j} - 5\hat{k}$ (B) $-2(\hat{i}+\hat{j}+\hat{k})$ & $8\hat{i} - \hat{j} - 4\hat{k}$

(C) $+2(\hat{i}+\hat{j}+\hat{k})$ & $4\hat{i} - 5\hat{j} - 8\hat{k}$ (D) none

B-9. For two particular vectors $\vec{A}$ and $\vec{B}$ it is known that $\vec{A}\times\vec{B} = \vec{B}\times\vec{A}$. What must be true about the two vectors?

(A) At least one of the two vectors must be the zero vector.

(B) $\vec{A}\times\vec{B} = \vec{B}\times\vec{A}$ is true for any two vectors.

(C) One of the two vectors is a scalar multiple of the other vector.

(D) The two vectors must be perpendicular to each other.

B-10. For a non zero vector $\vec{A}$ if the equations $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ and $\vec{A}\times\vec{B} = \vec{A}\times\vec{C}$ hold simultaneously, then:

(A) $\vec{A}$ is perpendicular to $\vec{B}-\vec{C}$ (B) $\vec{A} = \vec{B}$

(C) $\vec{B} = \vec{C}$ (D) $\vec{C} = \vec{A}$

B-11. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then the vectors $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ are:

(A) non-collinear (B) coplanar (C) perpendicular (D) parallel

B-12. Unit vector perpendicular to the plane of the triangle ABC with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ of the vertices A, B, C, is (where Δ is the area of the triangle ABC).

(A) $\dfrac{(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})}{\Delta}$ (B) $\dfrac{(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})}{2\Delta}$

(C) $\dfrac{(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})}{4\Delta}$ (D) $\dfrac{(\vec{a}\times\vec{b} + \vec{c}\times\vec{b} + \vec{c}\times\vec{a})}{2\Delta}$

Section (C) : Line

C-1. The equation of line perpendicular to the line $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and passing through (-2, -5, 7) is

(A) $\dfrac{x+2}{14} = \dfrac{y+5}{123} = \dfrac{z-7}{104}$ (B) $\dfrac{x+2}{14} = \dfrac{y+5}{137} = \dfrac{z-7}{-204}$

(C) $\dfrac{x+2}{76} = \dfrac{y+5}{137} = \dfrac{z-7}{-254}$ (D) none of these

C-2. The angle between the lines x - 1 = y - 2 = z - 3 and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{2}$ is

(A) $\cos^{-1}\left(\dfrac{5}{3}\right)$ (B) $\cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right)$ (C) $\cos^{-1}\left(\dfrac{3}{5}\right)$ (D) none of these

C-3. The length of perpendicular from the point (-1, 2, -2)) on the line $\dfrac{x+1}{2} = \dfrac{y-2}{-3} = \dfrac{z+2}{4}$ is

(A) $\sqrt{29}$ (B) $\sqrt{6}$ (C) $\sqrt{21}$ (D) none of these

C-4. A line through (1, 1, 0) is parallel to z-axis, the equation of the line is

(A) $\dfrac{x-1}{0} = \dfrac{y-1}{0} = \dfrac{z-0}{1}$ (B) $\dfrac{x}{1} = \dfrac{y-1}{1} = \dfrac{z-0}{0}$

(C) $\dfrac{x-1}{1} = \dfrac{y-1}{1} = \dfrac{z-1}{2}$ (D) none of these

C-5. The projection of the line of joining of the points (0, 0, 0) and (2, 2, 2) on the line $\dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z}{0}$ is

(A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $3\sqrt{2}$ (D) 2

C-6. The vertices of a triangle are A (1, 1, 2), B(4, 3, 1) and C(2, 3, 5). A vector representing the internal bisector of the angle A is :

(A) $\hat{i}+\hat{j}+2\hat{k}$ (B) $2\hat{i}-2\hat{j}+\hat{k}$ (C) $2\hat{i}+2\hat{j}-\hat{k}$ (D) $2\hat{i}+2\hat{j}+\hat{k}$

C-7. Equation of the angle bisector of the angle between the lines $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$ & $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{-1}$ is :

(A) $\frac{x-1}{2}=\frac{y-2}{2}$; $z-3=0$ (B) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

(C) $x-1=0$; $\frac{y-2}{1}=\frac{z-3}{1}$ (D) $\frac{x-1}{2}=\frac{y-2}{3}$; $z-3=0$

C-8. Shortest distance between the lines $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{1}$ and $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$ is equal to

(A) $\sqrt{14}$ (B) $\sqrt{7}$ (C) $\sqrt{2}$ (D) none of these

C-9. The shortest distance between the lines $\frac{x}{1}=\frac{y}{0}=\frac{z-1}{0}$ and y-axis is

(A) 0 (B) 1 (C) 2 (D) 3

Section (D) : STP, Tetrahedron, VTP, LI/LD

D-1 Given $\vec{a}=x\hat{i}+y\hat{j}+2\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$, $\vec{c}=\hat{i}+2\hat{j}$; $(\vec{a}\,\hat{}\,\vec{b})=\frac{\pi}{2}$, $\vec{a}\cdot\vec{c}=4$, then

(A) $[\vec{a}\,\vec{b}\,\vec{c}]^2=|\vec{a}|$ (B) $[\vec{a}\,\vec{b}\,\vec{c}]=|\vec{a}|$ (C) $[\vec{a}\,\vec{b}\,\vec{c}]=0$ (D) $[\vec{a}\,\vec{b}\,\vec{c}]=|\vec{a}|^2$

D-2. The value of $\left[(\vec{a}+2\vec{b}-\vec{c})\ (\vec{a}-\vec{b})\ (\vec{a}-\vec{b}-\vec{c})\right]$ is equal to the box product:

(A) $[\vec{a}\ \vec{b}\ \vec{c}]$ (B) $2[\vec{a}\ \vec{b}\ \vec{c}]$ (C) $3[\vec{a}\ \vec{b}\ \vec{c}]$ (D) $4[\vec{a}\ \vec{b}\ \vec{c}]$

D-3. If $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ is perpendicular to both $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ (where $\vec{a},\vec{b},\vec{c}$ are unit vectors) and $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & b_2 & c_3 \end{vmatrix}^{x^2+x+5}=1$ for all real x, then the angle between $\vec{a}$ and $\vec{b}$, is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

D-4. If three distinct vectors $\vec{x}=a\hat{i}+b\hat{j}+c\hat{k}$, $\vec{y}=b\hat{i}+c\hat{j}+a\hat{k}$ and $\vec{z}=c\hat{i}+a\hat{j}+b\hat{k}$ are coplanar then maximum value of $\vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{z}+\vec{z}\cdot\vec{x}$ is

(A) 0 (B) a + b + c (C) 9 (D) none of these

D-5. Let $\vec{a} = x\hat{i}+12\hat{j}-\hat{k}$, $\vec{b} = 2\hat{i}+2x\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}+\hat{k}$. If the ordered set $\left[\vec{b}\ \vec{c}\ \vec{a}\right]$ is left handed, then :

(A) $x \in (2, \infty)$ (B) $x \in (-\infty, -3)$ (C) $x \in (-3, 2)$ (D) $x \in \{-3, 2\}$

D-6. If $\vec{a}$ = i + j – k, $\vec{b}$ = i – j + k, $\vec{c}$ is a unit vector such that $\vec{c}.\vec{a} = 0$, $[\vec{c}\vec{a}\vec{b}] = 0$ then a unit vector $\vec{d}$ perpendicular to both $\vec{a}$ and $\vec{c}$ is

(A) $\frac{1}{\sqrt{6}}$(2i – j + k) (B) $\frac{1}{\sqrt{2}}$(j + k) (C) $\frac{1}{\sqrt{2}}$(i + j) (D) $\frac{1}{\sqrt{2}}$(i + k)

D-7. Vector of length 3 unit which is perpendicular to $\hat{i}+\hat{j}+\hat{k}$ and lies in the plane of $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-3\hat{j}$, is

(A) $\frac{3}{\sqrt{6}}(\hat{i}-2\hat{j}+\hat{k})$ (B) $\frac{3}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$ (C) $\frac{3}{\sqrt{114}}(7\hat{i}+8\hat{j}+\hat{k})$ (D) $\frac{3}{\sqrt{114}}(-7\hat{i}+8\hat{j}-\hat{k})$

D-8✍ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent vectors, then which one of the following set of vectors is linearly dependent?

(A) $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ (B) $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$ (C) $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ (D) $\vec{a}+2\vec{b}+3\vec{c}$, $\vec{b}-\vec{c}+\vec{a}$, $\vec{a}+\vec{c}$

D-9. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the points A, B, C and D respectively in three dimensional space no three of A, B, C, D are collinear and satisfy the relation $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d} = \vec{0}$, then which of the following is **INCORRECT**

(A) A, B, C and D are coplanar

(B) The line joining the points B and D divides the line joining the point A and C in the ratio 2 : 1.

(C) The line joining the points A and C divides the line joining the points B and D in the ratio 1 : 1.

(D) the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are linearly dependents.

D-10✍ Vector $\vec{x}$ satisfying the relation $\vec{A}.\vec{x} = c$ and $\vec{A}\times\vec{x} = \vec{B}$ is

(A) $\frac{c\vec{A}-(\vec{A}\times\vec{B})}{|\vec{A}|}$ (B) $\frac{c\vec{A}-(\vec{A}\times\vec{B})}{|\vec{A}|^2}$ (C) $\frac{c\vec{A}+(\vec{A}\times\vec{B})}{|\vec{A}|^2}$ (D) $\frac{c\vec{A}-2(\vec{A}\times\vec{B})}{|\vec{A}|^2}$

Section (E) : Plane

E-1. The equation of a plane which passes through (2, – 3, 1) & is perpendicular to the line joining the points (3, 4, – 1) & (2, – 1, 5) is given by:

(A) x + 5y – 6z + 19 = 0 (B) x – 5y + 6z – 19 = 0

(C) x + 5y + 6z + 19 = 0 (D) x – 5y – 6z – 19 = 0

E-2. The equation of the plane passing through the point (1, – 3, – 2) and perpendicular to planes x + 2y + 2z = 5 and 3x + 3y + 2z = 8, is

(A) 2x – 4y + 3z – 8 = 0 (B) 2x – 4y – 3z + 8 = 0

(C) 2x + 4y + 3z + 8 = 0 (D) 2x + 4y + 3z – 8 = 0

E-3. If line $\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ is parallel to the plane $\vec{r}\cdot(3\hat{i} - 2\hat{j} - m\hat{k}) = 14$, then the value of m is

(A) 2 (B) – 2

(C) 0 (D) can not be predicted with these informations

E-4✍ If a plane cuts off intercepts OA = a, OB = b, OC = c from the coordinate axes (where 'O' is the origin), then the area of the triangle ABC is equal to

(A) $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$ (B) $\frac{1}{2}$ (bc + ca + ab)

(C) $\frac{1}{2}$ abc (D) $\frac{1}{2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}$

E-5. The locus represented by xy + yz = 0 is

(A) A pair of perpendicular lines (B) A pair of parallel lines

(C) A pair of parallel planes (D) A pair of perpendicular planes

E-6. The distance of the point (– 1, – 5, – 10) from the point of intersection of the line, $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane, x – y + z = 5, is :

(A) 10 (B) 11 (C) 12 (D) 13

E-7. The reflection of the point (2, –1, 3) in the plane 3x – 2y – z = 9 is :

(A) $\left(\frac{26}{7}, \frac{15}{7}, \frac{17}{7}\right)$ (B) $\left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$ (C) $\left(\frac{15}{7}, \frac{26}{7}, \frac{-17}{7}\right)$ (D) $\left(\frac{26}{7}, \frac{17}{7}, \frac{-15}{7}\right)$

E-8. The distance of the point (1, –2, 3) from the plane x – y + z = 5 measured parallel to the line, $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$, is:

(A) 1 (B) 6/7 (C) 7/6 (D) 1/6

E-9. The distance of the point P(3, 8, 2) from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane 3x + 2y – 2z + 17 = 0 is

(A) 2 (B) 3 (C) 5 (D) 7

E-10. Given the vertices A (2, 3, 1), B (4, 1, –2), C (6, 3, 7) & D (–5, –4, 8) of a tetrahedron. The length of the altitude drawn from the vertex D is:

(A) 7 (B) 9 (C) 11 (D) 13

E-11✍ The values of 'λ' for which the two lines $\frac{x-1}{4} = \frac{y-2}{1} = \frac{z}{1}$ & $\frac{x+7}{\lambda} = \frac{y}{\lambda-6} = \frac{z+\lambda}{2}$ are coplaner

(A) –2, 8 (B) 2, –8 (C) 3, 5 (D) 1, 2

PART-III : MATCH THE COLUMN

1. **Column-I** **Column-II**

(A) P is point in the plane of the triangle ABC. pv's of A,B and C are $\vec{a},\vec{b}$ and $\vec{c}$ respectively with respect to P as the origin. If $(\vec{b}+\vec{c}).(\vec{b}-\vec{c})=0$ and $(\vec{c}+\vec{a}).(\vec{c}-\vec{a})=0$, then w.r.t. the triangle ABC,P is its — (p) centroid; (q) orthocentre

(B) If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the three non collinear points A,B and C respectively such that the vector $\vec{V}=\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is a null vector then w.r.t. the ΔABC, P is its — (r) Incentre

(C) If P is a point inside the ΔABC such that the vector $\vec{R}=(BC)\overrightarrow{PA}+(CA)(\overrightarrow{PB})+(AB)(\overrightarrow{PC})$ is a null vector then w.r.t. the ΔABC, P is its — (s) circumcentre

(D) If P is a point in the plane of the triangle ABC such that the scalar product $\overrightarrow{PA}.\overrightarrow{CB}$ and $\overrightarrow{PB}.\overrightarrow{AC}$ vanishes, then w.r.t. the ΔABC, P is its

2. Match the following set of lines to the corresponding type :

(A) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ & $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}$ — (p) parallel but not coincident

(B) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ & $\frac{x-5}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ — (q) intersecting

(C) $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}$ & $\frac{x-3}{-1}=\frac{y-4}{-1}=\frac{z-1}{1}$ — (r) skew lines

(D) $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{3}$ & $\frac{x}{2}=\frac{y+1}{3}=\frac{z}{1}$ — (s) Coincident

3. **Column – I** **Column – II**

(A) Twice of the area of the parallelogram constructed on the vectors $\vec{a}=\vec{p}+2\vec{q}$ and $\vec{b}=2\vec{p}+\vec{q}$, where $\vec{p}$ and $\vec{q}$ are unit vectors containing an angle of 30º, is : — (p) –3

(B) The points (0, –1, –1), (4, 5, 1), (3, 9, 4) and (–4, 4, k) are coplanar, then k = — (q) 2

(C) Let $\vec{a}$ & $\vec{b}$ be two non-zero perpendicular vectors. If a vector $\vec{x}$ satisfying the equation $\vec{x}\times\vec{b}=\vec{a}$ is $\vec{x}=\beta\vec{b}-\frac{1}{|\vec{b}|^2}.\vec{a}\times\vec{b}$ then β can be — (r) 4

(D) If $\vec{x}$ satisfying the conditions $\vec{b}.\vec{x}=\beta$ & $\vec{b}\times\vec{x}=\vec{a}$ is $\vec{x}=\frac{(\beta^2-12)\vec{b}}{|\vec{b}|^2}+\frac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ then β can be — (s) 3

Exercise # 2

PART-I : OBJECTIVE QUESTIONS

1✍ Let X be the midpoint of the side AB of triangle ABC. Let Y be the midpoint of CX. Let BY cut AC at Z. Then AZ : ZC =

(A) 1 : 2 (B) 2 : 1 (C) 1 : 3 (D) 3 : 1

2. ABC is triangle, right angled at A. The resultant of the forces acting along $\vec{AB}$, $\vec{AC}$ with magnitudes $\frac{1}{AB}$ and $\frac{1}{AC}$ respectively is the force along $\vec{AD}$, where D is the foot of the perpendicular from A onto BC. The magnitude of the resultant is-

(A) $\frac{(AB)(AC)}{AB+AC}$ (B) $\frac{1}{AB}+\frac{1}{AC}$ (C) $\frac{1}{AD}$ (D) $\frac{AB^2+AC^2}{(AB)^2+(AC)^2}$

3✍ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-zero, non coplanar vectors and $\vec{b}_1 = \vec{b} - \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$, $\vec{b}_2 = \vec{b} + \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$,

$\vec{c}_1 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b}$, $\vec{c}_2 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{b}_1\cdot\vec{c}}{|\vec{b}_1|^2}\vec{b}_1$, $\vec{c}_3 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{b}_2\cdot\vec{c}}{|\vec{b}_2|^2}\vec{b}_2$,

$\vec{c}_4 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a} - \frac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b}_1$, then the set of orthogonal vectors is

(A) $(\vec{a}, \vec{b}_1, \vec{c}_3)$ (B) $(\vec{a}, \vec{b}_1, \vec{c}_2)$ (C) $(\vec{a}, \vec{b}_1, \vec{c}_1)$ (D) $(\vec{a}, \vec{b}_2, \vec{c}_2)$

4. The position vectors of the vertices A, B and C of a triangle are $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$ and $\hat{i}+\hat{k}$ respectively. Find a unit vector lying in the plane of ABC and perpendicular to IA, where I is the incentre of the triangle.

(A) $\pm\frac{1}{2}(\hat{i}-\hat{j})$ (B) $\pm\frac{1}{2}(\hat{i}+\hat{j})$ (C) $\pm\frac{1}{\sqrt{5}}(2\hat{i}+\hat{j})$ (D) $\pm\frac{1}{\sqrt{5}}(2\hat{i}-\hat{j})$

5✍ If 3 non zero vectors $\vec{a},\vec{b},\vec{c}$ are such that $\vec{a}\times\vec{b} = 2(\vec{a}\times\vec{c})$, $|\vec{a}| = |\vec{c}| = 1$; $|\vec{b}| = 4$ the angle between $\vec{b}$ and $\vec{c}$ is $\cos^{-1}\frac{1}{4}$ then $\vec{b} = \ell\vec{c}+\mu\vec{a}$ where $|\ell| + |\mu|$ is -

(A) 6 (B) 5 (C) 4 (D) 0

6. Let the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$, $\overrightarrow{ST}$, $\overrightarrow{TU}$ and $\overrightarrow{UP}$ represent the sides of a regular hexagon.

STATEMENT-1 : $\overrightarrow{PQ} \times \left(\overrightarrow{RS}+\overrightarrow{ST}\right) \neq \vec{0}$

because

STATEMENT-2 : $\overrightarrow{PQ} \times \overrightarrow{RS} = \vec{0}$ and $\overrightarrow{PQ} \times \overrightarrow{ST} \neq \vec{0}$

(A) Statement-1 is True, Statement-2 is True ; Statement-2 **is** a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

7. If the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4}$ and $\frac{x+k}{3} = \frac{y-1}{2} = \frac{z-2}{h}$ are concurrent then

(A) h = – 2, k = – 6 (B) h = $\frac{1}{2}$, k = 2 (C) h = 6, k = 2 (D) h = 2, k = $\frac{1}{2}$

8. Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is :

(A) $-\hat{i} + \hat{j} + 2\hat{k}$ (B) $3\hat{i} - \hat{j} + \hat{k}$ (C) $3\hat{i} + \hat{j} - \hat{k}$ (D) $\hat{i} - \hat{j} - \hat{k}$

9. Consider the lines $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, then the equation of the line which

(A) bisects the angle between the lines is $\frac{x}{3} = \frac{y}{3} = \frac{z}{8}$

(B) bisects the angle between the lines is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

(C) passes through origin and is perpendicular to the given lines is x = y = – z

(D) none of these

10. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - \lambda^2\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - \lambda^2\hat{k}$ are coplanar, is

(A) zero (B) one (C) two (D) three

11. The volume of the parallelopiped constructed on the diagonals of the faces of the given rectangular parallelopiped is m times the volume of the given parallelopiped. Then m is equal to:

(A) 2 (B) 3 (C) 4 (D) 1

12✍ Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar unit vectors equally inclined to one another at an acute angle θ. Then $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$ in terms of θ is equal to:

(A) $(1 + \cos\theta)\sqrt{\cos 2\theta}$ (B) $(1 + \cos\theta)\sqrt{1 - 2\cos 2\theta}$

(C) $(1 - \cos\theta)\sqrt{1 + 2\cos\theta}$ (D) $(1 - \sin\theta)\sqrt{1 + 2\cos\theta}$

13✍ $$\frac{\left[\vec{R}\cdot\left(\vec{\beta}\times\left(\vec{\beta}\times\vec{\alpha}\right)\right)\right]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2} + \frac{\left[R.\left(\vec{\alpha}\times\left(\vec{\alpha}\times\vec{\beta}\right)\right)\right]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2} - \frac{\left[\vec{R}\vec{\alpha}\vec{\beta}\right]\left(\vec{\alpha}\times\vec{\beta}\right)}{|\vec{\alpha}\times\vec{\beta}|^2} =$$

(A) $\vec{R}$ (B) $-\vec{R}$ (C) $\vec{R}\times\vec{\alpha}$ (D) $2\vec{R}$

14. If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a} \parallel (\vec{b} \times \vec{c})$, then $(\vec{a} \times \vec{b}) . (\vec{a} \times \vec{c})$ is equal to

(A) $\vec{a}^2(\vec{b}.\vec{c})$ (B) $\vec{b}^2(\vec{a}.\vec{c})$ (C) $\vec{c}^2(\vec{a}.\vec{b})$ (D) $-\vec{a}^2(\vec{b}.\vec{c})$

15. If $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors such that $\vec{a} . \vec{b} \neq 0$, $\vec{b} . \vec{c} \neq 0$, then $\vec{a}$ and $\vec{c}$ are-

(A) Inclined at an angle of $\frac{\pi}{6}$ between them (B) Perpendicular

(C) Parallel (D) Inclined at an angle of $\frac{\pi}{3}$ between them

16✍ A variable plane at a distance of 1 unit from the origin cuts the co-ordinate axes at A, B and C. If the centroid D (x, y, z) of triangle ABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = K$, then the value of k is

(A) 9 (B) 3 (C) 1 (D) 1/3

17✍ Consider a tetrahedron with faces f_1, f_2, f_3, f_4. Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ be the vectors whose magnitudes are respectively equal to the areas of f_1, f_2, f_3, f_4 and whose directions are perpendicular to these faces in the outward direction. Then,

(A) $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4 = \vec{0}$ (B) $\vec{a}_1 + \vec{a}_3 = \vec{a}_2 + \vec{a}_4$

(C) $\vec{a}_1 + \vec{a}_2 = \vec{a}_3 + \vec{a}_4$ (D) $\vec{a}_1 + \vec{a}_2 - \vec{a}_3 + \vec{a}_4 = \vec{0}$

18. A line having direction ratios 3, 4, 5 cuts 2 planes $2x - 3y + 6z - 12 = 0$ and $2x - 3y + 6z + 2 = 0$ at point P & Q, then find length of PQ

(A) $\frac{35\sqrt{2}}{12}$ (B) $\frac{35\sqrt{2}}{24}$ (C) $\frac{35\sqrt{2}}{6}$ (D) $\frac{35\sqrt{2}}{8}$

19. The foot of the perpendicular from the point O(0, 0, 0) to the line of intersection of the planes $x + y + z = 4$ and $2x + y + 3z = 1$ is point A. Find the equation of line OA.

(A) $\frac{x}{8} = \frac{y}{29} = \frac{z}{-13}$ (B) $\frac{x}{8} = \frac{y}{29} = \frac{z}{13}$ (C) $\frac{x}{18} = \frac{y}{29} = \frac{z}{13}$ (D) None of these

20. The non zero value of 'a' for which the lines $2x - y + 3z + 4 = 0 = ax + y - z + 2$ and $x - 3y + z = 0 = x + 2y + z + 1$ are co-planar is :

(A) – 2 (B) 4 (C) 6 (D) 0

21. A line L_1 having dierection ratios 1, 0, 1 lies on xz plane. Now this xz plane is rotated about z-axis by an angle of 90°. Now the new position of L_1 is L_2. The angle between L_1 & L_2 is :

(A) 30° (B) 60° (C) 90° (D) 45°

PART-II : NUMERICAL QUESTIONS

1. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are on a circle of radius R whose centre is at origin and $\vec{c} - \vec{a}$ is perpendicular to $\vec{d} - \vec{b}$, such that $|\vec{d} - \vec{a}|^2 + |\vec{b} - \vec{c}|^2 = \lambda R^2$ then find λ.

2. If $a^2 + b^2 + c^2 = 9(x^2 + y^2 + z^2)$ then find the maximum value of $\frac{ax + by + cz}{x^2 + y^2 + z^2}$.

3. Four points P, A, B and C with position vectors as $\bar{r}, \bar{a}, \bar{b}$ and $\bar{c}$ respectively are in a plane such that P, A, B are collinear and satisfy $|\bar{a}| = |\bar{b}| = |\bar{c}|$ and $\bar{r}.\bar{c} = |\bar{c}|^2$. Find λ if $|\bar{r}-\bar{a}||\bar{r}-\bar{b}| = \lambda|\bar{r}-\bar{c}|^2$

4. If in a plane $A_1, A_2, A_3, ... A_{20}$ are the vertices of a regular polygon having 20 sides and O is its centre and $\sum_{i=1}^{19}(\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = \lambda(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$ then $|\lambda|$ is

5. In an equilateral ΔABC find the value of $\dfrac{|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2}{R^2}$ where P is any arbitrary point lying on its circumcircle, is

6. Let $\bar{u}$ and $\bar{v}$ are unit vectors and $\bar{w}$ is a vector such that $\bar{u} \times \bar{v} + \bar{u} = \bar{w}$ and $\bar{w} \times \bar{u} = \bar{v}$, Then find the value of [uvw].

7. If 'd' be the shortest distance between the lines $\dfrac{y}{b} + \dfrac{z}{c} = 1$; $x = 0$ and $\dfrac{x}{a} - \dfrac{z}{c} = 1$; $y = 0$ and if $\dfrac{\lambda}{d^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$ then λ is

8. If $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$, $D(\vec{d})$ are the position vector of cyclic quadrilateral then find the value of

$$\frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|}{[(\vec{b}-\vec{a}).(\vec{d}-\vec{a})]} + \frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|}{[(\vec{b}-\vec{c}).(\vec{d}-\vec{c})]}.$$

9. Given that $\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$ $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$, $\vec{w} = 2\hat{i} + j + 3\hat{k}$ and $(\vec{u}.\vec{R} - 10) + (\vec{v}.\vec{R} - 20)j + (\vec{w}.\vec{R} - 20)\hat{k} = 0$. If $\vec{R} = a\hat{i} + b\hat{j} + c\hat{k}$ then find a + b + c

10. If the circumcentre of the tetrahedron OABC is given by $\dfrac{\vec{a}^2(\vec{b} \times \vec{c}) + \vec{b}^2(\vec{c} \times \vec{a}) + \vec{c}^2(\vec{a} \times \vec{b})}{\alpha}$, where $\vec{a}, \vec{b}$ & $\vec{c}$ are the position vectors of the points A, B, C respectively relative to the origin 'O' such that $[\vec{a}\ \vec{b}\ \vec{c}] = 36$ then α is

11. A line $\dfrac{x+2}{1} = \dfrac{y-3}{2} = \dfrac{z-k}{3}$ cuts the y-z plane and the x-y plane at A and B respectively. If $\angle AOB = \dfrac{\pi}{2}$, then 2k, where O is the origin, is

12. If $\vec{r}$ represents the position vector of point R in which the line AB cuts the plane CDE, where position vectors of points A, B, C, D, E are respectively $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{c} = -4\hat{j} + 4\hat{k}$, $\vec{d} = 2\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{e} = 4\hat{i} + \hat{j} + 2\hat{k}$, then $\vec{r}^2$ is :

13. L is the equation of the straight line which passes through the point (2, –1, –1); is parallel to the plane 4x + y + z + 2 = 0 and is perpendicular to the line of intersection of the planes 2x + y = 0 = x – y + z. If the point $(3, \alpha, \beta)$ lies on line L, then $|\alpha + \beta|$ is

14✍ If the volume of tetrahedron formed by planes whose equations are y + z = 0, z + x = 0, x + y = 0 and x + y + z = 1 is t cubic unit, then the value of 3t is

15. Given three point on x – y plane on O(0, 0), A(1, 0) & B(–1, 0). Point P is moving on the plane satisfying the condition $(\overrightarrow{PA}.\overrightarrow{PB}) + 3(\overrightarrow{OA}.\overrightarrow{OB}) = 0$

If the maximum & minimum values of $|\overrightarrow{PA}||\overrightarrow{PB}|$ is M & m respectively then the value of $M^2 + m^2$ is

16. The lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ and 3x – 2y + z + 5 = 0 = 2x + 3y + 4z – k are coplanar, then the value of k is

17✍ About the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ the plane 3x + 4y + 6z + 7 = 0 is rotated till the plane passes through the origin. Now $4x + \alpha y + \beta z = 0$ is the equation of plane in new position. The value of $\alpha^2 + \beta^2$ is

18. The value of $\sec^3\theta$, where θ is the acute angle between the plane faces of a regular tetrahedron, is

19✍ A line L on the plane 2x + y – 3z + 5 = 0 is at a distance 3 unit from the point P(1, 2, 3). A spider starts moving from point A and after moving 4 units along the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-3}$ it reaches to point P. and from P it jumps to line L along the shortest distance and then moves 12 units along the line L to reach at point B. The distance between points A and B is

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. If perpendicular distances of point (a, b, c) from yz, zx and xy planes are in A.P. and distances from x, y and z axes are $\sqrt{13}$, $\sqrt{10}$ and $\sqrt{5}$ respectively then

(A) |a| = 1 (B) |b| = 2 (C) |c| = 3 (D) |a| = |b| = |c|

2. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle of $\cos^{-1}\frac{11}{14}$ and doubled in magnitude, then it becomes $4\hat{i} + (4x-2)\hat{j} + 2\hat{k}$. The value of 'x' **CANNOT** be :

(A) $-\frac{2}{3}$ (B) $\frac{2}{3}$ (C) $-\frac{20}{17}$ (D) 2

3. The vertices of a cube are (0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1) and (1, 1, 1). An insect starts from the origin and reaches the vertex (1, 1, 1) by crawling on the surface of the cube such that it travels the minimum distance. Which of the following can be the direction cosines of the pair of line segments constituting the possible path.

(A) $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$ and $\left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ (B) $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ and $\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$

(C) (1, 0, 0) and (0, 1, 1) (D) (0,1,0) and (1, 0, 1)

4✍ If a, b, c, x, y, z $\in$ R such that ax + by + cz = 2, then which of the following is always true

(A) $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq 4$
(B) $(x^2 + b^2 + z^2)(a^2 + y^2 + c^2) \geq 4$
(C) $(a^2 + y^2 + z^2)(x^2 + b^2 + c^2) \geq 4$
(D) $(a^2 + b^2 + z^2)(x^2 + y^2 + c^2) \geq 4$

5✍ The direction cosines of the lines bisecting the angle between the lines whose direction cosines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and the angle between these lines is θ, are

(A) $\dfrac{\ell_1+\ell_2}{\cos\frac{\theta}{2}}, \dfrac{m_1+m_2}{\cos\frac{\theta}{2}}, \dfrac{n_1+n_2}{\cos\frac{\theta}{2}}$
(B) $\dfrac{\ell_1+\ell_2}{2\cos\frac{\theta}{2}}, \dfrac{m_1+m_2}{2\cos\frac{\theta}{2}}, \dfrac{n_1+n_2}{2\cos\frac{\theta}{2}}$

(C) $\dfrac{\ell_1+\ell_2}{\sin\frac{\theta}{2}}, \dfrac{m_1+m_2}{\sin\frac{\theta}{2}}, \dfrac{n_1+n_2}{\sin\frac{\theta}{2}}$
(D) $\dfrac{\ell_1-\ell_2}{2\sin\frac{\theta}{2}}, \dfrac{m_1-m_2}{2\sin\frac{\theta}{2}}, \dfrac{n_1-n_2}{2\sin\frac{\theta}{2}}$

6. Which of the followings is/are correct :

(A) The angle between the two straight lines $\vec{r} = 3\hat{i} - 2\hat{j} + 4\hat{k} + \lambda(-2\hat{i} + \hat{j} + 2\hat{k})$ and

$\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k} + \mu(3\hat{i} - 2\hat{j} + 6\hat{k})$ is $\cos^{-1}\left(\dfrac{4}{21}\right)$

(B) $(\vec{r}\,.\,\hat{i})(\hat{i}\times\vec{r}) + (\vec{r}\,.\,\hat{j})(\hat{j}\times\vec{r}) + (\vec{r}\,.\,\hat{k})(\hat{k}\times\vec{r}) = \vec{0}$.

(C) The force determined by the vector $\vec{r} = (1, -8, -7)$ is resolved along three mutually perpendicular directions, one of which is in the direction of the vector $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$. Then the vector component of the force $\vec{r}$ in the direction of the vector $\vec{a}$ is $-\dfrac{7}{3}(2\hat{i}+2\hat{j}+\hat{k})$

(D) The cosine of the angle between any two diagonals of a cube is $\dfrac{1}{3}$.

7✍ Acute angle between the lines $\dfrac{x-1}{\ell} = \dfrac{y+1}{m} = \dfrac{z}{n}$ and $\dfrac{x+1}{m} = \dfrac{y-3}{n} = \dfrac{z-1}{\ell}$ where $\ell > m > n$, and ℓ, m, n are the roots of the cubic equation $x^3 + x^2 - 4x = 4$ is equal to :

(A) $\cos^{-1}\dfrac{3}{\sqrt{13}}$ (B) $\sin^{-1}\dfrac{\sqrt{65}}{9}$ (C) $2\cos^{-1}\sqrt{\dfrac{13}{18}}$ (D) $\tan^{-1}\dfrac{2}{3}$

8. The line $\dfrac{x-2}{3} = \dfrac{y+1}{2} = \dfrac{z-1}{-1}$ intersects the curve $xy = c^2$, z = 0 if c is equal to :

(A) –1 (B) $-\sqrt{5}$ (C) $\sqrt{5}$ (D) 1

9. The three lines $\dfrac{x-1}{3} = \dfrac{y-2}{2} = \dfrac{z-3}{1}$, $\dfrac{x-1}{5} = \dfrac{2y-4}{3} = \dfrac{3z-9}{1}$, $\dfrac{x-\lambda^2}{3} = \dfrac{y-2}{2} = \dfrac{z-3}{\lambda}$ are concurrent the value of λ may be :

(A) 1 (B) –1 (C) 2 (D) –2

10. Let $\vec{V} = 2\hat{i} + \hat{j} - \hat{k}$ & $\vec{W} = \hat{i} + 3\hat{k}$. If $\vec{U}$ is a unit vector, then the value of the scalar triple product $\left[\vec{U}\vec{V}\vec{W}\right]$ may be :

(A) $-\sqrt{59}$ (B) $\sqrt{10}+\sqrt{6}$ (C) $\sqrt{59}$ (D) $\sqrt{60}$

11. The coplanar points A , B , C , D are (2 – x , 2 , 2) , (2 , 2 – y , 2) , (2 , 2 , 2 – z) and (1 , 1 , 1) respectively, then

(A) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ (B) $\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 2$

(C) $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = 1$ (D) $\frac{x}{1-x} + \frac{y}{1-y} + \frac{z}{1-z} + 2 = 0$

12. A vector $\vec{v} = \lambda\left(a\hat{j} + b\hat{k}\right)$ is coplanar with the vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 2\hat{j} + \hat{k}$ and is orthogonal to the vector $-2\hat{i} + \hat{j} + \hat{k}$. It is given that the length of projection of $\vec{v}$ along the vector $\hat{i} - \hat{j} + \hat{k}$ is equal to $6\sqrt{3}$. Then the value of $\lambda^2 ab$ may be

(A) 81 (B) 9 (C) –9 (D) –81

13. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose length of projection on $\vec{a}$ is of $\sqrt{\frac{2}{3}}$, is

(A) 2i + 3j – 3k (B) 2i + 3j + 3k (C) – 2i – j + 5k (D) i – 5j + 3k

14. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b}) \cdot (\vec{c}\times\vec{d}) = \lambda$ and $\vec{a}.\vec{c} = \frac{\sqrt{3}}{2}$, then

(A) $\vec{a},\vec{b},\vec{c}$ are coplanar if $\lambda = 1$

(B) Angle between $\vec{b}$ and $\vec{d}$ is 30° if $\lambda = -1$

(C) angle between $\vec{b}$ and $\vec{d}$ is 150° if $\lambda = -1$

(D) If $\lambda = 1$ then angle between $\vec{b}$ and $\vec{c}$ is 60°

15. Let $\vec{a},\vec{b},\vec{c}$ be three mutually perpendicular vectors of equal magnitude. If the vector $\bar{x}$ satisfies the equation $\bar{a}\times\{\bar{x}-\bar{b})\times\bar{a}\} + \bar{b}\times\{(\bar{x}-\bar{c})\times\bar{b}\} + \bar{c}\times\{(\bar{x}-\bar{a})\times\bar{c}\} = 0$, then find $\bar{x}$.

(A) $\bar{x} = \frac{\bar{a}+\bar{b}+\bar{c}}{2}$ (B) $\bar{x} = \frac{\bar{a}-\bar{b}+\bar{c}}{2}$ (C) $\bar{x} = \frac{\bar{a}-\bar{b}+\bar{c}}{3}$ (D) $\bar{x} = \frac{\bar{a}+\bar{b}+\bar{c}}{3}$

16. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $-\frac{1}{\sqrt{3}}$, is

(A) $4\hat{i} - \hat{j} + 4\hat{k}$ (B) $3\hat{i} + \hat{j} - 3\hat{k}$ (C) $2\hat{i} + \hat{j} - 2\hat{k}$ (D) $4\hat{i} + \hat{j} - 4\hat{k}$

17. $\hat{a}$ and $\hat{b}$ are two given unit vectors at right angle. The unit vector equally inclined with $\hat{a}$, $\hat{b}$ and $\hat{a} \times \hat{b}$ will be:

(A) $-\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}+\hat{a}\times\hat{b}\right)$ (B) $\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}+\hat{a}\times\hat{b}\right)$

(C) $\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}-\hat{a}\times\hat{b}\right)$ (D) $-\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}-\hat{a}\times\hat{b}\right)$

18✍ A variable line passing through the point P(0, 0, 2) always makes angle 60º with z-axis, intersects the plane x + y + z = 1. The locus of point of intersection of the line and the plane is $x^2 + y^2 = a(z - b)^2$ then

(A) a = 3 (B) b = 2 (C) a = 2 (D) b = 3

19. The position vectors of the angular points of a tetrahedron are $A(3\hat{i}-2\hat{j}+\hat{k})$, $B(3\hat{i}+\hat{j}+5\hat{k})$, $C(4\hat{i}+3\hat{k})$ and $D(\hat{i})$. Then the acute angle between the lateral face ADC and the base face ABC is :

(A) $\tan^{-1}\frac{5}{2}$ (B) $\tan^{-1}\frac{2}{5}$ (C) $\tan^{-1}\frac{1}{5}$ (D) $2\tan^{-1}\left(\frac{2+\sqrt{25}}{5}\right)$

20. The line $\frac{x-4}{k} = \frac{y-2}{1} = \frac{z-k^2}{2}$ lies in the plane 2x – 4y + z = 1 . Then the value of k cannot be :

(A) 1 (B) – 1 (C) 2 (D) –2

21. If the π-plane $7x + (\alpha + 4)y + 4z - r = 0$ passing through the points of intersection of the planes $2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$ and is perpendicular to the plane $3x - y - 2z = 4$ and $\left(\frac{12}{\beta}, \frac{-78}{\beta}, \frac{57}{\beta}\right)$ is image of point (1, 1, 1) in π–plane, then

(A) $\alpha = 9$ (B) $\beta = -117$ (C) $\alpha = -9$ (D) $\beta = 117$

22✍ The planes 2x – 3y – 7z = 0, 3x – 14y – 13z = 0 and 8x – 31y – 33z = 0

(A) pass through origin (B) intersect in a common line

(C) form a triangular prism (D) pass through infinite the many points

23. Two lines are

$L_1 : \frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{2}$; $L_2 : \frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{2}$

Equation of line passing through (2, 1, 3) and equally inclined to L_1 & L_2 is/are

(A) $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z-3}{-3}$ (B) $\frac{x-3}{1} = \frac{y-2}{1} = \frac{z-5}{2}$

(C) $\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-3}{3}$ (D) $\frac{x}{2} = \frac{y+1}{2} = \frac{z-6}{-3}$

24. A line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ intersects the plane x – y + 2z + 2 = 0 at point A. The equation of the straight line passing through A lying in the given plane and at minimum inclination with the given line is/are

(A) $\frac{x+1}{1} = \frac{y+1}{5} = \frac{z+1}{2}$ (B) $5x - y + 4 = 0 = 2y - 5z - 3$

(C) $5x + y - 5z + 1 = 0 = 2y - 5z - 3$ (D) $\frac{x+2}{1} = \frac{y+6}{5} = \frac{z+3}{2}$

25. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. If the position vectors of the vertices of the base ABC are A(1, 0, 1), B(2,0, 0) and C(0, 1, 0), then position vectors of the vertex A_1 can be:

(A) (2, 2, 2) (B) (0, 2, 0) (C) (0, –2, 2) (D) (0, –2, 0)

PART - IV : COMPREHENSION

Comprehension # 1

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$, then the two systems are called Reciprocal System of vectors and $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

1. Find the value of $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}'$.

(A) $\vec{0}$ (B) $\vec{a} + \vec{b} + \vec{c}$ (C) $\vec{a} - \vec{b} + \vec{c}$ (D) $\vec{a} + \vec{b} - \vec{c}$

2. If $\vec{a}, \vec{b}, \vec{c}$ are non–coplanar and $\vec{d}$ is any vector, then $[\vec{d}\ \vec{b}\ \vec{c}]\,\vec{a} + [\vec{d}\ \vec{c}\ \vec{a}]\,\vec{b} + [\vec{d}\ \vec{a}\ \vec{b}]\,\vec{c} =$

(A) $-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$ (B) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$ (C) $2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$ (D) $-2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

3. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar non-zero vectors and $\vec{r}$ is any vector in space, then

$(\vec{a} \times \vec{b}) \times (\vec{r} \times \vec{c}) + (\vec{b} \times \vec{c}) \times (\vec{r} \times \vec{a}) + (\vec{c} \times \vec{a}) \times (\vec{r} \times \vec{b})$ is equal to

(A) $2[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{r}$ (B) $3[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{r}$ (C) $[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{r}$ (D) $4[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{r}$

Comprehension # 2

Consider a plane $\pi : \vec{r}.\vec{n} = d$ (where $\vec{n}$ is not a unit vector). There are two points A($\vec{a}$) and B($\vec{b}$) lying on the same side of the plane.

4. If foot of perpendicular from A and B to the plane π are P and Q respectively, then length of PQ be:

(A) $\frac{|(\vec{b}-\vec{a}).\vec{n}|}{|\vec{n}|}$ (B) $|(\vec{b}-\vec{a}).\vec{n}|$ (C) $\frac{|(\vec{b}-\vec{a}) \times \vec{n}|}{|\vec{n}|}$ (D) $|(\vec{b}-\vec{a}) \times \vec{n}|$

5✍ Reflection of A($\vec{a}$) in the plane π has the position vector:

(A) $\vec{a} + \frac{2}{(\vec{n})^2}(d - \vec{a}.\vec{n})\vec{n}$ (B) $\vec{a} - \frac{1}{(\vec{n})^2}(d - \vec{a}.\vec{n})\vec{n}$

(C) $\vec{a} + \frac{2}{(\vec{n})^2}(d + \vec{a}.\vec{n})\vec{n}$ (D) $\vec{a} + \frac{2}{(\vec{n})^2}\vec{n}$

6. If a plane π_1 is drawn from the point A($\vec{a}$) and another plane π_2 is drawn from point B($\vec{b}$) parallel to π, then the distance between the planes π_1 and π_2 is :

(A) $\frac{|(\vec{a}-\vec{b}).\vec{n}|}{|\vec{n}|}$ (B) $|(\vec{a}-\vec{b}).\vec{n}|$ (C) $|(\vec{a}-\vec{b})\times\vec{n}|$ (D) $\frac{|(\vec{a}-\vec{b})\times\vec{n}|}{|\vec{n}|}$

Comprehension # 3

A pyramid having square base ABCD & other vertex E with A(0, 0, 0), B(4, 0, 0), C(4, 0, 4) & D(0, 0, 4) & E(2, 6, 6)

7. Volume of the pyramid is :

(A) 32 (B) 16 (C) 8 (D) 4

8. Centroids of faces EAB, EBC, ECD & EDA are

(A) Non-coplanar

(B) Coplanar but the plane is not parallel to base plane

(C) Coplanar & plane is parallel to base plane

(D) Co-linear

9. The distance of the plane EBC from ortho-centre of ΔABD is :

(A) 2 (B) 5 (C) $\frac{12}{\sqrt{10}}$ (D) $\sqrt{10}$

Comprehension # 4

General equation of a sphere is given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, where $(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere

10. A variable plane passes through a fixed point (1, 2, 3). The locus of the foot of the perpendicular drawn from origin to this plane is:

(A) $x^2 + y^2 + z^2 - x - 2y - 3z = 0$

(B) $x^2 + 2y^2 + 3z^2 - x - 2y - 3z = 0$

(C) $x^2 + 4y^2 + 9z^2 + x + 2y + 3 = 0$

(D) $x^2 + y^2 + z^2 + x + 2y + 3z = 0$

11. The co-ordinates of the centre and the radius of the circle $x + 2y + 2z = 15$, $x^2 + y^2 + z^2 - 2y - 4z = 11$ are

(A) (4, 3, 1), $\sqrt{5}$ (B) (3, 4, 1), $\sqrt{6}$ (C) (1, 3, 4), $\sqrt{7}$ (D) none of these

12. Equation of the sphere with centre on the positive z-axis which passes through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3 is :

(A) $x^2 + y^2 + z^2 - 6x - 4 = 0$

(B) $x^2 + y^2 - 6y - 4 = 0$

(C) $x^2 + y^2 + z^2 - 6z - 4 = 0$

(D) $x^2 + y^2 - 6x - 6y - 4 = 0$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a **[IIT-JEE-2010, Paper-1, (3, –1), 84]**

(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square

2✍ If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}}$, then the value of

$(2\vec{a} + \vec{b}) \cdot [(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})]$ is **[IIT-JEE-2010, Paper-1, (3, 0), 84]**

3. Two adjacent sides of a parallelogram ABCD are given by **[IIT-JEE-2010, Paper-2, (5, –2), 79]**

$\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD′. If AD′ makes a right angle with the side AB, then the cosine of the angle α is given by

(A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

4. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is **[IIT-JEE-2010, Paper-1, (3, –1), 84]**

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$ (C) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

5✍ The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is **[IIT-JEE-2010, Paper-1, (3, –1), 84]**

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

6. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

[IIT-JEE-2010, Paper-1, (3, 0), 84]

7. If the distance of the point P(1, –2, 1) from the plane x + 2y – 2z = α, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is **[IIT-JEE-2010, Paper-2, (5, –2), 79]**

(A) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (B) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$ (C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (D) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

8✍ Match the statements in **Column-I** with those in **Column-II.** **[IIT-JEE-2010, Paper-2, (8, 0), 79]**

Column-I **Column-II**

(A) A line from the origin meets the lines (p) – 4

$$\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1} \text{ and } \frac{x-\frac{8}{3}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$$

at P and Q respectively. If length PQ = d, then d^2 is

(B) The values of x satisfying (q) 0

$\tan^{-1}(x + 3) - \tan^{-1}(x - 3) = \sin^{-1}\left(\frac{3}{5}\right)$ are

(C) Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}\cdot\vec{b} = 0$, (r) 4

$(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$ and $2|\vec{b}+\vec{c}| = |\vec{b}-\vec{a}|$. If $\vec{a} = \mu\vec{b} + 4\vec{c}$

then possible value of μ are

(D) Let f be the function on [$-\pi$, π] given by (s) 5

f(0) = 9 and f(x) = $\dfrac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ for $x \neq 0$. The value

of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ is

(t) 6

9. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\hat{i} - 3\hat{j} + 3\hat{k}$ (B) $-3\hat{i} - 3\hat{j} - \hat{k}$ (C) $3\hat{i} - \hat{j} + 3\hat{k}$ (D) $\hat{i} + 3\hat{j} - 3\hat{k}$

10*. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

(A) $\hat{j}-\hat{k}$ (B) $-\hat{i}+\hat{j}$ (C) $\hat{i}-\hat{j}$ (D) $-\hat{j}+\hat{k}$

11. Let $\vec{a}=-\hat{i}-\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}\,.\,\vec{a}=0$, then the value of $\vec{r}\,.\,\vec{b}$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

12. Match the statements given in **Column-I** with the values given in **Column-II**

Column-I		**Column-II**	
(A)	If $\vec{a}=\hat{j}+\sqrt{3}\,\hat{k}$, $\vec{b}=-\hat{j}+\sqrt{3}\,\hat{k}$ and $\vec{c}=2\sqrt{3}\,\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p)	$\frac{\pi}{6}$
(B)	If $\int_a^b (f(x)-3x)dx = a^2-b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q)	$\frac{2\pi}{3}$
(C)	The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r)	$\frac{\pi}{3}$
(D)	The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $\lvert z\rvert = 1$, $z \neq 1$ is given by	(s)	π
		(t)	$\frac{\pi}{2}$

13. The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1, –1, 4) with the plane 5x – 4y – z = 1. If S is the foot of the perpendicular drawn from the point T(2, 1,4) to QR, then the length of the line segment PS is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

14✍. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

$|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$,

then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is

15. The equation of a plane passing through the line of intersection of the planes x + 2y + 3z = 2 and x – y + z = 3 and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 5x – 11y + z = 17 (B) $\sqrt{2}x+y=3\sqrt{2}-1$

(C) $x + y + z = \sqrt{3}$ (D) $x-\sqrt{2}y = 1-\sqrt{2}$

16. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}| = \sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 0 (B) 3 (C) 4 (D) 8

17*. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are) **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(A) y + 2z = –1 (B) y + z = –1 (C) y – z = –1 (D) y – 2z = –1

18. Let $\overrightarrow{PR} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ} = \hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT} = \hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) 5 (B) 20 (C) 10 (D) 30

19. Perpendicular are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane x + y + z = 3. The feet of perpendiculars lie on the line **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

20.* A line l passing through the origin is perpendicular to the lines

$l_1 : (3 + t)\,\hat{i} + (-1 + 2t)\,\hat{j} + (4 + 2t)\,\hat{k},\ -\infty < t < \infty$

$l_2 : (3 + 2s)\,\hat{i} + (3 + 2s)\,\hat{j} + (2 + s)\,\hat{k},\ -\infty < s < \infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are)

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

(A) $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ (B) (–1, ,–1, 0) (C) (1, 1, 1) (D) $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$

21✍. Consider the set of eight vectors V = $\{a\hat{i}+b\hat{j}+c\hat{k} : a,b,c \in \{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

22.* Two lines L_1 : x = 5, $\frac{y}{3-\alpha} = \frac{z}{-2}$ and L_2 : x = α, $\frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 1 (B) 2 (C) 3 (D) 4

23. Match List I with List II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	List - I		List - II
P.	Volume of parallelopiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	1.	100
Q.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	2.	30
R.	Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	3.	24
S.	Area of a paralelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	4.	60

Codes :

	P	Q	R	S		P	Q	R	S
(A)	4	2	3	1	(B)	2	3	1	4
(C)	3	4	1	2	(D)	1	4	3	2

24. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ the equation of the plane passing through the point of intersection of lines L_1 and L_2, and perpendicular to planes P_1 and P_2.

Match List - I with List- II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	List- I/		List- II
P.	a =	1.	13
Q.	b =	2.	–3
R.	c =	3.	1
S.	d =	4.	–2

Codes :

	P	Q	R	S		P	Q	R	S
(A)	3	2	4	1	(B)	1	3	4	2
(C)	3	2	1	4	(D)	2	4	1	3

25*✍ Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$.

If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a nonzero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(A) $\vec{b} = (\vec{b}.\vec{z})(\vec{z} - \vec{x})$ (B) $\vec{a} = (\vec{a}.\vec{y})(\vec{y} - \vec{z})$ (C) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a} = (\vec{a}.\vec{y})(\vec{z} - \vec{y})$

26. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines y = x, z = 1 and y = –x, z = –1. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) $\sqrt{2}$ (B) 1 (C) –1 (D) $-\sqrt{2}$

27. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$.

If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

28. **List I** **List II**

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

P. Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2 - 1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ equals — **1.** 1

Q.✍ Let $A_1, A_2, \ldots, A_n$ (n > 2) be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, k = 1, 2,...., n. If $\left|\sum_{k=1}^{n-1}(\overrightarrow{a_k} \times \overrightarrow{a_{k+1}})\right| = \left|\sum_{k=1}^{n-1}(\overrightarrow{a_k}.\overrightarrow{a_{k+1}})\right|$, then the minimum value of n is — **2.** 2

R. If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line x + y = 8, then the value of h is — **3.** 8

S. Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is — **4.** 9

	P	Q	R	S		P	Q	R	S
(A)	4	3	2	1	(B)	2	4	3	1
(C)	4	3	1	2	(D)	2	4	1	3

29*✍ In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relation is (are) true ?

[JEE (Advanced) 2015, P-1 (4, –2)/ 88]

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

30*✍ In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M? **[JEE (Advanced) 2015, P-1 (4, –2)/ 88]**

(A) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$ (C) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (D) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

31*. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$ and $\vec{b}.\vec{c} = 24$, then which of the following is(are) true? **[JEE (Advanced) 2015, P-1 (4, –2)/ 88]**

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$ (B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$

(C) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$ (D) $\vec{a}.\vec{b} = -72$

32. **Column-I** **Column-II**

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

(A) In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\,\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $|\alpha|$ is (are) — (P) 1

(B) Let a and b be real numbers such that the function

$$f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$$ is differentiable for all $x \in R$.

Then possible value(s) of a is (are) — (Q) 2

(C) Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are) — (R) 3

(D) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5,q, b is an arithmetic progression, then the value(s) of $|q - a|$ is (are) — (S) 4

(T) 5

33. **Column-I** **Column-II**

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

(A)	In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	(P)	1
(B)	In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	(Q)	2
(C)	In R^2, let $\sqrt{3}\,\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	(R)	3
(D)	Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = \lvert\alpha x - 1\rvert + \lvert\alpha x - 2\rvert + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	(S)	5
		(T)	6

34. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r}), (\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x, y and z, respectively, then the value of $2x + y + z$ is

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

35. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP= 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then-

[JEE (Advanced) 2016, Paper-1 (4, –2) / 62]

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$.

(B) the equaiton of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

36. Let P be the image of the point (3, 1, 7) with respect to the plane x – y + z = 3. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ is

[JEE (Advanced) 2016, Paper-2 (3, –1) / 62]

(A) x + y – 3z = 0 (B) 3x + z = 0 (C) x – 4y + 7z = 0 (D) 2x – y = 0

37✍ Let $\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ be a unit vector in $\mathbb{R}^3$ and $\hat{w}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$. Given that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $|\hat{u}\times\vec{v}|=1$ and $\hat{w}.(\hat{u}\times\vec{v})=1$. Which of the following statement(s) is(are) correct?

[JEE (Advanced) 2016, Paper-2 (4, –2) / 62]

(A) There is exactly one choice for such $\vec{v}$ (B) There are infinitely many choice for such $\vec{v}$

(C) If $\hat{u}$ lies in the xy-plane then $|u_1|=|u_2|$ (D) If $\hat{u}$ lies in the xz-plane then $2|u_1|=|u_3|$

38✍ Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP}.\overrightarrow{OQ}+\overrightarrow{OR}.\overrightarrow{OS}=\overrightarrow{OR}.\overrightarrow{OP}+\overrightarrow{OQ}.\overrightarrow{OS}=\overrightarrow{OQ}.\overrightarrow{OR}+\overrightarrow{OP}.\overrightarrow{OS}$. Then the triangle PQR has S as its

[JEE (Advanced) 2017, Paper-2 (3, –1) / 61]

(A) incentre (B) orthocenter (C) circumcentre (D) centroid

39. The equation of the plane passing through the point (1,1,1) and perpendicular to the planes 2x + y – 2z = 5 and 3x – 6y – 2z = 7, is- **[JEE (Advanced) 2017, Paper-2 (3, –1) / 61]**

(A) 14x + 2y + 15z = 31 (B) 14x + 2y – 15z = 1

(C) –14x + 2y + 15z = 3 (D) 14x – 2y + 15z = 27

Comprehension (Q.40 & Q.41)

Let O be the origin, and $\overrightarrow{OX},\overrightarrow{OY},\overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR},\overrightarrow{RP},\overrightarrow{PQ}$, respectively, of a triangle PQR. **[JEE (Advanced) 2017, Paper-2 (3, 0) / 61]**

40. $\left|\overrightarrow{OX}\times\overrightarrow{OY}\right|=$

(A) sin(Q + R) (B) sin(P + R) (C) sin 2R (D) sin(P + Q)

41. If the triangle PQR varies, then the minimum value of cos(P + Q) + cos(Q + R) + cos(R + P) is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

42. Let P_1 : 2x + y – z = 3 and P_2 : x + 2y + z = 2 be two planes. Then, which of the following statement(s) is (are) TRUE ? **[JEE (Advanced) 2018, Paper-1 (4, –2) / 60]**

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, –1

(B) The line $\frac{3x-4}{9}=\frac{1-3y}{9}=\frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60°

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$

43. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b}=0$. For some x, y ∈ $\mathbb{R}$, let $\vec{c}=x\vec{a}+y\vec{b}+(\vec{a}\times\vec{b})$. If $|\vec{c}|$ = 2 and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ______ **[JEE (Advanced) 2018, Paper-1 (3, 0) / 60]**

44✍ Let P be a point in the first octant, whose image Q in the plane x + y = 3 (that is, the line segment PQ is perpendicular to the plane x + y = 3 and the mid-point of PQ lies in the plane x + y = 3) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____ . **[JEE (Advanced) 2018, Paper-2 (3, 0) / 60]**

45. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let S$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p}=\overrightarrow{SP}$, $\vec{q}=\overrightarrow{SQ}$, $\vec{r}=\overrightarrow{SR}$ and $\vec{t}=\overrightarrow{ST}$, then the value of $\left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right|$ is _____ .

[JEE (Advanced) 2018, Paper-2 (3, 0) / 60]

46. Let L_1 and L_2 denotes the lines

$$\vec{r}=\hat{i}+\lambda(-\hat{i}+2\hat{j}+2\hat{k}),\lambda\in\mathbb{R}$$

and $$\vec{r}=\mu(2\hat{i}-\hat{j}+2\hat{k}),\mu\in\mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ? **[JEE(Advanced)-2019, Paper-1 (4,–1)]**

(A) $\vec{r}=\frac{1}{3}(2\hat{i}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(B) $\vec{r}=\frac{2}{9}(2\hat{i}-\hat{j}+2\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(C) $\vec{r}=t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(D) $\vec{r}=\frac{2}{9}(4\hat{i}+\hat{j}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

47. Three lines are given by

$\vec{r}=\lambda\hat{i},\ \lambda\in\mathbb{R}$ $\quad\vec{r}=\mu(\hat{i}+\hat{j}),\mu\in\mathbb{R}$ and $\quad\vec{r}=\nu(\hat{i}+\hat{j}+\hat{k}),\nu\in\mathbb{R}$.

Let the lines cut the plane x + y + z = 1 at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals ___ **[JEE(Advanced)-2019,Paper-1 (3, 0]**

48. Three lines $L_1:\vec{r}=\lambda\hat{i},\ \lambda\in\mathbb{R}$, $L_2:\vec{r}=\hat{k}+\mu\hat{j},\ \mu\in\mathbb{R}$ and $L_3:\vec{r}=\hat{i}+\hat{j}+v\hat{k},\ v\in\mathbb{R}$ are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear ? **[JEE(Advanced)-2019, Paper-2 (4, –1)]**

(A) $\hat{k}+\hat{j}$ (B) $\hat{k}$ (C) $\hat{k}+\frac{1}{2}\hat{j}$ (D) $\hat{k}-\frac{1}{2}\hat{j}$

49✍ Let $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ be two vectors. Consider a vector $\vec{c}=\alpha\vec{a}+\beta\vec{b}$, $\alpha,\beta\in\mathbb{R}$. If the projection of $\vec{c}$ on the vector $(\vec{a}+\vec{b})$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c}-(\vec{a}\times\vec{b})\right).\vec{c}$ equals

[JEE(Advanced)-2019, Paper-2 (3, 0)]

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$, then the value of $(2\vec{a} - \vec{b}) \cdot [(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})]$ is:

[AIEEE 2011, I, (4, –1), 120]

(1) – 5 (2) –3 (3) 5 (4) 3

2. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying : $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to : **[AIEEE 2011, I, (4, –1), 120]**

(1) $\vec{b} - \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{d}}\right)\vec{c}$ (2) $\vec{c} + \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (3) $\vec{b} + \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (4) $\vec{c} - \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$

3. If the vector $p\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + q\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + r\hat{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of pqr – (p+q+r) is-

[AIEEE 2011, II, (4, –1), 120]

(1) 2 (2) 0 (3) –1 (4) –2

4✍ Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is : **[AIEEE 2011, II, (4, –1), 120]**

(1) $\vec{a}$ (2) $\vec{c}$ (3) $\vec{0}$ (4) $\vec{a} + \vec{c}$

5. If the angle between the line x = $\frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane x + 2y + 3z = 4 is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals:

[AIEEE 2011, I, (4, –1), 120]

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) $\frac{2}{5}$ (4) $\frac{5}{3}$

6. **Statement-1 :** The point A(1, 0, 7) is the mirror image of the point B(1, 6, 3) in the line :

$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ **[AIEEE 2011, I, (4, –1), 120]**

Statement-2 : The line : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining A(1, 0, 7) and B(1, 6, 3).

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(3) Statement-1 is true, Statement-2 is false.
(4) Statement-1 is false, Statement-2 is true.

7. The distance of the point (1, –5, 9) from the plane x – y + z = 5 measured along a straight line x = y = z is : **[AIEEE 2011, II, (4, –1), 120]**

(1) $10\sqrt{3}$ (2) $5\sqrt{3}$ (3) $3\sqrt{10}$ (4) $3\sqrt{5}$

8. The length of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is :

[AIEEE 2011, II, (4, –1), 120]

(1) $\sqrt{29}$ (2) $\sqrt{33}$ (3) $\sqrt{53}$ (4) $\sqrt{66}$

9. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is : **[AIEEE-2012, (4, –1)/120]**

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

10. A equation of a plane parallel to the plane x – 2y + 2z – 5 = 0 and at a unit distance from the origin is :

[AIEEE 2012, (4, –1), 120]

(1) x – 2y + 2z – 3 = 0 (2) x – 2y + 2z + 1 = 0 (3) x – 2y + 2z – 1 = 0 (4) x – 2y + 2z + 5 = 0

11. If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to :

[AIEEE 2012, (4, –1), 120]

(1) – 1 (2) $\frac{2}{9}$ (3) $\frac{9}{2}$ (4) 0

12. Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{q}$, $\overrightarrow{AD} = \vec{p}$ and ∠BAD be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by :

[AIEEE-2012, (4, –1)/120]

(1) $\vec{r} = 3\vec{q} - \frac{3(\vec{p}\,.\,\vec{q})}{(\vec{p}\,.\,\vec{p})}\vec{p}$ (2) $\vec{r} = -\vec{q} + \left(\frac{\vec{p}\,.\,\vec{q}}{\vec{p}\,.\,\vec{p}}\right)\vec{p}$ (3) $\vec{r} = \vec{q} - \left(\frac{\vec{p}\,.\,\vec{q}}{\vec{p}\,.\,\vec{p}}\right)\vec{p}$ (4) $\vec{r} = -3\vec{q} + \frac{3(\vec{p}\,.\,\vec{q})}{(\vec{p}\,.\,\vec{p})}\vec{p}$

13. Distance between two parallel planes 2x + y + 2z = 8 and 4x + 2y + 4z + 5 = 0 is

[AIEEE - 2013, (4, –1),360]

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3) $\frac{7}{2}$ (4) $\frac{9}{2}$

14. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k can have

[AIEEE - 2013, (4, –1),360]

(1) any value (2) exactly one value (3) exactly two values (4) exactly three values

15. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is **[AIEEE - 2013, (4, –¼),360]**

(1) $\sqrt{18}$ (2) $\sqrt{72}$ (3) $\sqrt{33}$ (4) $\sqrt{45}$

16. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane 2x – y + z + 3 = 0 is the line :

[JEE(Main) 2014, (4, – 1), 120]

(1) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (2) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

(3) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (4) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

17. The angle between the lines whose direction cosines satisfy the equations $\ell + m + n = 0$ and $\ell^2 = m^2 + n^2$ is

[JEE(Main) 2014, (4, – 1), 120]

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

18. If $\left[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\right]=\lambda\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$ then λ is equal to **[JEE(Main) 2014, (4, – 1), 120]**

(1) 0 (2) 1 (3) 2 (4) 3

19. The distance of the point (1,0,2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x – y + z = 16, is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $2\sqrt{14}$ (2) 8 (3) $3\sqrt{21}$ (4) 13

20. The equation of the plane containing the line 2x – 5y + z = 3, x + y + 4z = 5 and parallel to the plane x + 3y + 6z = 1, is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) 2x + 6y + 12z = 13 (2) x + 3y + 6z = –7 (3) x + 3y + 6z = 7 (4) 2x + 6y + 12z = –13

21. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$.

If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $\frac{2\sqrt{2}}{3}$ (2) $\frac{-\sqrt{2}}{3}$ (3) $\frac{2}{3}$ (4) $\frac{-2\sqrt{3}}{3}$

22. If the line, $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane, $\ell x + my - z = 9$, then $\ell^2 + m^2$ is equal to

[JEE(Main) 2016, (4, – 1), 120]

(1) 18 (2) 5 (3) 2 (4) 26

23. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\sqrt{3}}{2}\left(\vec{b}+\vec{c}\right)$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is :- **[JEE(Main) 2016, (4, – 1), 120]**

(1) $\frac{5\pi}{6}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{2}$ (4) $\frac{2\pi}{3}$

24. The distance of the point (1, –5, 9) from the plane x – y + z = 5 measured along the line x = y = z is : **[JEE(Main) 2016, (4, – 1), 120]**

(1) $\frac{20}{3}$ (2) $3\sqrt{10}$ (3) $10\sqrt{3}$ (4) $\frac{10}{\sqrt{3}}$

25. If the image of the point P(1, –2, 3) in the plane, 2x + 3y – 4z + 22 = 0 measured parallel to the line, $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to: **[JEE(Main) 2017, (4, – 1), 120]**

(1) $3\sqrt{5}$ (2) $2\sqrt{42}$ (3) $\sqrt{42}$ (4) $6\sqrt{5}$

26. The distantce of the point (1, 3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is :-

[JEE(Main) 2017, (4, – 1), 120]

(1) $\frac{10}{\sqrt{74}}$ (2) $\frac{20}{\sqrt{74}}$ (3) $\frac{10}{\sqrt{83}}$ (4) $\frac{5}{\sqrt{83}}$

27. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c}-\vec{a}|$ = 3, $\left|(\vec{a}\times\vec{b})\times\vec{c}\right|$ = 3 and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30°. Then $\vec{a}\cdot\vec{c}$ is equal to : **[JEE(Main) 2017, (4, – 1), 120]**

(1) $\frac{1}{8}$ (2) $\frac{25}{8}$ (3) 2 (4) 5

28. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b}$ = 24, then $|\vec{u}|^2$ is equal to - **[JEE(Main) 2018, (4, – 1), 120]**

(1) 315 (2) 256 (3) 84 (4) 336

29. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, x + y + z = 7 is : **[JEE(Main) 2018, (4, – 1), 120]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\sqrt{\frac{2}{3}}$ (4) $\frac{2}{\sqrt{3}}$

30. If L_1 is the line of intersection of the planes 2x – 2y + 3z – 2 = 0, x – y + z + 1 = 0 and L_2 is the line of intersection of the planes x + 2y – z – 3 =0, 3x – y + 2z – 1 = 0, then the distance of the origin from the plane, containing the lines L_1 and L_2 is : **[JEE(Main) 2018, (4, – 1), 120]**

(1) $\frac{1}{3\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{4\sqrt{2}}$

31. If the lines x = ay + b, z = cy + d and x=a'z + b', y = c'z + d' are perpendicular, then :

[JEE(Main) 2019, 9 Jan_Shift-2 (4, – 1), 120]

(1) cc' + a + a' = 0 (2) aa' + c + c' = 0 (3) ab' + bc' + 1 = 0 (4) bb' + cc' + 1 = 0

32. Let $\vec{a} = \hat{i} + \hat{j} + \sqrt{2}\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + \sqrt{2}\hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} + \sqrt{2}\hat{k}$ be three vectors such that the projection vector of $\vec{b}$ on $\vec{a}$ is $\vec{a}$. If $\vec{a} + \vec{b}$ is perpendicular to $\vec{c}$, then $|\vec{b}|$ is equal to :

[JEE(Main) 2019, 9 Jan_Shift-2 (4, – 1), 120]

(1) $\sqrt{22}$ (2) 4 (3) $\sqrt{32}$ (4) 6

33. Let P be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and R be any point (2, 1, 6). Then the image of R in the plane P is : **[JEE(Main) 2020, 7 Jan_Shift-1 (4, – 1), 100]**

(1) (6, 5, –2) (2) (4, 3, 2) (3) (3, 4, –2) (4) (6, 5, 2)

34. A vector $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k} (\alpha, \beta \in R)$ lies in the plane of the vectros $\vec{b} = \hat{i} + \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} + 4\hat{k}$. If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then: **[JEE(Main) 2020, 7 Jan_Shift-1 (4, – 1), 100]**

(1) $\vec{a}.\hat{i} + 1 = 0$ (2) $\vec{a}.\hat{i} + 3 = 0$ (3) $\vec{a}.\hat{k} + 4 = 0$ (4) $\vec{a}.\hat{k} + 2 = 0$

35. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three units vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. If $\lambda = \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$ and $\vec{d} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$, then the ordered pair, $(\lambda, \vec{d})$ is equal to :

[JEE(Main) 2020, 7 Jan_Shift-2 (4, – 1), 100]

(1) $\left(-\frac{3}{2}, 3\vec{a} \times \vec{b}\right)$ (2) $\left(-\frac{3}{2}, 3\vec{c} \times \vec{b}\right)$ (3) $\left(\frac{3}{2}, 3\vec{b} \times \vec{c}\right)$ (4) $\left(\frac{3}{2}, 3\vec{a} \times \vec{c}\right)$

36. If the foot of the perpendicular drawn from the point (1, 0, 3) on a line passing through (α, 7, 1) is $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, then α is equal to_____ **[JEE(Main) 2020, 7 Jan_Shift-2 (4, – 1), 100]**

Answers

Exercise # 1

PART-I

Section (A)

A-1. 7

A-2. –12

A-3 $k\sqrt{4+2\sqrt{2}}$

A-4. $\sqrt{6}$

A-6. OP : PD = 3 : 2

A-7. ± (2/3, –2/3, –1/3)

A-8. $\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$ or $\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

Section (B)

B-1. $\pi/3$

B-2. – 169

B-4. 60º

B-5. $\vec{r} = \hat{j} + 2\hat{k}$

B-6. (a) $-\hat{i} + \hat{j} + \hat{k}$ (b) $\frac{6}{\sqrt{19}}(6\hat{i} - \hat{j} + \hat{k})$

(c) $\frac{2\pi}{3}$

B-9. $\cos^{-1}\left(\frac{1}{6}\right)$

B-10. $\frac{10}{\sqrt{6}}$

B-11. $2\sqrt{2} - 2$

B-12. 13

B-14. $3\left(-\hat{i} + \hat{j} + \hat{k}\right)$

B-15. (b) 5 unit sq.

B-16. $\pm(4\hat{i} - 6\hat{j} + 12\hat{k})$

Section (C)

C-1 36

C-2. $\sqrt{26}$

C-3. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}, \frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

C-4. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(\hat{j} - \hat{k})$, where t is parameter

C-5. (9, 5, –2)

C-7. $\frac{6}{\sqrt{5}}$ unit

C-8. A = (3, 8, 3), B = (–3, – 7, 6), AB = $3\sqrt{30}$

Section (D)

D-1. $\sin\alpha \cos\alpha$

D-2. $\frac{6\sqrt{\sqrt{6}-2}}{2}$

D-3. (a) Coplanar (b) Non-coplanar

D-4 1 or –1

D-5 0

D-6. $\vec{v} = 0$

D-7. 1/2 unit^3

D-8. – 100

D-10. (i) No (ii) Yes

D-11. x = 1

D-12. $\frac{\pi}{2}$ & $\frac{\pi}{3}$

D-13. $\vec{x} = \vec{q} - \frac{(\vec{p}\cdot\vec{q})\vec{p}}{2|\vec{p}|^2}$

Section (E)

E-1. (i) $4x - y - z + 1 = 0$

(ii) $x + 2y + 3z - 2 = 0$

(iii) $x + y + z - 4 = 0$

(iv) $x + y + z - 6 = 0$

E-2. 3 : 2, (0,13/5, 2)

E-3. $\pi/2$

E-4. $\sin^{-1}\frac{4}{\sqrt{30}}$

E-5. $11x - y - 3z = 35$

E-6. 0

E-7. $8x - 13y + 15z + 13 = 0$

E-8. $x+y \pm \sqrt{2}\, z = 1$

E-9. $x - y + 3z - 2 = 0$; (3, 1, 0) ; $\sqrt{11}$

E-10. $x^2 + y^2 + z^2 = 9$

E-11. $\frac{x-4}{9} = \frac{y+1}{-1} = \frac{z-7}{-3}$

E-12. 7

E-13. $4x + 3y + 5z = 50$

E-14. 1

E-15. (i) $\vec{r}\,.\,\hat{n} = \pm p$ (ii) $\vec{r}\,.\,(\vec{a}\,q - p\,\vec{b}) = 0$

PART-II

Section (A)

A-1.	(B)	**A-2.**	(B)
A-3.	(C)	**A-4.**	(B)
A-5.	(C)	**A-6.**	(B)
A-7.	(B)	**A-8.**	(B)
A-9.	(C)	**A-10.**	(C)

Section (B)

B-1.	(B)	**B-2.**	(D)
B-3.	(D)	**B-4**	(A)
B-5.	(C)	**B-6.**	(A)
B-7.	(C)	**B-8.**	(A)
B-9.	(C)	**B-10.**	(C)
B-11.	(D)	**B-12.**	(B)

Section (C)

C-1.	(B)	**C-2.**	(B)
C-3.	(D)	**C-4.**	(A)
C-5.	(B)	**C-6.**	(D)
C-7.	(A)	**C-8.**	(C)
C-9.	(B)		

Section (D)

D-1	(D)	**D-2.**	(C)
D-3.	(D)	**D-4.**	(A)
D-5.	(C)	**D-6.**	(B)
D-7.	(D)	**D-8.**	(B)
D-9.	(B)	**D-10.**	(B)

Section (E)

E-1.	(A)	**E-2.**	(A)
E-3.	(A)	**E-4.**	(A)
E-5.	(D)	**E-6.**	(D)
E-7.	(B)	**E-8.**	(A)
E-9.	(D)	**E-10.**	(C)
E-11.	(A)		

PART-III

1. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

2. (A) → (q), (B) → (p), (C) → (s), (D) → (r)

3. (A) → (s), (B) → (r), (C) → (pqrs), (D) → (pr)

Exercise # 2

PART-I

1.	(B)	**2.**	(C)
3.	(B)	**4.**	(A)
5.	(A)	**6.**	(C)
7.	(D)	**8.**	(C)
9.	(C)	**10.**	(C)
11.	(A)	**12.**	(C)
13.	(B)	**14.**	(A)
15.	(C)	**16.**	(A)
17.	(A)	**18.**	(A)
19.	(A)	**20.**	(A)
21.	(B)		

PART-II

1.	4	**2.**	3
3.	1	**4.**	19
5.	6	**6.**	1
7.	4	**8.**	0
9.	6	**10.**	72
11.	9	**12.**	18
13.	6	**14.**	2
15.	34	**16.**	4
17.	32	**18.**	27
19.	13		

PART - III

1.	(A), (B), (C)	**2.**	(A), (B), (C)
3.	(A), (B)	**4.**	(A), (B), (C), (D)
5.	(B), (D)	**6.**	(A), (B), (C), (D)
7.	(B), (C)	**8.**	(B), (C)
9.	(B)	**10.**	(A), (B), (C)
11.	(A), (B)	**12.**	(A), (D)
13.	(A), (C)	**14.**	(A), (C)
15.	(A)	**16.**	(A)
17.	(A), (B)	**18.**	(A), (B)
19.	(A), (D)	**20.**	(B), (C), (D)
21.	(A), (D)	**22.**	(A), (B), (D)
23.	(A), (B), (C) (D)	**24.**	(A), (B), (C) , (D)
25.	(A), (D)		

PART - IV

1.	(A)	**2.**	(B)
3.	(A)	**4.**	(C)
5.	(A)	**6.**	(A)
7.	(A)	**8.**	(C)
9.	(C)	**10.**	(A)
11.	(C)	**12.**	(C)

Exercise # 3

PART - I

1. (A) **2.** 5

3. (B) **4.** (C)

5. (A) **6.** 6

7. (A)

8. (A) $\rightarrow$ (t), (B) $\rightarrow$ (p, r), (C) $\rightarrow$ (q) (JEE given q, s) (D) $\rightarrow$ (r)

9. (C)

10. (A), (D)

11. 9

12. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s),(D) $\rightarrow$ (t)

13.	(A)	**14.**	3
15.	(A)	**16.**	(C)
17.	(B)	**18.**	(C)
19.	(D)	**20.**	(B), (D)
21.	32	**22.**	(A), (D)
23.	(C)	**24.**	(A)
25.	(A), (B), (C)	**26.**	(C)
27.	(4)	**28.**	(A)

29. (B), (D)

30. (A), (B)

31. (A), (C), (D)

32. (A) $\rightarrow$ P,Q; (B)$\rightarrow$ P, Q; (C)$\rightarrow$P,Q,S,T; (D) $\rightarrow$ Q, T

33. (A) $\rightarrow$ P,R,S ; (B) $\rightarrow$ P ; (C) $\rightarrow$ P,Q ; (D) $\rightarrow$ S, T

34. **BONUS**

35. (B), (C), (D)

36. (C)

37. (B), (C)

38.	(B)	**39.**	(A)
40.	(D)	**41.**	(B)
42.	(C), (D)	**43.**	3
44.	8	**45.**	0.5
46.	(A), (B), (D)	**47.**	0.75
48.	(C), (D)	**49.**	18.00

PART - II

1.	(1)	**2.**	(4)
3.	(4)	**4.**	(3)
5.	(1)	**6.**	(2)
7.	(1)	**8.**	(3)
9.	(3)	**10.**	(1)
11.	(3)	**12.**	(2)
13.	(3)	**14.**	(3)
15.	(3)	**16.**	(3)
17.	(3)	**18.**	(2)
19.	(4)	**20.**	(3)
21.	(1)	**22.**	(3)
23.	(1)	**24.**	(3)
25.	(2)	**26.**	(3)
27.	(3)	**28.**	(4)
29.	(3)	**30.**	(1)
31.	(2)	**32.**	(4)
33.	(1)	**34.**	(4)
35.	(1)	**36.**	4

SUBJECTIVE QUESTIONS

1. Position vector of A, B and C are $\hat{i}+\hat{j},\ 3\hat{i}+\hat{j}$ and $2\hat{i}+2\hat{j}+\hat{k}$ respectively and P is a point such that $|\overrightarrow{CP}|=|\overrightarrow{CA}|$ and $[\overrightarrow{PA}\ \overrightarrow{PB}\ \overrightarrow{PC}]$ = 0. If $|\overrightarrow{PA}|=\sqrt{\frac{3}{2}}|\overrightarrow{PB}|$, then find the position vector of point P.

2. Find the equation of a line in xz plane equally inclined with x and z axis which is at a unit distance from the line $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z}{1}$.

3. Find the condition for the equations $\vec{r}\times\vec{a}=\vec{b},\ \vec{r}\times\vec{c}=\vec{d}$ to be consistent and then solve them.

4. In a ΔABC, points D, E and F are taken on the sides BC, CA and AB respectively such that $\frac{BD}{DC}=\lambda$ and $\frac{CE}{AE}=\mu$. A parallelogram AFDE is completed. Using vector method, find the value of λ such that area of parallelogram AFDE is one third of the area of ΔABC.

5. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}$ cos t + $\hat{b}$ sin t. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. Find $\hat{u}$ and M

6. Prove that the direction cosines of a line equally inclined to three mutually perpendicular lines having D.C.'s as ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 ; ℓ_3, m_3, n_3 are $\frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$.

7. R and r are the circum–radius and in–radius of a regular tetrahedron respectively in terms of the length k of each edge. If $R^2 + r^2 = \frac{p}{q}k^2$, where $p, q \in I$ then absolute minimum value of p + q is

8. In a ΔABC, let M be the mid point of segment AB and let D be the foot of the bisector of $\angle C$. Then prove that $\frac{ar(\Delta\ CDM)}{ar(\Delta ABC)} = \frac{1}{2}\ \frac{a-b}{a+b} = \frac{1}{2}\tan\frac{A-B}{2}\cot\frac{A+B}{2}$.

9. If A_1, A_2, A_3 and B_1, B_2, B_3 are two sets of collinear points, then using vector method, show that the points of intersection of the pair of lines (A_1B_2, A_2B_1), (A_1B_3, A_3B_1) and (A_2B_3, A_3B_2) are collinear.

10. In a ΔABC, prove that distance between centroid and circumcentre is $\sqrt{R^2-\left(\frac{a^2+b^2+c^2}{9}\right)}$ where R is the circumradius and a, b, c denotes the sides of ΔABC.

11. Consider the following linear equations

$ax + by + cz = 0 \qquad bx + cy + az = 0 \qquad cx + ay + bz = 0$

Column I	**Column II**
(A) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	(p) the equations represent planes meeting only at a single point.
(B) $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	(q) the equations represent the line $x = y = z$.
(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	(r) the equations represent identical planes
(D) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	(s) the equations represent the whole of the three dimensional space

12. The line $\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$ is the hypotenuse of an isosceles right angle triangle whose opposite vertex is (7, 2, 4) find the equation of remaining sides

13. If D, E, F be three point on BC, CA, AB respectively of a ΔABC. Such that the line AD, BE, CF are concurrent then find the value of $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF}$.

14. Without expanding the determinant, Prove that

$$\begin{vmatrix} na_1+b_1 & na_2+b_2 & na_3+b_3 \\ nb_1+c_1 & nb_2+c_2 & nb_3+c_3 \\ nc_1+a_1 & nc_2+a_2 & nc_3+a_3 \end{vmatrix} = (n^3 + 1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

15. The length of edge of a regular tetrahedron D-ABC is 'a'. Point E & F are taken on the edges AD and BD respectively. Such that E divide $\overrightarrow{DA}$ and F divide $\overrightarrow{BD}$ in the ratio 2 : 1 each. The area of ΔCEF is equal to $\frac{\lambda\sqrt{3}}{36}a^2$, then value of λ is :

16. Find the minimum distance of the point (1, 1, 1) from the plane $x + y + z = 1$ measured perpendicular to the line $\frac{x-x_1}{1} = \frac{y-y_1}{2} = \frac{z-z_1}{3}$

17. AB , AC and AD are three adjacent edges of a parallelopiped . The diagonal of the parallelopiped passing through A and directed away from it is vector $\vec{a}$. The vector area of the faces containing vertices A , B , C and A , B , D are $\vec{b}$ and $\vec{c}$ respectively i.e. $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{b}$ and $\overrightarrow{AD} \times \overrightarrow{AB} = \vec{c}$. If projection of each edge AB and AC on diagonal vector $\vec{a}$ is $\frac{|\vec{a}|}{3}$, then find the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of $\vec{a}, \vec{b}, \vec{c}$ and $|\vec{a}|$.

18. Lengths of two opposite edges of a tetrahedron are a and b. Shortest distance between these edges is d and the angle between them is θ. Prove that its volume is (1/6) abd sin θ.

19. If $\vec{r}.\vec{a} = 0$, $\vec{r}.\vec{b} = 1$ and $[\vec{r}\ \vec{a}\ \vec{b}] = 1$, $\vec{a}\ \vec{b} \neq 0$ and $|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = 1$ then find $\vec{r}$ in terms of $\vec{a}$ & $\vec{b}$

20. If $\vec{a}$ & $\vec{b}$ are two non collinear vector $\vec{a}\,.\,\vec{b} \neq 0$ $\underbrace{\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}.............\times(\vec{a}}_{2016\text{ times}}\times(\vec{a}\times\vec{b})) = \lambda(\vec{a}\times\vec{b})$

21. Find the locus of the centroid of the tetrahedron of constant volume $64K^3$, formed by the three co-ordinates planes and a variable plane

22. Let $\vec{u}$ & $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that

$\left|(\vec{u}\times\vec{v})\,.\,\vec{w}\right| \leq \frac{1}{2}$ and the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

23. Let OABC is a regular tetrahedron and P is any point in space. If edge length of tetrahedron is 1 unit, find the least value of $2(PA^2 + PB^2 + PC^2 + PO^2)$.

24. The lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$, $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar and these determine a single plane if $\alpha\gamma \neq \beta\delta$. Find the equation of the plane in which they lie.

25. Consider the plane E : $\vec{r} = \begin{bmatrix}-1\\1\\1\end{bmatrix} + \lambda\begin{bmatrix}1\\2\\0\end{bmatrix} + \mu\begin{bmatrix}1\\0\\1\end{bmatrix}$

F is a plane containing the point A (–4, 2, 2) and parallel to E. Suppose the point B is on the plane E, such that B has a minimum distance from point A. If C(–3, 0, 4) lies in the plane F. Then find the area of ΔABC.

26. If A $(\vec{a})$, B $(\vec{b})$ and C $(\vec{c})$ are three non collinear points and origin does not lie in the plane of the points A, B and C, then for any point P $(\vec{p})$ in the plane of the ΔABC, prove that ;

(i) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{p}\,.\left(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right)$

(ii) A point $\vec{v}$ is on plane of ΔABC such that vector $\overrightarrow{ov}$ is $\perp$ to plane of ΔABC. Then show that

$\vec{v} = \frac{[\vec{a}\ \vec{b}\ \vec{c}](\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{4\Delta^2}$, where $\vec{\Delta}$ is the vector area of the Δ ABC.

27. Prove that the line $\frac{x}{1} = \frac{y}{1} = \frac{z-1}{-2}$ lies in the plane x + y + z = 1. Find the lines in the plane through the point (0, 0, 1) which are inclined at an angle $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$ with the line.

28. Find the equation of the sphere which has centre at the origin and touches the line 2(x + 1) = 2 – y = z + 3.

29. A mirror and a source of light are situated at the origin O and at a point on OX (x-axis) respectively. A ray of light from the source strikes the mirror and is reflected. If the Drs of the normal to the plane are 1, –1, 1, then find d.c's of the reflected ray.

30. A variable plane $\ell x + my + nz = p$ (where ℓ, m, n are direction cosines) intersects with co-ordinate axes at points A, B and C respectively show that the foot of normal on the plane from origin is the orthocentre of triangle ABC and hence find the coordinates of circumcentre of triangle ABC.

31. Find the maximum distance between the point P(0, 0, 3) and the circle

$x^2 + y^2 - 2\sqrt{5}\,x - 4y + 8 = 0;\ z = 0.$

32. Find the co-ordinates of the vertices of a square inscribed in the triangle with vertices A(0,0,0), B(2,1,3) and C(3,0,0), given that two of its vertices are on the side AC.

Answers

1. $3\hat{i}+3\hat{j}+2\hat{k}$ or $\frac{19}{9}\hat{i}+\frac{7}{9}\hat{j}+\frac{7}{9}\hat{k}$

2. $\frac{x-\sqrt{6}+1}{1}=\frac{y-0}{0}=\frac{z}{-1}$ and $\frac{x-1-\sqrt{2}}{1}=\frac{y-0}{0}=\frac{z-0}{1}$

3. Condition of consistency is $\vec{a}.\vec{d}+\vec{b}.\vec{c}=0$ and $\vec{r}=\frac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\left\{-\left(\vec{b}.\vec{d}\right)\vec{a}+\left(\vec{a}.\vec{d}\right)\vec{b}+\left(\vec{b}.\vec{b}\right)\vec{c}\right\}$

4. $\lambda = 2\pm\sqrt{3}$

5. $\hat{u} = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + \hat{a}\,.\,\hat{b})^{1/2}$

7. 17

11. (A) $\to$ (r), (B) $\to$ (q), (C) $\to$ (p), (D) $\to$ (s)

12. $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$; $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

13. 1

15. 11

16. $\frac{2}{3}\sqrt{21}$ units.

17. $\overrightarrow{AB}-\overrightarrow{AD}=3\frac{\vec{c}\times\vec{a}}{|\vec{a}|^2}$; $\overrightarrow{AC}=\frac{1}{3}\vec{a}+\frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2}+\frac{3(\vec{b}\times\vec{a})}{|\vec{a}|^2}$; $\overrightarrow{AD}=\frac{1}{3}\vec{a}+\frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2}-\frac{3(\vec{c}\times\vec{a})}{|\vec{a}|^2}$

19. $\vec{r} = -(\vec{a}.\vec{b})\,\vec{a} + |\vec{a}|^2\,\vec{b} + (\vec{a}\times\vec{b})$

20. $|\vec{a}|^{2016}$

21. $xyz = 6k^3$

23. 3

24. $x - 2y + z = 0$

25. $\frac{9}{2}$

26.

28. $9(x^2 + y^2 + z^2) = 5$

29. d.c's of the reflected ray are $\left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right\rangle$

30. $\left(\frac{p-\ell^2p}{2\ell}, \frac{p-m^2p}{2m}, \frac{p-n^2p}{2n}\right)$

31. 5

32. $P \equiv \left(6\left(\sqrt{10}-3\right),\ 0,\ 0\right)$ $Q \equiv \left(6\left(\sqrt{10}-3\right),\ 3\left(\sqrt{10}-3\right),\ 9\left(\sqrt{10}-3\right)\right)$

$R \equiv \left(3\left(4-\sqrt{10}\right),\ 3\left(\sqrt{10}-3\right),\ 9\left(\sqrt{10}-3\right)\right)$ $S \equiv \left(3\left(4-\sqrt{10}\right),\ 0,\ 0\right)$.

Self Assessment Paper

JEE ADVANCED

Maximum Marks : 62 **Total Time : 1:00 Hr**

SECTION-1 : ONE OPTION CORRECT (Marks - 12)

1. If $\vec{v}_1 = \hat{i} + \hat{j} + \hat{k}$ and $\vec{v}_2 = a\hat{i} + b\hat{j} + c\hat{k}$ where a, b, c, $\in \{-2, -1, 0, 1, 2\}$, then number of possible non-zero vectors $\vec{v}_2$ perpendicular to $\vec{v}_1$ is

(A) 10 (B) 13 (C) 15 (D) 18

2. If the distance between the planes represented by the equation $(x - 2y - 2z)^2 - 6(x - 2y - 2z) + 10 - k = 0$ is $\frac{4}{3}$ where $k > 1$, then the value of k is

(A) 2 (B) 3 (C) 4 (D) 5

3. If $[\vec{a}\ \vec{b}\ \vec{c}] = 2$, then $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \times (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \times (\vec{b} \times \vec{d})$ =

(A) $-5\vec{d}$ (B) $-3\vec{d}$ (C) $-4\vec{d}$ (D) $3\vec{d}$

4. Given $\vec{a} = \frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}$, $\vec{b} = \frac{\sqrt{3}}{2}\hat{j} + \frac{\hat{k}}{2}$, $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$, then volume of parallelopiped with coterminus edges

$\vec{u} = (\vec{a}.\vec{a})\vec{a} + (\vec{a}.\vec{b})\vec{b} + (\vec{a}.\vec{c})\vec{c}$ $\quad \vec{v} = (\vec{a}.\vec{b})\vec{a} + (\vec{b}.\vec{b})\vec{b} + (\vec{b}.\vec{c})\vec{c}$ $\quad \vec{w} = (\vec{a}.\vec{c})\vec{a} + (\vec{b}.\vec{c})\vec{c} + (\vec{c}.\vec{c})\vec{c}$

(A) $(2 \cos 30^\circ - \sin 30^\circ)^3$ (B) $(2 \cos 60^\circ - \sin 60^\circ)^3$
(C) $2 \cos 45^\circ - \sin 60^\circ)^3$ (D) $(2 \sin 45^\circ - \cos 60^\circ)^3$

SECTION-2 : ONE OR MORE THAN ONE CORRECT (Marks - 32)

5. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be unit vectors in x-y plane, each one is internal angle bisector of the four quadrants. Which of the following is/are incorrect?

(A) $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = 0$

(B) There exist i, j with $1 \le i < j \le 4$ such that $\vec{v}_i + \vec{v}_j$ is in first quadrant

(C) There exist i, j with $1 \le i < j < k \le 4$ such that $\vec{v}_i + \vec{v}_j + \vec{v}_k$ is in second quadrant

(D) There exist i, j with $1 \le i < j \le 4$ such that $\vec{v}_i.\vec{v}_j > 0$

6. Let H be the orthocentre of an acute-angled triangle ABC and O be its circumcentre, then $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC}$ is

(A) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HO}$ (B) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HO}$

(C) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 3\overrightarrow{HO}$ (D) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = -2(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$

7. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 5\hat{i} + \hat{j} - \hat{k}$ be three vectors. The region formed by the set of points whose position vectors '$\vec{r}$' satisfy the equations $\vec{r}.\vec{a} = 5$ and $|\vec{r} - \vec{b}| + |\vec{r} - \vec{c}| = 4$ is ellipse whose

(A) Area is $\sqrt{2}\pi$ (B) Area is 2π (C) Latus rectum is $\frac{1}{2}$ (D) Latus rectum is $\frac{1}{\sqrt{2}}$

8. Let A be the set of vectors $\vec{a}$ = (a_1, a_2, a_3) satisfying $\left(\sum_{i=1}^{3} \frac{a_i}{2^i}\right)^2 = \sum_{i=1}^{3} \frac{a_i^2}{2^i}$, then

(A) $a_1 = a_2 = a_3 = 0$ (B) A contains exactly one element
(C) $a_1a_2a_3 = 0$ (D) A has infinitely many elements

9. Let $\vec{v} = a\hat{i} + b\hat{j} + \frac{3}{\sqrt{2}}\hat{k}$ be a position vector such that $|\vec{v} - \hat{i}| = |\vec{v} - 2\hat{i}| = |\vec{v} - \hat{j}|$, then

(A) a = $\frac{3}{2}$ (B) b = $\frac{3}{2}$ (C) $|\vec{v}| = 3$ (D) $a \neq b$

10. A plane cuts the rectangular prism, having x + y = 0, x – y = 0 and x = 1 as equation of its faces, in a section which form equilateral triangle, then direction ratio of normal can be

(A) (1, $\sqrt{2}$, 0) (B) (–1, $\sqrt{2}$, 0) (C) ($\sqrt{2}$, 0, 1) (D) (–$\sqrt{2}$, 0, 1)

11. The position vector of the vertices A, B and C of a tetrahedron ABCD are (1, 1, 1), (1, 0,0) and (3, 0, 0) respectively. The altitude from vertices D to the opposite face ABC meets the median through A of ΔABC at point E. If AD = 4 and volume of tetrahedron = $\frac{2\sqrt{2}}{3}$, then

(A) altitude from vertex D is 2 (B) there is only one possible position for point E
(C) there are two possible position for point E (D) vector $\hat{j} - \hat{k}$ is normal to the plane ABC

12. A straight line cuts the sides AB, AC and AD of a parallelogram ABCD at points P, Q and R respectively. If $\overrightarrow{AP} = \lambda_1\overrightarrow{AB}$, $\overrightarrow{AR} = \lambda_2\overrightarrow{AD}$ and $\overrightarrow{AQ} = \frac{\lambda_3}{2}\overrightarrow{AC}$, where λ_1, λ_2, λ_3 are positive scalars, then

(A) λ_1, λ_2, λ_3 are in AP (B) λ_1, λ_2, λ_3 are in HP
(C) λ_1, λ_3, λ_2 are in HP (D) λ_1, $\lambda_2 \geq \lambda_3^2$

SECTION-3 : NUMERICAL VALUE TYPE (Marks - 18)

13. Consider two lines

L_1 : = x – 1= $\frac{y-2}{2}$ = $\frac{z-3}{3}$ & L_1 : $\frac{x-2}{3}$ = y – 3 = $\frac{z-1}{2}$.

If the plane x + by + cz + d = 0 parallel to the lines L_1, L_2 and equidistant from both L_1 and L_2 then absolute value of $\frac{b}{c+d}$ is

14. If the reflection of point $\vec{a}$ in the plane $\vec{r}.\vec{n} = q$ is $\alpha\vec{a} + \frac{\beta}{|\vec{n}|^2}(q + r\vec{a}.\vec{n})$, then $(\alpha - \gamma)^{-\beta}$ equals

15. Two adjacent sides OA and OB of a rectangle OACB are represented by $\vec{a}$ and $\vec{b}$ respectively, where O is origin.

If $16|\vec{a} \times \vec{b}| = 3(|\vec{a}| + |\vec{b}|)^2$ and 'θ' is the angle between the diagonals OC and AB, then non-negative square root of the reciprocal of the sum of greatest value and the product of all possible values of tan $\frac{\theta}{2}$ equals

16. If 3x + 4y + z = 5, x, y z $\in$ R then the reciprocal of the minimum value of $(x^2 + y^2 + z^2)$ equals

17. If the distance between points P and Q is 'd' and the projections of PQ on the coordinate planes are d_1, d_2, d_3 respectively, then $\frac{d^2}{d_1^2 + d_2^2 + d_3^2}$ equals

18. If the line with direction cosines proportional to 2, 1, 2 meets each of the lines x = y + 1 = z at P and x – 2y + 1 = 0 & y – z = 0 at θ, then PQ equals

Answers

1.	(D)	**2.**	(D)	**3.**	(C)	**4.**	(A)
5.	(B), (C), (D)	**6.**	(B), (D)	**7.**	(A), (C)	**8.**	(A), (B), (C)
9.	(A), (B), (C)	**10.**	(C), (D)	**11.**	(A), (C), (D)	**12.**	(C), (D)
13.	0.50	**14.**	0.25	**15.**	0.50	**16.**	1.04
17.	0.50	**18.**	3.00				