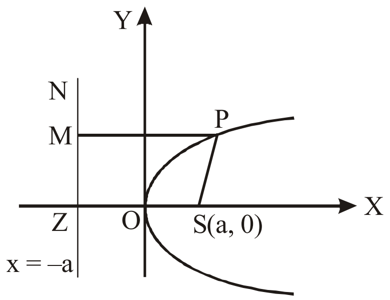
PARABOLA

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix). Eccentricity of parabola is 1.

Standard Equation of a parabola :

Let S be the focus and ZN is the directrix of the parabola.
From S, draw SZ perpendicular to the directrix.
Let O be the middle point of ZS. Take O as the origin and OS as x-axis and OY perpendicular to OS as the y-axis.
Let ZS = 2a, then ZO = OS = a
Now, S ≡ (a, 0) and the equation of ZN is x = –a or x + a = 0.
Let P(x, y) be any point on the parabola.
∴ PS = PM (by definition of parabola).



$$\Rightarrow \quad \sqrt{(x-a)^2+(y-0)^2} = \frac{|x+a|}{\sqrt{1^2+0}}$$

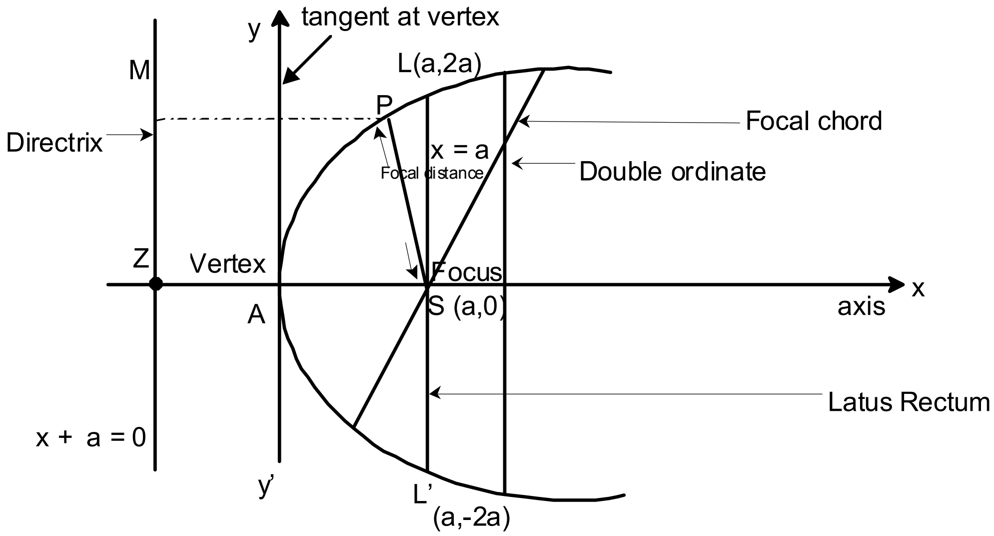
$$\Rightarrow \quad \sqrt{(x-a)^2+y^2} = |x+a|$$

or $\quad (x-a)^2 + y^2 = (x+a)^2$

or $\quad x^2 - 2ax + a^2 + y^2 = x^2 + 2xa + a^2$

or $\quad y^2 = 4ax \qquad$ which is the required equation.

Terms related to Parabola :



(1) **Axis :** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola. For the parabola $y^2 = 4ax$, x-axis is the axis.
Since equation has even power of y therefore the parabola is symmetric about x-axis i.e. about its axis.

(2) **Vertex :** The point of intersection of a parabola and its axis is called the vertex of the Parabola. For the parabola $y^2 = 4ax$, O(0, 0) is the vertex.
The vertex is the middle point of the focus and the point of intersection of axis and directrix.

(3) **Focal Distance :** The distance of any point P (x, y) on the parabola from the focus is called the focal length (distance) of point P.
The focal distance of P = the perpendicular distance of the point P from the directrix.

(4) **Double Ordinate :** The chord which is perpendicular to the axis of Parabola or parallel to Directrix is called double ordinate of the Parabola.

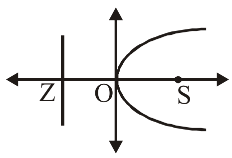
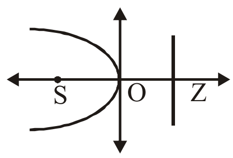
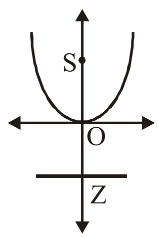
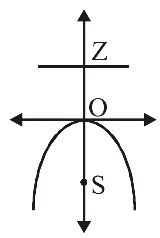
(5) **Focal Chord :** Any chord of the parabola passing through the focus is called Focal chord.

(6) **Latus Rectum :** If a double ordinate passes through the focus of parabola then it is called as latus rectum. The extremities of the latus rectum are L (a, 2a) and L'(a, –2a). Since LS = L'S = 2a, therefore length of the latus rectum LL' = 4a.

(7) **Parametric Equation of Parabola :** The parametric equation of Parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$. Hence any point on this parabola is $(at^2, 2at)$ which is also called as 't' point.

Note:
(i) The length of the latus rectum = 2 × perpendicular distance of focus from the directrix.
(ii) If $y^2 = lx$ then length of the latus rectum = l.
(iii) Two parabolas are said to be equal if they have same latus rectum.
(iv) The ends of a double ordinate of a parabola can be taken as $(at^2, 2at)$ and $(at^2, -2at)$.
(v) Parabola has no centre, but circle, ellipse, hyperbola have centre.

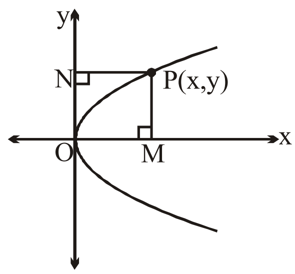
Other Standard parabola :

Equation of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
(a) Graphs				
(b) Eccentricity	e = 1	e = 1	e = 1	e = 1
(c) Focus	S(a, 0)	S(–a, 0)	S(0, a)	S(0, –a)
(d) Equation of directrix	x + a = 0	x – a = 0	y + a = 0	y – a = 0
(e) Equation of axis	y = 0	y = 0	x = 0	x = 0
(f) Vertex	O(0, 0)	O(0, 0)	O(0, 0)	O(0, 0)
(g) Extremities of latusrectum	(a, ±2a)	(–a, ±2a)	(±2a, a)	(±2a, –a)
(h) Length of latusrectum	4a	4a	4a	4a
(i) Equation of tangent at vertex	x = 0	x = 0	y = 0	y = 0
(j) Parametric coordinates of any point on parabola	$P(at^2, 2at)$	$P(-at^2, 2at)$	$P(2at, at^2)$	$P(2at, -at^2)$

Equation of parabola with respect to two perpendicular lines :

Let P(x, y) is any point on the parabola then equation of parabola $y^2 = 4ax$ is consider as

$$(PM)^2 = 4a(PN)$$



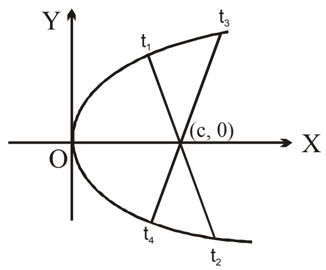
i.e. (The distance of P from its axis)^2 = (latus-rectum) (The distance of P from the tangent at its vertex) where P is any point on the parabola.

CHORD :

Line joining any two points on the parabola is called its chords.

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, lie on the parabola then equation of chord is

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$



$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\Rightarrow \quad (t_1 + t_2)\, y = 2x + 2at_1 t_2$$

If this chord meet the x-axis at point (c, 0) then from above equation

$$c + at_1t_2 = 0 \ \text{ i.e. } t_1t_2 = -c/a.$$

Note:

(i) If the chord joining t_1, t_2 & t_3, t_4 pass through a point (c, 0) on the axis, then $t_1t_2 = t_3t_4 = -c/a$.

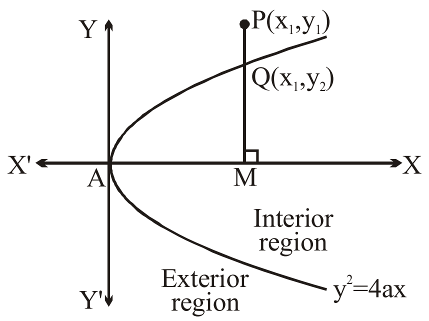
(ii) If PQ is a focal chord then $t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$. which is required relation.

Hence if one extremity of a focal chord is $(at^2, 2at)$ then the other extremity will be $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Position of a point with respect to a parabola $y^2 = 4ax$:

Let $P(x_1, y_1)$ be a point. From P draw PM $\perp$ AX (on the axis of parabola) meeting the parabola $y^2 = 4ax$ at $Q(x_1, y_2)$ where $Q(x_1, y_2)$ lie on the parabola therefore

$$y_2^2 = 4ax_1 \qquad \ldots(1)$$



Now, P will be outside, on or inside the parabola $y^2 = 4ax$ according as

$$PM >, =, \text{ or } < QM$$

$$\Rightarrow \quad (PM)^2 >, =, \text{ or } < (QM)^2$$

$$\Rightarrow \quad y_1^2 >, =, \text{ or } < y_2^2$$

$$\Rightarrow \quad y_1^2 >, =, \text{ or } < 4ax_1 \qquad \text{(from(1))}$$

Hence $y_1^2 - 4ax_1 >, =, \text{ or } < 0$

Hence in short, equation of parabola $S(x, y) = y^2 - 4ax$.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the parabola.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the parabola.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the parabola.

This result holds true for circle, parabola and ellipse.

Note :

(i) *The length of focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.*

(ii) $\because \quad \left| t + \frac{1}{t} \right| \geq 2$ *for all* $t \neq 0 \qquad (\because AM \geq GM)$

$\therefore \quad a\left(t + \frac{1}{t}\right)^2 \geq 4a$

$\Rightarrow$ *Length of focal chord $\geq$ latus rectum*

i.e., The length of smallest focal chord of the parabola is 4a, which is the latus rectum of a parabola.

Note : *If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus rectum is $\frac{4l_1l_2}{l_1 + l_2}$.*

INTERSECTION BETWEEN THE LINE AND PARABOLA :

Let the parabola be $y^2 = 4ax$ ... (i)
and the given line be $y = mx + c$ ... (ii)
then line may cut, touch or does not meet parabola.

The points of intersection of the line (1) and the parabola (2) will be obtained by solving the two equations simultaneously. By solving equation (i) and (ii), we get

$my^2 - 4ay + 4ac = 0$

this equation has two roots and its nature will be decided by the discriminant $D = 16a(a - cm)$

Now, if $D > 0$ i.e., $c < \frac{a}{m}$, then line intersect the parabola at two distinct points.

If $D = 0$ i.e., $c = \frac{a}{m}$, then line touches the parabola. **(It is condition of tangency)**

If $D < 0$ i.e., $c > \frac{a}{m}$, then line neither touch nor intersect the parabola.

EQUATION OF TANGENT :

1. Point Form :

Equation of parabola is $y^2 = 4ax$ ....(1)
Let $P \equiv (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points on parabola (1), then

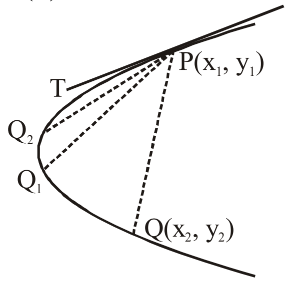
$y_1^2 = 4ax_1$ ....(2)

and $y_2^2 = 4ax_2$ ....(3)

Subtracting (2) from (3) then

$y_2^2 - y_1^2 = 4a(x_2 - x_1)$

or $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}$ ....(4)



Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad ....(5)$$

From (4) and (5), then

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \qquad ......(6)$$

Now for tangent at P, Q $\rightarrow$ P, i.e., $x_2 \rightarrow x_1$ and $y_2 \rightarrow y_1$ then equation (6) becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1)$$

or $\quad yy_1 - y_1^2 = 2ax - 2ax_1$

or $\quad yy_1 = 2ax + y_1^2 - 2ax_1$

or $\quad yy_1 = 2ax + 4ax_1 - 2ax_1$ $\qquad$ [From (2)]

or $\quad yy_1 = 2ax + 2ax_1$

which is the required equation of tangent at (x_1, y_1).

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + yx_1}{2}$ and without changing the constant (if any) in the equation of curve.

This method (standard substitution) is apply to all conic when point lie on the conic.

2. Slope Form :

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \qquad ....(1)$$

Since m is the slope of the tangent then

$$m = \frac{2a}{y_1} \quad \text{or} \quad y_1 = \frac{2a}{m}$$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

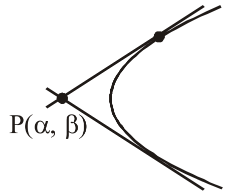
$$y_1^2 = 4ax_1 \quad \text{or} \quad \frac{4a^2}{m^2} = 4ax_1 \quad \Rightarrow \quad x_1 = \frac{a}{m^2}.$$

Substituting the values of x_1 and y_1 in (1), we get

$$y = mx + \frac{a}{m} \qquad ....(2)$$

Thus, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m, where m is the slope of the tangent and the co-ordinates of the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Two tangents can be drawn from a point P (α, β) to a parabola if P lies outside the parabola :



Let the parabola be $y^2 = 4ax$ ....(1)
Let P (α, β) be the given point

The equation of a tangent to parabola (1) is $y = mx + \frac{a}{m}$ ....(2)

If line (2) passes through P (α, β), then $\beta = m\alpha + \frac{a}{m}$ or $m^2\alpha - \beta m + a = 0$ ....(3)

There will be two tangents to parabola (1) from P (α, β) if roots of equation (3) are real and distinct i.e., D > 0 i.e. if $\beta^2 - 4\alpha a > 0 \Rightarrow P(\alpha, \beta)$ lies outside parabola (1).

We can also find the angle between two tangents from point P(α, β) using the formula

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

3. Parametric Form :

We have to find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'

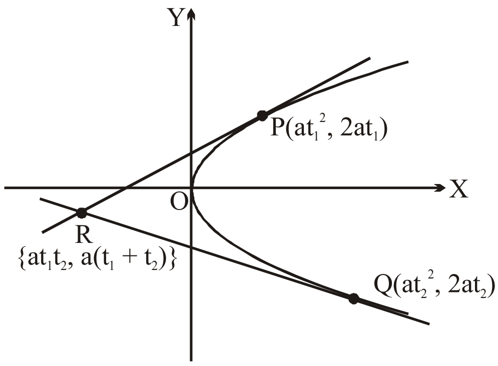
Since the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad ....(1)$$

replacing x_1 by at^2 and y_1 by 2at, then (1) becomes

$$y(2at) = 2a(x + at^2)$$
$$ty = x + at^2$$

Point of intersection of tangents at any two points on the parabola :



Let the given parabola be $y^2 = 4ax$ and two points on the parabola are

$$P = (at_1^2, 2at_1) \text{ and } Q = (at_2^2, 2at_2)$$

Equation of tangents at

$$P(at_1^2, 2at_1) \text{ and } Q(at_2^2, 2at_2)$$

are $t_1y = x + at_1^2$ ....(1)

and $t_2y = x + at_2^2$ ....(2)

Solving these equations we get

$$x = at_1t_2,\ y = a(t_1 + t_2)$$

Thus, the co-ordinates of the point of intersection of tangents at

$P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $R(at_1t_2, a(t_1 + t_2))$.

Note :

(i) The Arithmetic mean of the y-co-ordinates of P and Q $\left(\text{i.e., } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y - co-ordinate of the point of intersection of tangents at P and Q on the parabola.

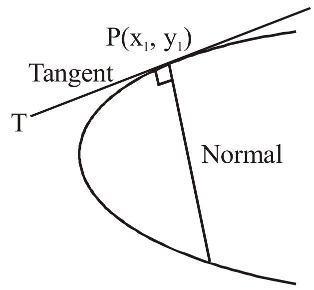
(ii) The Geometric mean of the x-co-ordinates of P and Q $\left(\text{i.e., } \sqrt{at_1^2 \times at_2^2} = at_1t_2\right)$ is the x co-ordinate of the point of intersection of tangents at P and Q on the parabola.

EQUATION OF NORMALS :

1. Point form :

Since the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad ....(1)$$



The slope of the tangent at $(x_1, y_1) = 2a/y_1$.

$\therefore$ Slope of the normal at $(x_1, y_1) = -y_1/2a$

Hence the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. Slope form :

The equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad ....(1)$$

Since m is the slope of the normal

then $\quad m = -\frac{y_1}{2a} \quad$ or $\quad y_1 = -2am$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$y_1^2 = 4ax_1 \quad$ or $\quad 4a^2m^2 = 4ax_1$

$\therefore \quad x_1 = am^2$

Substituting the values of x_1 and y_1 in (1) we get

$$y + 2am = m(x - am^2) \quad ....(2)$$

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal. The co-ordinates of the point of contact are $(am^2, -2am)$.

Hence $\quad y = mx + c$ will be normal to parabola. If and only if $c = -2am - am^3$

3. Parametric form :

Equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad ....(1)$$

Replacing x_1 by at^2 and y_1 by $2at$ then (1) becomes

$y - 2at = -t(x - at)^2$

or $\quad y = -tx + 2at + at^3$

THREE SUPPLEMENTARY RESULTS:

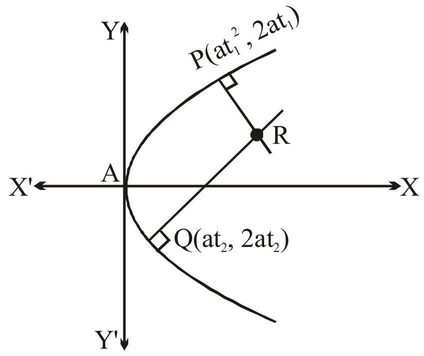
(a) Point of intersection of normals at any two points on the parabola :

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$

The equations of the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1x + 2at_1 + at_1^3 \qquad ....(1)$$

and $$y = -t_2x + 2at_2 + at_2^3 \qquad ....(2)$$



Hence point of intersection of above normals will be obtained by solving (1) and (2), we get

$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$

$$y = -at_1t_2(t_1 + t_2)$$

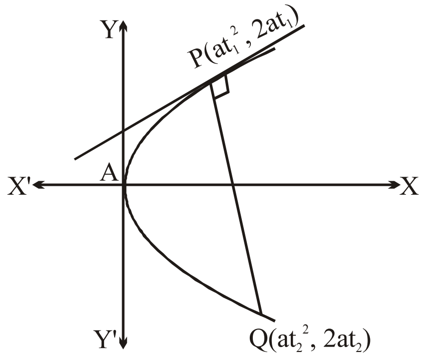
If R is the point of intersection then it is

$$R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$

(b) Relation between t_1 and t_2 if normal at t_1 meets the parabola again at t_2 :

Let the parabola be $y^2 = 4ax$, equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \qquad ....(1)$$



Since normal meet the parabola again at $Q(at_2^2, 2at_2)$

$\therefore \quad 2at_2 = -at_1t_2^2 + 2at_1 + at_2^3$

$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow \quad a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$

$\because \quad a(t_2 - t_1) \neq 0 \qquad (\because t_1 \text{ and } t_2 \text{ are different})$

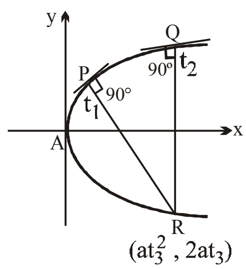
$\therefore \quad 2 + t_1(t_2 + t_1) = 0$

$\therefore \quad t_2 = -t_1 - \dfrac{2}{t_1}$

(c) If normal to the parabola $y^2 = 4ax$ drawn at any point $(at^2, 2at)$ meet the parabola at t_3 then

$$t_3 = -t - \frac{2}{t}$$

$\Rightarrow \quad t^2 + t\,t_3 + 2 = 0 \qquad ...(i)$



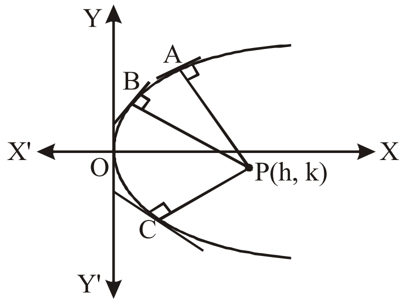
It has two roots t_1 & t_2. Hence there are two such point $P(t_1)$ & $Q(t_2)$ on the parabola from where normals are drawn and which meet parabola at $R(t_3)$

$\Rightarrow \quad t_1 + t_2 = -t_3 \quad \& \quad t_1t_2 = 2$

Thus the line joining $P(t_1)$ & $Q(t_2)$ meet x-axis at $(-2a, 0)$

CO-NORMAL POINTS :

Maximum three normals can be drawn from a point to a parabola and their feet (points where the normal meet the parabola) are called co-normal points.



Let P(h, k) be any given point and $y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k) then

$$k = mh - 2am - am^3$$

$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \qquad$ ......(i)

This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0, \qquad$ ...(ii)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - h)}{a}, \qquad \text{...(iii)}$$

$$m_1m_2m_3 = -\frac{k}{a} \qquad \text{...(iv)}$$

Hence for any given point P(h, k), (i) has three real or imaginary roots. Corresponding to each of these three roots, we have each normal passing through P(h, k). Hence we have three normals PA, PB and PC drawn through P to the parabola.

Points A, B, C in which the three normals from P(h, k) meet the parabola are called co-normal points.

Properties of co-normal points :

(1) The algebraic sum of the slopes of three concurrent normals is zero. This follows from equation (ii).

(2) The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero.

Let the ordinates of A, B, C be y_1, y_2, y_3 respectively then

$$y_1 = -2am_1,\ y_2 = -2am_2 \text{ and } y_3 = -2am_3$$

$\therefore$ Algebraic sum of these ordinates is

$$\begin{aligned} y_1 + y_2 + y_3 &= -2am_1 - 2am_2 - 2am_3 \\ &= -2a(m_1 + m_2 + m_3) \\ &= -2a \times 0 \qquad \{\text{from equation (ii)}\} \\ &= 0 \end{aligned}$$

(3) If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) is real then $h > 2a$.

When normals are real, then all the three roots of equation (i) are real and in that case

$$m_1^2 + m_2^2 + m_3^2 > 0 \qquad \text{(for any values of } m_1, m_2, m_3)$$

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2 \quad (m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow \quad (0)^2 - \dfrac{2(2a - h)}{a} > 0$

$\Rightarrow \quad h - 2a > 0$

or $\quad h > 2a$

(4) The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.
If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = \left(\frac{x_1+x_2+x_3}{3}, 0\right)$$

Since $y_1 + y_2 + y = 0$ (from result-2). Hence the centroid lies on the x-axis, which is the axis of the parabola also.

Now $\frac{x_1+x_2+x_3}{3} = \frac{1}{3}(am_1^2 + am_2^2 + am_3^2) = \frac{a}{3}\left(m_1^2 + m_2^2 + m_3^2\right)$

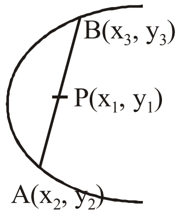
$$= \frac{a}{3}\{(m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)\}$$

$$= \frac{a}{3}\left\{(0)^2 - 2\left\{\frac{2a-h}{a}\right\}\right\} = \frac{2h-4a}{3}$$

$\therefore$ Centroid of ΔABC is $\left(\frac{2h-4a}{3}, 0\right)$

CHORD OF THE PARABOLA y^2=4ax WHOSE MIDDLE POINT IS GIVEN:

Equation of the parabola is $y^2 = 4ax$ ....(1)
Let AB be a chord of the parabola whose middle point is $P(x_1, y_1)$.
Equation of chord AB is $y - y_1 = m(x - x_1)$ ....(2)
where m = slope of AB
Let $A = (x_2, y_2)$ and $B = (x_3, y_3)$.
Since A and B lie on parabola (1)



$\therefore\ y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$

$\therefore\ y_2^2 - y_3^2 = 4a(x_2 - x_3)$ or $\frac{y_2 - y_3}{x_2 - x_3} = \frac{4a}{y_2 + y_3}$ ....(3)

But $P(x_1, y_1)$ is the middle point of AB $y_2 + y_3 = 2y_1$

$\therefore$ From (3), $\frac{y_2 - y_3}{x_2 - x_3} = \frac{4a}{2y_1} = \frac{2a}{y_1}$

$\therefore$ Slope of AB i.e., $m = \frac{2a}{y_1}$ ....(4)

From (2), equation of chord AB is $y - y_1 = \frac{2a}{y_1}(x - x_1)$

or $yy_1 - y_1^2 = 2ax - 2ax_1$ or $yy_1 - 2ax = y_1^2 - 2ax_1$

or $yy_1 - 2a(x - x_1) = y_1^2 - 4ax_1$ [Subtracting $2ax_1$ from both sides] ....(5)
(5) is the required equation. In usual notations, equation (5) can be written as $T = S_1$.

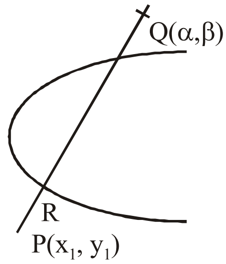
The same result holds true for circle, ellipse and hyperbola also.

PAIR OF TANGENTS :

Let the parabola be $y^2 = 4ax$ ....(1)

Let $P(x_1, y_1)$ be a point outside the parabola.

Let a chord of the parabola through the point $P(x_1, y_1)$ cut the parabola at R and let Q (α, β) be an arbitrary point on line PR. Let R divide PQ in the ratio $\lambda : 1$,



then $R = \left(\dfrac{\lambda\alpha + x_1}{\lambda + 1}, \dfrac{\lambda\beta + y_1}{\lambda + 1}\right)$.

Since R lies on parabola (1), therefore,

$$\left(\frac{\lambda\beta + y_1}{\lambda + 1}\right)^2 - 4a\left(\frac{\lambda a + x_1}{\lambda + 1}\right) = 0$$

or $(\lambda\beta + y_1)^2 - 4a(\lambda\alpha + x_1)(\lambda + 1) = 0$

or $(\beta^2 - 4a\alpha)\lambda^2 + 2[\beta y_1 - 2a(\alpha + x_1)]\lambda + (y_1^2 - 4ax_1) = 0$ ....(2)

Line PQ will become tangent to parabola (1) if roots of equation (2) are equal or if

$$4[\beta y_1 - 2a(\alpha + x_1)]^2 = 4(\beta^2 - 4a\alpha)(y_1^2 - 4ax_1)$$

Hence, locus of Q (α, β) i.e. equation of pair of tangents from P (x_1, y_1) is

$$[yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$$

$\Rightarrow$ $\mathbf{SS_1 = T^2}$

where S, S_1 and T have usual meanings.

The same result holds true for circle, ellipse and hyperbola also.

CHORD OF CONTACT OF POINT WITH RESPECT TO A PARABOLA :

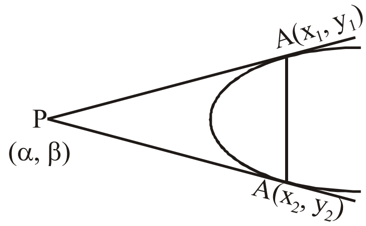
Two tangents PA and PB are drawn to parabola, then line joining AB is called the chord of contact to the parabola with respect to point P.

Let the parabola be $y^2 = 4ax$ ....(1)

Let $P(\alpha, \beta)$ be a point outside the parabola.

Let PA and PB be the two tangents from $P(\alpha, \beta)$ to parabola (1).

Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$



Equation of the tangent PA is $yy_1 = 2a(x + x_1)$ ....(2)

Equation of the tangent PB is $yy_2 = 2a(x + x_2)$ ....(3)

Since lines (2) and (3) pass through $P(\alpha, \beta)$, therefore

$\beta y_1 = 2a(a + x_1)$ ....(4)

and $\beta y_2 = 2a(a + x_2)$ ....(5)

Now we consider the equation $y\beta = 2a(x + \alpha)$ ……(6)

From (4) and (5), it follows that line (6) passes through A (x_1, y_1) and B (x_2, y_2).

Hence (6) is the equation of line AB which is the chord of contact of point $P(\alpha, \beta)$ with respect to parabola (1) i.e, chord of contact is $\mathbf{y\beta = 2a(x + \alpha)}$

The same result holds true for circle, ellipse and hyperbola also.

DIAMETER OF A PARABOLA :

Diameter of a conic is the locus of middle points of a series of its parallel chords.

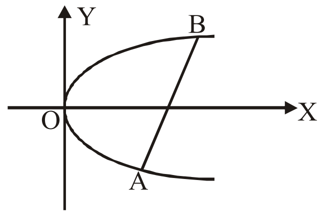
Equation of diameter of a parabola :

Let the parabola be $y^2 = 4ax$ ....(1)

Let AB be one of the chords of a series of parallel chords having slope m.

Let P(α, β) be the middle point of chord AB, then equation of AB will be $T = S_1$.

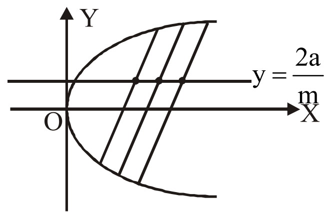
or $y\beta - 2a(x + \alpha) = \beta^2 - 4\alpha a$ ....(2)



Slope of line (2) $= \dfrac{2a}{\beta}$

but slope of line (1) i.e. line AB is m.

$\therefore \dfrac{2a}{\beta} = m$ or $\beta = \dfrac{2a}{m}$



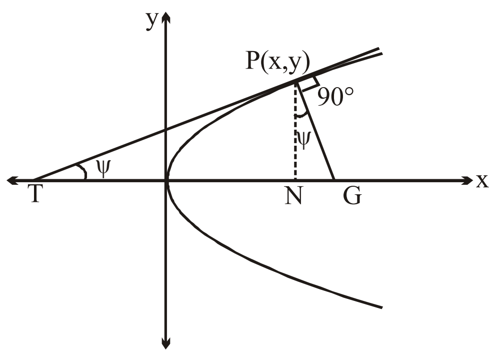
Hence locus of P(α, β) i.e. equation of diameter (which is the locus of a series of a parallel chords having slope m) is

$$y = \frac{2a}{m} \quad ....(3)$$

Clearly line (3) is parallel to the axis of the parabola. Thus a diameter of a parabola is parallelto its axis.

Length of tangent, subtangent, normal and sub-normal :

Let the parabola is $y^2 = 4ax$. Let the tangent at any point P(x, y) meet the axis of parabola at T and G respectively and tangent makes an angle ψ with x-axis.



$\therefore \quad \tan\psi = \left(\dfrac{dy}{dx}\right)_{P(x,y)}$ and PN = y

$\therefore \quad$ PT = length of tangent = PN cosec ψ = y cosec ψ

PG = length of normal = y sec ψ

TN = length of sub-tangent = PN cot ψ = y cot ψ

NG = length of sub-normal = y tan ψ

Properties of Parabola :

(1) Circle described on the focal length (distance) as diameter touches the tangent at the vertex.

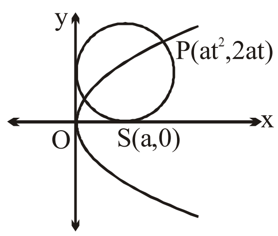
Equation of the circle described on SP as diameter is

$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$

Solve it with y-axis i.e. x = 0, we get

$y^2 - 2aty + a^2t^2 = 0 \Rightarrow (y - at)^2 = 0$

circle touches y-axis at (0, at).



(2) Circle described on the focal chord as diameter touches directrix

Equation of the circle described on PQ as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

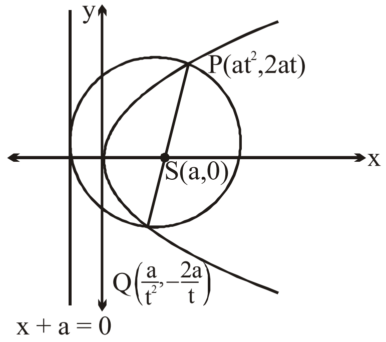
Solving it with $x = -a$

$$(-a - at^2)\left(-a - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

$\Rightarrow$ $$y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right)^2 = 0$$

$\Rightarrow$ $$\left[y - a\left(t - \frac{1}{t}\right)\right]^2 = 0$$

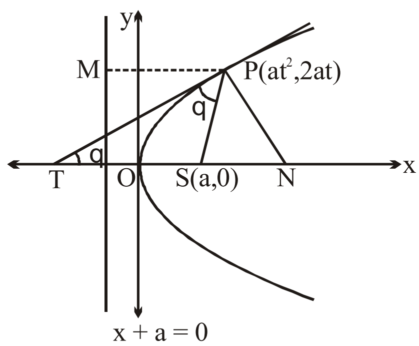
$\Rightarrow$ circle touches the directrix.



(3) Tangent at P is

$yt = x + at^2$, meet x-axis at T, then $T(-at^2, 0)$
Normal at P is $y + xt = 2at + at^3$, meet x-axis at N, then $N(2a + at^2, 0)$

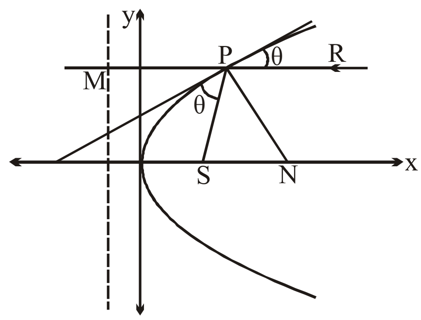
$\Rightarrow$ $ST = SN = a + at^2 = PM = PS$
$\Rightarrow$ $\angle PTS = \angle TPS = \theta$
$\because$ $TS = PS = PM \Rightarrow \angle TPM = \theta$



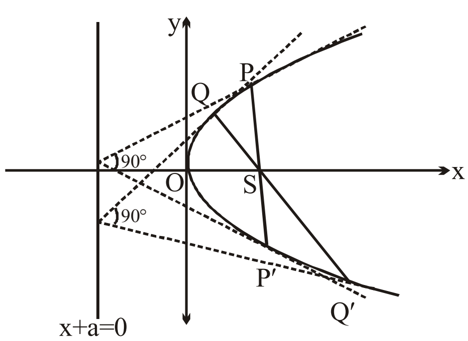
Tangent and Normal at any point P bisect the angle between PS and PM internally and externally. This property leads to the *reflection property of parabola.*

Circle circumscribing the triangle formed by any tangent normal and x-axis, has its centre at focus.
If we extend MP, then from figure $\angle RPN = \angle SPN = 90 - \theta$

Thus ray parallel axis meet parabola at P and after reflection from P it passes through the focus.



(4) The tangents at the extremities of a focal chord intersect at right angles on the directrix.



(5) The portion of tangent to the parabola intercepted between the directrix and the curve subtends a right angle at the focus.

tangent at P(at^2, 2at) is $yt = x + at^2$ meet the directrix at $x = -a \Rightarrow Q\left(-a, \frac{at^2 - a}{t}\right)$ and S(a, 0).

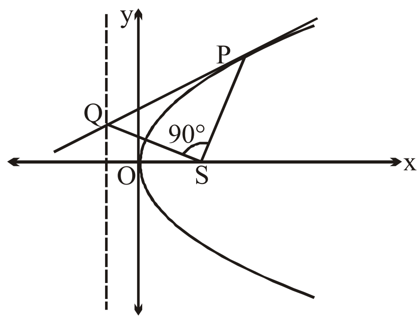
Slope at SP = $\frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = m_1$

Slope at SQ = $\frac{\frac{at^2 - a}{t} - 0}{-a - 0} = \frac{t^2 - 1}{-2t} = m_2$.

$\Rightarrow$ $m_1 m_2 = -1$

$\Rightarrow$ $SP \perp SQ$

$\Rightarrow$ $\angle PSQ = 90°$



(6) Tangent at P is $yt = x + at^2$ ......(i)

Line perpendicular to above line is $xt + y = \lambda$

and passes through (a, 0) gives $\lambda = at$

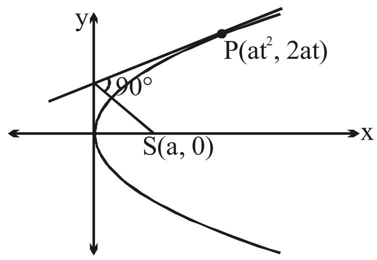
$\therefore$ perpendicular line will be $xt + y = at$ .....(ii)

Solve (i) and (ii), we get

$x = 0$

i.e., these two line intersect at y-axis i.e. tangent at the vertex.

The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at vertex.



(7) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being (– a, 0) & (3a, 0).

(8) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(9) The orthocentre of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co–ordinates $-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3)$.

(10) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Exercise

1. Latus rectum of the parabola whose focus is (3, 4) and whose tangent at vertex has the equation $x + y = 7 + 5\sqrt{2}$ is

(A) 5 (B) 10

(C) 20 (D) 15

2. Directrix of a parabola is x + y = 2. If it's focus is origin, then latus rectum of the parabola is equal to

(A) $\sqrt{2}$ units (B) 2 units

(C) $2\sqrt{2}$ units (D) 4 units

3. Which one of the following equations represents parametrically, parabolic profile ?

(A) x = 3 cost ; y = 4 sint

(B) $x^2 - 2 = -\cos t$; $y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$

(D) $x = \sqrt{1-\sin t}$; $y = \sin\frac{t}{2} + \cos\frac{1}{2}$

4. The point of intersection of the curves whose parametric equations are $x = t^2 + 1$, $y = 2t$ and $x = 2s$, $y = 2/s$ is given by

(A) (1, –3) (B) (2, 2)

(C) (–2, 4) (D) (1, 2)

5. If the line x – 1 = 0 is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of 'k' is

(A) 1/8 (B) 8

(C) 4 (D) 1/4

6. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

(A) x = – a (B) x = – a/2

(C) x = 0 (D) x = a/2

7. If $(t^2, 2t)$ is one end of a focal chord of the parabola $y^2 = 4x$ then the length of the focal chord will be

(A) $\left(t+\frac{1}{t}\right)^2$ (B) $\left(t+\frac{1}{t}\right)\sqrt{\left(t^2+\frac{1}{t^2}\right)}$

(C) $\left(t-\frac{1}{t}\right)\sqrt{\left(t^2+\frac{1}{t^2}\right)}$ (D) none

8. Locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$ is

(A) 2x – 1 = 0 (B) 2x + 3 = 0

(C) 2y + 3 = 0 (D) 2x + 5 = 0

9. Two common tangents to the circle $x^2 + y^2 = 2a^2$ and parabola $y^2 = 8ax$ are

(A) x = ±(y + 2a) (B) y = ±(x + 2a)

(C) x = ±(y + a) (D) y = ± (x + a)

10. The locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other is

(A) $y^2 = \frac{9}{2}ax$ (B) $y^2 = \frac{9}{4}ax$

(C) $y^2 = 9ax$ (D) $x^2 = 4ay$

11. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(A) 3y = 9x + 2 (B) y = 2x + 1

(C) 2y = x + 8 (D) y = x + 2

12. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then

(A) $t_2 = -t_1 - \frac{2}{t_1}$ (B) $t_2 = -t_1 + \frac{2}{t_1}$

(C) $t_2 = t_1 - \frac{2}{t_1}$ (D) $t_2 = t_1 + \frac{2}{t_1}$

13. Locus of the intersection of the tangents at the ends of the normal chords of the parabola $y^2 = 4ax$ is

(A) $(2a + x)y^2 + 4a^3 = 0$

(B) $(x + 2a)y^2 + 4a^2 = 0$

(C) $(x + 2a)y^2 + 4a^2 = 0$

(D) none

14. If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then

(A) $x_1 = x_2$ (B) $x_1 = y_2$

(C) $y_1 = y_2$ (D) $x_2 = y_1$

15. The equation of a straight line passing through the point (3, 6) and cutting the curve $y = \sqrt{x}$ orthogonally is

(A) 4x + y – 18 = 0 (B) x + y – 9 = 0

(C) 4x – y – 6 = 0 (D) none

16. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are
(A) (3, 2) (B) (2, 4)
(C) (3, 4) (D) (4, 1)

17. From the point (4, 6) a pair of tangent lines are drawn to the parabola, $y^2 = 8x$. The area of the triangle formed by these pair of tangent lines & the chord of contact of the point (4, 6) is
(A) 2 (B) 4
(C) 8 (D) none

18. Let PSQ be the focal chord of the parabola, $y^2 = 8x$. If the length of SP = 6 then, l(SQ) is equal to(where S is the focus)
(A) 3 (B) 4
(C) 6 (D) none

19. T is a point on the tangent to a parabola $y^2 = 4ax$ at its point P. TL and TN are the perpendiculars on the focal radius SP and the directrix of the parabola respectively. Then
(A) SL = 2 (TN) (B) 3 (SL) = 2 (TN)
(C) SL = TN (D) 2 (SL) = 3 (TN)

20. The equation of the common tangent touching the circle$(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is
(A) $\sqrt{3}\,y = 3x + 1$ (B) $\sqrt{3}\,y = -(x + 3)$
(C) $\sqrt{3}\,y = x + 3$ (D) $\sqrt{3}\,y = -(3x + 1)$

21. AB, AC are tangents to a parabola $y^2 = 4ax$. p_1 p_2 and p_3 are the lengths of the perpendiculars from A, B and C respectively on any tangent to the curve, then p_2, p_1, p_3 are in
(A) A.P. (B) G.P.
(C) H.P. (D) none of these

22. The tangent and normal at P (t), for all real positive t, to the parabola $y^2 = 4ax$ meet the axis of the parabola in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through the points P, T and G is
(A) $\cot^{-1} t$ (B) $\cot^{-1} t^2$
(C) $\tan^{-1} t$ (D) $\sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$

23. Through the vertex O of the parabola, $y^2 = 4ax$ two chords OP and OQ are drawn and the circles on OP and OQ as diameter intersect in R. If q_1, q_2 and f are the angles made with the axis by the tangent at P and Q on the parabola and by OR then the value of $\cot q_1 + \cot q_2$ equals
(A) – 2tanf (B) – 2tan(p – f)
(C) 0 (D) 2cotf

24. The locus of the vertices of the family of parabolas

$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is **[AIEEE 2006]**

(A) $xy = \frac{105}{64}$ (B) xy = 3/4
(C) $xy = \frac{35}{16}$ (D) $xy = \frac{64}{105}$

25. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is parapendicular to the given tangent is **[AIEEE 2007]**
(A) (–1, 1) (B) (0, 2)
(C) (2, 4) (D) (–2, 0)

26. A parabola has the origin as its focus and the line x = 2 as the directrix. Then the vertex of the parabola is a **[AIEEE 2008]**
(A) (0, 2) (B) (1, 0)
(C) (0, 1) (D) (2, 0)

27. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the lo-cus of P is **[AIEEE 2010]**
(A) x=1 (B) 2x + 1=0
(C) x= –1 (D) 2x – 1=0

28. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : **[AIEEE 2014]**
(A) $\frac{1}{2}$ (B) $\frac{3}{2}$
(C) $\frac{1}{8}$ (D) $\frac{2}{3}$

29. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P i **[AIEEE 2015]**
(A) $y^2 = 2x$ (B) $x^2 = 2y$
(C) $x^2 = y$ (D) $y^2 = x$

30. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is: **[JEE MAIN 2016]**

(A) $x^2 + y^2 - x + 4y - 12 = 0$

(B) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

(C) $x^2 + y^2 - 4x + 9y + 18 = 0$

(D) $x^2 + y^2 - 4x + 8y + 12 = 0$

31. The centres of those circle which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on: **[JEE MAIN 2016]**

(A) an ellipse which is not a circle

(B) a hyperbola

(C) a parabola

(D) a circle

32. Latus rectum of the parabola whose focus is (3, 4) and whose tangent at vertex has the equation $x + y = 7 + 5\sqrt{2}$ is -

(A) 5 (B) 10

(C) 20 (D) 15

33. If $(t^2, 2t)$ is one end of a focal chord of the parabola $y^2 = 4x$ then the length of the focal chord will be-

(A) $\left(t + \frac{1}{t}\right)^2$

(B) $\left(t + \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$

(C) $\left(t - \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$ (D) none

34. From the focus of the parabola $y^2 = 8x$ as centre, a circle is described so that a common chord of the curves is equidistant from the vertex and focus of the parabola. The equation of the circle is -

(A) $(x - 2)^2 + y^2 = 3$ (B) $(x - 2)^2 + y^2 = 9$

(C) $(x + 2)^2 + y^2 = 9$ (D) $x^2 + y^2 - 4x = 0$

35 Tangents are drawn from the point (–1, 2) on the parabola $y^2 = 4x$. The length , these tangents will intercept on the line x = 2 :

(A) 6 (B) $6\sqrt{2}$

(C) $2\sqrt{6}$ (D) none of these

36. From the point (4, 6) a pair of tangent lines are drawn to the parabola, $y^2 = 8x$. The area of the triangle formed by these pair of tangent lines & the chord of contact of the point (4, 6) is

(A) 2 (B) 4

(C) 8 (D) none

37. If the tangent at the point P (x_1,y_1) to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q & R, then the mid point of QR is -

(A) $(x_1 + b, y_1 + b)$ (B) $(x_1 - b, y_1 - b)$

(C) (x_1, y_1) (D) $(x_1 + b, y_1)$

38. Two parabolas $y^2 = 4a(x - l_1)$ and $x^2 = 4a(y - l_2)$ always touch one another , the quantities l_1 and l_2 are both variable. Locus of their point of contact has the equation -

(A) $xy = a^2$ (B) $xy = 2a^2$

(C) $xy = 4a^2$ (D) none

Answer key

Differential Equation

1.	D	2.	C	3.	B	4.	C	5.	C	6.	D	7.	C
8.	A	9.	D	10.	C	11.	A	12.	A	13.	A	14.	C
15.	C	16.	A	17.	A	18.	C	19.	A	20.	A	21.	B
22.	B	23.	C	24.	B	25.	B	26.	D	27.	A	28.	B
29.	C	30.	A	31.	A	32.	A	33.	C	34.	A		

Circle

1.	B	2.	A	3.	D	4.	A	5.	A	6.	B	7.	D
8.	C	9 .	A	10.	B	11.	C	12.	A	13.	C	14.	C
15.	A	16.	C	17.	C	18.	A	19.	D	20.	A	21.	A
22.	A	23.	C	24.	A	25 .	B	26.	C	27.	A	28.	D
29.	D	30	A	31.	A	32.	C	33.	A	34.	D	35.	C
36.	C	37.	C	38.	A	39.	A	40.	D	41.	A		

Parabola

1.	C	2.	C	3.	B	4.	B	5.	C	6.	C	7.	A
8.	D	9.	B	10.	A	11.	D	12.	A	13.	A	14.	C
15.	A	16.	C	17.	A	18.	A	19.	C	20.	D	21.	B
22.	C	23.	A	24.	A	25.	D	26.	B	27.	C	28.	A
29.	B	30.	D	31.	C	32.	C	33.	A	34.	B	35.	B
36.	A	37.	C	38.	C								