02

Inverse Circular Functions

2.01 Introduction

If $\sin\theta = x$ then we say x is the sin of θ and θ is the sine inverse of x. This statement is written mathematically as $\theta = \sin^{-1} x$ or $\theta = \arcsin x$ is read as sine inverse x.

2.02 Inverse circular functions

We know that $\sin\theta, \cos\theta, \tan\theta$ are trigonometrical circular functions, which for every value of θ gives a fixed value

If $\sin\theta = x$ then $\theta = \sin^{-1} x$

$\sin^{-1} x$ is said to be an inverse circular function. The simialr inverse function are

$\cos^{-1} x, \tan^{-1} x,$ $\cos^{-1} x$ and $\cot^{-1} x$

Note:

1. In functions $\sin^{-1} x, \cos^{-1} x$, -1 is not a exponent but representation of an inverse function as $(\sin x)^{-1} = \dfrac{1}{\sin x}$ therefore $\sin^{-1} x \neq (\sin x)^{-1}$

2. $\sin^{-1} x$ denotes an angle whereas $\sin\theta$ denotes a number where θ is an angle.

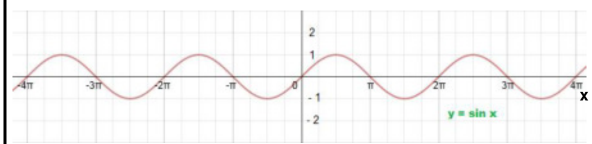
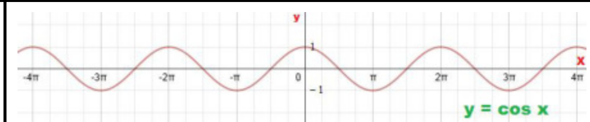
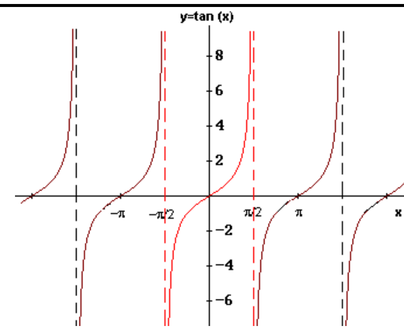
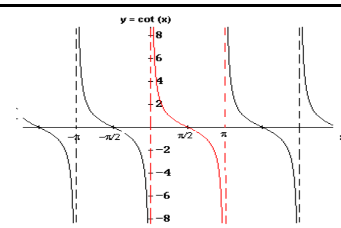
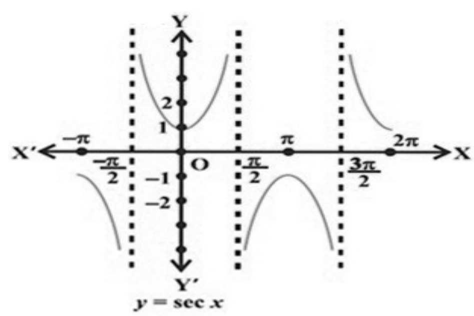
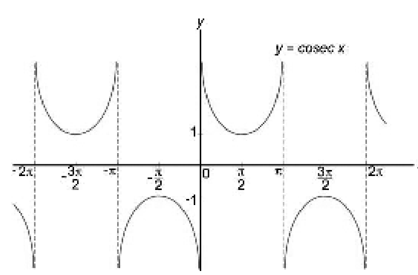
Inverse circular function:

To find the inverse to f i.e. f^{-1} the function f must be one-one onto.

It is clear from the study of trigonometric function that normally they are not bijective. Therefore it is not possible to find their inverse under normal conditions, but on restricting the domain of these function, they become one-one onto and we can easily derive their inverse, under these conditions.

the domain and range of inverse trigonometric function can be understood by the following table-

Table 2.1

Function $y = f(x)$	Domain	Range	Curve
$\sin x$	$x \in R$ or $\ldots\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\ldots$	$y \in [-1, 1]$	
$\cos x$	$x \in R$ or $\ldots[-\pi, 0], [0, \pi], [\pi, 2\pi]\ldots$	$y \in [-1, 1]$	
$\tan x$	$x \in R - (2n+1)\frac{\pi}{2}, \ \forall \ n \in Z$ or $\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\ldots$	$y \in R$	
$\cot x$	$x \in R - n\pi \ \ \forall \ n \in Z$ or $\ldots(-\pi, 0), (0, \pi), (\pi, 2\pi)\ldots$	$y \in R$	
$\sec x$	$x \in R - (2n+1)\frac{\pi}{2} \ \ \forall \ n \in Z$ or $\ldots[-\pi, 0] - \{-\pi/2\}$, $[0, \pi] - \{\pi/2\}$, $[\pi, 2\pi] - \{3\pi/2\}$	$y \in (-\infty, -1] \cup [1, \infty)$ i.e. range does not exist between -1 and 1	
$\text{cosec}\, x$	$x \in R - n\pi \ \ \forall \ n \in Z$ $\ldots[-3\pi/2, -\pi/2] - \{-\pi\}$, $[-\pi/2, \pi/2] - \{0\}$, $[\pi/2, 3\pi/2] - \{\pi\}, \ldots$	$y \in (-\infty, -1] \cup [1, \infty)$ i.e. range does not exist between -1 and 1	

Analysing the above table we see that

(i) Circular functiosn are not bijective in their entire domain.

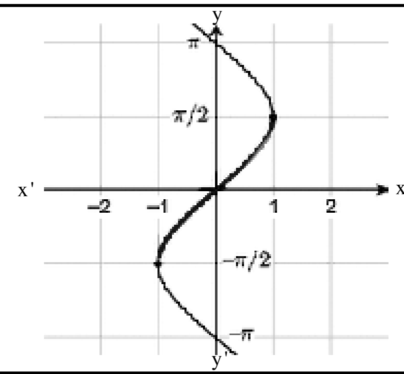
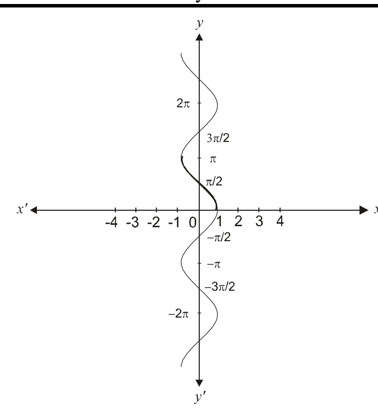
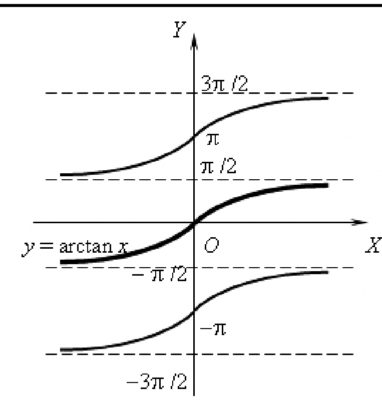
(ii) $\tan, \cot, \sec, \text{cosec}$ functions are not defined on some of its points in their domain.

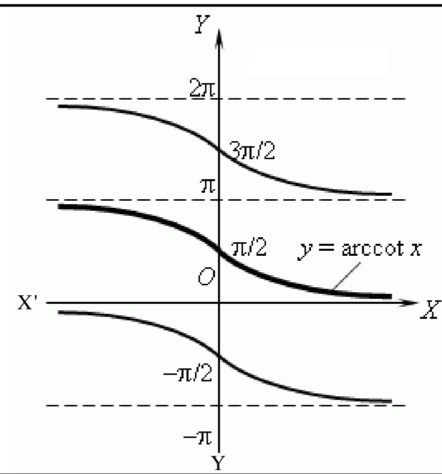
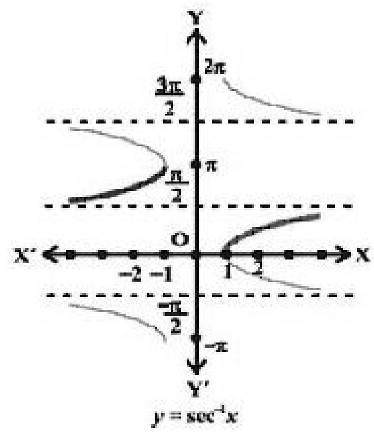
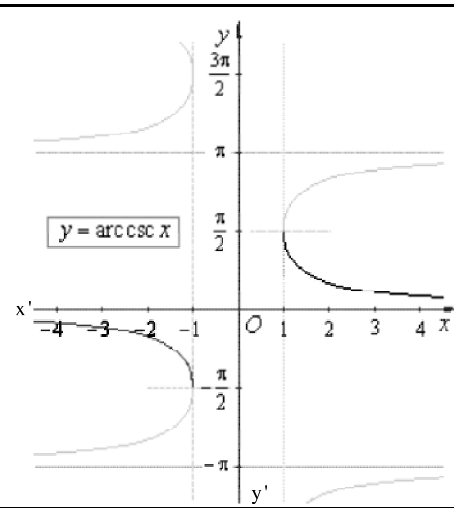
(iii) In sine and cosine functions range is restricted to [–1, 1] whereas in sec and cosec functions range do not lie between the interval (–1, 1)

Now if we have to find the inverse of these functions then we have to restrict their domain and make them one-one onto, for that from the above table the trigonometric fucntions become bijective itself by restricting their domain to any of the given intervals and then their inverse can be found out.

The following table shows the domain and range of inverse trigonometric functions under these bounded conditions. Every interval of range have a branch of inverse function. In these branches there is a principal branch, their range and shape represented by dark black colour.

Table 2.2

Function $y =$	Domain	Range	Curve
$\sin^{-1} x$	$x = [-1, 1]$	$\ldots\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \ldots$	
$\cos^{-1} x$	$x \in [-1, 1]$	$\ldots[-\pi, 0]$; $[0, \pi]$; $[\pi, 2\pi], \ldots$	
$\tan^{-1} x$	$x \in R$	$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$ **Note:** function is not defined on $\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$	

$\cot^{-1} x$	$x \in R$	$...(-\pi, 0)$; $(0, \pi)$; $(\pi, 2\pi),...$ **Note:** function is not defined on $...-\pi, 0, \pi, 2\pi...$	
$\sec^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$...[-\pi, 0]-\{-\pi/2\}$; $[0, \pi]-\{\pi/2\}$, $[\pi, 2\pi]-\{3\pi/2\},...$ **Note:** function is not defined on $...-\pi/2, \pi/2, 3\pi/2,...$	
$\text{cosec}^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$...[-3\pi/2, -\pi/2]-\{-\pi\}$; $[-\pi/2, \pi/2]-\{0\}$; $[\pi/2, 3\pi/2]-\{\pi\},...$ **Note:** function is not defined on $...-\pi, 0, \pi,...$	

Note : If $y = f(x)$ then we get $x = f^{-1}(y)$ i.e. in the graph of trigonometric functions if we interchange the X and Y-axis then we get the graph of inverse trigonometric functions.

(i) If the branch of inverse circular function is not defined then we mean the principal branch of the function only.

(ii) If the value of inverse circular functions lies in the principal branch then that value is termed as its principal value, See table 2.3

General values

We know that $\sin\theta = \sin\left\{n\pi + (-1)^n \theta\right\}$, where $n \in Z$ is the set of integers.

Now if $\sin^{-1} x = \theta$ then the general value of $\sin^{-1} x$ is $n\pi + (-1)^n \sin^{-1} x$ and is denoted by $\sin^{-1} x$

Thus $\quad \text{Sin}^{-1}x = n\pi + (-1)^n \sin^{-1} x, \; n \in Z$

Similarly $\quad \text{Cos}^{-1}x = 2n\pi \pm \cos^{-1} x, \; n \in Z$

$\text{Tan}^{-1}x = n\pi + \tan^{-1} x$ etc.

where by $\text{Cos}^{-1}x, \text{Tan}^{-1}x$ we mean the general value of $\cos^{-1} x, \tan^{-1} x$. Similarly $\text{Sec}^{-1}x, \; \text{Cosec}^{-1}x, \; \text{Cot}^{-1}x$ we mean the general value of $\sec^{-1} x, \; \operatorname{cosec}^{-1}x, \cot^{-1} x$

Principal value

The Principal value of inverse circular function is the smallest positive or negative value of θ which satisifies the equation $\sin\theta = x, \cos\theta = x$ For example $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ, \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. We denote this by $\sin^{-1} x, \; \cos^{-1} x, \tan^{-1} x$ etc.

The intervals of inverse circular functions are different:

Table 2.3

Function	Principal Value	Domain
$y = \sin^{-1} x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$-1 \le x \le 1$
$y = \cos^{-1} x$	$0 \le y \le \pi$	$-1 \le x \le 1$
$y = \tan^{-1} x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$-\infty < x < \infty$
$y = \sec^{-1} x$	$0 < y \le \pi, \; y \ne \frac{\pi}{2}$	$(-\infty < x \le -1) \cup (1 \le x < \infty)$
$y = \operatorname{cosec}^{-1}x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \; y \ne 0$	$(-\infty < x \le -1) \cup (1 \le x < \infty)$
$y = \cot^{-1} x$	$0 < y < \pi$	$-\infty < x < \infty$

Note: (i) If $x > 0$ then the principal vlaues of all inverse circular functions lie in the first quadrant $[0, \pi/2]$

(ii) If $x < 0$ then the principal vlaues of $\sin^{-1} x, \tan^{-1} x$ and $\operatorname{cosec}^{-1}x$ lie in the fourth quadrant $[-\pi/2, 0]$ whereas the values of $\cot^{-1} x, \sec^{-1} x$ lie in the second quadrant $[\pi/2, \pi]$

2.03 Relation between Inverse Circular Functions

Let $\theta = \sin^{-1} x$ then $\sin\theta = x$ then $\cos\theta = \sqrt{1-x^2}$ $\left(\because \sin^2\theta + \cos^2\theta = 1\right)$

$$\theta = \cos^{-1}\sqrt{1-x^2}$$

Similarly $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-x^2}}{x} \Rightarrow \theta = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \theta = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x} \Rightarrow \theta = \operatorname{cosec}^{-1}\frac{1}{x}$$

$$\therefore \quad \sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\frac{1}{x}$$

Note: The validity of these formulae is for certain interval.

2.04 Properties of inverse circular functions

(i) $\sin\left(\sin^{-1}x\right) = x, \quad -1 \le x \le 1$ and $\sin^{-1}\left(\sin\theta\right) = x, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Proof : $\because \sin^{-1}x = \theta$ then $\sin\theta = x$ [by definition]

putting the vlaue of θ, we have $\sin\left(\sin^{-1}x\right) = x$

again if $\sin\theta = x, -1 \le x \le 1$

then $\theta = \sin^{-1}x, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}; \theta = \sin^{-1}\left(\sin\theta\right)$

thus from the given table for the values of x and θ we have

$$\cos\left(\cos^{-1}x\right) = x \qquad \cos^{-1}\left(\cos\theta\right) = \theta$$

$$\tan\left(\tan^{-1}x\right) = x \qquad \tan^{-1}\left(\tan\theta\right) = \theta$$

$$\cot\left(\cot^{-1}x\right) = x \qquad \cot^{-1}\left(\cot\theta\right) = \theta$$

$$\sec\left(\sec^{-1}x\right) = x \qquad \sec^{-1}\left(\sec\theta\right) = \theta$$

$$\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right) = x \qquad \operatorname{cosec}^{-1}\left(\operatorname{cosec}\theta\right) = \theta$$

Note: $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \ne \frac{2\pi}{3}$ Since the principal value of $\sin^{-1}x$ is not $\frac{2\pi}{3}$

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$$

(ii) $\sin^{-1}\frac{1}{x}=\cos ec^{-1}x, \quad R\sim(-1,1)$

Note: $\sin^{-1}\frac{1}{x}=\theta\Rightarrow\sin\theta=\frac{1}{x}\Rightarrow\cos ec\theta=x\Rightarrow\theta=\cos ec^{-1}x\Rightarrow\sin^{-1}\frac{1}{x}=\cos ec^{-1}x$

$$\cos ec^{-1}x=\sin^{-1}\frac{1}{x},\ x\le-1,\ x\ge1$$

$$\cos^{-1}x=\sec^{-1}\frac{1}{x},\ x\le-1,\ x\ge1$$

$$\sec^{-1}x=\cos^{-1}\frac{1}{x},\ x\le-1,\ x\ge1$$

$$\tan^{-1}x=\cot^{-1}\frac{1}{x},\ x>0 \quad \text{and} \quad \cot^{-1}x=\tan^{-1}\frac{1}{x},\ x>0$$

(iii) $\sin^{-1}(-x)=-\sin^{-1}x$ and $\cos^{-1}(-x)=\pi-\cos^{-1}x, -1\le x\le1$

Proof : $\sin^{-1}(-x)=\theta\Rightarrow -x=\sin\theta\Rightarrow x=-\sin\theta=\sin(-\theta)$

or $\quad \sin^{-1}x=-\theta=-\sin^{-1}(-x)$

or $\quad \sin^{-1}(-x)=-\sin^{-1}x$

Similarly if $\quad \cos^{-1}(-x)=\theta$ तो $x=-\cos\theta$

or $\quad x=\cos(\pi-\theta)$

∴ $\quad \cos^{-1}x=\pi-\theta$

or $\quad \cos^{-1}x=\pi-\cos^{-1}(-x)$

or $\quad \cos^{-1}(-x)=\pi-\cos^{-1}x$

Similarly $\quad \tan^{-1}(-x)=-\tan^{-1}x, \quad \text{cosec}^{-1}(-x)=-\text{cosec}^{-1}x$

$$\sec^{-1}(-x)=\pi-\sec^{-1}x,\ \cot^{-1}(-x)=\pi-\cot^{-1}x$$

2.05 Other important standrad formulae

(i) To Prove that:

(a) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right\}$

(b) $2\sin^{-1}x=\sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$

(c) $3\sin^{-1}x=\sin^{-1}\left\{3x-4x^3\right\}$

Proof : (a) Let $\sin^{-1} x = \theta_1 \quad \sin\theta_1 = x$ and $\sin^{-1} y = \theta_2$

i.e. $\sin\theta_2 = y$ then $\cos\theta_1 = \sqrt{1-\sin^2\theta_1} = \sqrt{1-x^2}$

similarly $\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-y^2}$

we know that

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$$

or $$\theta_1 \pm \theta_2 = \sin^{-1}(\sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2)$$

$\therefore$ $$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$$

(b) Let $\sin^{-1} x = \theta$ i.e. $\sin\theta = x$

$\therefore$ $$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2x\sqrt{1-x^2}$$

$\Rightarrow$ $$2\theta = \sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$$

$$2\sin^{-1} x = \sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$$

(c) We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore$ $$3\theta = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

or $$3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$$

(ii) To Prove that

(a) $$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$$

(b) $$2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$$

(c) $$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

Proof : (a) Let $\cos^{-1} x = \theta_1$ i.e. $\cos\theta_1 = x$

and $\cos^{-1} y = \theta_2$ i.e. $\cos\theta_2 = y$

then $\sin\theta_1 = \sqrt{1-x^2}$ and $\sin\theta_2 = \sqrt{1-y^2}$

Now we know that

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$$

or $$\theta_1 \pm \theta_2 = \cos^{-1}(\cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2)$$

$\therefore$ $$\cos^{-1} x \pm \cos^{-1} y = \left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

(b) Let $\cos^{-1} x = \theta$ i.e. $\cos\theta = x$ $\therefore$ $\cos 2\theta = \left(2\cos^2\theta\right) - 1 = 2x^2 - 1$

or $$2\theta = \cos^{-1}\left(2x^2 - 1\right)$$

or $$2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right)$$

(c) We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ $\therefore 3\theta = \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$

or $$3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right)$$

(iii) To Prove that

(a) $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(b) $$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

(c) $$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(d) $$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(e) $$3\tan^{-1} x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Proof : (a) Let $\tan^{-1} x = \theta_1$ i.e., $\tan\theta_1 = x$ and $\tan^{-1} y = \theta_2$ i.e., $\tan\theta_2 = y$

We know that

$$\tan\left(\theta_1 + \theta_2\right) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy}$$

or $$\theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

or $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ can be proved in a similar manner as (a)

(c) Now $$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y}{1-xy}\right)+\tan^{-1}z$$

$$=\tan^{-1}\left[\frac{\{(x+y)/(1-xy)\}+z}{1-z\{(x+y)/(1-xy)\}}\right]$$ [(*a*) से]

$$=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(d) Let $\tan^{-1}x=\theta$ i.e. $\tan\theta=x$

$\therefore$ $$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2x}{1-x^2}$$

or $$2\theta=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

or $$2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(e) we know that $$\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

$\therefore$ $$3\theta=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

or $$3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

(iv) **To prove that**

(a) $$\cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right)$$

(b) $$\cot^{-1}x-\cot^{-1}y=\cot^{-1}\left(\frac{xy+1}{y-x}\right).$$

Proof : (a) Let $\cot^{-1}x=\theta_1$ and $\cot^{-1}y=\theta_2$

then $\cot\theta_1=x,\qquad \cot\theta_2=y$

we know that $$\cot(\theta_1+\theta_2)=\frac{\cot\theta_1\cot\theta_2-1}{\cot\theta_1+\cot\theta_2}$$

or $$\theta_1 + \theta_2 = \cot^{-1}\left(\frac{\cot\theta_1 \cot\theta_2 - 1}{\cot\theta_1 + \cot\theta_2}\right)$$

or $$\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left(\frac{xy-1}{x+y}\right).$$

(b) $\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{xy+1}{y-x}\right)$ can be proved as (a)

(iv) To Prove that

(a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (b) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ (c) $\sec^{-1} x + \cos ec^{-1} x = \frac{\pi}{2}$.

Proof : (a) Let $\sin^{-1} x = \theta$ then $\sin^{-1} x = \theta \Rightarrow x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\Rightarrow \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

(b) Let $\tan^{-1} x = \theta$ then $\tan^{-1} x = \theta \Rightarrow x = \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$

$$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - \theta$$

$$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\Rightarrow \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

(c) Let $\sec^{-1} x = \theta$ then $\sec^{-1} x = \theta \Rightarrow x = \sec\theta = \cos ec\left(\frac{\pi}{2} - \theta\right)$

$$\Rightarrow \cos ec^{-1} x = \frac{\pi}{2} - \theta$$

$$\Rightarrow \cos ec^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

$$\Rightarrow \sec^{-1} x + \cos ec^{-1} x = \frac{\pi}{2}.$$

Illustrative Examples

Example 1. Find the principal value of

(a) $\sin^{-1}\left(-\frac{1}{2}\right)$ (b) $\tan^{-1}\left(-\sqrt{3}\right)$ (c) $\sec^{-1}\left(\sqrt{2}\right)$.

Solution : (a) Let $\sin^{-1}\left(-\frac{1}{2}\right)=\theta,\quad \sin\theta=-\frac{1}{2}$

since the principal vlaue of $\sin^{-1}x$ lies in the interval $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$

$\therefore \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

But $\sin\theta$ is negative

$\therefore \quad -\frac{\pi}{2}\le\theta\le 0$

$\Rightarrow \quad \sin\theta=-\frac{1}{2}=-\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right)\Rightarrow\theta=-\frac{\pi}{6}$

$\therefore$ the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$

(b) Let $\tan^{-1}\left(-\sqrt{3}\right)=\theta, \quad \Rightarrow \quad \tan\theta=-\sqrt{3}$

since the principal value of $\tan^{-1}x$ lies in the interval $-\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}$

$\therefore \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

But $\tan\theta$ is negative

$\therefore \quad -\frac{\pi}{2}<\theta<0$

$\Rightarrow \quad \tan\theta=-\sqrt{3}=-\tan\frac{\pi}{3}=\tan\left(-\frac{\pi}{3}\right) \quad\Rightarrow\quad \theta=-\frac{\pi}{3}$

$\therefore$ the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\pi/3$

(c) Let $\sec^{-1}\left(\sqrt{2}\right)=\theta, \quad\Rightarrow\quad \sec\theta=\sqrt{2}$

Here since $x\ge 1$ i.e. for $1\le x$ the principal value $\sec^{-1}x$ lies in the interval $0\le\sec^{-1}x<\frac{\pi}{2}$

$\therefore \quad 0<\theta\le\frac{\pi}{2}$

$\Rightarrow \qquad \sec\theta = \sqrt{2} = \sec\pi/4 \quad \Rightarrow \quad \theta = \pi/4$

$\therefore$ Thus the principal value of $\sec^{-1}(\sqrt{2})$ is $\pi/4$

Example 2. Prove that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$

Solution : L.H.S. $= 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

$$= 2\left(2\tan^{-1}\frac{1}{5}\right) - \left(\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right)$$

$$= 2\tan^{-1}\frac{2/5}{1-1/25} - \tan^{-1}\frac{1/70-1/99}{1+1/70\times 1/99}$$

$$= 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{29}{6931}$$

$$= \tan^{-1}\frac{2\times 5/12}{1-25/144} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right]$$

$$= \tan^{-1}\frac{28561}{28561} = \tan^{-1}(1) = \frac{\pi}{4} = \text{(RHS)}$$

Example 3. Prove that

$$2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$$

Solution : Let $\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} = \theta$

$\therefore \qquad \tan\theta = \sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}$

$\Rightarrow \qquad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$= \frac{1-\frac{a-b}{a+b}\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}} = \frac{b\left(1+\tan^2\frac{x}{2}\right)+a\left(1-\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}$$

$$= \frac{b + a\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}{a + b\dfrac{1-\tan^2(x/2)}{1-\tan^2(x/2)}} \qquad \text{[dividing Nr and Dr by } 1+\tan^2 x/2 \text{]}$$

$$= \frac{b + a\cos x}{a + b\cos x}$$

$$\Rightarrow \qquad 2\theta = \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right)$$

$$\therefore \qquad 2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} = \cos^{-1}\frac{b + a\cos x}{a + b\cos x}.$$

Example 4. Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}.$

Solution : Let $\frac{1}{2}\cos^{-1}\frac{a}{b} = \theta,$ तब $\cos 2\theta = \frac{a}{b}$

L.H.S. $= \tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$$

$$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}$$

$$= 2\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \frac{2}{\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)} = \frac{2}{\cos 2\theta} = \frac{2b}{a} = \text{R.H.S.}$$

Example 5. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$ then Prove that $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha.$

Solution : Given $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$

$$\cos^{-1}\left\{\frac{x}{a}.\frac{y}{b}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha$$

$$\Rightarrow \quad \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

Example 6. Solve the following equation :

$$\cos^{-1}\frac{1-a^2}{1+a^2}+\cos^{-1}\frac{1-b^2}{1+b^2}=2\tan^{-1}x.$$

Solution : Let $a=\tan\theta, b=\tan\phi$, $\quad\Rightarrow\quad \theta=\tan^{-1}a, \phi=\tan^{-1}b$

$$\therefore \quad \frac{1-a^2}{1+a^2}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$$

$$\Rightarrow \quad \frac{1-b^2}{1+b^2}=\frac{1-\tan^2\phi}{1+\tan^2\phi}=\cos 2\phi$$

from the given equation

$$\cos^{-1}(\cos 2\theta)+\cos^{-1}(\cos 2\phi)=2\tan^{-1}x$$

$$\Rightarrow \quad 2\theta+2\phi=2\tan^{-1}x$$

$$\Rightarrow \quad \theta+\phi=\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}a+\tan^{-1}b=\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\frac{a+b}{1-ab}=\tan^{-1}x$$

$$\therefore \quad x=\frac{a+b}{1-ab}.$$

Example 7. Prove that

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]=\sqrt{\frac{x^2+1}{x^2+2}}.$$

Solution : Let $\cot^{-1} x = \theta,$ then $\cot\theta = x$

If $\quad \cot\theta = x,\quad \sin\theta = \dfrac{1}{\operatorname{cosec}\theta} = \dfrac{1}{\sqrt{\cot^2\theta + 1}} = \dfrac{1}{\sqrt{x^2+1}}$

$$\theta = \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{x^2+1}}$$

L.H.S. $\quad = \cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1} x\right)\right\}\right]$

$$= \cos\left[\tan^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+1}}\right)\right\}\right]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

We know that $\tan\phi = \dfrac{1}{\sqrt{1+x^2}}$ then $\cos\phi = \dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

$\therefore$ L.H.S. $\quad = \cos\left(\cos^{-1}\dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right)$

$$= \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}} = \text{R.H.S.}$$

Example 8. Solve the following equation :

$$\tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{a^2-x+1}.$$

Solution : $\quad \tan^{-1}\dfrac{1}{a-1} - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{a^2-x+1}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{\dfrac{1}{a-1} - \dfrac{1}{x}}{1 + \dfrac{1}{(a-1)x}}\right) = \tan^{-1}\dfrac{1}{a^2-x+1}$

$\Rightarrow \quad \dfrac{x-a+1}{ax-x+1} = \dfrac{1}{a^2-x+1}$

$\Rightarrow \quad (x-a+1)\left(a^2-x+1\right) = ax-x+1$

$$\Rightarrow \quad xa^2 - a^3 - x^2 + a^2 + x - a = 0$$

$$\Rightarrow \quad a^2(x-a) - (x+a)(x-a) + (x-a) = 0$$

$$\Rightarrow \quad (x-a)\left[a^2 - (x+a) + 1\right] = 0$$

$$\Rightarrow \quad (x-a)\left(a^2 - x - a + 1\right) = 0$$

$$\Rightarrow \quad x = a \text{ and } x = a^2 - a + 1.$$

Example 9. Solve the following equation :

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

Solution :

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

$$\Rightarrow \quad \left(\frac{\pi}{2} - \cos^{-1} x\right) + \left(\frac{\pi}{2} - \cos^{-1} 2x\right) = \frac{\pi}{3}$$

$$\Rightarrow \quad \cos^{-1} x + \cos^{-1} 2x = \frac{2\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}\left[x.2x - \sqrt{1-x^2}\sqrt{1-4x^2}\right] = \frac{2\pi}{3}$$

$$\Rightarrow \quad 2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad 2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2} = -\frac{1}{2}$$

$$\Rightarrow \quad 2x^2 + \frac{1}{2} = \sqrt{1-x^2}\sqrt{1-4x^2}$$

$$\Rightarrow \quad 4x^4 + \frac{1}{4} + 2x^2 = (1-x^2)(1-4x^2) \quad \text{[by squaring]}$$

$$\Rightarrow \quad 4x^4 + \frac{1}{4} + 2x^2 = 1 - 5x^2 + 4x^4$$

$$\Rightarrow \quad 7x^2 = \frac{3}{4} \quad \Rightarrow \quad x^2 = \frac{3}{28} \quad \Rightarrow \quad x = \pm\frac{1}{2}\sqrt{\frac{3}{7}}$$

But $x = -\frac{1}{2}\sqrt{\frac{3}{7}}$ does not satisfy the given equation

thus the solution is $x = \frac{1}{2}\sqrt{\frac{3}{7}}.$

Exercise 2.1

1. Find the principal value of the following angles:

(i) $\sin^{-1}(1)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\sec^{-1}\left(-\sqrt{2}\right)$

(iv) $\operatorname{cosec}^{-1}(-1)$ (v) $\cot^{-1}\left(-\sqrt{\frac{1}{3}}\right)$ (vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Prove that [from 2- 8]

2. $2\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

3. $\tan^{-1}\frac{17}{19}-\tan^{-1}\frac{2}{3}=\tan^{-1}\frac{1}{7}$

4. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$

5. $\sec^2\left(\tan^{-1}2\right)+\operatorname{cosec}^2\left(\cot^{-1}3\right)=15$

6. $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}$

7. $\tan^{-1}\sqrt{\frac{ax}{bc}}+\tan^{-1}\sqrt{\frac{bx}{ca}}+\tan^{-1}\sqrt{\frac{cx}{ab}}=\pi$, where $a+b+c=x$

8. $\frac{1}{2}\tan^{-1}x=\cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}}$.

9. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then prove that $x^2+y^2+z^2+2xyz=1$.

10. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then prove that $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.

(Hint : If $A+B+C=\pi$ then $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$)

11. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then prove that $xy+yz+zx=1$.

12. If $\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}+\frac{1}{3}\tan^{-1}\frac{3z-z^3}{1-3z^2}=5\pi$, then prove that $x+y+z=xyz$.

13. If $\sec^{-1}\left(\sqrt{1+x^2}\right)+\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+y^2}}{y}\right)+\cot^{-1}\left(\frac{1}{2}\right)=3\pi$, then prove that $x+y+z=xyz$.

14. Prove that $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

15. If $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P., then prove that $y^2(x+z)+2y(1-xz)-x-z=0$

16. If the roots of $x^3+px^2+qx+p=0$ are α, β, γ, then prove that (except one situation) $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=n\pi$ and also find the situation when it does not happen.

Solve the following equation [Q. 17 to 25]

17. $\sec^{-1}\left(\frac{x}{a}\right)-\sec^{-1}\left(\frac{x}{b}\right)=\sec^{-1}b-\sec^{-1}a$

18. $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$

19. $\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$

20. $\tan^{-1}\frac{x+7}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi-\tan^{-1}7$

21. $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cot^{-1}x=\frac{\pi}{4}$

22. $3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$

23. $\sin 2\left[\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]=0$

24. $\tan^{-1}\left(\frac{1}{4}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{6}\right)+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{4}$

25. $\sin^{-1}x-\sin^{-1}y=\frac{2\pi}{3}; \quad \cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$.

Miscellaneous Exercise - 2

1. The principal value of $\tan^{-1}(-1)$ is

   (a) 45° (b) 135° (c) -45° (d) -60°.

2. $2\tan^{-1}(1/2)$ equals

   (a) $\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$ (c) $\cos^{-1}\left(\frac{5}{3}\right)$ (d) $\cos^{-1}\left(\frac{1}{2}\right)$.

3. If $\tan^{-1}(3/4)=\theta$ then the vlaue of $\sin\theta$ is

   (a) $\frac{5}{3}$ (b) $\frac{3}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{4}$.

4. The value of $\cot\left[\tan^{-1}\alpha + \cot^{-1}\alpha\right]$ is

(a) 1 (b) ∞ (c) 0 (d) None of these

5. If $\sin^{-1}\left(\frac{1}{2}\right) = x$, then the general value of x is

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $n\pi \pm \frac{\pi}{6}$ (d) $n\pi + (-1)^n \frac{\pi}{6}$.

6. The value of $2\tan\left(\tan^{-1}x + \tan^{-1}x^3\right)$ is

(a) $\frac{2x}{1-x^2}$ (b) $1+x^2$ (c) $2x$ (d) None of these

7. If $\tan^{-1}(3x) + \tan^{-1}(2x) = \frac{\pi}{4}$, then the value of x is

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{10}$ (d) $\frac{1}{2}$.

8. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) π.

9. If $\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1}x$, then the value of x is

(a) -1 (b) 0 (c) 1 (d) $-\frac{1}{2}$.

10. If $\cot^{-1}x + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$, then the value of x is

(a) 1 (b) 3 (c) $\frac{1}{3}$ (d) None of these

11. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then find the value of x

12. Find the value of $\cos\left[(\pi/2) + \sin^{-1}(1/3)\right]$

13. If $\sin^{-1}(3/4) + \sec^{-1}(4/3) = x$, then find the value of x

14. Find the value of $\sin^{-1}(4/5) + 2\tan^{-1}(1/3)$

15. If $\sin^{-1}(5/x) + \sin^{-1}(12/x) = 90^\circ$, then find the value of x

16. Prove that : $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$.

17. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then prove that $x+y+z+=xyz$.

18. Prove that : $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^2 A)=0$.

19. Prove that : $\tan^{-1}x=2\tan^{-1}\left[\operatorname{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)\right]$.

20. If $\phi=\tan^{-1}\frac{x\sqrt{3}}{2K-x}$ and $\theta=\tan^{-1}\frac{2x-K}{K\sqrt{3}}$, then prove that the value of $\phi-\theta$ is 30°

21. Prove that : $2\tan^{-1}\left[\tan(45^\circ-\alpha)\tan\frac{\beta}{2}\right]=\cos^{-1}\left(\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right)$.

Important Points

1. If $\sin\theta=x$ then $\theta=\sin^{-1}x$ and $\sin^{-1}x=\theta$ then $\sin\theta=x$.

2. $\sin(\sin^{-1}x)=x$, $\sin^{-1}(\sin x)=x$; $\cos(\cos^{-1}x)=x$, $\cos^{-1}(\cos x)=x$ etc.

3. (i) The principal value of $\sin^{-1}x, \tan^{-1}x, \cot^{-1}x, \operatorname{cosec}^{-1}x$ is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

   (ii) The principal value of $\cos^{-1}x$ and $\sec^{-1}x$ lies from 0 to π

4. (i) $\sin^{-1}(-x)=-\sin^{-1}x$, $\tan^{-1}(-x)=-\tan^{-1}x$, $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x$

   (ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x$, $\sec^{-1}(-x)=\pi-\sec^{-1}x$, $\cot^{-1}(-x)=\pi-\cot^{-1}x$

5. (i) $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x}$, $\cos^{-1}x=\sec^{-1}\frac{1}{x}$, $\tan^{-1}x=\cot^{-1}\frac{1}{x}$

   (ii) $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}$, $\sec^{-1}x=\cos^{-1}\frac{1}{x}$, $\cot^{-1}x=\tan^{-1}\frac{1}{x}$

6. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$, $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$

7. (i) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right)$

   (ii) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

8. $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}=\tan^{-1}\frac{2x}{1-x^2}$

9. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x.\sqrt{1-y^2} \pm y.\sqrt{1-x^2}\right)$

10. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-x^2}\right)$

11. (i) $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ (ii) $2\cos^{-1}x = \cos^{-1}\left(2x^2-1\right)$

12. (i) $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right)$ (ii) $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right)$

(iii) $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

Answers

Exercise 2.1

1. (i) $\frac{\pi}{2}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $-\frac{\pi}{2}$ (v) $\frac{2\pi}{3}$ (vi) $\frac{\pi}{6}$

17. $x = ab$ 18. $x = \tan\left(\frac{\pi}{12}\right)$ 19. $x = 0, 3, \frac{-2}{3}$ 20. $x = 11 \pm 4\sqrt{6}$

21. $x = 3$ 22. $x = 2$ 23. $x = \pm 1, \pm\left(1 \pm \sqrt{2}\right)$ 24. $x = \frac{-461}{9}$

25. $x = \frac{1}{2}, y = 1$

Miscellaneous Exericse - 2

1. (c) 2. (a) 3. (b) 4. (c) 5. (d) 6. (a) 7. (a)
8. (c) 9. (c) 10. (c) 11. 1/2 12. –1/3 13. π/2 14. π/2
15. 13