5 Mathematical Induction

KEY FACTS

1. A statement $T(n)$ is true for all $n \in N$, where N is the set of natural numbers, provided:
(*i*) $T(1)$ is true (*ii*) $T(k)$ is true $\Rightarrow T(k+1)$ is true.

2. The proof of a proposition $T(n)$ by the method of mathematical induction consists of the following steps:

(***a***) **Step I: (Basic Step):** Actual verification of the proposition $[T(1)]$, $[T(2)]$, etc., for particular positive integral values of n say $n = 1, 2, \ldots$.

(***b***) **Step II: (Induction Step):** Assuming the proposition to be true for some positive integral value k of n *i.e.*, $T(k)$ and then proving that it is true for the value $(k + 1)$ which is the next higher integer, *i.e.*, proving $T(k + 1)$ true whenever $T(k)$ holds.

SOLVED EXAMPLES

Type I: Summation of Series

Ex. 1. Using the method of induction, show that $1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1)$, for all $n \in N$.

Sol. Let $T(n) = 1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1)$

***Basic Step*:** For $n = 1$,

LHS $= T(1) = 1$, RHS $= \frac{1}{2} \times 1 \times 2 = 1 \Rightarrow$ LHS = RHS $\Rightarrow$ $T(1)$ is true.

***Induction Step*:** Assume that $T(k)$ is true, *i.e.*,

$$1 + 2 + 3 + \ldots + k = \frac{1}{2} k(k+1)$$

To obtain $T(k + 1)$, we add $(k + 1)$th term $= (k + 1)$ to both the sides, *i.e.*,

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{1}{2} k(k+1) + (k+1)$$

$$\Rightarrow 1 + 2 + 3 + \ldots + k + (k+1) = (k+1)\left(\frac{k}{2} + 1\right)$$

$$\Rightarrow 1 + 2 + 3 + \ldots + k + (k+1) = \frac{1}{2}(k+1)(k+2)$$

$\Rightarrow$ Thus the statement $T(n)$ is true for $n = k + 1$ under the assumption that $T(k)$ is true. Therefore, by the principle of mathematical induction, the statement is true for every +ve integer n.

Ex. 2. Prove that for all +ve integral values of n, $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Sol. Let $T(n)$ be the statement: $1 + 3 + 5 + ... + (2n - 1) = n^2$

Basic Step: For $n = 1$, LHS = 1, RHS = $1^2 \Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true

***Induction* Step:** Assume that $T(k)$ is true, *i.e.*,

$1 + 3 + 5 + ... + (2k - 1) = k^2$

To obtain $T(k + 1)$, add the $(k + 1)$th term $= 2(k + 1) - 1 = 2k + 2 - 1 = 2k + 1$ to both the sides. Then,

$1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = k^2 + 2k + 1 \Rightarrow 1 + 3 + 5 + ...$ to $(k + 1)$ terms $= (k + 1)^2$

Thus the statement is true for $n = k + 1$ under the assumption that statement is true for $n = k$

Therefore, the statement $1 + 3 + 5 + ...$ to n terms $= n^2$ for every positive integer n.

Ex. 3. Prove that for every natural number n.

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Sol. Let $T(n)$ be the statement,

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2 (n+1)^2}{4}$$

Basic Step: For $n = 1$, $1^3 = \dfrac{1^2 (1+1)^2}{4} = 1 \Rightarrow T(1)$ is true.

Induction Step: Let $T(k)$ hold for a natural number k, that is

$$1^3 + 2^3 + 3^3 + ... + k^3 = \frac{k^2 (k+1)^2}{4}$$

Now, to obtain $T(k + 1)$, add the $(k + 1)$th term $= (k + 1)^3$ to both the sides of $T(k)$, *i.e.*,

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{k^2 (k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4(k+1)] = \frac{(k+1)^2 (k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

Hence $T(k + 1)$ is true, whenever $T(k)$ is true.

Ex. 4. Use the principal of mathematical induction to prove the following statement true for all $n \in N$.

$$x + 4x + 7x + ... + (3n - 2)x = \frac{1}{2} n (3n - 1)x.$$

Sol. Let $T(n)$ be the statement:

$$x + 4x + 7x + ... + (3n - 2)x = \frac{1}{2} n (3n - 1)x$$

Basic Step: For $n = 1$,

$$x = \frac{1}{2} \times 1 \times (3 \times 1 - 1) \times x \Rightarrow x = x \Rightarrow T(1) \text{ is true.}$$

Induction Step: Assume $T(k)$ holds for a natural number k, *i.e.*,

$$x + 4x + 7x + ... + (3k - 2)x = \frac{1}{2} k (3k - 1) x$$

Now to show that $T(k + 1)$ holds, add the $(k + 1)$th term $= [3(k + 1) - 2]\, x = (3k + 1)x$ to both the sides of $T(k)$, *i.e.*,

$$x + 4x + 7x + ... + (3k - 2)\, x + (3k + 1)\, x = \frac{1}{2} k (3k - 1)\, x + (3k + 1)\, x$$

$$= \frac{1}{2}[k(3k-1)x + 2(3k+1)x] = \frac{1}{2}[(3k^2+5k+2)x] = \frac{1}{2}(k+1)(3k+2)x$$

$$= \frac{1}{2}(k+1)[3(k+1)-1]x$$

$\Rightarrow$ $T(k+1)$ is true, whenever $T(k)$ is true.

Ex. 5. Prove by the method of mathematical induction that $a + (a+d) + (a+2d) + ... + (a+(n-1)d) = \frac{n}{2}\{2a+(n-1)d\}$ for all $n \in N$, where $a, d \in R$.

Sol. Let $T(n)$ be the statement

$$a + (a+d) + (a+2d) + ... + (a+(n-1)d) = \frac{n}{2}[2a+(n-1)d]$$

Basic Step: For $n = 1$, LHS $= a$, RHS $= \frac{1}{2}[2a] = a$

$\Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true.

Induction Step: Let $T(k)$ hold true, *i.e.*,

$$a + (a+d) + (a+2d) + ... + (a+(k-1)d) = \frac{k}{2}[2a+(k-1)d]$$

Now to show that $T(k+1)$ holds true, we add the $(k+1)$th term, *i.e.*, $a + \{(k+1)-1\}d = a + kd$ to both the sides of $T(k)$, *i.e.*,

$$a + (a+d) + (a+2d) + ... + (a+(k-1)d) + (a+kd)$$

$$= \frac{k}{2}[2a+(k-1)d] + (a+kd) = ak + \frac{k(k-1)d}{2} + a + kd$$

$$= a(k+1) + \frac{1}{2}\{k(k-1)d + 2kd\} = (k+1)a + \frac{1}{2}\{k^2d + kd\}$$

$$= (k+1)a + \frac{1}{2}k(k+1)d = \frac{(k+1)}{2}[2a + \{(k+1)-1\}d].$$

Thus, $T(k+1)$ is true, whenever $T(k)$ is true.

Ex. 6. Using the method of mathematical induction, show that for all $n \in N$, $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{(1-r)}$, $r \neq 1$.

Sol. Let $T(n)$ be the statement:

$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{(1-r)}, \; r \neq 1$$

Basic Step: For $n = 1$, $\Rightarrow$ LHS $= a$, RHS $= \frac{a(1-r^1)}{1-r} = a.$

$\Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true.

Induction Step: Let the statement hold true for $n = k$, *i.e.*, let $T(k)$ be true, *i.e.*, $a + ar + ar^2 + ... + ar^{k-1} = \frac{a(1-r^k)}{1-r}$

Then to show $T(k+1)$ holds, add the $(k+1)$th term $= ar^{(k+1)-1} = ar^k$ to both the sides of $T(k)$, *i.e.*,

$$a + ar + ar^2 + ... + ar^{k-1} + ar^k = \frac{a(1-r^k)}{(1-r)} + ar^k$$

$$= \frac{a - ar^k + ar^k(1-r)}{(1-r)} = \frac{a - ar^k + ar^k - ar^{k+1}}{(1-r)} = \frac{a - ar^{k+1}}{(1-r)} = \frac{a(1-r^{k+1})}{(1-r)}$$

Thus, $T(k + 1)$ is true, whenever $T(k)$ holds true.

Ex. 7. Prove that for all +ve integral values of n, $\frac{1}{1.2} + \frac{1}{2.3} + \ldots.. + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Sol. Let $T(n)$ be the statement: $\frac{1}{1.2} + \frac{1}{2.3} + \ldots.. + \frac{1}{n(n+1)} = \frac{n}{n+1}$

***Basic Step*:** For $n = 1$, $\frac{1}{1\cdot 2} = \frac{1}{1+1} \Rightarrow T(1)$ is ture.

***Induction Step*:** Assume $T(k)$ is true, *i.e.*, $\frac{1}{1.2} + \frac{1}{2.3} + \ldots.. + \frac{1}{k(k+1)} = \frac{k}{k+1}$

To obtain $T(k + 1)$, add the $(k + 1)$th term, *i.e.*, $\frac{1}{(k+1)(k+2)}$ to both sides of $T(k)$. Then,

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots.. + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{\mathbf{(k+1)}}{\mathbf{(k+2)}}$$

Thus $T(k + 1)$ is true with the assumption that $T(k)$ is true. Hence the statement $T(n)$ holds for all positive integral values of n.

Ex. 8. Prove by the principle of induction that
$1.4.7 + 2.5.8 + 3.6.9 + \ldots + n(n+3)(n+6) = \frac{n}{4}(n+1)(n+6)(n+7)$.

Sol. Let $T(n)$ denote the given statement

$$1.4.7 + 2.5.8 + 3.6.9 + \ldots + n(n+3)(n+6) = \frac{n}{4}(n+1)(n+6)(n+7)$$

***Basic Step*:** For $n = 1$, LHS $= 1.4.7 = 28$

RHS $= \frac{1}{4}(1+1)(1+6)(1+7) = 28$

$\Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true.

***Induction Step*:** Assume $T(k)$ is true for all $k \in N$, *i.e.*,

$$1.4.7 + 2.5.8 + 3.6.9 + \ldots + k(k+3)(k+6) = \frac{k}{4}(k+1)(k+6)(k+7)$$

Now we shall show that $T(k + 1)$ is also true.

To obtain $T(k + 1)$ add the $(k + 1)$th term, *i.e.*, $(k+1)(k+1+3)(k+1+6) = (k+1)(k+4)(k+7)$ to both the sides of $T(k)$. Then,

$$1.4.7 + 2.5.8 + 3.6.9 + \ldots + k(k+3)(k+6) + (k+1)(k+4)(k+7)$$

$$= \frac{k}{4}(k+1)(k+6)(k+7) + (k+1)(k+4)(k+7) = (k+1)(k+7)\left[\frac{k}{4}(k+6)+(k+4)\right]$$

$$= \frac{(k+1)(k+7)}{4}[k^2 + 6k + 4k + 16] = \frac{(k+1)}{4}(k+7)(k^2+10k+16)$$

$$= \frac{(k+1)}{4}(k+7)(k+2)(k+8) = \frac{1}{4}(k+1)(k+2)(k+7)(k+8)$$

$\Rightarrow T(k + 1)$ is true, assuming $T(k)$ is true.

$\Rightarrow T(n)$ is true for all $n \in N$.

Ex. 9. Prove by the principle of mathematical induction that $n < 2^n$ for all $n \in N$.

Sol. Let the statement $T(n) = n < 2^n$.

Basic Step: For $n = 1$, $1 < 2^1 \Rightarrow T(1)$ is true.

Induction Step: Let $T(k)$ be true $\Rightarrow k < 2^k$ for all $k \in N$.

$k < 2^k \Rightarrow 2k < 2.2^k$

$\Rightarrow 2k < 2^{k+1} \quad \Rightarrow \quad (k + k) < 2^{k+1}$

$\Rightarrow (k + 1) \le (k + k) < 2^{k+1}$ $\qquad (\because k \in N \Rightarrow k \ge 1)$

$\Rightarrow (k + 1) < 2^{k+1}$

$\Rightarrow T(k + 1)$ is true, whenever $T(k)$ is true.

$\therefore$ $T(n)$ is true $\forall\, n \in N$.

Ex. 10. Prove by the principle of mathematical induction that $1 + 2 + 3 + ... + n < \dfrac{(2n+1)^2}{8}$.

Sol. Let $T(n)$ be the statement

$$1 + 2 + 3 + ... + n < \frac{(2n+1)^2}{8}$$

Basic Step: For $n = 1$, we have $\dfrac{(2n+1)^2}{8} \dfrac{(2 \times 1+1)^2}{8} = \dfrac{9}{8} > 1 \Rightarrow T(1)$ is true.

Induction Step: Let $T(k)$ be true. Then,

$$1 + 2 + ... + k < \frac{(2k+1)^2}{8}$$

Now we need to prove $T(k + 1)$ to be true. To obtain $T(k + 1)$ add the $(k + 1)$th term $= (k + 1)$ to both the sides of $T(k)$. Then,

$$1 + 2 + 3 + ... + k + (k + 1) < \frac{(2k+1)^2}{8} + (k + 1)$$

$$\Rightarrow \ 1 + 2 + 3 + ... + k + (k + 1) < \frac{4k^2 + 12k + 9}{8} = \frac{(2k+3)^2}{8}$$

$$\Rightarrow \ 1 + 2 + 3 + ... + k + (k + 1) < \frac{[2\,(k+1)+1]^2}{8}$$

Hence $T(k + 1)$ is true, whenever $T(k)$ is true $\Rightarrow T(n)$ is true for all $n \in N$.

Ex. 11. Using the principle of mathematical induction prove that $1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + n\underline{|n} = \underline{|n+1} - 1$ for all $n \in N$.

Sol. Let the statement $T(n)$ be

$$1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + n\underline{|n} = \underline{|n+1} - 1$$

Basic Step: For $n = 1$, LHS $= 1\underline{|1} = 1 \times 1 = 1$

RHS $= \underline{|1+1} - 1 = \underline{|2} - 1 = 2 - 1 = 1 \ \Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true.

Induction Step: Assume the statement $T(k)$ to be true for $n = k$, $k \in N$. Then,

$$1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + k\underline{|k} = \underline{|k+1} - 1$$

Now we need to prove $T(k + 1)$ to be true.

To obtain $T(k + 1)$, add the $(k + 1)$th term $= (k + 1)\ \underline{|k+1}$ to both sides of $T(k)$, *i.e.*,

$$\Rightarrow \ 1\underline{|1} + 2\underline{|2} + 3\underline{|3} + ... + k\underline{|k} + (k+1)\underline{|k+1} = \underline{|k+1} - 1 + (k + 1)\ \underline{|k+1}$$

$$= \underline{|k+1} + (k+1)\,\underline{|k+1} - 1$$
$$= \underline{|k+1}\,(1+k+1) - 1$$
$$= \underline{|k+1}\,(k+2) - 1$$
$$= \underline{|k+2} - 1 = \underline{|(k+1)+1} - 1$$

$\Rightarrow$ The result is true for $n = k + 1$

$\Rightarrow$ $T(k + 1)$ is true on the assumption that $T(k)$ is true.

$\Rightarrow$ $T(n)$ holds for all $n \in N$.

Ex. 12. If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$. **(*WBJEE 2008*)**

Sol. Let the statement $T(n)$ be: If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

Basic Step: For $n = 1$, $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^1 = \begin{bmatrix} 1+2\times1 & -4\times1 \\ 1 & 1-2\times1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow T(1) \text{ is true}$$

Induction Step: Assume $T(k)$ to be true, *i.e.*,

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

Now we need to show $T(k + 1)$ is true. $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

$$\therefore A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k+2k-1 \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}. \Rightarrow T(k+1) \text{ is true, whenever } T(k) \text{ is true.}$$

PRACTICE SHEET–1

1. Let $S(k) = 1 + 3 + 5 + ... + (2k - 1) = 3 + k^2$. Then, which of the following is true?
(*a*) $S(1)$ is correct
(*b*) $S(k) \Rightarrow S(k + 1)$
(*c*) $S(k) \not\Rightarrow S(k + 1)$
(*d*) Principle of mathematical induction can be used to prove the above formula. **(*AIEEE 2004*)**

2. If 'n' be any positive integer, then $n(n + 1)(2n + 1)$ is
(*a*) an odd integer (*b*) an integral multiple of 6
(*c*) a perfect square (*d*) None of these
(*EAMCET 2005*)

3. For all natural number n, $2 + 4 + 6 + ... + 2n$ equals
(*a*) $2(n + 1)$ (*b*) $\frac{1}{2} n(n+2)$
(*c*) $n(n + 1)$ (*d*) $(n + 2)(n + 4)$

4. For all $n \in N$, the sum of the series $1^2 + 2^2 + 3^2 + ... + n^2$ is equal to
(*a*) $\frac{n}{3}(n+1)(2n+1)$ (*b*) $\frac{n}{6}(n+1)(n+3)$
(*c*) $\frac{n}{3}(2n-1)(n+2)$ (*d*) $\frac{n}{6}(n+1)(2n+1)$

5. For all $n \in N$, the sum $1.3 + 3.5 + 5.7 + ... + (2n - 1)(2n + 1)$ equals
(*a*) $\frac{n(2n^2+3n+1)}{6}$ (*b*) $\frac{n(4n^2+6n-1)}{3}$
(*c*) $\frac{1}{6} n(n^2+4)$ (*d*) $\frac{1}{3} n^2(4n^2+5)$

ANSWERS

1. (*b*) **2.** (*b*) **3.** (*c*) **4.** (*d*) **5.** (*b*)

HINTS AND SOLUTIONS

1. $S(k) = 1 + 3 + 5 + ... + (2k - 1) = 3 + k^2$

Putting $k = 1$ on both the sides, we get

LHS = 1, RHS = 3 + 1 = 4 $\Rightarrow$ LHS $\neq$ RHS $\Rightarrow S(1)$ is not true.

Assume $S(k) = 1 + 3 + 5 + ... + (2k - 1) = 3 + k^2$ is true. Then,

To find $S(k + 1)$, add the $(k + 1)$th term $= (2(k + 1) - 1) = 2k + 1$ on both the sides of $S(k)$.

$\therefore S(k + 1) = 1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = 3 + k^2 + 2k + 1$

$\Rightarrow 1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = 3 + (k + 1)^2$

$\Rightarrow S(k + 1)$ is also true.

$\therefore S(k) \Rightarrow S(k + 1)$ is true.

2. When $n = 1$, $n(n + 1)(2n + 1) = (1)(2)(3) = 6$, which is an integral multiple of 6. It is neither an odd integer nor a perfect square. Using the principle of mathematical induction, we shall now show that the expression $n(n + 1)(2n + 1)$ is an integral multiple of 6 $\forall\, n \in N$. Assume $T(n) = n(n + 1)(2n + 1) = 6x$ where $x \in N$.

Basic Step: $T(1)$ is true as shown above.

Induction Step: Let $T(k)$ be true for all $k \in N$.

$\Rightarrow \quad k(k + 1)(2k + 1) = 6x$, where $x \in N$. .....(*i*)

For $T(k + 1)$, we replace k by $(k + 1)$ in the given expression, *i.e.*,

$$\begin{aligned} T(k + 1) &= (k + 1)(k + 2)(2(k + 1) + 1) \\ &= (k + 1)(k + 2)((2k + 1) + 2) \\ &= (k + 1)(k + 2)(2k + 1) + 2(k + 1)(k + 2) \\ &= k(k + 1)(2k + 1) + 2(k + 1)(2k + 1) + 2(k + 1)(k + 2) \\ &= k(k + 1)(2k + 1) + 2(k + 1)[(2k + 1) + (k + 2)] \\ &= 6x + 2(k + 1)(3k + 3) = 6x + 6(k + 1)^2 \\ &= 6(x + (k + 1)^2) \\ &= 6 \times \text{a positive integer} \end{aligned}$$

$\therefore T(k)$ is true $\Rightarrow T(k + 1)$ is true.

$\therefore T(n)$ is true for all $n \in N$.

3. Let $S_n = 2 + 4 + 6 + ... + 2n$

When $n = 1$, $S_n = 2$

Now from the options given, when $n = 1$,

$2(n+1) = 4,\ \frac{1}{2} n(n+2) = \frac{3}{2},\ n(n+1) = 2,\ (n+2)(n+4) = 15$

$\therefore \quad S_n \neq 2(n + 1),\ S_n \neq \frac{1}{2} n(n + 2),\ S_n \neq (n + 2)(n + 4)$ for $n = 1$

$S_n = n(n + 1)$ for $n = 1$

$\therefore$ We need to prove $2 + 4 + 6 + ... + 2n = n(n + 1)\ \forall\, n \in N$.

Let $T(n) = 2 + 4 + 6 + ... + 2n = n(n + 1)$

Basic Step: For $n = 1$, LHS $= 2 \times 1 = 2$, RHS $= 1 \times (1 + 1) = 2$

$\Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true.

Induction Step: Assume $T(k)$ is true, *i.e.*,

$2 + 4 + 6 + ... + 2k = k(k + 1)$

To obtain $T(k + 1)$, we add the $(k + 1)$th term, *i.e*, $2(k + 1)$ to both the sides of $T(k)$, *i.e.*,

$2 + 4 + 6 + ... + 2k + 2(k + 1)$

$= k(k + 1) + 2(k + 1)$

$= (k + 1)(k + 2) = (k + 1)((k + 1) + 1)$

Thus the statement $T(n)$ is true for $n = k + 1$, whenever it is true for $n = k$.

Therefore by the principle of mathematical induction it is true for all $n \in N$.

4. Let $S_n = 1^2 + 2^2 + 3^2 + ... + n^2$.

When $n = 1$, $S_n = 1^2 = 1$.

Now from the options given, **when $n = 1$,**

$\frac{n}{3}(n + 1)(2n + 1) = \frac{1}{3} \times 2 \times 3 = 2$

$\frac{n}{6}(n + 1)(n + 3) = \frac{1}{6} \times 2 \times 4 = \frac{4}{3}$

$\frac{n}{3}(2n - 1)(n + 2) = \frac{1}{3} \times 1 \times 3 = 1$

$\frac{n}{6}(n + 1)(2n + 1) = \frac{1}{6} \times 2 \times 3 = 1$

$\therefore S_n \neq \frac{n}{3}(n + 1)(2n + 1),\ S_n \neq \frac{n}{6}(n + 1)(n + 3)$ for $n = 1$

$S_n = \frac{n}{3}(2n - 1)(n + 2)$ and $S_n = \frac{n}{6}(n + 1)(2n + 1)$ for $n = 1$

When $n = 2$, $S_n = 1^2 + 2^2 = 5$

$\frac{n}{3}(2n - 1)(n + 2) = \frac{2}{3} \times 3 \times 4 = 8 \neq S_n$

$\frac{n}{6}(n + 1)(2n + 1) = \frac{2}{6} \times 3 \times 5 = 5$

$\therefore S_n = \frac{n}{6}(n + 1)(2n + 1)$ when $n = 2$.

Clearly $S_n = \frac{n}{6}(n + 1)(2n + 1)$ for $n = 1$, and $n = 2$.

$\therefore$ We shall show by mathematical induction that

$\mathbf{1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n}{6}(n + 1)(2n + 1)}$ **for all $n \in N$.**

Let $T(n) = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n}{6}(n + 1)(2n + 1)$

Basic Step: For $n = 1$, LHS $= 1^2 = 1$

RHS $= \frac{1}{6} \times 2 \times 3 = 1$

$\Rightarrow$ LHS = RHS $\Rightarrow T(1)$ is true.

Induction Step: Let $T(k)$ be true for all $k \in N$.

$\Rightarrow 1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k}{6}(k + 1)(2k + 1)$

To obtain $T(k + 1)$, add the kth term $= (k + 1)^2$ to both the sides of $T(k)$ *i.e.*,

$1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2$

$= \frac{k}{6}(k + 1)(2k + 1) + (k + 1)^2$

$= \frac{1}{6}(k + 1)[k(2k + 1) + 6(k + 1)]$

$= \frac{1}{6}(k + 1)[2k^2 + k + 6k + 6]$

$= \frac{1}{6}(k + 1)(2k^2 + 7k + 6)$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$

$= \frac{1}{6}(k + 1)(k + 2)[2(k + 1) + 1]$

$\Rightarrow T(k + 1)$ is true, whenever $T(k)$ is true, $k \in N$

$\Rightarrow T(n)$ is true for all $n \in N$.

5. Let $S_n = 1.3 + 3.5 + 5.7 + ... + (2n – 1)(2n + 1)$

When $n = 1$, $S_n = 1.3 = 3$

From the given options, when $n = 1$,

$\frac{n(2n^2 + 3n + 1)}{6} = \frac{1 \times (2 + 3 + 1)}{6} = 1 \neq S_n$

$\frac{n(4n^2 + 6n - 1)}{3} = \frac{1 \times (4 + 6 - 1)}{3} = \frac{9}{3} = 3 = S_n$

$\frac{n(n^2 + 4)}{6} = \frac{1 \times 5}{6} = \frac{5}{6} \neq S_n$

$\frac{n^2(4n^2 + 5)}{3} = \frac{1 \times (4 + 5)}{3} = \frac{9}{3} = 3 = S_n$

When $n = 2$, $S_n = 1.3 + 3.5 = 3 + 15 = 18$

$\frac{n(4n^2 + 6n - 1)}{3} = \frac{2(4 \times 4 + 12 - 1)}{3} = 18 = S_n$

$\frac{n^2(4n^2 + 5)}{3} = \frac{4 \times (4 \times 4 + 5)}{3} = \frac{4 \times 21}{3} = 28 \neq S_n.$

$\therefore$ **$S_n = \frac{n}{3}(4n^2 + 6n - 1)$ for $n = 1$ and $n = 2$**

Now we shall show that $S_n = \frac{n}{3}(4n^2 + 6n – 1)$ for all $n \in N$.

Using the principle of mathematical induction.

Let $T(n) = 1.3 + 3.5 + 5.7 + ... + (2n – 1)(2n + 1)$

$= \frac{n}{3}(4n^2 + 6n – 1)$

***Basic Step*:** For $n = 1$, LHS $= 1.3 = 3$

$$\text{RHS} = \frac{1 \times (4 + 6 - 1)}{3} = \frac{9}{3} = \mathbf{3.}$$

$\therefore$ LHS = RHS $\Rightarrow T(1)$ is true.

***Induction Step*:** Assume $T(n)$ to be true for $n = k$, $k \in N$

$\Rightarrow 1.3 + 3.5 + 5.7 + ... + (2k – 1)(2k + 1) = \frac{k}{3}(4k^2 + 6k – 1)$

To obtain $T(k + 1)$,

add the $(k + 1)$th term $= [(2(k + 1) – 1)(2(k + 1) + 1)]$

$= (2k + 1)(2k + 3)$ to both the sides of $T(k)$, *i.e.*,

$1.3 + 3.5 + 5.7 + ... + (2k – 1)(2k + 1) + (2(k + 1) – 1)(2(k + 1) + 1)$

$= \frac{1}{3}k(4k^2 + 6k – 1) + (2k + 1)(2k + 3)$

$= \frac{1}{3}k(4k^2 + 6k – 1) + (4k^2 + 8k + 3)$

$= \frac{1}{3}\{4k^3 + 6k^2 – k + 12k^2 + 24k + 9\}$

$= \frac{1}{3}\{4k^3 + 18k^2 + 23k + 9\}$

$= \frac{1}{3}(k + 1)(4k^2 + 14k + 9)$

$= \frac{1}{3}(k + 1)\{4(k + 1)^2 + 6(k + 1) – 1\}$

$\therefore T(k + 1)$ is true, whenever $T(k)$ is true.

Hence, by the principle of mathematical induction, $T(n)$ is true for all $n \in N$.

$\Rightarrow 1.3 + 3.5 + 5.7 + ... + (2n – 1)(2n + 1)$

$= \frac{1}{3}n(4n^2 + 6n – 1)$ is true $\forall n \in N$.

Type II: Proving Divisibility

Ex. 1. Prove that $x^n – y^n$ is divisible by $x – y$, when n is a + ve integer.

Sol. Let $T(n)$ be the statement: $x^n – y^n$ is divisible by $x – y$.

***Basic Step*:** For $n = 1$, $x^1 – y^1 = x – y$ is divisible by $(x – y) \Rightarrow T(1)$ is true

***Induction Step*:** Assume that $T(k)$ is true, *i.e.*, for $k \in N$

$x^k – y^k$ is divisible by $(x – y)$

Now, we prove $T(k + 1)$ is true.

$x^{k+1} – y^{k+1} = x^k . x – y^k . y = x^k . x – x^k . y + x^k .y – y^k . y$ (Adding and subtracting $x^k.y$)

$= x^k(x – y) + y(x^k – y^k)$

Since $x^k(x – y)$ is divisible by $(x – y)$ and $(x^k – y^k)$ is divisible by $(x – y)$ (By induction step, *i.e.*, assuming $T(k)$ is true), therefore,

$x^{k+1} - y^{k+1} = x^k (x - y) + y (x^k - y^k)$ is divisible by $(x - y)$

$\Rightarrow T(k + 1)$ is true, whenever $T(k)$ is true.

$\Rightarrow T(n)$ holds for all positive integral values of n.

Ex. 2. Prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for any positive integer *n*.

Sol. Let $T(n)$ be the statement: $3^{2n+2} - 8n - 9$ is divisible by 64.

Basic Step: For $n = 1$, $3^{2 \times 1+2} - 8 \times 1 - 9 = 81 - 17 = 64$ which is divisible by 64.

$\Rightarrow T(1)$ holds.

Induction Step: Let $T(k)$, $k \in N$ hold, *i.e.*,

$3^{2k+2} - 8k - 9$ is divisible by 64.

Then, $T(k + 1) = 3^{2(k+1)+2} - 8(k + 1) - 9 = 3^2 . 3^{2k+2} - 8k - 17$

$= 9 (3^{2k+2} - 8k - 9) + 64k + 64 = 9. T(k) + 64 (k + 1)$

$\Rightarrow T(k + 1)$ is divisible by 64, whenever $T(k)$ is divisible by 64.

$\Rightarrow T(n)$ is true for every natural number n.

Ex. 3. Use the principle of mathematical induction to prove, for all $n \in N$, $10^{2n-1} + 1$ is divisible by 11.

Sol. Let the given statement $T(n) = 10^{2n-1} + 1$ be a multiple of 11

$\Rightarrow \quad 10^{2n-1} + 1 = M(11)$

Basic Step: For $n = 1$, $10^{2 \times 1 - 1} + 1 = 10 + 1 = 11$ which is divisible by 11.

Induction Step: Assume that $T(k) = 10^{2k-1} + 1$ is divisible by 11.

$\Rightarrow \quad 10^{2k-1} + 1 = M(11) \; \forall \; n \in N$ .....(*i*)

Then, we now show that $T(k + 1)$ is true.

$T(k + 1) = 10^{2(k+1)-1} + 1 = 10^{2k-1+2} + 1 = 10^2 . 10^{2k-1} + 1$

$= 100 (M(11) - 1) + 1$ (From (*i*))

$= 100 . M(11) - 100 + 1 = 100 . M(11) - 99$

$\Rightarrow T(k + 1)$ is divisible by 11, when $T(k)$ is divisible by 11.

$\Rightarrow T(n)$ holds true for all $n \in N$.

Ex. 4. If *n* is an integer, $n \geq 1$, then show that $3^{2^n} - 1$ is divisible by 2^{n+2}. ***(IIT)***

Sol. Let $T(n)$ be the statement: $3^{2^n} - 1$ is divisible by 2^{n+2}

Basic Step: For $n = 1$,

$3^{2^1} - 1 = 8$ and $2^{n+2} = 8 \Rightarrow T(1)$ is true

Induction Step: Assume $T(k)$ to be true, *i.e.*,

$T(k) = 3^{2^k} - 1$ is divisible by 2^{k+2}

$= 3^{2^k} - 1 = m. 2^{k+2}$ when $m \in N$ ...(*i*)

$= 3^{2^k} = m. 2^{k+2} + 1$

Now we need to prove that $T(k + 1)$ holds true.

$\therefore \; 3^{2^{k+1}} - 1 = 3^{2^k . 2} - 1 = (m . 2^{k+2} + 1)^2 - 1$ (using (*i*))

$= m^2 (2^{k+2})^2 + 2m . 2^{k+2} + 1 - 1 = 2^{k+2} (m^2 . 2^{k+2} + 2m)$

$\Rightarrow T(k + 1) = 3^{2^{k+1}} - 1$ is divisible by 2^{k+2}, whenever $T(k)$ holds.

Thus $3^{2^n} - 1$ is divisible by 2^{n+2} for all integers $n \geq 1$.

PRACTICE SHEET–2

1. For all $n \in N$, $(2^{3n} - 1)$ will be divisible by

(*a*) 25 (*b*) 8 (*c*) 7 (*d*) 3

(*WBJEE 2010*)

2. If n is a positive integer, then $n^3 + 2n$ is divisible by

(*a*) 2 (*b*) 6 (*c*) 15 (*d*) 3

(*Karnataka CET 2009*)

3. For each $n \in N$, $49^n + 16n - 1$ is divisible by

(*a*) 3 (*b*) 29 (*c*) 19 (*d*) 64

(*BCECE 2009*)

4. If n is a positive integer, then $5^{2n+2} - 24n - 25$ is divisible by

(*a*) 574 (*b*) 576 (*c*) 675 (*d*) 575

(*Kerala CEE 2009*)

5. For all integers $n \geq 1$, which of the following is divisible by 9?

(*a*) $8^n + 1$ (*b*) $10^n + 1$

(*c*) $4^n - 3n + 1$ (*d*) $3^{2n} + 3n + 1$

(*EAMCET 2006*)

6. For all $n \in N$, $2^{3n} - 7n - 1$ is divisible by

(*a*) 64 (*b*) 36 (*c*) 49 (*d*) 25

(*AIEEE 2006*)

7. $10^n + 3(4^{n+2}) + 5$ is divisible by (for all $n \in N$)

(*a*) 5 (*b*) 7 (*c*) 9 (*d*) 13

(*Kerala PET 2005*)

8. For all natural numbers n, the expression $2.7^n + 3.5^n - 5$ is divisible by

(*a*) 16 (*b*) 24 (*c*) 20 (*d*) 21

(*IIT 1985*)

ANSWERS

1. (*c*) **2.** (*d*) **3.** (*d*) **4.** (*b*) **5.** (*c*) **6.** (*c*) **7.** (*c*) **8.** (*b*)

HINTS AND SOLUTIONS

1. For $n = 1$, $2^{3n} - 1 = 2^3 - 1 = 8 - 1 = 7$, which is divisible by 7, and not divisible by any other alternative given.

$\therefore$ We shall prove $2^{3n} - 1$ divisible by 7 for all $n \in N$.

Let $T(n) = 2^{3n} - 1$ is divisible by 7.

Basic Step: For $n = 1$, $T(1) = 2^3 - 1 = 8 - 1 = 7$ is divisible by 7 is true.

Induction Step: Assume $T(k)$ to be true, *i.e.*,

$T(k) = 2^{3k} - 1$ is divisible by 7

$\Rightarrow 2^{3k} - 1 = 7m, m \in N$

$\Rightarrow 2^{3k} = 7m + 1$ ...(*i*)

Now $2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^3 \,.\, 2^{3k} - 1 = 8.2^{3k} - 1$ ...(From (*i*))

$= 8.(7m + 1) - 1 = 56m + 7 = 7(8m + 1)$

$\Rightarrow 2^{3(k+1)} - 1$ is divisible by 7

$\therefore T(k + 1)$ is true whenever $T(k)$ is true.

$\Rightarrow$ **$2^{3n} - 1$ is divisible by 7 for all $n \in N$.**

2. For $n = 1$, $n^3 + 2n = 1 + 2 = 3$ which is divisible by 3 and none of the other given alternatives.

$\therefore$ We shall prove $n^3 + 2n$ divisible by 3 for all $n \in N$.

Let $T(n) = n^3 + 2n$ is divisible by 3.

Basic Step: For $n = 1$, $T(1) = n^3 + 2n = 1 + 2 = 3$ is divisible by 3 is true.

Induction Step: Assume $T(k)$ to be true, *i.e.*,

$T(k) = k^3 + 2k$ is divisible by 3

$= k^3 + 2k = 3m$, where $m \in N$. ...(*i*)

Now we need to prove that $T(k + 1)$ holds true, *i.e.*, $(k + 1)^3 + 2(k + 1)$ is divisible by 3.

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + (3k^2 + 3k + 3)$

$= 3m + 3(k^2 + k + 1)$ (From (*i*))

$\Rightarrow T(k + 1) = (k + 1)^3 + 2(k + 1)$ is divisible by 3, whenever $T(k) = k^3 + 2k$ is divisible by 3.

$\Rightarrow$ **$n^3 + 2n$ is divisible by 3 $\forall$ $n \in N$.**

3. For $n = 1$, $49^1 + 16 \times 1 - 1 = 49 + 15 = 64$

$\therefore$ For $n = 1$, $49^n + 16n - 1$ is divisible by 64 and not by any of the other given alternatives.

$\therefore$ We shall prove using mathematical induction, that $49^n + 16n - 1$ is divisible by 64 $\forall$ $n \in N$.

Let $T(n)$ be the statement: $49^n + 16n - 1$ is divisible by 64

Basic Step: For $n = 1$, $T(1)$ is divisible by 64 as proved above.

Induction Step: Assume $T(k)$ to be true *i.e.*,

$T(k) = 49^k + 16k - 1$ is divisible by 64, *i.e.*,

$49^k + 16k - 1 = 64m, m \in N.$ ...(*i*)

$\therefore \quad T(k + 1) = 49^{k+1} + 16(k + 1) - 1$

$= 49.\, 49^k + 16k + 16 - 1$

$= 49.\, 49^k + 16k + 15$

$= 49(49^k + 16k - 1) - 48(16k) + 64$

$= 49(64m) - 12(64k) + 64$

$= 64(49m - 12k + 1)$

$\Rightarrow 49^{k+1} + 16(k + 1) - 1$ is divisible by 64.

$\Rightarrow T(k + 1)$ is true whenever $T(k)$ is true.

$\Rightarrow$ **$49^n + 16n - 1$ is divisible by 64 $\forall$ $n \in N$.**

4. For $n = 1$, $5^{2n+2} - 24n - 25 = 5^4 - 24 - 25 = 625 - 49 = 576$ which is divisible by 576 and none of the other given alternative.

$\therefore$ To prove: $5^{2n+2} - 24n - 25$ is divisible by 576 using mathematical induction.

Let $T(n)$ be the statement: $5^{2n+2} - 24n - 25$ is divisible by 576 $\forall\ n \in N$.

Basic Step: For $n = 1$, $T(1) = 5^4 - 24 - 25 = 576$ which is divisible by 576.

$\Rightarrow T(1)$ is true.

Induction Step: Assume $T(k)$ where $n = k$, $k \in N$ to be true *i.e.*,

$T(k) = 5^{2k+2} - 24k - 25$ is divisible by 576 is true, *i.e.*,

$5^{2k+2} - 24k - 25 = 576m,\ m \in N$ ....(*i*)

$\therefore\ T(k+1) = 5^{2(k+1)+2} - 24(k+1) - 25$

$= 5^{2k+2} . 25 - 24k - 24 - 25$

$= 5^{2k+2} . 25 - 24k - 49$

$= 25(5^{2k+2} - 24k - 25) + 24.(24k) + 576$

$= 25.(576m) + 576k + 576$ (From (*i*))

$= 576(25m + k + 1)$

$\Rightarrow 2^{2(k+1)+2} - 24(k+1) - 25$ is divisible by 576

$\Rightarrow T(k+1)$ is true, whenever $T(k)$ is true.

$\Rightarrow$ **$5^{2n+2} - 24k - 25$ is divisible by 576 $\forall\ n \in N$.**

5. For $n = 1$,

$8^n + 1 = 8^1 + 1 = 9$ divisible by 9

$10^n + 1 = 10^1 + 1 = 11$ **not** divisible by 9

$4^n - 3n - 1 = 4 - 3 - 1 = 0$ divisible by 9

$3^{2n} + 3n + 1 = 13$ **not** divisible by 9

For $n = 2$

$8^n + 1 = 8^2 + 1 = 65$ not divisible by 9

$4^n - 3n - 1 = 4^2 - 3 \times 2 - 1 = 16 - 6 - 1 = 9$ divisible by 9

$\therefore$ We need to prove $4^n - 3n - 1$ to be divisible by 9 $\forall\ n \in N$. using mathematical induction.

Let $T(n)$: $4^n - 3n - 1$ is divisible by 9,

Basic Step: $T(1) = 0$ which is divisible by 9 $\Rightarrow T(1)$ is true.

Induction Step: Assume $T(k)$ to be true, *i.e.*,

$4^k - 3k - 1$ is divisible by 9 $k \in N$

$\Rightarrow 4^k - 3k - 1 = 9m,\ m \in N$ ...(*i*)

$\therefore\ 4^{k+1} - 3(k+1) - 1 = 4.4^k - 3k - 3 - 1 = 4.4^k - 3k - 4$

$= 4(4^k - 3k - 1) + 9k = 4.9m + 9k = 9(4m + k)$

$\Rightarrow 4^{k+1} - 3(k+1) - 1$ is divisible by 9

$\Rightarrow T(k+1)$ is true whenever $T(k)$ is true, $k \in N$

$\Rightarrow$ **$4^n + 3n - 1$ is divisible by 9 $\forall\ n \in N$.**

6. For $n = 1$, $2^{3n} - 7n - 1 = 2^3 - 7 - 1 = 8 - 8 = 0$ which is divisible by all the given alternatives.

For $n = 2$, $2^{3n} - 7n - 1 = 2^6 - 7 \times 2 - 1 = 64 - 14 - 1 = 49$, which is divisible by only 49 out of the given alternatives.

$\therefore$ We need to prove $2^{3n} - 7n - 1$ is divisible by 49 $\forall\ n \in N$.

Let $T(n)$ be the statement: $2^{3n} - 7n - 1$ is divisible by 49

Basic Step: For $n = 1$, $2^{3n} - 7n - 1 = 0$, divisible by 49

$\Rightarrow T(1)$ is true.

Induction Step: Assume $T(k)$ is true $\forall\ k \in N$, *i.e.*,

$2^{3k} - 7k - 1$ is divisible by 49, *i.e.*,

$2^{3k} - 7k - 1 = 49m,\ m \in N$

Now $2^{3(k+1)} - 7(k+1) - 1 = 2^{3k} . 2^3 - 7k - 7 - 1$

$= 8.2^{3k} - 7k - 8 = 8(.2^{3k} - 7k - 1) + 49k$

$= 8.\ 49m + 49k = 49(8m + k)$

$\Rightarrow 2^{3(k+1)} - 7(k+1) - 1$ is divisible by 49

$\Rightarrow T(k+1)$ is true whenever $T(k)$ is true

$\Rightarrow$ **$2^{3n} - 7n - 1$ is divisible by 49 $\forall\ n \in N$.**

7. For $n = 1$, $10^n + 3(4^{n+2}) + 5 = 10 + 3 \times 4^3 + 5$

$= 10 + 192 + 5 = 207$

which is divisible by only 9 and none of the other given alternatives.

$\therefore$ We need to prove $10^n + 3(4^{n+2}) + 5$ is divisible by 9 $\forall\ n \in N$.

Let $T(n)$ be the statement $10^n + 3(4^{n+2}) + 5$ is divisible by 9.

Basic Step: For $n = 1$, $T(1)$ holds true as prove above.

Induction Step: Assume $T(k)$ to be true, $k \in N$ *i.e.*,

$10^k + 3(4^{k+2}) + 5$ is divisible by 9, *i.e.*,

$10^k + 3(4^{k+2}) + 5 = 9m,\ m \in N$ ....(*i*)

Now, $10^{k+1} + 3(4^{k+1+2}) + 5$

$= 10^{k+1} + 3(4^{k+3}) + 5$

$= 10.\ 10^k + 12.\ 4^{k+2} + 5$

$= 4(10^k + 3(4^{k+2}) + 5) + 6\ .\ 10^k - 15$

$= 4.(9m) + 6(10^k - 1) - 9$

$= 4.(9m) + 6.(9x) - 9$ ($\because\ 10^k - 1$ is always divisible by 9)

$= 9(4m + 6x - 1)$

$\Rightarrow 10^{k+1} + 3(4^{(k+1)+2}) + 5$ is divisible by 9.

$\Rightarrow T(k+1)$ is true whenever $T(k)$ is true, $\forall\ k \in N$

$\Rightarrow$ **$10^n - 3(4^{n+2}) + 5$ is divisible by 9 $\forall\ k \in N$.**

8. When $n = 1$, $2.7^n + 3.5^n - 5 = 2.7 + 3.5 - 5 = 24$ which is divisible by 24 and none of the other given alternatives.

$\therefore$ We need to prove $2.7^n + 3.5^n - 5$ is divisible by 24 $\forall$ $n \in N$.

Let $T(n)$ be the statement $2.7^n + 3.5^n - 5$ is divisible by 24.

$T(1)$ holds true as shown above.

Assume $T(k)$ to be true, *i.e.*, $2.7^k + 3.5^k - 5$ is divisible by 24, *i.e.*,

$2.7^k + 3.5^k - 5 = 24m,\ m \in N$ ....(*i*)

Now

$2.7^{k+1} + 3.5^{k+1} - 5 = 2.7.7^k + 3.5.5^k - 5$

$= (2.7^k + 3.5^k - 5) + 12(7^k) + 12(5^k)$

$= 24m + 12(7^k + 5^k)$

[Now 7^k and 5^k, $k \in N$ being both odd, their sum is even.
Let $7^k + 5^k = 2x,\ x \in N$]

$= 24m + 12(2x)$; $m, x \in N = 24(m + x)$

$\Rightarrow 2.7^{k+1} + 3.5^{k+1} - 5$ is divisible by 24

$\Rightarrow T(k+1)$ is true whenever $T(k)$ is true, $k \in N$.

$\Rightarrow$ **$2.7^n + 3.5^n - 5$ is divisible by 24 for all $n \in N$.**