NOTES

M A T H E M A T I C S

FUNDAMENTALS OF MATHEMATICS

INTRODUCTION TO INEQUALITIES

What you already know

- Number Theory
- Sets

What you will learn

- Representation of inequalities
- Properties of inequalities
- Solving Linear inequalities

Whenever there is a comparison, there exists an **inequality**.
Example - Roshan is taller than Michael can be written mathematically as Height of Roshan > Height of Michael.

Symbols

1. **Strict inequality:** The relations when a is not equal to b, are known as strict inequalities

(i) less than (<)

Example $1 < 3$

Strict Inequality

- "Less Than" (<)
  $a < b$ means "a is strictly less than b"
- "Greater Than" (>)
  a > b means "a is strictly greater than b"

(ii) greater than (>)

Example $-2 > -5$

Note: Quantity towards which the mouth of the inequality is opening, is larger than the other.

2. Non-strict inequality

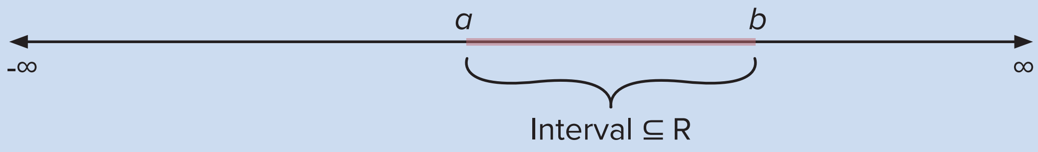
(i) The notation $a \le b$ means a is less than or equal to b or a is at most b.

(ii) The notation $a \ge b$ means a is greater than or equal to b or a is at least b.

"Less than or Equal to" (≤)	"Greater than or Equal to" (≥)
$a \le b$ means "a is less than or equal to b" or a is atmost b or a is not greater than b	$a \ge b$ means "a is greater than or equal to b" or a is atleast b or a is not less than b

Intervals as subsets of R

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in R$ such that $a < b$.



We can define four types of intervals as follows:

Name	Representation	Description
Open Interval	$(a, b) = \{x : a < x < b\}$ (a, b) 	$\{x : a < x < b\}$ i.e. both end points are not included.
Closed Interval	$[a, b] = \{x : a \le x \le b\}$ $[a, b]$ 	$\{x : a \le x \le b\}$ i.e. both end points are also included. This is possible only when both a and b are finite.
Open - Closed Interval	$x \in (a, b] = \{x : a < x \le b\}$ $(a, b]$ 	$\{x : a < x \le b\}$ i.e. a is excluded and b is included
Closed - Open Interval	$x \in [a, b) = \{x : a \le x < b\}$ $[a, b)$ 	$\{x : a \le x < b\}$ i.e. a is included and b is excluded.

Quick Query

Represent the following:

(a) Real numbers strictly greater than -2

(b) Set of real numbers less than or equal to 0

(c) Real numbers greater than 100 and less than 1000

Any real number is either positive or negative or zero.
Between two real numbers, there are infinitely many real numbers.

Properties of Inequality

Property – I (Converse)

We can swap a and b over if we make sure the symbol still "points at" the smaller value.

- If $a > b$ then $b < a$
- If $a < b$ then $b > a$

Example: Alex is older than Billy, so Billy is younger than Alex.

Property – II (Transitivity)

When we link up inequalities in order, we can "jump over" the middle inequality.

If $a > b$ and $b > c$, then $a > c$

Example:

If Alex is older than Billy and Billy is older than Carol, then Alex must be older than Carol.

Property – III (Addition/Subtraction)

If $a \leq b \Rightarrow a + c \leq b + c$. Also, if $a \leq b \Rightarrow a - c \leq b - c$. The inequality relation is preserved under addition or subtraction of a common real constant.

Example: If $x \leq y$, then $x + 10 \leq y + 10$ and $x - 10 \leq y - 10$, both of these inequalities will be preserved.

Property – IV (Multiplication/Division)

Case I ($c > 0$): $a \geq b \Rightarrow ac \geq bc$ and $a \geq b \Rightarrow (\frac{a}{c}) \geq (\frac{b}{c})$

Example: $(10 > 0) : 5 > 4 \Rightarrow 10 \times 5 > 10 \times 4$ and $\frac{5}{10} > \frac{4}{10}$

Case II ($c < 0$): If $a \geq b \Rightarrow ac \leq bc$ and $a \geq b \Rightarrow (\frac{a}{c}) \leq (\frac{b}{c})$

Example: $(-5 < 0) : 2 > 1 \Rightarrow -5 \times 2 < -5 \times 1$ and $\frac{-2}{5} < \frac{-1}{5}$

The inequality relation is preserved under multiplication or division by a positive constant, but is reversed when a negative constant is involved.

Solve

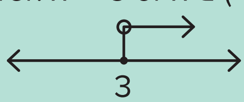
1. Solve $\frac{(6 - 2x)}{3} < 4$

Step 1:
Use Property IV (Multiplication/Division):
$\frac{(6 - 2x)}{3} < 4$
Multiplying by 3
$(6 - 2x) < 12$
Dividing both sides by 2
$(3 - x) < 6$

Step 2:
Use Property – III (Addition/Subtraction)
Subtracting both sides by 3
$3 - 3 - x < 6 - 3$

$\Rightarrow -x < 3$

Step 3:
Again use Property 4 (Multiplication/Division)
Multiplying both sides from -1, and reversing the inequality relation, $x > -3$
Answer: $x > -3$ or $x \in (-3 , \infty)$



Concept Check

1. Solve $7x + 5 \le 5x + 15$ when (a) $x \in N$, (b) $x \in W$, (c) $x \in Z$, (d) $x \in R$.

Solve

2. If $a = \frac{(x+3)}{4}$ and $b = \frac{(2x+1)}{3}$ and $b < \frac{7}{3} < 2a$, then, find the range of values of x.

Step 1:

Solve $b < \frac{7}{3}$

$\frac{2x+1}{3} < \frac{7}{3}$

Multiplying by 3 on both sides

$2x + 1 < 7$

$\Rightarrow 2x < 6$

$\Rightarrow x < 3 = A$

Step 2:

Solve $\frac{7}{3} < 2a$

$\Rightarrow \frac{7}{3} < \frac{x+3}{2}$

$\Rightarrow 14 < 3x + 9$

$\Rightarrow 3x > 5$

$\Rightarrow B = x > \frac{5}{3}$

Step 3:

Taking $A \cap B$

$A \cap B = \{ x \in R; \frac{5}{3} < x < 3 \}$

Property – V (Reciprocal [only for non-zero reals])

When a and b are both positive or both negative, taking reciprocal changes the direction of the inequality relation.

When a and b are of opposite signs, taking reciprocal retains the inequality relation as it is.

I. $0 < a \le b \Rightarrow (\frac{1}{a}) \ge (\frac{1}{b}) > 0$, for example, $0 < 5 < 7 \Rightarrow (\frac{1}{5}) > (\frac{1}{7}) > 0$

II. $a \le b < 0 \Rightarrow 0 > (\frac{1}{a}) \ge (\frac{1}{b})$, for example, $-7 < -5 < 0 \Rightarrow 0 > \frac{-1}{7} > \frac{-1}{5}$

III. $a < 0 < b \Rightarrow (\frac{1}{a}) < 0 < (\frac{1}{b})$. for example, $-7 < 0 < 5 \Rightarrow \frac{-1}{7} < 0 < \frac{1}{5}$

Example: Alex and Billy both complete a journey of 1 kilometre.

Alex runs at 6 km/h and Billy walks at 4 km/h.

Alex's speed is greater than Billy's speed: $6 > 4$

But time taken by Alex is less than Billy: $\frac{1}{6} < \frac{1}{4}$

10 minutes < 15 minutes

Property – VI (Squaring)

When a and b are both positive, squaring retains the inequality relation.

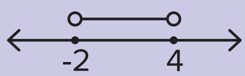
When a and b are both negative, squaring changes the direction of the inequality relation.

When a and b are of opposite signs, nothing can be concluded about the direction of the inequality relation, after squaring.

I. If $a, b > 0$; $a \le b \Rightarrow a^2 \le b^2$, for example, $7, 5 > 0$; $5 \le 7 \Rightarrow 25 \le 49$

II. If $a, b < 0$; $a \le b \Rightarrow a^2 \ge b^2$, for example, $-7, -5 < 0$; $-7 \le -5 \Rightarrow 49 \ge 25$

III. If $a < 0 < b$ then nothing can be concluded, for example, $-7 < 5$ and $(-7)^2 > 5^2$. But $-2 < 5$ and $(-2)^2 < (5)^2$.



Solve

Solve for x : $\sqrt{(x-4)} > 5$

Step 1:
Check for feasible region = A
$\sqrt{(x-4)}$ exists iff
$x - 4 \geq 0$
i.e. $x \geq 4$
$\Rightarrow x \in [4, \infty) = A$

Step 2:
Use Squaring property
If a, b > 0; $a \leq b \Rightarrow a^2 \leq b^2$
$\sqrt{(x-4)} > 5 \Rightarrow x - 4 > 25$
$x > 29$
$x \in (29, \infty) = B$

Step 3:
Taking Intersection of A and B
$A \cap B = [4, \infty) \cap (29, \infty)$
$\therefore$ Solution set = $A \cap B = (29, \infty)$

Property – VII

(A) If $ab > 0$ then following cases are possible
Case i - both numbers are positive: $a > 0$ and $b > 0$
Case ii - both numbers are negative: $a < 0$ and $b < 0$
The union of the above two cases is our final answer
Example: $(8 \times 6) > 0$ and $(-8) \times (-6) > 0$

(B) If $ab < 0$ then both numbers should be of opposite sign
Case i - $a > 0$ and $b < 0$
Case ii - $a < 0$ and $b > 0$
The union of the above two cases is our final answer
Example: $(-8) \times 6 < 0$ and $8 \times (-6) < 0$

Solve

1. Solve $(x + 2)(x - 4) < 0$

Step 1:
Case 1 - $a > 0$ and $b < 0$
$(x + 2) > 0$ and $(x - 4) < 0$
$\Rightarrow x > -2$ and $x < 4$
Taking Intersection, as both inequalities need to be simultaneously satisfied
$-2 < x < 4$

Step 2:
Case 2 - $a < 0$ and $b > 0$
$(x + 2) < 0$ and $(x - 4) > 0$
$\Rightarrow x < -2$ and $x > 4$
Taking Intersection, as both inequalities need to be simultaneously satisfied
$x \in \{\ \}$

Step 3:
Taking union of Step 1 and Step 2
$x = (-2,4) \cup \{\ \} = (-2, 4)$
Hence $x \in (-2, 4)$

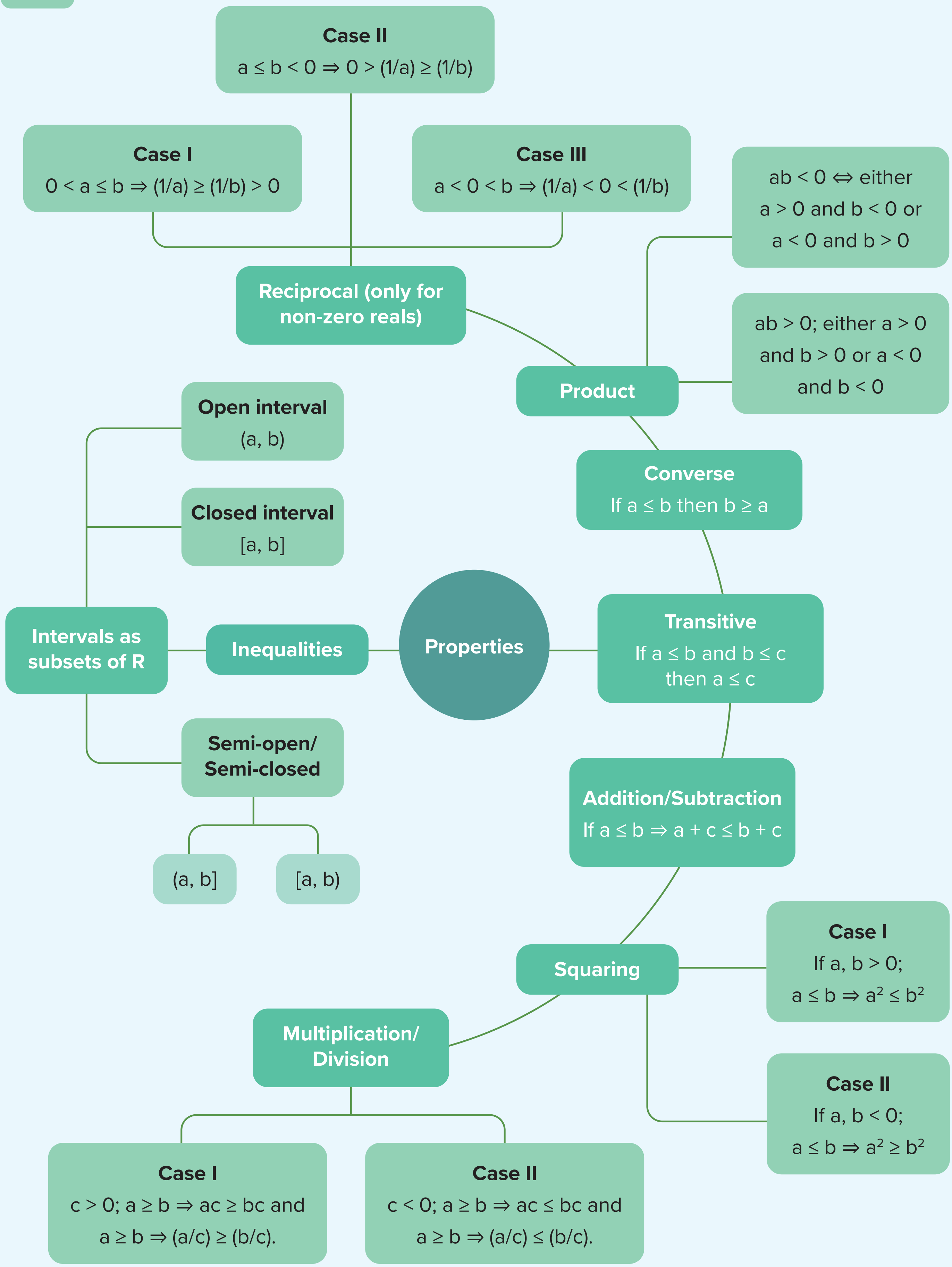
1. $\dfrac{\text{Any real number}}{0}$ = undefined or $\notin$ R.
2. Never cancel the terms involving variables.
3. In an inequality, cross multiplication of terms involving variables is not allowed.

Summary

Key takeaways

1. Intervals are subsets of R and are commonly used in solving inequalities.
2. The inequality relation is preserved under addition or subtraction of a common real constant.
3. The inequality relation is preserved under multiplication or division by a positive constant, but is reversed when a negative constant is involved.
4. Taking the reciprocal of both a and b can change the direction of the inequality.
5. The inequality relation is preserved under squaring for positive a and b, is reversed for negative a and b and remains inconclusive for a and b with different signs.
6. Product of a and b is positive, only when both a and b have the same sign. Similarly, the product of a and b is negative, when a and b are of different signs.

Mind map



Case II
$a \le b < 0 \Rightarrow 0 > (1/a) \ge (1/b)$
Case I
$0 < a \le b \Rightarrow (1/a) \ge (1/b) > 0$
Case III
$a < 0 < b \Rightarrow (1/a) < 0 < (1/b)$
$ab < 0 \Leftrightarrow$ either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$
Reciprocal (only for non-zero reals)
$ab > 0$; either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$
Product
Open interval
(a, b)
Closed interval
[a, b]
Converse
If $a \le b$ then $b \ge a$
Intervals as subsets of R
Inequalities
Properties
Transitive
If $a \le b$ and $b \le c$ then $a \le c$
Semi-open/ Semi-closed
(a, b]
[a, b)
Addition/Subtraction
If $a \le b \Rightarrow a + c \le b + c$
Squaring
Case I
If $a, b > 0$; $a \le b \Rightarrow a^2 \le b^2$
Multiplication/ Division
Case II
If $a, b < 0$; $a \le b \Rightarrow a^2 \ge b^2$
Case I
$c > 0$; $a \ge b \Rightarrow ac \ge bc$ and $a \ge b \Rightarrow (a/c) \ge (b/c)$.
Case II
$c < 0$; $a \ge b \Rightarrow ac \le bc$ and $a \ge b \Rightarrow (a/c) \le (b/c)$.

Self-Assessment

1. If $\frac{x - 1}{x} \geq 2$ then solve for x.
2. Solve the following system of inequalities: $\frac{(5 - 9x)}{(3x - 1)} \geq 0$, $x + 2 > 0$
3. Find the maximum value of m, if $mx + 2 < 1 - 4m$, $m \in Z$, and $x = 2$ is in the solution set of inequality.

Answers

Quick Query

1. (a) $x \in (-2, \infty)$ (b) $x \in (-\infty, 0]$ (c) $x \in (100, 1000)$

Concept Check

1. Solve $7x + 5 \leq 5x + 15$ when (a) $x \in N$, (b) $x \in W$, (c) $x \in Z$, (d) $x \in R$.

$7x + 5 \leq 5x + 15$

$\Rightarrow 2x \leq 10 \Rightarrow x \leq 5$

(a) $x \in N$ (Set of Natural Numbers)
$x \in \{1, 2, 3, 4, 5\}$

(b) $x \in W$ (Set of Whole Numbers)
$x \in \{0, 1, 2, 3, 4, 5\}$

(c) $x \in Z$ (Set of Integers)
$x \in \{..., -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

(d) $x \in R$ (Set of Real numbers)
$x \in \{x \in R; x \leq 5\} = (-\infty, 5]$

Self-Assessment

1.

Step 1:
Subtracting 2 from both sides

$\Rightarrow \frac{(x - 1)}{x} - 2 \geq 0$

$\Rightarrow \frac{-(x + 1)}{x} \geq 0$

$\Rightarrow \frac{(x + 1)}{x} \leq 0$

Step 2:
Solving for Case 1, $a \geq 0$ $b < 0$

$x + 1 \geq 0$ and $x < 0$

$\Rightarrow x \in [-1, 0)$

Step 3:
Solving for case 2, $a \leq 0$ and $b > 0$

$x + 1 \leq 0$ and $x > 0$

$\Rightarrow x \in \phi$

The final solution set is the union of the above two cases,
that is $x \in [-1, 0)$

2.

Step 1:
Solve I = $x + 2 > 0$
$\Rightarrow x > -2$
Solution Set of I = $(-2, \infty)$

Step 2:
Solve II = $\frac{(5 - 9x)}{(3x - 1)} \geq 0$
We know $\frac{a}{b} \geq 0$, only when a and b are of same sign and $b \neq 0$, so taking two cases $a, b > 0$ and $a, b < 0$

Step 3:
Solving for Case 1 $a > 0$ and $b > 0$
$5 - 9x \geq 0$ and $3x - 1 > 0$
$x \leq \frac{5}{9}$ and $x > \frac{1}{3}$
$\therefore x \in (\frac{1}{3}, \frac{5}{9}]$

Step 4:
Solving for Case 2
$a \leq 0$ and $b < 0$
$5 - 9x \leq 0$ and $3x - 1 < 0$
$x \geq \frac{5}{9}$ and $x < \frac{1}{3}$
$\therefore x \in \{\}$
Hence Solution Set II = $(\frac{1}{3}, \frac{5}{9}]$

Step 5:
Final Solution will be
(Solution Set of I) ∩ (Solution Set of II)
$x \in (\frac{1}{3}, \frac{5}{9}]$

3.

Step 1:
$x = 2$ satisfies the inequality, hence
$2m + 2 < 1 - 4m$

Step 2:
Solving inequality in m
$2m + 2 < 1 - 4m$
$\Rightarrow 6m < -1$
$\Rightarrow m \in (-\infty, \frac{-1}{6})$

Step 3:
Given $m \in Z$ and maximum value of m is required
$\Rightarrow m = -1$

FUNDAMENTALS OF MATHEMATICS

WAVY CURVE METHOD

What you will learn

Wavy curve method
Solving polynomial inequalities

What you already know

Inequality representation
Properties of inequalities

We can use conventional algebraic methods to solve any inequality, but the algebraic methods are too tedious and time-consuming. so, we look for an alternative, the wavy curve method.

The wavy curve method (also known as the method of intervals) is a strategy used to solve inequalities of the form $\frac{f(x)}{g(x)} > 0$ (<0, ≥, or ≤). The method uses the fact that $\frac{f(x)}{g(x)}$ changes sign at critical points.

Critical points are the value of x at which each individual factor is equal to zero.

For Example:
$f(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, has {1, 4} as critical points.

Steps for Wavy Curve Method/Method of Intervals

Step: 1
In the given inequality, take everything to the LHS, and make RHS equal to zero.

$$\frac{(x - 1)}{x} \geq 2$$

$$\Rightarrow \frac{(x - 1)}{x} - 2 \geq 0$$

Step: 2
Factorise LHS to the maximum possible levels.

$$\frac{(x - 1)}{x} - 2 \geq 0$$

$$\frac{x - 1 - 2x}{x} \geq 0$$

$$\Rightarrow \frac{x + 1}{x} \leq 0$$

Step: 3
Equate the linear factor(s) to zero, and get the critical point(s).

$$\frac{(x + 1)}{x} \leq 0$$

Critical points: $x + 1 = 0$ and $x = 0$

∴ $x = -1$ **and 0 are the critical points**

Step: 4

Plot the critical points on the real number line.

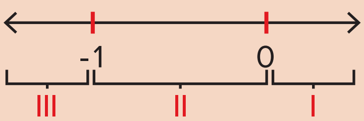
Step: 5

Decide about the inclusion/exclusion of critical points in the solution set based on the value of the defined expression at the critical point(s) by denoting a solid circle/hollow circle.

For $\frac{(x+1)}{x}$, x = -1 will be included and represented by a solid circle and x=0 will be excluded, as $\frac{(x+1)}{x}$ is not defined at x = 0. It will be represented by a hollow circle

Step: 6

Divide the real number line into disjoint regions based on the placement of critical points, denoting right-most region as region I, and then moving from right to left with regions II, III, etc.

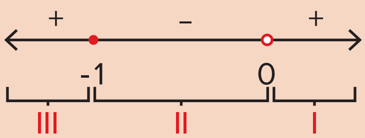


Step: 7

Determine the sign of the expression.

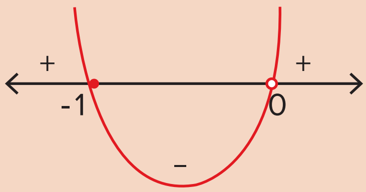
Take any real value of x greater than the right-most critical point and put it in the expression to get the sign in I.

Repeat the same to get the sign of the expression in II and III.



Step: 8

Find out the solution set of the inequality as per region's sign and solid circle(s) on critical point(s).



Final solution set

$\Rightarrow x \in [-1, 0)$

Solve the inequality $x^2 - 12x + 35 \geq 0$.

Step: 1

Factorise $x^2 - 12x + 35$

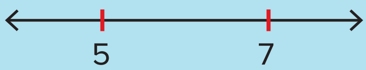
$x^2 - 7x - 5x + 35$

$(x - 7)(x - 5)$

Step: 2

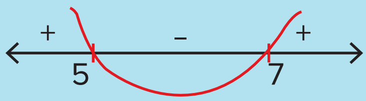
Plot critical points

Critical points = 5, 7



Step: 3

Use wavy curve method



$\therefore x \in (-\infty, 5] \cup [7, \infty)$

Consider the inequality $(x - 2)^2 (x^2 + x - 42) < 0$.

Step: 1

Factorise

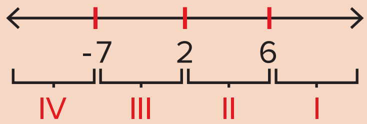
$(x - 2)^2 (x^2 + x - 42) < 0$

$(x - 2)^2 (x - 6)(x + 7) < 0$

Step: 2

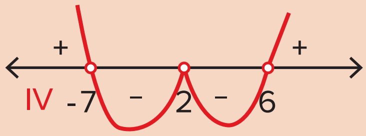
Plot the critical points

Critical Points $\Rightarrow x = -7, 2, 6$



Step: 3

Plotting critical points and sign determination:



Step: 4

Find the solution set of the inequality as per region's sign and solid circle(s) on critical point(s).

$x \in (-7, 6) - \{2\}$

Shortcut Method for Finding Signs of the Regions

1. Manually find the sign of the right-most region.
2. For odd power of the linear factor(s), corresponding to each critical point(s), change the sign. For even power of the factor(s), do not change the sign.

Say, the linear factor corresponding to the critical point 'a' is $(x - a)^n$. If n = even = 2k, then the sign of the expression remains the same about 'a'. If n = odd = 2k + 1, then the sign of the expression will change about 'a'.

For the factor $(x - a)^n$. If n is even, then the sign of the expression remains the same. If n is odd, then the sign of the expression will change about 'a'.

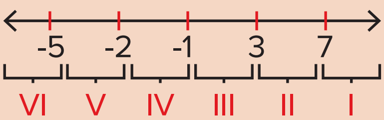
Solve $\dfrac{[(x - 3)^3 (x + 2)^9 (x + 5)^5]}{[(x + 1)(x - 7)^7]} < 0$

Step: 1

Identify and plot the critical points.

$$\frac{[(x-3)^3 (x+2)^9 (x+5)^5]}{[(x+1)(x-7)^7]} < 0$$

Here x = -5, -2, -1, 3, 7



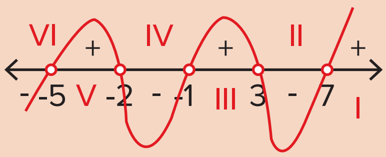
Step: 2

For $(x - a)^n$, if n is even, then the sign of the expression remains the same. If n is odd, then the sign of the expression will change about 'a'

Hence, sign of the expression will change about x = -5, -2, -1, 3, 7

Step: 3

Final solution



$x \in (-\infty, -5) \cup (-2, -1) \cup (3, 7)$

Concept Check 1

Solve the following inequalities

1. $\dfrac{(3x - x^2)}{(x + 4)^2} \le 0$ 2. $x^3 - 8 \ge 0$ 3. $\dfrac{(x^2 - 8x + 15)}{(x - 3)} < 0$

Summary sheet

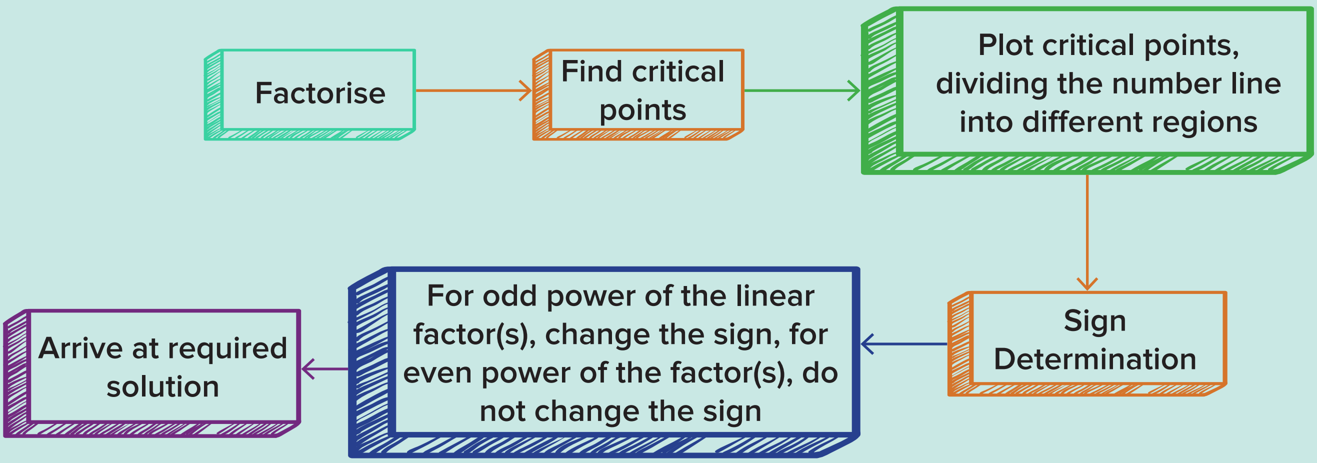
Key Takeaways

The wavy curve method (also known as the method of intervals) is a strategy used to solve inequalities of the form $\dfrac{f(x)}{g(x)} > 0$ (< 0, $\ge$, or $\le$).

Critical points are the values of x at which each individual factor is equal to zero. For Example: $f(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, has {1, 4} as critical points.

For odd power of the linear factor(s), corresponding to each critical point(s), change the sign. For even power of the factor(s), do not change the sign.

Mind Map



Self-Assessment

Solve the following inequalities

1. $x^3 + 6x^2 + 12x + 8 > 0$ 2. $\dfrac{(x^2 - 11x + 28)}{(-x^2 + x - 1)} \le 0.$ 3. $\dfrac{[x^{2020}(x - 17)^{2019}(x - 13)^{2018}]}{[(x + 7)^{2021}(x - 31)^{2022}]} \ge 0.$

Answers

Concept Check

1. $\dfrac{(3x - x^2)}{(x + 4)^2} \leq 0$

Step: 1

Factorise

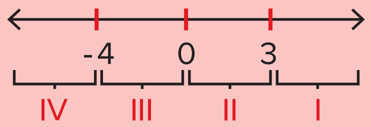
$\dfrac{(3x - x^2)}{(x + 4)^2} \leq 0$

$\Rightarrow \dfrac{x(3 - x)}{(x + 4)^2} \leq 0$

Step: 2

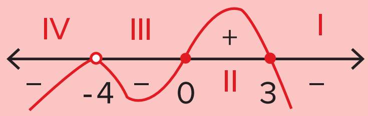
Plot the critical points

Critical Points $x = -4, 0, 3$



Step: 3

Sign determination and arriving at solution



Solution set:

$(-\infty, -4) \cup (-4, 0] \cup [3, \infty)$

2. $x^3 - 8 \geq 0$

Step: 1

Factorise

$x^3 - 8$

$\Rightarrow (x - 2)(x^2 + 2x + 4) \geq 0$

$(x-2)\,((x + 1)^2 + 3) \geq 0$

Here, $(x + 1)^2 + 3$ can not factorise into real factors, as the quadratic equation has no real solution.

Also $(x + 1)^2 + 3 \geq 3 > 0 \;\forall\; x \in R$

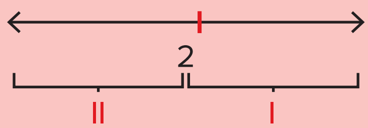
$(x - 2)(x^2 + 2x + 4) \geq 0$

$(x - 2) \geq 0$ (a.b > 0, then both a and b will have the same sign)

Step: 2

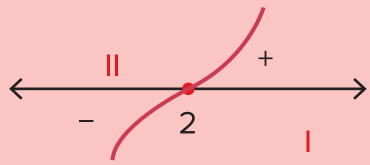
Plot the critical points

Critical Point $x = 2$



Step: 3

Sign determination and arriving at solution



$x \in [2, \infty)$

3. $\dfrac{(x^2 - 8x + 15)}{(x - 3)} < 0$

Step: 1

Factorise

$\dfrac{(x^2 - 8x + 15)}{(x - 3)} < 0$

$\dfrac{(x - 3)\,(x - 5)}{(x - 3)}$

For $x \neq 3$, the expression becomes $x - 5 < 0$.

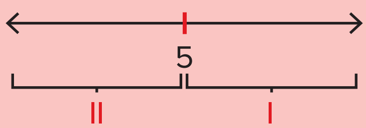
Step: 2

Plot the critical point

Here, the final inequality is

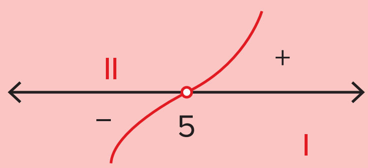
$x - 5 < 0$

Critical point $x = 5$



Step: 3

Sign determination and arriving at the solution



$x \in (-\infty, 5) - \{3\}$

Self-Assessment

1. $x^3 + 6x^2 + 12x + 8 > 0$

Step: 1

Factorise

$x^3 + 6x^2 + 12x + 8 > 0$

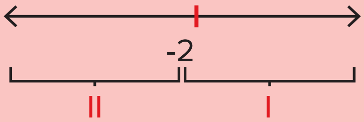
$\Rightarrow x^3 + 3(2)(x)^2 + 3(2)^2(x) + 2^3 > 0$

$\Rightarrow (x + 2)^3 > 0$

Step: 2

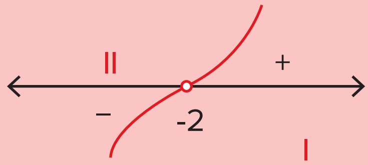
Plot the critical point

Critical point x = -2



Step: 3

Sign determination and arriving at solution



As for odd powers, the sign changes

$x \in (-2, \infty)$

2. $\dfrac{(x^2 - 11x + 28)}{(-x^2 + x - 1)} \leq 0$

Step: 1

Factorise

Here $x^2 - 11x + 28 = (x - 7)(x - 4)$

Also, $-x^2 + x - 1$ can not be factorised, as it has no real roots.

Observe, $-x^2 + x - 1 = -\left((x - \frac{1}{2})^2 + \frac{3}{4}\right) < 0 \ \forall\, x \in \mathbb{R}$

$\therefore \dfrac{X^2 - 11x + 28}{-x^2 + x - 1} \leq 0 = (x - 7)(x - 4) \geq 0$

$(\because -x^2 + x - 1 < 0 \ \forall\, x \in \mathbb{R})$

$\left(\dfrac{\text{“Something”}}{(-\mathbf{ve})} \leq 0\text{, iff “Something”} \geq 0\right)$

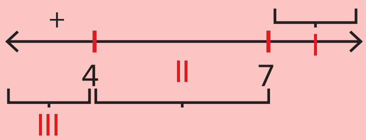
Step: 2

Plot the critical point

Plot the critical points

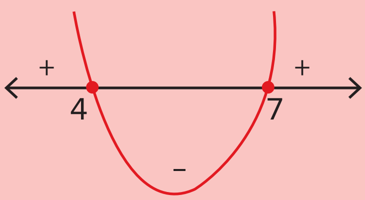
$(x - 7)(x - 4) \geq 0$

x = 4, 7 are the critical points



Step: 3

Sign determination and arriving at the solution



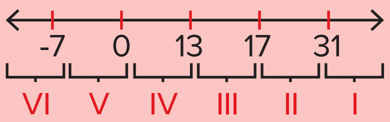
$x \in (-\infty, 4] \cup [7, \infty)$

3. $\dfrac{[x^{2020}(x-17)^{2019}(x-13)^{2018}]}{[(x+7)^{2021}(x-31)^{2022}]} \geq 0$

Step: 1

Find the critical points

Critical points are $x = -7, 0, 13, 17, 31$



Step: 2

Sign determination

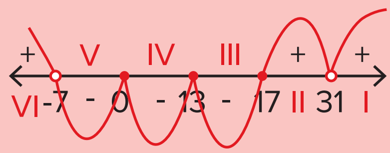
For even power, sign remains the same and for odd power, sign changes.

Sign wil change for x = -7, 17

Sign will remain same for x = 0, 13, 31

Step: 3

Final solution



$\therefore x \in \{0, 13\} \cup (-\infty, -7)$
$\cup [17, 31) \cup (31, \infty)$