8 Quadratic Equations

Introduction

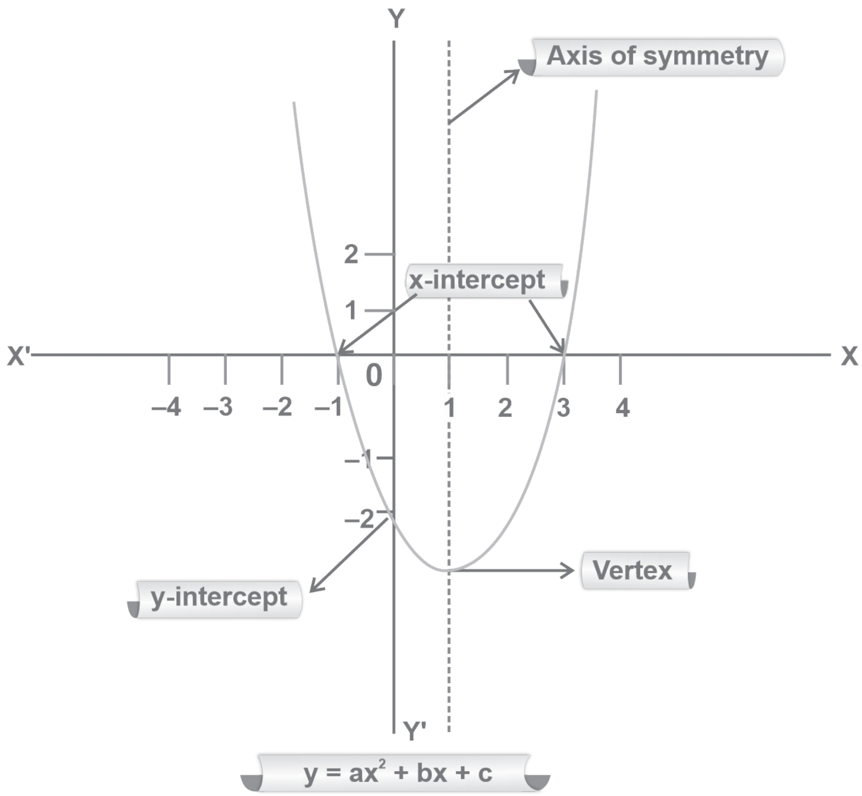
In this chapter, we will learn about different methods of solving quadratic equations. Direct questions are asked on this topic in CAT every year. Every year, two or three problems from this chapter are asked in CAT and other examinations.

Definition

Any equation of the form $ax^2 + bx + c = 0$ (degree 2) is called quadratic equation. Every quadratic equation has 2 roots, say α and β.

Physical Representation:

Quadratic equations always gives parabola.



Solution to Quadratic Equation:

Solution of $ax^2 + bx + c = 0$, is the value of x which satisfies the equation.

1. Factorisation Method

Given equation $ax^2 + bx + c = 0$

We try to split coefficient of midle term 'b' in two parts (say p and q) such that $b = p + q$ and $pq = ac$

Example 1:

Solve: $2x^2 + 5x + 3 = 0$

Solution: x $= -1, -\frac{3}{2}$

Now we need to break 5 into two parts such that the product of these parts is $2 \times 3 = 6$

Factors of 6	Parts (where sum = 5)
1×6	$1 + 6 \neq 5$
2×3	$2 + 3 = 5$

So we will choose 2×3 as $2 + 3 = 5$

$$2x^2 + 5x + 3 = 0$$

$$2x^2 + (2x + 3x) + 3 = 0$$

$$2x^2 + 2x + 3x + 3 = 0$$

$$2x(x + 1) + 3(x + 1) = 0$$

$$(x + 1)(2x + 3) = 0$$

$\Rightarrow \quad x = -1, -\frac{3}{2}$ are two roots.

But there is limitation to this method. Sometimes it becomes difficult for us to factorize, specially when roots are imaginary or irrational.

Previous Years' Question

What is the number of real number solutions of the equation

$$x^2 - 7|x| - 18 = 0?$$

(A) 2 (B) 4

(C) 3 (D) 1

Previous Years' Question

$x^2 - 9x + |k| = 0$ has real roots. How many integer values can 'k' take?

(A) 40 (B) 21

(B) 20 (D) 41

2. Discriminant Method

This is most commonly used method. Here, two roots are given by

$$\alpha, \beta = \frac{-b \pm \sqrt{D}}{2a}$$

Where D (discriminant) $= b^2 - 4ac$

- If $D > 0$, then the roots are real
- If $D < 0$ then roots are imaginary (of the form $a \pm ib$)
- If $D = 0$, then roots are real and equal. These are $x = -\frac{b}{2a}$

Key Points

Using the value of D, we can identify the nature of roots.

Roots of Quadratic Equation and their properties:

$ax^2 + bx + c = 0$

Let roots $= \alpha, \beta$

Sum of the roots $= S = \alpha + \beta = -\frac{b}{a}$

Product of the roots $= P = \alpha\beta = \frac{c}{a}$

- If the sum of roots $= \alpha + \beta = 0$, then the roots equal, but with opposite signs.

Key Points

Imaginary roots always exist in pair. If one is $a + ib$, then other in $a - ib$

Example 2:

Solve: $x^2 + 5x + 1 = 0$

Solution:

Product of roots $= \alpha\beta = \frac{c}{a} = 1$

$$x = \frac{-5 \pm \sqrt{21}}{2}$$

$$\Rightarrow \alpha = \frac{-5 + \sqrt{21}}{2}, \beta = -\frac{-5 - \sqrt{21}}{2}$$

Reciprocal of $\alpha = \frac{1}{\alpha} = \frac{1}{\frac{-5+\sqrt{21}}{2}}$

$= \frac{2}{-5+\sqrt{21}} \times \frac{\sqrt{21}+5}{\sqrt{21}+5}$ (Rationalising)

$= \frac{2\left(5+\sqrt{21}\right)}{(21-25)}$

$= \frac{2\left(5+\sqrt{21}\right)}{-4} = -\left(\frac{5+\sqrt{21}}{2}\right) = \beta$

$\Rightarrow$ Roots are reciprocal of each other.

Previous Years' Question

Equation $x^2 + 5x - 7 = 0$ has roots a and b. Equation $2x^2 + px + q = 0$ has roots a + 1 and b + 1. Find p + q.

(A) 6 (B) 0

(C) –16 (D) 2

Example 3:

Solve: $x^2 - 4 = 0$

Solution:

Sum of the roots $= \alpha + \beta = -\frac{b}{a} = 0$

Now, $x^2 - 4 = 0$

$X^2 = 4$

$X = \pm 2$

$\Rightarrow$ Roots are of opposite signs if $\alpha + \beta = 0$

Finding quadratic equation when roots or some property is given:

- If roots are given, say α and β, then we can write quadratic equation as $X^2 - Sx + P = 0$

 Where, S = Sum of roots $= \alpha + \beta$

 P = Product of roots $= \alpha\beta$

Previous Years' Question

$x^2 - 11x + |p| = 0$ has integer roots. How many integer values can 'p' take?

(A) 6 (B) 4

(C) 8 (D) More than 8

Example 4:

If the roots of quadratic equation are 2 and –3, then find the equation.

Solution: $x^2 + x - 6 = 0$

Let $\alpha = 2$, $\beta = -3$

S = Sum of roots $= \alpha + \beta = -1$

P = Product of roots $= \alpha\beta = -6$

Therefore, required quadratic equation is

$X^2 - Sx + P = 0$

$\Rightarrow x^2 - (-1)x + (-6) = 0$

$\Rightarrow x^2 + x - 6 = 0.$

- If we need to find a quadratic equation whose roots are reciprocal of the roots of the equation $ax^2 + bx + c = 0$:

Let α, β are the roots of $ax^2 + bx + c = 0$, then the equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is given by $cx^2 + bx + a = 0$

Previous Years' Question

How many real solutions are there for the equation $x^2 - 7|x| - 30 = 0$?

(A) 3 (B) 1

(C) 2 (D) none

Example 5:

If the equation $X^2 + x - 6 = 0$ has roots, 2, −3, then find the equation whose roots are $\frac{1}{2}, \frac{-1}{3}$

Solution: $6x^2 - x - 1 = 0$

2, −3 are roots of the equation $x^2 + x - 6 = 0$

Then equation with roots $\frac{1}{2}, \frac{-1}{3}$ is given by

$$-6x^2 + x + 1 = 0$$

$\Rightarrow$ $6x^2 - x - 1 = 0$ ...(1)

We can prove this also:

S = Sum of roots $= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

P = Product of roots $= \frac{1}{2} \times \left(-\frac{1}{3}\right) = -\frac{1}{6}$

Therefore, equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \frac{1}{6}x - \frac{1}{6} = 0$$

$\Rightarrow$ $6x^2 - x - 1 = 0$ ...(2)

Therefore, equation 1 and equation 2 both are the same.

Previous Years' Question

If $(3x + 2y - 22)^2 + (4x - 5y + 9)^2 = 0$ and $5x - 4y = 0$, then find the value of $x + y$.

(A) 7 (B) 9

(C) 11 (D) 13

Previous Years' Question

If $f(y) = x^2 + (2p + 1)x + p^2 - 1$ and x is a real number, then for what values of 'p' the function becomes 0?

(A) $p > 0$ (B) $p > -1$

(C) $p \geq -54$ (D) $p \leq 34$

Previous Years' Question

What is the largest positive integer such that $\frac{n^2 + 7n + 12}{n^2 - n - 12}$ is also positive integer?

Practice Exercise - 1

Level of Difficulty – 1

1. How many real solution exists for the equation $x^2 - |x| - 6 = 0$?

(A) 0

(B) 1

(C) 2

(D) 4

2. What is number of integer values that p can take is that $x^2 - 8x + |p| = 0$ has real roots?

(A) 30

(B) 31

(C) 32

(D) 36

3. Given that $(2x + 3y + 10)^2 + (3x - 7y - 8)^2 = 0$ and $7x - y + 12 = 0$, find the value of $x - y$.

(A) 0

(B) 1

(C) 3

(D) 8

4. How many integer values x can take where $(x+2)^{x^2+11x+18} = 1$?

(A) 1

(B) 2

(C) 4

(D) 6

5. If 5p, –2p are roots of the quadratic equation $x^2 - qx + r = 0$, where p is an integer, then which of the following can be the possible value of $2q^2 + \frac{r}{2}$?

(A) 273

(B) 2057

(C) 2160

(D) 1872

Level of Difficulty – 2

6. The sum of distinct roots of $x^2 + 4x + 4 = |x^2 + 6x + 8|$ is given by

(A) 4

(B) –5

(C) –6

(D) –8

7. If α and β are roots of the equation $x^2 - 7x + 10 = 0$ and $(\alpha - 1), (\beta - 1)$ are the roots of equation $3x^2 - bx + c = 0$, then the find the value of $(b - c)$.

(A) 3

(B) 2

(C) 7

(D) 4

8. Product of roots of a quadratic equation is twice the sum of its roots. If one of the roots is 3 more than the other root, then find the sum of the roots.

(A) 9 or 1

(B) –9 or 1

(C) 9 or –1

(D) 6 or –2

9. Find the largest positive integer m such that $\frac{m^2 + 3m - 10}{m^2 - 5m + 6}$ is also positive integer.

(A) 9

(B) 11

(C) 13

(D) 14

10. Shagun and Ravi were given a quadratic equation to solve. Shagun by mistake wrongly copied the coefficient of x. Shagun got (–3, 4) as roots. Ravi was engrossed in some work and made the mistake in copying constant term and got (6, –2) as roots. Find the correct roots.

(A) 2, 6

(B) 2, –6

(C) –2, –6

(D) –2, 6

Level of Difficulty – 3

11. Given that sum of reciprocal of roots of $x^2 - px + q = 0$ is 8. Find the sum of reciprocal of roots of $x^2 - qx + p = 0$.

(A) 1.25

(B) 0.25

(C) 0.125

(D) 0.125

12. How many integers values 'K' can take, if $x^2 - 5x + |K| = 0$, has real and distinct roots?

(A) 12

(B) 13

(C) 14

(D) 15

13. If $x^2 + ax + b = 0$ has roots a and b, then (a + b) is equal to? (Given that $a \neq 0$ and $b \neq 0$)

(A) 0

(B) 1

(C) –1

(D) –2

14. The arithmetic mean of roots of quadratic equation is 10 and geometric mean is 6. Find the equation.

(A) $x^2 - 20x + 36 = 0$

(B) $x^2 + 20x + 36 = 0$

(C) $x^2 - 20x - 36 = 0$

(D) $x^2 + 20x - 36 = 0$

15. One of the root of the equation $x^2 + bx + 21 = 0$ is 7. Find the value of c in the equation $x^2 + bx + c = 0$ which has equal roots.

(A) 10

(B) 15

(C) 22

(D) 25

Solutions

1. (C)

$x^2 - |x| - 6 = 0$

We will solve the above equation after considering two cases:

Case I: When $x < 0$

$\Rightarrow \quad |x| = -x$

Equation becomes

$X^2 + x - 6 = 0$

$\Rightarrow \quad x^2 + 3x - 2x - 6 = 0$

$\Rightarrow \quad x(x + 3) - 2(x + 3) = 0$

$\Rightarrow \quad x = -3, 2$

But $x < 0 \Rightarrow x = -3$ only

Case II: When $x > 0$

$|x| = x$

$\Rightarrow \quad x^2 - x - 6 = 0$

$x^2 - 3x + 2x - 6 = 0$

$x(x - 3) + 2(x - 3) = 0$

$x = 3, -2$

But $x > 0 \Rightarrow x = 3$ only

Combining both the cases: $X = +3, -3$

2. **(B)**

$x^2 - 8x + |p| = 0$

$D = b^2 - 4ac = 64 - 4|p|$

For real roots, $64 - 4|p| > 0 \Rightarrow |p| < 16$

$\Rightarrow -16 < p < 16$

Therefore p can take values: –15, –14, –13, –12, –1, 0, 1,………… 13, 14 and 15 Thus, number of integer values that p can take in 31.

3. **(A)**

$(2x + 3y + 10)^2 + (3x - 7y - 8)^2 = 0$

$(2x + 3y + 10)^2 \geq 0$ and $(3x - 7y - 8)^2 \geq 0$

But R.H.S = 0 (given)

$\Rightarrow 2x + 3y + 10 = 0$ ...(1)

and $3x - 7y - 8 = 0$

Multiply $7x - y + 12 = 0$ by 3 and add it to equation 1:

$$\begin{array}{r} 2x + 3y + 10 = 0 \\ 21x - 3y + 36 = 0 \\ \hline 23x + 46 = 0 \end{array}$$

$$x = \frac{-43}{23} = -2$$

Put $x = -2$ in $7x - y + 12 = 0$

$\Rightarrow y = -2$

Now $x - y = -2 + 2 = 0$.

4. **(C)**

For $(x+2)^{x^2+11x+18} = 1$, this can be achieved if $x + 2 = 1$ because

(1) any number = 1

OR

(2) if $x + 2 = -1$ and $x^2 - 11x + 18$ = even number, then $(-1)^{\text{even number}} = 1$

OR

(3) if $x + 2$ can be any non-zero number and $x^2 - 11x + 18 = 0$ so that $(\text{number})^0 = 1$

Case I: $x + 2 = 1 \Rightarrow x = -1$

Case II: $x + 2 = -1 \Rightarrow x = -3$

Let us see $x^2 - 11x + 18$ is even or not at $x = -3$

$X^2 - 11x + 18 = (-3)^2 - 11(-3) + 18$

$= 60$ (even)

Therefore $x = -3$ is possible

Case III: $x^2 - 11x + 18 = 0$

$\Rightarrow x^2 - 9x - 2x + 18 = 0$

$x(x - 9) - 2(x - 9) = 0$

$x = 2, 9$ which are possible

Therefore total number of possible values are –1, –3, 2, 9 (4 values)

5. **(D)**

Given equation $x^2 - qx + r = 0$

Given roots = 5p, –2p

$\Rightarrow$ Sum of roots = 5p – 2p = +q (since the sum of the roots of the above equation = –(–q)/1 = q)

$\Rightarrow q = 3p$ ...(1)

$\Rightarrow$ Product of roots = 5p × –2p = r (similarly product of the roots of the above equation = r/1 = r)

$\Rightarrow r = -10p^2$

Therefore, $2q^2 + \frac{r}{2} = 2 \times 9p^2 + \frac{1}{2} \times (-10p^2)$

$= 18p^2 - 5p^2 = 13p^2$

Now, from the given option, we need to choose the one which is multiple of 13 and contain perfect square also.

We will pick options one by one:

273 = 13 × 3 × 7 is a multiple of 13 but 21 is not a perfect square.

105 = 13 × 5 × 17 is also a multiple of 13, but 85 is not a perfect square.

In $2160 = 15 \times (12)^2$, 144 is perfect square of 12 but number is not multiple of 13.

Last option $1872 = 13 \times 144 = 13 \times (12)^2$ is only option which is multiple of 13 as well as perfect square (=144).

6. (B)

$x^2 + 4x + 4 = |x^2 + 6x + 8|$ ...(1)

Two cases arises:

Case I: when $(x^2 + 6x + 8) \geq 0$

$\Rightarrow$ $|x^2 + 6x + 8| = x^2 + 6x + 8$

Therefore (1) becomes:

$x^2 + 4x + 4 = x^2 + 6x + 8$

$\Rightarrow$ $2x = -4$

$\Rightarrow$ $x = -2$ ...(2)

Case II: when $(x^2 + 6x + 8) < 0$

$\Rightarrow$ $|x^2 + 6x + 8| = -x^2 - 6x - 8$

Therefore (1) becomes:

$x^2 + 4x + 4 + x^2 + 6x + 8 = 0$

$\Rightarrow$ $2x^2 + 10x + 12 = 0$

$2x^2 + 6x + 4x + 12 = 0$

$2x(x + 3) + 4(x + 3) = 0$

$(2x + 4)(x + 3) = 0$

$x = -2, -3$, but here $x = -2$ is not possible because at $x = -2$,

$x^2 + 6x + 8 < 0$, which does not satisfy

All the possible values of x are -2 and -3

Hence, the sum of the distinct roots $= -2 -3 = -5$

7. (A)

Let us solve $x^2 - 7x + 10 = 0$ first

$\Rightarrow$ $x^2 - 5x - 2x + 10 = 0$

$x(x - 5) -2(x - 5) = 0$

$x = 2, 5$

$\Rightarrow$ $\alpha = 2, \beta = 5$

For equation $3x^2 - bx + c = 0$, roots are $\alpha - 1$ and $\beta - 1$

Sum of roots $= \frac{b}{3} = \alpha - 1 + \beta - 1$

$= \frac{b}{3} = 2 - 1 + 5 - 1$

$\Rightarrow$ $b = 15$

Product of roots $= (\alpha - 1)(\beta - 1)$

$\Rightarrow$ $\frac{c}{3} = \alpha\beta - \alpha - \beta + 1$

$\Rightarrow$ $\frac{c}{3} = 10 - 2 - 5 + 1$

$\Rightarrow$ $c = 4 \times 3 = 12$

Therefore, $b - c = 15 - 12 = 3$

8. (C)

Let one of the root = a

Therefore, other root = a + 3

Now, sum of roots $= a + a + 3 = 2a + 3$

Product of roots $= a(a + 3) = a^2 + 3a$

It is given that,

Product of roots = 2 (Sum of roots)

$a^2 + 3a = 2(2a + 3)$

$\Rightarrow$ $a^2 + 3a - 4a - 6 = 0$

$\Rightarrow$ $a^2 - a - 6 = 0$

$a^2 - 3a + 2a - 6 = 0$

$\Rightarrow$ $a = 3, -2$

Therefore roots are (3, 6) or (–2, 1) so, sum of roots will be 9 or –1.

9. (B)

$\frac{m^2 + 3m - 10}{m^2 - 5m + 6} = 1 + \frac{8m - 16}{m^2 - 5m + 6}$

$= 1 + \frac{8(m-2)}{(m-2)(m-3)}$ ...(1)

Here, $m \neq 2$ otherwise denominator in equation (1) become zero.

Also, for (1) to be positive integer, $\frac{8(m-2)}{(m-2)(m-3)}$ has to be positive integer.

For this $\frac{8}{m-3} = 1$

$\Rightarrow$ $8 = m - 3$

$\Rightarrow$ $m = 11$

If $m > 11$ then $\frac{8}{m-3}$ will be fraction.

This will result in making $\frac{m^2 + 3m - 10}{m^2 - 5m + 6}$ a fraction.

Hence, the maximum value of m = 11.

10. (D)

Shaguns roots are (–3, 4)

Sum of roots $= -3 + 4 = 1$

Product of roots = $-3 \times 4 = -12$

$\Rightarrow$ equation will be $x^2 - x - 12 = 0$

The coefficient of x is wrong but the constant term is correct.

$\Rightarrow$ Product = -12 is correct ....(1)

Ravi's roots are (8, −4)

Sum of roots = 4

Products of roots = −32

Therefore, Ravi's equation will be $x^2 - 4x - 32 = 0$

But his constant term is wrong

Only coefficient of x is correct

$\Rightarrow$ Sum of roots = +4 ...(2)

From (1) and (2)

Correct equation is

$x^2 - 4x - 12 = 0$

$x^2 - 6x + 2x - 12 = 0$

$x(x - 6) + 2(x - 6) = 0$

$x = -2, 6$

11. **(C)**

Let α, β be roots of the equation

$x^2 - px + q = 0$

$\Rightarrow$ sum of the roots = $-(-p) = p$

Product of roots = q

Sum of reciprocal of roots $= \frac{1}{\alpha} + \frac{1}{\beta}$

$\Rightarrow$ Sum of reciprocal of roots

$= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{p}{q}$ ...(1)

Let a, b be roots of equation

$x^2 - qx + p = 0$

Sum of roots = q

Product of the roots = p

Sum of reciprocal of its roots $= \frac{1}{a} + \frac{1}{b}$

$= \frac{a+b}{ab} = \frac{q}{p}$ ...(2)

We know that $\frac{p}{q} = 8$ (given)

$\Rightarrow \frac{q}{p} = \frac{1}{8} = 0.125$

12. **(B)**

$x^2 - 5x + |K| = 0$

For roots to be real and distinct:

$D = b^2 - 4ac > 0$

$\Rightarrow$ $25 - 4|K| > 0$

$\Rightarrow$ $4|K| < 25$

$\Rightarrow$ $|K| < 6.25$ or $-6.25 < K < 6.25$

But k is an integer

Therefore K can take values

−6, −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6

Total 13 values.

13. **(C)**

Given equation is $x^2 + ax + b = 0$

Given roots are a, b

Sum of the roots = $a + b = -a$ (from the given equation)

$\Rightarrow$ $b = -2a$ ...(1)

Product of the roots = $ab = b$ (from the given equation)

$\Rightarrow$ $a = 1$ ...(2)

Put $a = 1$ in equation (1):

$\Rightarrow$ $b = -2$

Now when $a = 1$, $b = -2$

$a + b = -1$

14. **(A)**

Let the roots be a and b.

Arithmetic Mean (A.M.)

$= \frac{a+b}{2} = 10$ (given)

$\Rightarrow$ $\frac{a+b}{2} = 10$

$\Rightarrow$ $a + b = 20$ ...(1)

Geometric Mean (G.M.) = $\sqrt{ab} = 6$ (given)

$\Rightarrow$ $ab = 36$ ...(2)

Therefore, Sum of the roots

$= a + b = 20$

Product of the roots = $ab = 36$

Required equation will be

$x^2 - 20x + 36 = 0$

Alternative Method:

A.M. = 10, G.M. = 6

Required equation is given by

$x^2 - 2(\text{A.M. of roots})x + (\text{G.M of roots})2 = 0$

$x^2 - 2(10)x + (6)^2 = 0$

$x^2 - 20x + 36 = 0$

15. (D)

7 is one of the root of

$x^2 + bx + 21 = 0$ ...(1)

Let 'a' be the other root.

Sum of the roots = 7 + a = −b (from (1)) ...(2)

Product of the roots = 7a = 21 (from (1))

$\Rightarrow$ a = 3

Put a = 3 in (2):

7 + 3 = −b

$\Rightarrow$ b = −10

Now, b = −10, therefore second equation $x^2 + bx + c = 0$ becomes

$x^2 - 10x + c = 0$

It is given that $x^2 - 10x + c = 0$ has equal roots.

$\Rightarrow$ Discriminant = D = 0

$\Rightarrow$ $b^2 - 4ac = 0$

$\Rightarrow$ $(-10)^2 - 4(1)c = 0$

$\Rightarrow$ 4c = 100

$\Rightarrow$ c = 25

Alternate Method:

7 is one of the root of $x^2 + bx + 21 = 0$, then it must satisfy this equation.

$\Rightarrow$ $(7)^2 + b(7) + 21 = 0$

$\Rightarrow$ 49 + 7b +21 = 0

$\Rightarrow$ 7b + 70 = 0

$\Rightarrow$ 7b = −70

$\Rightarrow$ b = −10

Now b = −10, therefore second equation $x^2 + bx + c = 0$ becomes

$x^2 - 10x + c = 0$

It is given that $x^2 - 10x + c = 0$ has equal roots.

$\Rightarrow$ Discriminant = D = 0

$\Rightarrow$ $b^2 - 4ac = 0$

$\Rightarrow$ $-(10)^2 - 4(1)c = 0$

$\Rightarrow$ 4c = 100

$\Rightarrow$ c = 25

Practice Exercise - 2

Level of Difficulty – 1

1. What is the value of x?

Statement-1 : $9x^4 = 144$

Statement-2 : $x^2 - 3x - 10 = 0$

(A) If the question can be answered by using statement-1 alone not by statement-2 alone.

(B) If the question can be answered by using statement-2 alone not by statement-1 alone.

(C) If the question can be answered by using either of the statements alone

(D) If the question cannot be answered by using one of the statements alone but can be answered by combining both the statements

(E) If the question cannot be answered even by combining both the statements

2. If α and β are the roots of the equation $x^2 - 12x + 20 = 0$, find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

(A) 0.063

(B) 0.252

(C) 0.126

(D) 0.120

3. If α and β are the roots of the quadratic equation $3x^2 + 4x + 5 = 0$, the value of $\left(\frac{1}{3\alpha+4} + \frac{1}{3\beta+4}\right)$ is?

(A) $\frac{5}{12}$

(B) $\frac{1}{5}$

(C) $\frac{4}{15}$

(D) None of these

4. Find the real root of the equation $3^{4x} + 3^{2x+2} - 36 = 0$

(A) 2

(B) $\log_2 3$

(C) $\log_3 2$

(D) ½

5. For how many integral values of 'r', the quadratic equation $x^2 - 12x + |r| = 0$ has real and distinct roots?

(A) 70

(B) 71

(C) 72

(D) 73

6. Given that α and β be the roots of the equation $x^2 - Kx + 3 = 0$ and $\left(\alpha + \frac{1}{\beta}\right)$ and $\left(\beta + \frac{1}{\alpha}\right)$ are the roots of the equation $x^2 - Px + a = 0$. What is the value of a?

(A) $\frac{8}{3}$

(B) $\frac{17}{3}$

(C) $\frac{16}{3}$

(D) $\frac{14}{3}$

7. If 'm' is the root of the equation $x^2 + 3m + 7 = 0$, then the value of m (m + 1) (m + 2) (m + 3) is?

8. Find the product of the minimum value of $x^2 - 5x + 6$ and the minimum value of $x^2 - 7x + 12$.

(A) $\frac{1}{8}$

(B) $\frac{1}{16}$

(C) $\frac{1}{4}$

(D) $\frac{1}{64}$

9. The values of p satisfying $\log_4 (p^2 + 50p + 637) = 2$ are?

(A) −26, −27

(B) −23, 27

(C) 23, −27

(D) −23, −27

10. If the equation $2a^2 + (m + 4) a + 2m = 0$ has real and equal roots. Find the value of m?

(A) 9

(B) 6

(C) 3

(D) 4

Level of Difficulty – 2

11. If the roots of the equation $ax^2 + bx + b = 0$ are in the ratio $\alpha : \beta$, then

$$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}} = ?$$

(A) 0

(B) 1

(C) 2

(D) −1

12. The average monthly income per member of a family is (40 − x) thousand rupees and the number of members in the family is (5x + 40). What is the average monthly income (in rupees) per member, when the total income of the family is maximum in a particular month?

13. If '2' lies b/w the roots of f(x), where $f(x) = x^2 - 3k - 28$, then find which of the following must be true for k?

(A) $k < 7$

(B) $k < -8$

(C) $k > -7$

(D) $k > -8$

14. On the occasion of his daughter's birthday, Raj planned to buy 'x' number of chocolates for y rupees (x and y are integers greater than 1). If Raj had bought 15 more chocolates, the total amount paid by him will be Rs. 3 and would have saved Re. 0.40 per dozen chocolates. What is the value of x?

(A) 12

(B) 13

(C) 14

(D) 15

15. Given a quadratic equation $ax^2 + bx + c = 0$. If $a \neq b \neq c$ and $(a, b, c) \in \{1, 2, 3\}$. Find how many quadratic equations with real roots can be formed?

(A) 0

(B) 1

(C) 2

(D) 3

16. In a quadratic equation $ax^2 + bx + 4 = 0$ (where x is a real number), find the total number of ordered pairs (a, b) possible such that the roots of the quadratic equation are equal. Also given that a and b are integers and a lies from – 100 to 100 (including both – 100 and 100)

(A) 20

(B) 18

(C) 10

(D) 9

17. If log3, log(3^x–1) and log(3^x+1) are in arithmetic progression, then x lies in which of the following intervals?

(A) (3, 4)

(B) (2, 3)

(C) (1, 2)

(D) (0, 1)

18. If $a + 1 = a^2$ and $a > 0$, find the value of $5a^4$.

(A) $7+\frac{\sqrt{5}}{2}$

(B) $5+\frac{\sqrt{3}}{2}$

(C) $\frac{17+\sqrt{5}}{2}$

(D) $\frac{35+15\sqrt{5}}{2}$

19. If the roots of the quadratic equations $(p - q) x^2 + (q - r) x + (r - p) = 0$ are equal, then find the value of 2p.

(A) –(r + q)

(B) (r – q)

(C) (q – r)

(D) (q + r)

20. Roots of the equation $x^2 - ax + 16 = 0$ are real and distinct. The difference between the roots is not more than 15. Find the number of integral values that 'a' can take.

Level of Difficulty – 3

21. If the roots of the equation $x^2 + bx + c = 0$ are α and β (where $\alpha > \beta$), find the equation whose roots are α^2 and $-\beta^2$

(A) $x^2 + b\sqrt{b^2 - 4ac}\ x + c = 0$

(B) $x^2 + (b\sqrt{b^2 - 4c})x - c^2 = 0$

(C) $x^2 + 4ax + c^2 = 0$

(D) $9x^2 + 4bcx + c^2 = 0$

22. If x is a positive real number and its given that $25^{(2x-1)} - 5^{(2x+1)} + 100 = 0$ then find the correct range, which represents the sum (K) of all the possible values of x. (Take log 5 = 0.7)

(A) 0 < K < 1

(B) 1 < K < 2

(C) 2 < K < 3

(D) 3 < K < 4

23. If a and b are roots of the equation $2x^2 - 5x + 4 = 0$, then find the value of $a^2 + b^2 + a^3 + b^3 + a^4 + b^4 + \ldots\ldots$ up to infinity.

(A) $-\frac{11}{2}$

(B) $-\frac{13}{4}$

(C) $\frac{9}{2}$

(D) $\frac{11}{2}$

24. If α and β are the roots of $3x^2 + 27x + 15 = 0$, where $\alpha > \beta$, then find the value of

$$61\left(\frac{37}{2\alpha+9}\right)^2 + 61\left(\frac{13}{2\beta+9}\right)^2 = ?$$

Write your answer here.

25. Let 'p' and 'q' be positive integers. If $x^2 + px + 2q = 0$ and $x^2 + 2qx + p = 0$ have real roots, then the smallest possible value of p + q is?

(A) 2

(B) 4

(C) 5

(D) 6

26. If α and are the roots of quadratic equation $x^2 - 7x + 5 = 0$, then form the quadratic equation whose roots are $(\alpha + \beta)^3$ and $(\alpha - \beta)^3$. Also given $\alpha > \beta$

(A) $x^2 - 344x + 343 = 0$

(B) $x^2 - 468x + 42875 = 0$

(C) $x^2 - 468x + 48275 = 0$

(D) None of these

27. If the sum of the roots of the quadratic equation $px^2 + qx + r = 0$ is equal to the sum of the squares of their reciprocals, then qr^2, rp^2, pq^2 are in?

(A) Geometric progression

(B) Arithmetic progression

(C) Harmonic progression

(D) AP and GP combined.

28. The roots of the quadratic equation $x^2 - 4x + 10$ are m and n. Find the value of $(1 + m + n^2) \times (1 + n + m^2)$.

(A) 35

(B) 55

(C) 43

(D) 70

29. If a and b are real numbers, then the least possible value of the expression: $3(a - 3)^2 + (b - 2)^2 - 2(a - 2)^2$ is?

(A) −6

(B) 5

(C) −3

(D) −2

30. Let one of the roots of the equations $ax^2 - bx + c = 0$ is $\left(7+4\sqrt{3}\right)$, where a, b, c are rational numbers a≠0, if $b = c^3$ then $3a^2 = ?$

(A) 42

(B) 45

(C) 4

(D) 19

Solution

1. **(D)**

From Statement-1:

$9x^4 = 144$

or, $x^4 = 16$

$x = \pm 2$

We are getting two values of x, statement-1 alone is not sufficient.

From Statement-2:

$x^2 - 3x - 10 = 0$

$x^2 - 5x + 2x - 10 = 0$

$x(x - 5) + 2(x - 5) = 0$

$(x - 5)(x + 2) = 0$

or, $x = 5, -2$

Again, we are getting two values of x, statement-2 alone is not sufficient.

Now, if we combine statement-1 and statement-2, then we are getting a unique value of x i.e. '-2'.

Hence, option (D) is the correct choice.

2. **(C)**

Since, given equation is: $x^2 - 12x + 20 = 0$

Here, a = 1, b = −12 and c = 20

Now,

$$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{(\alpha\beta)^3}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$$

$$=\frac{(\alpha+\beta)^3-3\left(\frac{c}{a}\right)\times\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)3}$$

$$=\frac{\left(\frac{12}{1}\right)^3-3\times\left(\frac{20}{1}\right)\times\left(\frac{12}{1}\right)}{(20)^3}=0.126$$

Hence, option (C) is the correct answer.

3. **(C)**

Sum of the roots = $\alpha+\beta=-\frac{4}{3}$

Product of the roots = $\alpha\beta=\frac{5}{3}$

$$\frac{1}{3\alpha+4}+\frac{1}{3\beta+4}$$

$$\frac{3\beta+4+3\alpha+4}{(3\alpha+4)\ (3\beta+4)}=\frac{3(\alpha+\beta)+8}{9\alpha\beta+12(\alpha+\beta)+16}$$

$$=\frac{3\left(-\frac{4}{3}\right)+8}{9\left(\frac{5}{3}\right)+12\left(-\frac{4}{3}\right)+16}=\frac{4}{15}$$

Hence, option (C) is the correct answer.

4. **(D)**

$3^{4x}+3^{2x+2}-36=0$

Let $t=3^{2x}$

$\Rightarrow$ $t^2+t\,(9)-36=0$

$t^2+12t-3t-36=0$

$t=-12, 3$

$t=3^{2x}$ is always positive

$\Rightarrow$ $3^{2x}=-12$ is not a real solution.

$\Rightarrow$ $3^{2x}=3$ in the only real solution.

$\Rightarrow$ **2x = 1 or x = 1/2**

Hence, option (D) is the correct answer.

5. **(B)**

$x^2-12x+|r|=0$

For real and distinct roots: $D>0$

$\Rightarrow$ $b^2-4ac>0$

$144-4|r|>0$

$\Rightarrow$ $4|r|<144$

$|r|<36$

$\Rightarrow$ $-36<r<36$

Which means r can take values

−35, −34,....... −1, 0, 1,......., 34, 35

Total possible values = 71

Hence, option (B) is the correct answer.

6. **(C)**

$x^2-Kx+3=0$

Product of the roots = $\alpha\times\beta=3$

Also, $x^2-Px+a=0$

Product of the roots

$$=\left(\alpha+\frac{1}{\beta}\right)\times\left(\beta+\frac{1}{\alpha}\right)=a$$

$$\Rightarrow\left(\frac{\alpha\beta+1}{\beta}\right)\times\left(\frac{\alpha\beta+1}{\alpha}\right)=a$$

$\frac{(\alpha\beta+1)^2}{\alpha\beta}=a$, now put 〈 × β = 3

$$\frac{(3+1)^2}{3}=a\Rightarrow\frac{16}{3}=a$$

$$\boxed{a=\frac{16}{3}}$$

Hence, option (C) is the correct answer.

7. **35**

$m(m+1)(m+2)(m+3)$

$=[m(m+3)][(m+1)(m+2)]$

$=(m^2+3m)(m^2+3m+2)$

Since ‘m’ is the root of the equation

$x^2+3x+7=0$

$\Rightarrow$ $m^2+3m+7=0$

$\Rightarrow$ $m^2+3m=-7$

Now, $(m^2+3m)(m^2+3m+2)$

$=-7\times(-7+2)=35.$

8. **(B)**

Since we know that in a general quadratic expression ax^2+bx+c, if the coefficient of x^2 is positive (or a > 0), then, the minimum value = $\frac{4ac - b^2}{4a}$

∴ Minimum value of

$$x^2 - 5x + 6 = \frac{4 \times 1 \times 6 - (-5)^2}{4 \times 1}$$

$$= \frac{24 - 25}{4} = -\frac{1}{4}$$

Also, the minimum value of $x^2 - 7x + 12$

$$= \frac{4 \times 1 \times 12 - (-7)^2}{4 \times 1} = \frac{48 - 49}{4} = -\frac{1}{4}$$

Therefore, the product

$$= \left(-\frac{1}{4}\right) \times \left(-\frac{1}{4}\right) = \frac{1}{16}$$

Hence option (B) is the correct answer.

9. (D)

We have: $\log_4 (p^2 + 50p + 637) = 2$

Using logarithm property

⇒ $p^2 + 50p + 637 = 4^2$

⇒ $p^2 + 50p + 621 = 0$

⇒ $p^2 + 27p + 23p + 621 = 0$

⇒ $(p + 23)(p + 27) = 0$

⇒ $p = -23, -27$

10. (D)

Since we know that if the roots are real and equal, then its discriminant is equal to zero.

∴ D = 0 or $b^2 - 4ac = 0$

$(m + 4)^2 - 4 \times 2 \times 2m = 0$

$m^2 + 8m + 16 - 16m = 0$

$m^2 - 8m + 16 = 0$

∴ $m^2 - 8m + (4)^2 = 0$

$(m - 4)^2 = 0$

$m = 4$

Hence option (D) is the correct answer.

then $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}} = ?$

11. (A)

Let the roots of the equation $ax^2 + bx + b = 0$ be αx and βx

∴ $\alpha x + \beta x = -\frac{b}{a}$

Also, $\alpha x \times \beta x = \frac{b}{a}$

$\alpha\beta x^2 = \frac{b}{a}$

∴ $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}}$

$$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{b}{a}}$$

$$= \frac{(\alpha + \beta)x}{\sqrt{\alpha\beta x^2}} + \sqrt{\frac{b}{a}}$$

$$= \frac{-\frac{b}{a}}{\sqrt{\frac{b}{a}}} + \sqrt{\frac{b}{a}}$$

$$= -\frac{\sqrt{\frac{b}{a}} \times \sqrt{\frac{b}{a}}}{\sqrt{\frac{b}{a}}} + \sqrt{\frac{b}{a}}$$

$$= -\sqrt{\frac{b}{a}} + \sqrt{\frac{b}{a}}$$

= 0

Hence, option (A) is the correct answer.

12. 24,000

Total income of the family (in thousands)

= (40 – x) (5x + 40)

= 5(x + 8) (40 – x)

= 5 [320 + 32x – x^2]

= 5 [576 – $(x - 16)^2$]

Maximum possible total income will occur at x = 16.

∴ Average monthly Income per member

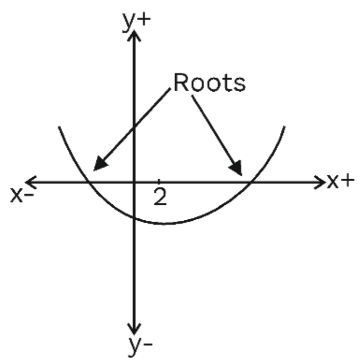
= (40 – x) = (40 – 16)

= 24 thousand Rupees

13. (D)

Given $f(x) = x^2 - 3k - 28$

Here the coefficient of x^2 is 1 i.e. > 0, this implies the graph will be an upward parabola.



Since '2' lies b/w the roots, which implies the value of f(x) at x = 2 is less than 0.

Therefore, $f(2) < 0$

$4 - 3k - 28 \quad < \quad 0$

$- 3K - 24 \quad < \quad 0$

$-3k < \quad 24$

$K \quad > \quad -8$

Hence, option (D) is the correct answer.

14. (D)

According to the question; x, y > 1

Also (x + 15) chocolates costs Rs. 3 that means y < 3

$\Rightarrow$ The only possible integer value of y = 2 = 200 paise.

Now, each chocolate cost can be compared.

It is given that

$$\frac{200}{x} - \frac{300}{x+15} = \frac{40}{12}$$

$$\Rightarrow \quad \frac{20}{x} - \frac{30}{x+15} = \frac{1}{3}$$

$$\Rightarrow \quad 20x + 300 - 30x = \frac{1}{3}\left(x^2 + 15x\right)$$

$\Rightarrow$ $x^2 + 45x - 900 = 0$

$\Rightarrow$ $x = 15, -60$

But $x \neq -60$

Therefore x = 15

Hence, option (D) is the correct answer.

15. (C)

We have, $ax^2 + bx + c = 0$

Now, condition for real roots:

$b^2 - 4ac \geq 0$ or $b^2 \geq 4ac$

Now we will take different values of b from the set.

Taking b = 1, not possible

Taking b = 2, not possible

Taking b = 3, then a = 1

and c = 2 or a = 2 and c = 1

Therefore, only 2 quadratic equations can be formed.

Hence, option (C) is the correct answer.

16. (A)

For equal roots $b^2 - 4ac = 0$

$b^2 - 4 \times a \times 4 = 0$

or $b^2 = 16a$

or $a = (b/4)^2$ -------------I

Now in I, R.H.S is a perfect square, so L.H.S must also be a perfect square and must also be positive as perfect squares are never negative.

So, possible values of a are 1, 4, 9, 16, 25 ...81 and 100 and accordingly the values of b would be ± 4, ± 8, ± 12 – – – and ± 40

Hence, 20 ordered pairs of (a, b) are possible

Hence, option (A) is the correct answer.

17. (C)

Given: log3, $\log(3^x - 1)$ and $\log(3^x + 1)$ are in arithmetic progression.

Since log3, $\log(3^x - 1)$ and $\log(3^x + 1)$ are in arithmetic progression

$2.\log(3^x - 1) = \log 3 + \log(3^x + 1)$

$\Rightarrow \quad \log(3^x - 1)^2 = \log\{3(3^x + 1)\}$

$\Rightarrow\ (3^x - 1)^2 = 3(3^x + 1)$

$\Rightarrow\ 3^{2x} - 2.3^x + 1 = 3.3^x + 3$

Putting $3^x = y$

$y^2 - 5y - 2 = 0$

Solving above equation, $y = \frac{(\sqrt{33}+5)}{2}$,

$\left\{\text{Neglecting}\quad y \neq \frac{(-\sqrt{33}+5)}{2}\right\}$

$y = \frac{(\sqrt{33}+5)}{2}$

$= \left\{\frac{(\text{a value between 5 and 6})+5}{2}\right\}$

= a value between 5 and 6

$\Rightarrow\ 3^x = \frac{(\sqrt{33}+5)}{2} =$ a value between 5 and 6, which lies between 3^1 and 3^2

Hence, x belongs to (1, 2)

Hence, option (C) is the correct answer.

18. (D)

$a^2 = a + 1$ (is given in the question)

$\therefore\ (a^2)^2 = (a + 1)^2$

$a^4 = a^2 + 2a + 1$ ...(1)

Since $a^2 = a + 1$ (is given)

Put this value of $a^2 = a + 1$ in equation (1)

$\therefore\ a^4 = a + 1 + 2a + 1$

$a^4 = 3a + 2$

$\therefore\ 5a^4 = 5(3a + 2)$

$5a^4 = 15a + 10$ ...(2)

We can find the value 'a' from the equation

$a^2 = a + 1 \qquad \left(\text{Use } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$

$a^2 - a - 1 = 0$

$\therefore\ a = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times 1\times(-1)}}{2\times 1}$

$a = \frac{1 \pm \sqrt{1+4}}{2\times 1}$

$a = \frac{1 \pm \sqrt{5}}{2}$

Since a > 0

$\therefore\ a = \frac{1+\sqrt{5}}{2}$

Put the value of 'a' in equation (2)

$5a^4 = 15a + 10$

$= 15\left(\frac{1+\sqrt{5}}{2}\right) + 10$

$= \frac{15}{2} + \frac{15\sqrt{5}}{2} + 10$

$$\boxed{5a^4 = \frac{35 + 15\sqrt{5}}{2}}$$

Hence, option (D) is the correct answer.

19. (D)

Here, a = (p – q), b = (q – r) and c = (q – r)

$\therefore\ D = b^2 - 4ac$

$= (q - r)^2 + 4(p - q)(r - p)$

$= q^2 + r^2 - 2qr - 4(pr - p^2 - qr + pq)$

$= q^2 + r^2 - 2qr - 4pr + 4p^2 + 4qr - 4pq$

$= (-2p)^2 + q^2 + r^2 + 2(-2p)q + 2qr + 2r(-2p)$

$= (-2p + q + r)^2$

For real and equal roots, we must have D = 0

$\Rightarrow\ (-2p + q + r)^2 = 0$

$\Rightarrow\ -2p + q + r = 0$

$\Rightarrow\ 2p = q + r$

20. 18

Let α and β be the roots of quadratic equation

$x^2 - ax + 16 = 0$

We have, $\alpha + \beta = a$ and $\alpha\beta = 16$

Since, roots are real and distinct, D > 0

$\Rightarrow\ a^2 - 4\ 16 > 0$

$\Rightarrow\ |a| > 8$

$\Rightarrow\ a < -8$ or $a > 8$ ...(1)

Now, $(\alpha-\beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow\ (\alpha - \beta)^2 = a^2 - 64$

Given, $\alpha - \beta \leq 15$

$\Rightarrow\ a^2 - 64 \leq 225$

$\Rightarrow$ $a^2 \leq 225 + 64$

$\Rightarrow$ $a^2 \leq 289$

$\Rightarrow$ $|a| \leq 17$

$\Rightarrow$ $-17 \leq a \leq 17$ ...(2)

From (1) and (2), possible integral values of 'a' are – 9, – 10,........., –16 and –17 and also from 9, 10, 11,........., 16 and 17 = a total of 18 values are possible.

Level of Difficulty – 3

21. (B)

The given quadratic equation is ($x^2 + bx + c = 0$) and its roots are α and β.

$\therefore$ Sum of roots $(\alpha + \beta) = -\frac{b}{1} = -b$ and

product of roots $(\alpha \times \beta) = \frac{c}{1} = c$

Since the roots of the new equations are α^2 and $-\beta^2$.

$\therefore$ The sum of the roots of the new equation

$= \alpha^2 - \beta^2$

$= (\alpha + \beta) \times (\alpha - \beta)$

$= (\alpha + \beta) \times \sqrt{(\alpha - \beta)^2}$

$= (\alpha + \beta) \times \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta}$

$= (\alpha + \beta) \times \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta + 2\alpha\beta - 2\alpha\beta}$

$= (\alpha + \beta) \times \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Put the values of $(\alpha + \beta)$ and $(\alpha . \beta)$ in the above:

$= -b \times \sqrt{(-b)^2 - 4 \times c} = -b\sqrt{b^2 - 4ac}$

Also, the product of the roots of new equation

$= \alpha^2 \times (-\beta^2) = -\alpha^2\beta^2 = -(\alpha\beta)^2 = -c^2$

Therefore, the new equation is

$x^2 + (b\sqrt{b^2 - 4c})x - c^2 = 0$

Hence, option (B) is the correct answer.

22. (C)

$25^{(2x-1)} - 5^{(2x+1)} + 100 = 0$

$\Rightarrow$ $5^{(4x-2)} - 5^{(2x+1)} + 100 = 0$

$\Rightarrow$ $\frac{5^{4x}}{25} - (5^{2x} \times 5) + 100 = 0$

Let's assume $5^{2x} = y$

$\Rightarrow$ $\frac{y^2}{25} - 5y + 100 = 0$

$\Rightarrow$ $y^2 - 125y + 2500 = 0$

$\Rightarrow$ $(y - 100)(y - 25) = 0$

$\Rightarrow$ $Y = 100$ or 25

$\Rightarrow$ $5^{2x} = 25$ or $5^{2x} = 100$

$x = 1$ or $x = \log_{25}100$

$= \log_5 10 = \frac{1}{0.7} = 1.428$

Thus, the sum of all possible values of $x = K = 1 + 1.428 = 2.428$ which lies between 2 and 3

Hence, option (C) is the correct answer.

23. (A)

$2x^2 - 5x + 4 = 0$

Sum of roots $= \frac{5}{2} = a + b$

Product of roots $= \frac{4}{2} = 2 = ab$

Required sum

$= a^2 + b^2 + a^3 + b^3 + a^4 + b^4 + ...$

$= (a^2 + a^3 + a^4 + ...) + (b^2 + b^3 + b^4 + ...)$

$= \frac{a^2}{1-a} + \frac{b^2}{1-b}$

(Sum to infinite terms in G.P.)

$= \frac{a^2(1-b) + b^2(1-a)}{(1-a)(1-b)}$

$= \frac{a^2 - a^2b + b^2 - ab^2}{1 - b - a + ab}$

$= \frac{a^2 + b^2 - ab(a+b)}{1 - (a+b) + ab}$

We have values for (a +b) and ab.

To find $a^2 + b^2$:

$(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow \quad a^2 + b^2 = (a + b)^2 - 2ab$

$\Rightarrow \quad a^2 + b^2 = \frac{25}{4} - 2(2) = \frac{9}{4}$

$\Rightarrow$ Required Sum = $\frac{\frac{9}{4} - 2\left(\frac{5}{2}\right)}{1 - \frac{5}{2} + 2} = \frac{-\frac{11}{4}}{\frac{1}{2}} = -\frac{11}{2}$

Hence, option (A) is the correct answer.

24. 1,538

As α and β are the roots of the equation

$3x^2 + 27x + 15 = 0$

$x^2 + 9x + 5 = 0$

$\therefore \quad \alpha^2 + 9\alpha + 5 = 0$

$\alpha^2 + 9\alpha = -5$

Therefore, $(2\alpha + 9)^2 = 4\alpha^2 + 36\alpha + 81$

$= 4\alpha(\alpha + 9) + 81$

$= 4(\alpha^2 + 9\alpha) + 81$

Put the value of $\alpha^2 + 9\alpha = -5$ in the above equation.

$(2\alpha + 9)^2 = 4(\alpha^2 + 9\alpha) + 81$

$= 4x(-5) + 81 = -20 + 81 = 61$

Similarly 'β' is also the root of the equation

$3x^2 + 27x + 15 = 0$

$\therefore \quad x^2 + 9x + 5 = 0$

$\beta^2 + 9\beta + 5 = 0$

$\beta^2 + 9\beta = -5$

Therefore, $(2\beta + 9)^2 = 4\beta^2 + 36\beta + 81$

$= 4(\beta^2 + 9\beta) + 81 = -20 + 81 = 61$

Hence the value of

$$\left(\frac{37}{2\alpha + 9}\right)^2 + \left(\frac{13}{2\beta + 9}\right)^2 = \frac{37^2}{61} + \frac{13^2}{61}$$

$$= \frac{1369}{61} + \frac{169}{61} = \frac{1538}{61}$$

Hence, $61\left(\frac{37}{2\alpha + 9}\right)^2 + 61\left(\frac{13}{2\beta + 9}\right)^2 = 1538$

25. (D)

If any quadratic equation has real roots then its discriminant must be greater than or equal to 'O' (D > 0)

$\therefore \quad x^2 + px + 2q = 0$

$[D = b^2 - 4ac > 0]$

$\therefore \quad p^2 - 4 \times 1 \times 2q > 0$

$p^2 - 8q > 0$

$\mathbf{p^2 > 8q}$ ....(i)

Again, Discriminant of quadratic equation

$x^2 + 2qx + p = 0$ is:

$(2q)^2 - 4 \times 1 \times p > 0$

(since roots are real)

$4q^2 - 4p > 0$

$4q^2 > 4p$

$\mathbf{q^2 > p}$ ....(ii)

Now, squaring both sides of the equation ....(ii)

$\therefore \quad (q^2)^2 > p^2$

$q^4 > p^2$

Since, $p^2 > 8q$ from equation 1.

$\therefore \quad q^4 > 8q$

$q^3 > 8$

$q > 2$

Again

$p^2 > 8q$

$P^2 > 8 \times 2$

$p^2 > 16$

$p \geq 4$

Since we have to find the smallest possible value of p + q.

Therefore, by taking p = 4 and q = 2. we will get the smallest value.

$\therefore \quad p + q = 4 + 2 = 6$

Hence, option (D) is the correct answer.

26. (B)

Given, and are the roots of quadratic equation $x^2 - 7x + 6 = 0$

$$\alpha + \beta = -\frac{(-7)}{1} = 7 \text{ and } \alpha\beta = \frac{6}{1} = 6 \quad ...(i)$$

$\therefore \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow \quad (\alpha - \beta)^2 = (7)^2 - 4(6) = 25$

$\Rightarrow \quad \alpha - \beta = 5 \quad ...(ii)$

The quadratic equation whose roots are $(\alpha + \beta)^3$ and $(\alpha - \beta)^3$

$= x^2 - [(\alpha + \beta)^3 + (\alpha - \beta)^3]\, x + (\alpha + \beta)^3.(\alpha - \beta)^3 = 0$

Using equations (i) and (ii), we get:

$x^2 - [(7)^3 + (5)^3]\, x + (7)^3.(5)^3 = 0$

$\Rightarrow \quad x^2 - (343 + 125)\, x + 343(125) = 0$

$\Rightarrow \quad x^2 - 468x + 42875 = 0$, is the required quadratic equation.

Hence, option (B) is the correct answer.

27. (B)

The given equation is $px^2 + qx + r = 0$

$\therefore \quad$ Sum of the roots $(\alpha + \beta) = -\dfrac{q}{p}$

Also, product of the roots $(\alpha\beta) = \dfrac{r}{p}$

Now, according to the condition given in the question

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$\alpha + \beta = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\alpha + \beta = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$-\frac{q}{p} = \frac{\left(-\frac{q}{p}\right)^2 - 2 \times \frac{r}{p}}{\left(\frac{r}{p}\right)^2}$$

$$-\frac{q}{p} = \frac{q^2 - 2pr}{r^2}$$

$-r^2q = pq^2 - 2p^2r$

$2p^2r - pq^2 = r^2q$

$p^2r - r^2q = pq^2 - p^2r$

Since this is the condition of Arithmetic progression.

Therefore, qr^2, p^2r, pq^2 are in A:P.

Hence option (B) is the correct answer.

28. (B)

For the quadratic equation

$x^2 - 4x + 10 = 0$

$$m + n = -\frac{(-4)}{1} = 4 \text{ and } m \times n = \frac{10}{1} = 10$$

Therefore, $(1 + m + n^2) \times (1 + n + m^2)$

$= 1 + n + m^2 + m + mn + m^3 + n^2 + n^3 + n^2m^2$

$= 1 + (m + n) + (m^2+n^2) + m^3 + n^3 + m^2n^2 + mn$ ------- Equation (1)

Since, $(m + n)^2 = m^2 + n^2 + 2mn$

$\therefore \quad (4)^2 = m^2 + n^2 + 2 \times 10$

$\quad m^2 + n^2 = 16 - 20 = -4$

Again, $(m + n)^3 = m^3 + n^3 + 3mn\,(m + n)$

$4^3 = m^3 + n^3 + 3 \times 10 \times 4$

$m^3 + n^3 = 64 - 120$

$m^3 + n^3 = -56$

Now, put all these values in Equation (1)

$(1 + m + n^2) \times (1 + n + m^2)$

$= 1 + (m + n) + (m^2 + n^2) + (m^3 + n^3) + (m \times n)^2 + m \times n$

$= 1 + 4 + (-4) + (-56) + 10^2 + 10$

$= 1 - 56 + 100 + 10 = 111 - 56 = 55$

Hence, option (B) is the correct answer.

29. (A)

$3(a - 3)^2 + (b - 2)^2 - 2(a - 2)^2$

$= 3(a^2 + 9 - 6a) + (b^2 + 4 - 4b) - 2(a^2 + 4 - 4a)$

$= 3a^2 + 27 - 18a + b^2 + 4 - 4b - 2a^2 - 8 + 8a$

$= a^2 - 10a + b^2 - 4b + 23$

$= (a^2 - 2 \times 5a + 25) + (b^2 - 2.2b + 4) + 23 - 25 - 4$

$= (a - 5)^2 + (b - 2)^2 - 6$

Since the minimum value for the square term is '0'.

$\therefore$ Least value of the above expression $= -6$

Hence, option (A) is the correct answer.

30. (A)

Since it is given in the question that a, b, c are rational numbers.

Also, rational numbers can be written in the form of $\frac{p}{q}$.

Thus, the sum of the roots and product of the roots must be rational numbers.

For this, both the roots must be conjugate in nature.

$\therefore$ First root $= 7 + 4\sqrt{3}$

2^{nd} root $= 7 - 4\sqrt{3}$

$\therefore$ Sum of the roots $= 7 + 4\sqrt{3} + 7 - 4\sqrt{3}$

$= 14$

Also, the product of the roots

$= \left(7 + 4\sqrt{3}\right) \times \left(7 - 4\sqrt{3}\right) = 49 - 48 = 1.$

Also, we know that: The sum of the roots

$= -\left(\frac{-b}{a}\right) = \frac{b}{a}$

and the product of the roots $= \frac{c}{a}$

Again, $\frac{b}{a} = 14$

$b = 14a$ ....(1)

and $\frac{c}{a} = 1$

$c = a$ ....(2)

Also, it is given in question $b = c^3$.

$\therefore$ $14a = a^3$

$a^2 = 14$

Therefore, $3a^2 = 3 \times 14 = 42$

Hence option (A) is the correct answer.