DAY EIGHT

Rotational Motion

Learning & Revision for the Day

- Concept of Rotational Motion
- Equation of Rotational Motion
- Moment of Force or Torque
- Angular Momentum
- Law of Conservation of Angular Momentum
- Equilibrium of a Rigid Bodies
- Rigid Body Rotation

Concept of Rotational Motion

In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

- Rotational motion is characterised by angular displacement $d\theta$ and angular velocity $\omega = \frac{d\theta}{dt}$.
- If angular velocity is not uniform, then rate of change of angular velocity is called the **angular acceleration**.

  Angular acceleration, $\alpha = \frac{d\omega}{dt}$.

  SI unit of angular acceleration is rad/s^2.
- Angular acceleration α and linear tangential acceleration $\mathbf{a}_t$ are correlated as $\mathbf{a}_t = \alpha \times \mathbf{r}$.

Equation of Rotational Motion

If angular acceleration α is uniform, then equations of rotational motion may be written as

(i) $\omega = \omega_0 + \alpha t$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

(iii) $\omega^2 - \omega_0^2 = 2\alpha\theta$

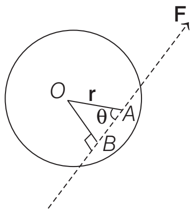
(iv) $\theta_{nth} = \omega_0 + \frac{\alpha}{2}(2n-1)$

Moment of Force or Torque

Torque (or moment of a force) is the turning effect of a force applied at a point on a rigid body about the axis of rotation.

Mathematically, torque, $\tau = \mathbf{r} \times \mathbf{F} = |\mathbf{r} \times \mathbf{F}|\,\hat{\mathbf{n}} = rF\sin\theta\,\hat{\mathbf{n}}$

where, $\hat{\mathbf{n}}$ is a unit vector along the axis of rotation. Torque is an axial vector and its SI unit is newton-metre (N-m).



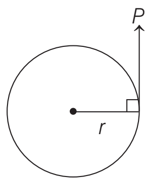
- The torque about axis of rotation is independent of choice of origin O, so long as it is chosen on the axis of rotation AB.
- Only normal component of force contributes towards the torque. Radial component of force does not contribute towards the torque.
- A torque produces angular acceleration in a rotating body. Thus, torque, $\tau = I\alpha$
- Moment of a couple (or torque) is given by product of position vector **r** between the two forces and either force **F**. Thus, $\tau = \mathbf{r} \times \mathbf{F}$
- If under the influence of an external torque, τ the given body rotates by $d\theta$, then work done, $dW = \tau \cdot d\theta$.
- In rotational motion, power may be defined as the scalar product of torque and angular velocity, i.e. Power $P = \tau \cdot \omega$.

Angular Momentum

The moment of linear momentum of a given body about an axis of rotation is called its angular momentum. If $\mathbf{p} = m\mathbf{v}$ be the linear momentum of a particle and **r** is its position vector from the point of rotation, then

Angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p} = rp\sin\theta\,\hat{\mathbf{n}} = mvr\sin\theta\,\hat{\mathbf{n}}$

where, $\hat{\mathbf{n}}$ is a unit vector in the direction of rotation. Angular momentum is an axial vector and its SI unit is kg-m^{2}s^{-1} or J-s.



- For rotational motion of a rigid body, **angular momentum** is equal to the product of angular velocity and moment of inertia of the body about the axis of rotation. Mathematically, $L = I\omega$.
- According to the second law of rotational motion, the rate of change of angular momentum of a body is equal to the external torque applied on it and takes place in the direction of torque. Thus,

$$\tau = \frac{dL}{dt} = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha \qquad \left[\because \alpha = \frac{d\omega}{dt}\right]$$

- Total effect of a torque applied on a rotating body in a given time is called angular impulse. **Angular impulse** is equal to total change in angular momentum of the system in given time. Thus, angular impulse,

$$J = \int_0^{\Delta t} \tau\, dt = \Delta L = L_f - L_i$$

- The angular momentum of a system of particles about the origin is $L = \sum^{n} r_i \times p_i$

Law of Conservation of Angular Momentum

According to the law of conservation of angular momentum, if no external torque is acting on a system, then total vector sum of angular momentum of different particles of the system remains constant.

We know that, $\dfrac{dL}{dt} = \tau_{\text{ext}}$

Hence, if $\tau_{\text{ext}} = 0$, then $\dfrac{dL}{dt} = 0 \Rightarrow L = \text{constant}$

Therefore, in the absence of any external torque, total angular momentum of a system must remain conserved.

Comparison of Linear and Rotational Motion

	Linear Motion	Rotational Motion
1.	Linear momentum, $p = mv$	Angular momentum, $L = I\omega$, $L = \sqrt{2IE}$
2.	Force, $F = ma$	Torque, $T = I\alpha$
3.	Electric energy, $E = \frac{1}{2}mv^2$	Rotational energy, $E = \frac{1}{2}I\omega^2$

Equilibrium of a Rigid Bodies

For mechanical equilibrium of a rigid body, two condition need to be satisfied.

1. Translational Equilibrium

A rigid body is said to be in translational equilibrium, if it remains at rest or moving with a constant velocity in a particular direction. For this, the net external force or the vector sum of all the external forces acting on the body must be zero,

i.e. $\mathbf{F} = 0$ or $F = \Sigma F_i = 0$

2. Rotational Equilibrium

A rigid body is said to be in rotational equilibrium, if the body does not rotate or rotates with constant angular velocity. For this, the net external torque or the vector sum of all the torques acting on the body is zero.

For the body to be in rotational equilibrium,

$$\tau_{\text{ext}} = 0, \frac{dL}{dt} = 0 \text{ or } \sum \tau_i = 0$$

Rigid Body Rotation

Spinning

When the body rotates in such a manner that its axis of rotation does not move, then its motion is called spinning motion.

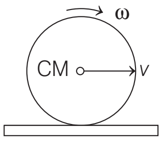
In spinning rotational kinetic energy is given by, $K_R = \frac{1}{2}I\omega^2$.

Rotational kinetic energy is a scalar having SI unit joule (J). Rotational kinetic energy is related to angular momentum as per relation,

$$K_R = \frac{L^2}{2I} \text{ or } L = \sqrt{2IK_R}$$

Pure Rolling Motion

Let a rigid body, having symmetric surface about its centre of mass, is being spined at a certain angular speed and placed on a surface, so that plane of rotation is perpendicular to the surface. If the body is simultaneously given a translational motion too, then the net motion is called **rolling motion.**



The total kinetic energy in rolling motion,

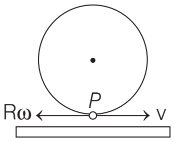
$$K_N = K_R + K_T$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mv^2\left(\frac{K^2}{R^2}\right)$$

$$K_N = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$$

Rolling Without Slipping

If the given body rolls over a surface such that there is no relative motion between the body and the surface at the point of contact, then the motion is called **rolling without slipping**.

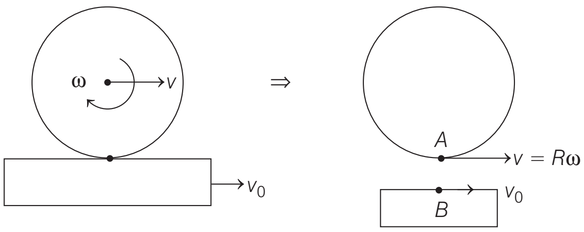


Impure Rolling Motion

In impure rolling motion, the point of contact of the body with the platform is not relatively at rest w.r.t. platform on which, it is performing rolling motion, as a result sliding occurs at point of contact.

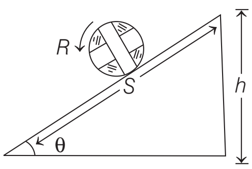
For impure rolling motion, $v_{AB} \neq 0$ i.e. $v - R\omega \neq v_0$

If platform is stationary, i.e. $v_0 = 0$, then $v \neq R\omega$



Rolling on an Inclined Plane

When a body of mass m and radius R rolls down on inclined plane of height h and angle of inclination θ, it loses potential energy. However, it acquires both linear and angular speeds and hence gain kinetic energy of translation and that of rotation.



By conservation of mechanical energy, $mgh = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$

- **Velocity at the lowest point** $v = \sqrt{\dfrac{2gh}{1 + \dfrac{K^2}{R^2}}}$
- **Acceleration in motion** From second equation of motion, $v^2 = u^2 + 2as$

  By substituting $u = 0$, $s = \dfrac{h}{\sin\theta}$ and $v = \sqrt{\dfrac{2gh}{1 + \dfrac{K^2}{R^2}}}$, we get

$$a = \frac{g\sin\theta}{1 + \dfrac{K^2}{R^2}}$$

- **Time of descent** From first equation of motion, $v = u + at$

  By substituting $u = 0$ and value of v and a from above expressions $t = \dfrac{1}{\sin\theta}\sqrt{\dfrac{2h}{g}\left[1 + \dfrac{K^2}{R^2}\right]}$

  From the above expressions, it is clear that,

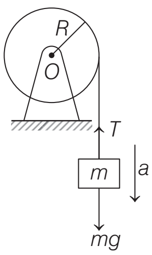
$$v \propto \frac{1}{\sqrt{1 + \dfrac{K^2}{R^2}}};\ a \propto \frac{1}{1 + \dfrac{K^2}{R^2}};\ t \propto \sqrt{1 + \dfrac{K^2}{R^2}}$$

Important Terms Related to Inclined Plane

- Here, factor $\left(\dfrac{K^2}{R^2}\right)$ is a measure of moment of inertia of a body and its value is constant for given shape of the body and it does not depend on the mass and radius of a body.
- Velocity, acceleration and time of descent (for a given inclined plane) all depends on $\dfrac{K^2}{R^2}$. Lesser the moment of inertia of the rolling body lesser will be the value of $\dfrac{K^2}{R^2}$. So, greater will be its velocity and acceleration and lesser will be the time of descent.

Rotation about Axis through Centre of Mass (Centroidal Rotation)

When an object of given mass is tied to a light string wound over a pulley whose moment of inertia is I and radius R as shown in the figure. The wheel bearing is frictionless and the string does not slip on the rim, then



Tension in the string is

$$T = \frac{I}{I + mR^2} \times mg$$

and acceleration, $a = \dfrac{mR^2}{I + mR^2} \cdot g$

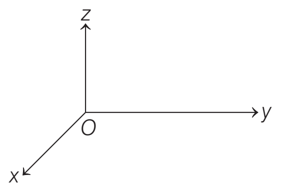
DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 When a body is projected at an angle with the horizontal in the uniform gravitational field of the earth, the angular momentum of the body about the point of projection, as it proceeds along the path

(a) remains constant
(b) increases continuously
(c) decreases continuously
(d) initially decreases and after reaching highest point increases

2 A force of $-F\hat{\mathbf{k}}$ acts on O, the origin of the coordinate system. The torque about the point $(1, -1)$ is



(a) $F(\hat{\mathbf{i}} - \hat{\mathbf{j}})$ (b) $-F(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (c) $F(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) $-F(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

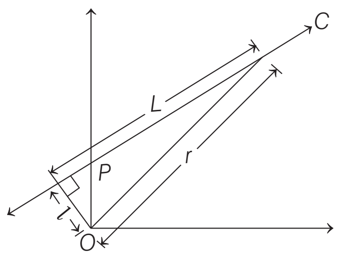
3 Angular momentum of the particle rotating with a central force is constant due to

(a) constant force (b) constant linear momentum
(c) zero torque (d) constant torque

4 A particle of mass $m = 3$ kg moves along a straight line $4y - 3x = 2$, where x and y are in metre, with constant velocity $v = 5\,\text{ms}^{-1}$. The magnitude of angular momentum about the origin is

(a) $12\,\text{kg-m}^2\,\text{s}^{-1}$ (b) $8.0\,\text{kg-m}^2\,\text{s}^{-1}$
(c) $6.0\,\text{kg-m}^2\,\text{s}^{-1}$ (d) $4.5\,\text{kg-m}^2\,\text{s}^{-1}$

5 A particle of mass m moves along line PC with velocity v as shown in the figure. What is the angular momentum of the particle about O?



(a) mvL (b) mvl (c) mvr (d) zero

6 A particle of mass 5 g is moving with a uniform speed of $3\sqrt{2}\,\text{cm s}^{-1}$ is in the xy-plane along the line $y = 2\sqrt{5}$ cm. The magnitude of its angular momentum about the origin in $\text{g-cm}^2\,\text{s}^{-1}$ is

(a) zero
(b) 30
(c) $30\sqrt{2}$
(d) $30\sqrt{10}$

7 A bob of mass m attached to an inextensible string of length l is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical support. About the point of suspension → **JEE Main 2014**

(a) angular momentum is conserved
(b) angular momentum changes in magnitude but not in direction
(c) angular momentum changes in direction but not in magnitude
(d) angular momentum changes both in direction and magnitude

8 A particle of mass 2 kg is moving such that at time t, its position, in metre, is given by $\mathbf{r}(t) = 5\hat{\mathbf{i}} - 2t^2\hat{\mathbf{j}}$. The angular momentum of the particle at $t = 2$ s about the origin in $\text{kg m}^{-2}\,\text{s}^{-1}$ is → **JEE Main (Online) 2013**

(a) $-80\hat{\mathbf{k}}$
(b) $(10\hat{\mathbf{i}} - 16\hat{\mathbf{j}})$
(c) $-40\hat{\mathbf{k}}$
(d) $40\hat{\mathbf{k}}$

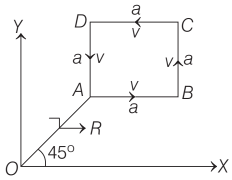
9 A uniform disc of radius a and mass m, is rotating freely with angular speed ω in a horizontal plane, about a smooth fixed vertical axis through its centre. A particle, also of mass m, is suddenly attached to the rim of the disc and rotates with it. The new angular speed is

(a) $\omega/6$
(b) $\omega/3$
(c) $\omega/2$
(d) $\omega/5$

10 If the radius of the earth contracts $\frac{1}{n}$ of its present day value, length of the day will be approximately.

(a) $\frac{24}{n}$ h (b) $\frac{24}{n^2}$ h
(c) $24\,n$h (d) $24\,n^2$h

11 A particle of mass m is moving along the side of a square of side a, with a uniform speed v in the X-Y plane as shown in the figure.



Which of the following statements is false for the angular momentum **L** about the origin? → **JEE Main 2016 (Offline)**

(a) $\mathbf{L} = -\frac{mv}{\sqrt{2}} R\,\hat{\mathbf{k}}$, when the particle is moving from A to B.

(b) $\mathbf{L} = mv\left(\frac{R}{\sqrt{2}} + a\right)\hat{\mathbf{k}}$, when the particle is moving from B to C.

(c) $\mathbf{L} = mv\left(\frac{R}{\sqrt{2}} - a\right)\hat{\mathbf{k}}$, when the particle is moving from C to D.

(d) $\mathbf{L} = \frac{mv}{\sqrt{2}} R\,\hat{\mathbf{k}}$, when the particle is moving from D to A.

12 A uniform rod AB of mass m and length l at rest on a smooth horizontal surface. An impulse P is applied to the end B. The time taken by the rod to turn through at right angle is

(a) $2\pi\frac{ml}{P}$ (b) $2\frac{\pi P}{ml}$ (c) $\frac{\pi}{12}\frac{ml}{P}$ (d) $\frac{\pi P}{ml}$

13 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad/s. The radius of the cylinder is 0.25 m. Then the kinetic energy associated with the rotation of the cylinder and the magnitude of angular momentum of the cylinder about its axis is respectively

(a) 3200 J, 62.5 J-s (b) 3125 J, 62.5 J-s
(c) 3500 J, 68 J-s (d) 3400 J, 63.5 J-s

14 A ring and a disc having the same mass, roll without slipping with the same linear velocity. If the kinetic energy of the ring is 8 J, that of the disc must be

(a) 2 J (b) 4 J (c) 6 J (d) 16 J

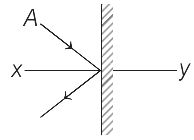
15 A round uniform body of radius R, mass M and moment of inertia I, rolls down (without slipping) an inclined plane making an angle θ with the horizontal. Then, its acceleration is

(a) $\frac{g\sin\theta}{1 + I/MR^2}$ (b) $\frac{g\sin\theta}{1 + MR^2/I}$ (c) $\frac{g\sin\theta}{1 - I/MR^2}$ (d) $\frac{g\sin\theta}{1 - MR^2/I}$

Direction (Q. Nos. 16-18) *Each of these questions contains two statements : Statement* I *and Statement* II. *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*), (*d*) *given here.*

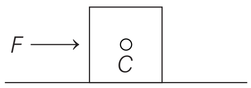
(a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

16 **Statement I** A disc A moves on a smooth horizontal plane and rebounds elastically from a smooth vertical wall (Top view is as shown in the figure) in this case about any point on line xy. The angular momentum of the disc remains conserved.



Statement II About any point in the plane, the torque experienced by disc is zero as gravity force and normal contact force balance each other.

17 **Statement I** A block is kept on a rough horizontal surface, under the action of a force F as shown in the figure. The torque of normal contact force about centre of mass is having zero value.



Statement II The point of application of normal contact force may pass through centre of mass.

18 **Statement I** Two cylinders, one hollow (metal) and the other solid (wood) with the same mass and identical dimensions are simultaneously allowed to roll without slipping down an inclined plane from the same height. The solid cylinder will reach the bottom of the inclined plane first.

Statement II By the principle of conservation of energy, the total kinetic energies of both the cylinders are equal, when they reach the bottom of the incline.

DAY PRACTICE SESSION 2

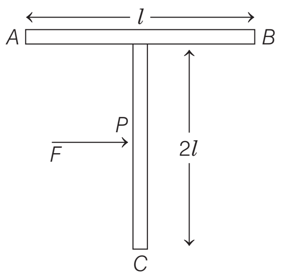
PROGRESSIVE QUESTIONS EXERCISE

1 A thin circular ring of mass m and radius R is rotating about its axis with a constant angular velocity ω. Two objects each of mass M are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with an angular velocity ω'

(a) $\frac{\omega(m+2M)}{m}$ (b) $\frac{\omega(m-2M)}{(m+2M)}$

(c) $\frac{\omega m}{(m+M)}$ (d) $\frac{\omega m}{(m+2M)}$

2 A T shaped object with dimensions as shown in the figure, is lying on a smooth floor. A force is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C.

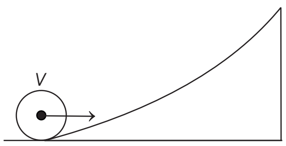


(a) $\frac{2}{3}l$ (b) $\frac{3}{2}l$ (c) $\frac{4}{3}l$ (d) l

3 A thin uniform rod of length l and mass m is swinging freely about horizontal axis passing through its end. Its maximum angular speed is ω. Its centre of mass rises to a maximum height of

(a) $\frac{1}{3}\frac{l^2\omega^2}{g}$ (b) $\frac{1}{6}\frac{l\omega}{g}$ (c) $\frac{1}{2}\frac{l^2\omega^2}{g}$ (d) $\frac{1}{6}\frac{l^2\omega^2}{g}$

4 A small object of uniform density rolls up a curved surface with an initial velocity v. It reached up to a maximum height of $\frac{3v^2}{4g}$ with respect to the initial position. The object is



(a) ring (b) solid sphere
(c) hollow sphere (d) disc

5 A ring of mass M and radius R is rotating about its axis with angular velocity ω. Two identical bodies each of mass m are now gently attached at the two ends of a diameter of the ring. Because of this, the kinetic energy loss will be **→ JEE Main (Online) 2013**

(a) $\frac{m(M+2m)}{M}\omega^2R^2$ (b) $\frac{Mm}{(M+2m)}\omega^2R^2$

(c) $\frac{Mm}{(M-2m)}\omega^2R^2$ (d) $\frac{(M+m)M}{(M+2m)}\omega^2R^2$

6 A hoop of radius r and mass m rotaing with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity of the centre of the hoop, when it ceases to slip? **→ JEE Main 2013**

(a) $\frac{r\omega_0}{4}$ (b) $\frac{r\omega_0}{3}$

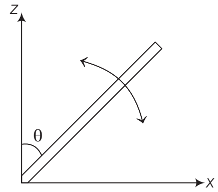
(c) $\frac{r\omega_0}{2}$ (d) $r\omega_0$

7 An annular ring with inner and outer radii R_1 and R_2 is rolling without slipping with a uniform angular speed. The ratio of the forces experienced by the two particles situated on the inner and outer parts of the ring, $\frac{F_1}{F_2}$ is

(a) $\frac{R_2}{R_1}$ (b) $\left(\frac{R_1}{R_2}\right)^2$

(c) 1 (d) $\frac{R_1}{R_2}$

8 A slender uniform rod of mass M and length l is pivoted at one end so that it can rotate in a vertical plane (see the figure). There is negligible friction at the pivot. The free end is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical, is



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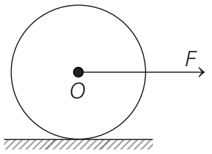
(a) $\frac{2g}{3l}\sin\theta$ (b) $\frac{3g}{2l}\cos\theta$ (c) $\frac{2g}{3l}\cos\theta$ (d) $\frac{3g}{2l}\sin\theta$

9 A horizontal force F acts on the sphere at its centre as shown. Coefficient of friction between ground and sphere is μ. What is the maximum value of F, for which there is no slipping?



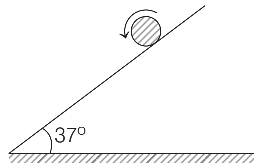
(a) $F \le \frac{5}{2}\mu mg$ (b) $F \le \frac{7}{2}\mu mg$

(c) $F \le \frac{9}{2}\mu mg$ (d) $F \le \frac{3}{2}\mu mg$

10 A solid uniform disc of mass m rolls without slipping down a fixed inclined plank with an acceleration a. The frictional force on the disc due to surface of the plank is

(a) $\frac{1}{4}ma$ (b) $\frac{3}{2}ma$ (c) ma (d) $\frac{1}{2}ma$

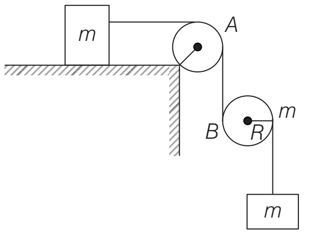
11 A cylinder having radius 0.4 m, initially rotating (at $t = 0$) with $\omega_0 = 54$ rad/s is placed on a rough inclined plane with $\theta = 37°$ having frictional coefficient $\mu = 0.5$. The time taken by the cylinder to start pure rolling is $[g = 10 \text{m/s}^2]$.



(a) 5.4 s (b) 2.4 s

(c) 1.4 s (d) 1.2 s

12 Angular acceleration of the Cylinder (B) shown in the figure is (all strings and pulley are ideal)



(a) $\frac{2g}{3R}$ (b) $\frac{2g}{5R}$

(c) $\frac{2g}{R}$ (d) $\frac{g}{2R}$

ANSWERS

SESSION 1	**1** (b)	**2** (c)	**3** (c)	**4** (c)	**5** (b)	**6** (d)	**7** (c)	**8** (a)	**9** (b)	**10** (b)
	11 (c,d)	**12** (c)	**13** (b)	**14** (c)	**15** (a)	**16** (a)	**17** (a)	**18** (b)		
SESSION 2	**1** (d)	**2** (c)	**3** (d)	**4** (d)	**5** (b)	**6** (c)	**7** (d)	**8** (d)	**9** (b)	**10** (d)
	11 (d)	**12** (b)								

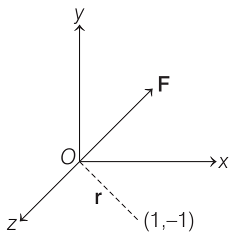
Hints and Explanations

SESSION 1

1 $\tau_0 = mg \times r_\perp$

As, $r_\perp$ is continuously increasing or torque is continuously increasing on the particle. Hence, angular momentum is continuously increasing.

2 $\tau = \mathbf{r} \times \mathbf{F}$



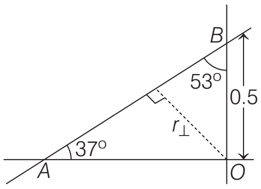
$$\tau = (\hat{\mathbf{i}} - \hat{\mathbf{j}}) \times (-F\hat{\mathbf{k}})$$
$$= F[(-\hat{\mathbf{i}} \times \hat{\mathbf{k}}) + (\hat{\mathbf{j}} \times \hat{\mathbf{k}})]$$
$$= F[\hat{\mathbf{j}} + \hat{\mathbf{i}}]$$
$$= F[\hat{\mathbf{i}} + \hat{\mathbf{j}}]$$

3 Torque due to central force is zero,

$$\tau = \frac{d}{dt}(L) = 0$$

$\therefore$ $L = \text{constant}$

4 Equation of straight line AB is



$$4y - 3x = 2$$

or $$y = \frac{3}{4}x + 0.5$$

$\therefore$ Slope $= \tan\theta = \frac{3}{4}$

or $\theta = 37°$

$$r_\perp = 0.5 \sin 53°$$
$$= 0.4 \text{ m}$$

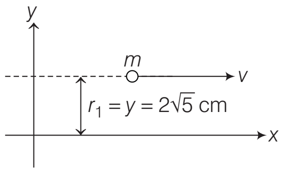
$\therefore$ Angular momentum $(L) = \text{m}v r_\perp$

or $L = 3 \times 5 \times 0.4$
$$= 6 \text{kg - m}^2\text{s}^{-1}$$

5 Angular momentum of particle about O,

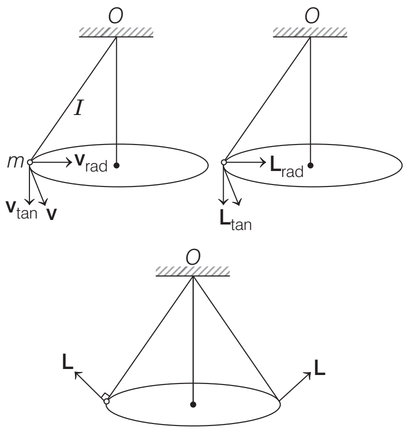
$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$
$$|\mathbf{L}| = mr\, v \sin\theta$$
$$= mv(r\sin\theta)$$
$$= mvl$$

6 $L = mvr_1$



$$= (5)(3\sqrt{2})(2\sqrt{5})$$
$$= 30\sqrt{10} \text{ g- cm}^2\text{s}^{-1}$$

7 Angular momentum of the pendulum about the suspension point O is



$\mathbf{L} = m\,(\mathbf{r} \times \mathbf{v})$

Then, $\mathbf{v}$ can be resolved into two components, radial component r_{rad} and tangential component r_{tan}. Due to v_{rad}, $\mathbf{L}$ will be tangential and due to v_{tan}, $\mathbf{L}$ will be radially outwards as shown. So, net angular momentum will be as shown in figure, whose magnitude will be constant $(|L| = mvl)$. But its direction will change as shown in the figure.

$$\mathbf{L} = m\,(\mathbf{r} \times \mathbf{v})$$

where, r = radius of circle.

8 Given, $m = 2$ kg, $\mathbf{r}(t) = 5\hat{\mathbf{i}} - 2t^2\hat{\mathbf{j}}$

Angular momentum $(L) = \mathbf{r} \times \mathbf{p}$

$\therefore$ Velocity, $v = \frac{dr}{dt} = \frac{d}{dt}(5\hat{\mathbf{i}} - 2t^2\hat{\mathbf{j}})$

$$= -8\hat{\mathbf{j}} \qquad (\text{at } t = 2\text{s})$$

$\therefore$ $p = mv$
$$= 2 \times (-8\hat{\mathbf{j}}) = -16\hat{\mathbf{j}}$$

Therefore,
$$L = \mathbf{r} \times \mathbf{p}$$
$$= (5\hat{\mathbf{i}} - 2t^2\hat{\mathbf{j}}) \times (-16)\hat{\mathbf{j}} \qquad (\text{at } t = 2\text{s})$$
$$= -80\hat{\mathbf{k}}$$

9 Conserving angular momentum

$$I_1\omega_1 = I_2\omega_2$$

$\therefore$ $$\omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{\left(\frac{ma^2}{2}\right)}{\left(\frac{ma^2}{2}\right) + ma^2}\omega = \frac{\omega}{3}$$

10 $I\omega = \text{constant}$

or $\frac{R^2}{T} = \text{constant}$

$$\left(\text{As, } I \propto R^2 \text{ and } \omega \propto \frac{1}{T}\right)$$

$\therefore$ $T \propto R^2$

As, $R' = \frac{1}{n}R$

$\therefore$ $T' = \frac{T}{n^2} = \frac{24}{n^2}$ h

11 For a particle of mass m is moving along the side of a square of side a. Such that Angular momentum $\mathbf{L}$ about the origin

$$= \mathbf{L} = \mathbf{r} \times \mathbf{p}$$
$$= rp\sin\theta\,\hat{\mathbf{n}}$$

or $\mathbf{L} = \mathbf{r}\,(\mathbf{p})\,\hat{\mathbf{n}}$

When a particle is moving from D to A,

$$\mathbf{L} = \frac{R}{\sqrt{2}}\,mv(-\hat{\mathbf{k}})$$

A particle is moving from A to B,

$$\mathbf{L} = \frac{R}{\sqrt{2}}\,mv(-\hat{\mathbf{k}})$$

and it moves from C to D,

$$\mathbf{L} = \left(\frac{R}{\sqrt{2}} + a\right) mv\,(\hat{\mathbf{k}})$$

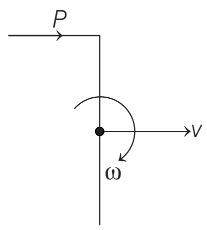
For B to C, we have

$$\mathbf{L} = \left(\frac{R}{\sqrt{2}} + a\right) mv\,(\hat{\mathbf{k}})$$

Hence, options (c) and (d) are incorrect.

12 Angular impulse $= P \times \frac{l}{2}$

= change in angular momentum



$$\therefore \quad \frac{Pl}{2} = I\omega = \left(\frac{ml^2}{12}\right)\omega$$

$$\therefore \quad \omega = \frac{6P}{ml}$$

Now, $t = \frac{\theta}{\omega} = \frac{\pi/2}{6P/ml}$

$$= \frac{\pi ml}{12P}$$

13 Moment of inertia of the solid cylinder about its axis of symmetry,

$$I = \frac{1}{2}MR^2$$
$$= \frac{1}{2} \times 20 \times (0.25)^2$$
$$= 10 \times 0.0625$$
$$= 0.625\,\text{kg-m}^2 = 62.5\,\text{J-s}$$

Kinetic energy associated with the rotation of the cylinder is given by

$$K = \frac{1}{2}I\omega^2$$
$$= \frac{1}{2} \times 0.625 \times (100)^2$$
$$= 0.3125 \times 10000 = 3125\,\text{J}$$

14 For ring, $\frac{K_R}{K_T} = 1$

For disc, $\frac{K_R}{K_T} = \frac{1}{2}$

For ring, total kinetic energy

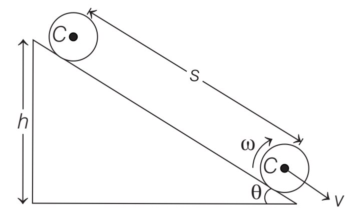
$= 2(K_T) = 8$ J (given)

Hence, $K_T = 4$ J

For disc, total kinetic energy $= \frac{3}{2}K_T$

$$= \frac{3}{2} \times 4 = 6\text{ J}$$

15 Assuming that no energy is used up against friction, the loss in potential energy is equal to the total gain in the kinetic energy.



i.e. $Mgh = \frac{1}{2}I\left(\frac{v^2}{R^2}\right) + \frac{1}{2}Mv^2$

or $\frac{1}{2}v^2\left(M + \frac{I}{R^2}\right) = Mgh$

or $v^2 = \frac{2Mgh}{M + I/R^2} = \frac{2gh}{1 + I/MR^2}$

If s be the distance covered along the plane, then

$$h = s\sin\theta$$

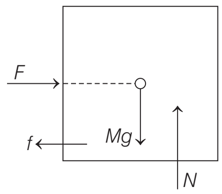
$$\therefore \quad v^2 = \frac{2gs\sin\theta}{1 + I/MR^2}$$

Now, $v^2 = 2as$

$$\therefore 2as = \frac{2gs\sin\theta}{1 + I/MR^2} \Rightarrow a = \frac{g\sin\theta}{1 + I/MR^2}$$

16 The forces experienced by disc are gravity and normal contact force. In addition to these, impact force (during collision) will act on the disc along line xy. Gravity and normal contact force balance each other (in terms of force and torque both), but impact force causes non-zero torque acting on disc about all points except the points lying on its line of action i.e. xy. So, angular momentum remains conserved about any point on xy.

17 Here, as the block is kept in equilibrium, the net torque experienced by the body about any point has to be zero. Here, due to F and Mg about C, the torque is zero but friction is providing non-zero torque in clockwise direction, now other force is N only which can produce non-zero force in anti-clockwise direction to make net torque zero and it is possible only when N does not pass through C.



18 In case of pure rolling on inclined plane,

$$a = \frac{g\sin\theta}{1 + I/mR^2}$$

$$I_{\text{Solid}} < I_{\text{Hollow}}$$

$$\therefore \quad a_{\text{Solid}} > a_{\text{Hollow}}$$

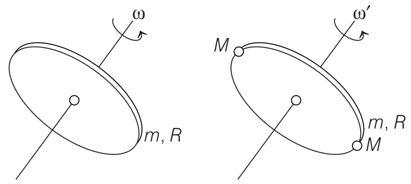
$\therefore$ Solid cylinder will reach the bottom first. Further, in case pure rolling on stationary ground, work done by friction is zero. Therefore, mechanical energy of both the cylinders will remain constant.

$\therefore (\text{KE})_{\text{Hollow}} = (\text{KE})_{\text{Solid}}$

$= \text{decrease in PE} = mgh$

SESSION 2

1 As, no external torque is acting on the system, angular momentum of system remains conserved.

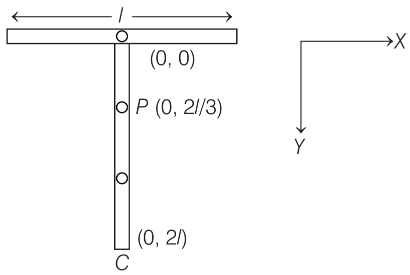


i.e. $I_1\omega = I_2\omega'$

$\Rightarrow \quad mR^2\omega = (mR^2 + 2MR^2)\omega'$

$\Rightarrow \quad \omega' = \left(\frac{m}{m + 2M}\right)\omega$

2 For pure translatory motion, net torque about centre of mass should be zero. Thus, **F** is applied at centre of mass of system.



$$OP = \frac{m_1 \times 0 + m_2 \times l}{m_1 + m_2}$$

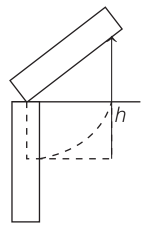
where, m_1 and m_2 are masses of horizontal and vertical section of the object. Assuming object is uniform,

$$m_2 = 2m_1$$

$$\Rightarrow \quad OP = \frac{2l}{3}$$

$$\therefore \; PC = \left(l - \frac{2l}{3} + l\right) = \left(2l - \frac{2l}{3}\right) = \frac{4l}{3}$$

3 If centre of mass rises to a maximum height, then loss in KE = Gain in PE, we get



$$\frac{1}{2}I\omega^2 = mgh$$

$$\frac{1}{2} \times \frac{1}{3}ml^2\omega^2 = mgh \Rightarrow h = \frac{1}{6}\frac{l^2\omega^2}{g}$$

4 $\frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = mg\left(\frac{3v^2}{4g}\right)$

$\therefore \qquad I = \frac{1}{2}mR^2$

Therefore, the body is a disc.

5 By conservation of angular momentum,

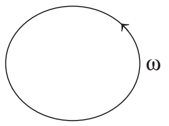
$I_1\omega_1 = I_2\omega_2$ ...(i)

where, $I_1 = mR^2$ ...(ii)

and $I_2 = 2mR^2 + MR^2$ ...(iii)

Now change in,

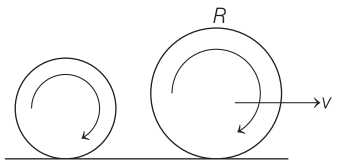
$\text{KE} = \frac{1}{2}L_1 I_1^2 - \frac{1}{2}L_2 I_2^2$ ...(iv)



Substituting the values from Eqs. (i), (ii) and (iii) into Eq. (iv), we get,

Change in KE $= \left(\frac{Mm}{M + 2m}\right)\omega^2 R^2$

6

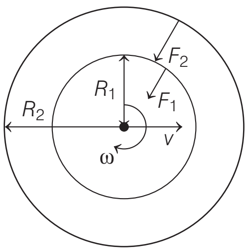


From conservation of angular momentum

$$mr^2\omega_0 = mvr + mr^2 \times \frac{v}{r}$$

$$\Rightarrow \qquad v = \frac{\omega_0 r}{2}$$

7 Since ω is constant, v would also be constant. So, no net force or torque is acting on ring. The force experienced by any particle is only along radial direction, or we can say the centripetal force.

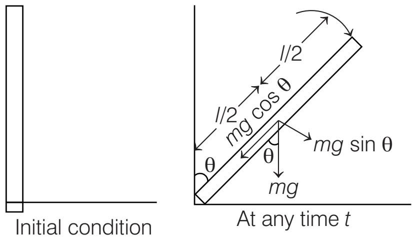


The force exprienced by inner part, $F_1 = m\omega^2 R_1$ and the force experienced by outer part,

$F_2 = m\omega^2 R_2$

$$\frac{F_1}{F_2} = \frac{R_1}{R_2}$$

8 As the rod rotates in vertical plane so a torque is acting on it, which is due to the vertical component of weight of rod.



Now, torque τ = force × perpendicular distance of line of action of force from axis of rotation

$$= mg \sin\theta \times \frac{l}{2}$$

Again, torque $\tau = I\alpha$

where, I = moment of inertia

$$= \frac{ml^2}{3}$$

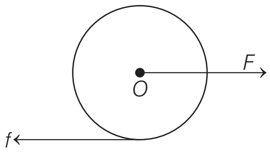
[Force and torque frequency along axis of rotation passing through in end]

α = angular acceleration

$$\therefore \quad mg \sin\theta \times \frac{l}{2} = \frac{ml^2}{3}\alpha$$

$$\therefore \qquad \alpha = \frac{3g \sin\theta}{2l}$$

9 Here, $F - f = ma$ ... (i)



and $\tau = I\alpha = I\left(\frac{a}{R}\right)$

$$\Rightarrow \quad f.R = \frac{2}{5}mR^2 \times \frac{a}{R}$$

or $f = \frac{2}{5}ma$ ...(ii)

From Eqs. (i) and (ii), we get

$$f = \frac{2}{7}F$$

or $\frac{2}{7}F \le \mu mg$

or $F \le \frac{7}{2}\mu mg$

Hence, (b) is the correct option.

10 Since, $a = \frac{g\sin\theta}{1 + I/mR^2}$

or $a = \frac{g\sin\theta}{1 + \frac{1}{2}}$;

[For uniform solid disc, $I = \frac{1}{2}mR^2$]

$$= \frac{2}{3}g\sin\theta$$

or $g\sin\theta = \frac{3}{2}a$

Now, force of friction $(f) = \frac{mg\sin\theta}{1 + \frac{mR^2}{I}}$

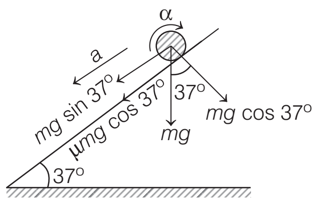
or $f = \frac{m(3a/2)}{1+2} = \frac{1}{2}ma$

Hence, (d) is the correct option.

11 $a = (\mu g\cos\theta) + (g\sin\theta)$

$= 0.5 \times 10 \times 0.8 + 10 \times 0.6$

or $a = 10\,\text{m/s}^2$



And angular acceleration (α)

$$= \frac{(\mu\ mg\cos\theta)R}{\frac{1}{2}mR^2}$$

or $\alpha = \frac{2\mu g\cos\theta}{R}$

or $\alpha = \frac{2 \times 0.5 \times 10 \times 0.8}{0.4} = 20\,\text{rad/s}^2$

Pure rolling will start, when

$v = R\omega$

where, $v = a \times t$ and $\omega = (\omega_0 - \alpha t)$

$\therefore \quad at = R(\omega_0 - \alpha t)$

or $10t = 0.4(54 - 20t)$

or $t = 1.2$ s

Hence, (d) is the correct option.

12 Linear acceleration of cylinder (B) is given by the relation

$$a = \frac{mR^2}{I + m'R^2} \cdot g$$

where, $I = \frac{1}{2}mR^2$ (for solid cylinder) and

$m' = 2m$

$$\therefore \quad a = \frac{mR^2}{\frac{1}{2}mR^2 + 2mR^2} \cdot g$$

$$= \frac{2mR^2}{5mR^2} \cdot g = \frac{2g}{5}$$

∴ Angular acceleration (α) of the cylinder B is

$$\alpha = \frac{a}{R} \text{ or } \alpha = \frac{2g}{5R}$$

Hence, (b) is the correct option.